Title: Algebraic construction of interacting QFT models on causal sets

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Series: Quantum Gravity

Date: August 22, 2019 - 2:30 PM

URL: http://pirsa.org/19080083

Abstract: In this talk I will report on recent progress in building QFT models on causal sets. The framework I'm using is that of perturbative algebraic quantum field theory (pAQFT). It was developed for rigorous study of perturbative QFT in the continuum, but can also be applied in the situation where spacetime is replaced by a discrete structure. Causality plays a key role in pAQFT, so it is natural to apply it to causets. Construction of models in this framework is purely algebraic and one does not need to specify a state or a Hilbert space representation until the final step, in which correlation functions are computed.

Algebraic construction of interacting QFT models on causal sets

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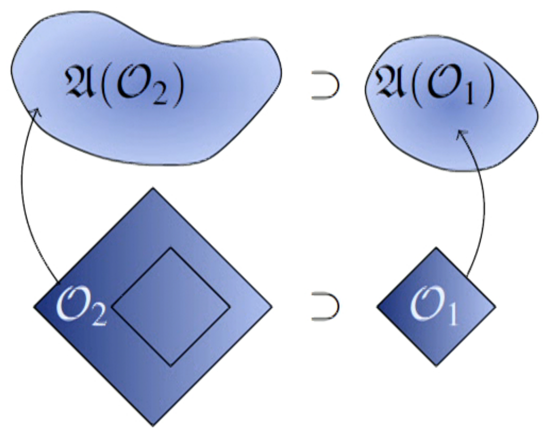
PI, 22.08.2019

[1] Based on joint work with Edmund Dable-Heath, Christopher J. Fewster and Nick Woods.

Outline of the talk

Algebraic quantum field theory

- A convenient framework to investigate conceptual problems in QFT is the Algebraic Quantum Field Theory.
- It started as the axiomatic framework of Haag-Kastler: a model is defined by associating to each region $\mathcal{O}$ of Minkowski spacetime an algebra $\mathfrak{A}(\mathcal{O})$ of observables that can be measured in $\mathcal{O}$.
- The physical notion of subsystems is realized by the condition of isotony, i.e.: $\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$. We obtain a net of algebras.



Algebraic QFT on curved spacetimes

- Algebraic approach generalizes also to curved spacetimes. Key feature: construction of the algebra of observables is separated from the choice of a state.
- The corresponding generalization of AQFT is called locally covariant quantum field theory [Hollands-Wald 01, Brunetti-Fredenhagen-Verch 01, Fewster-Verch 12,...].
- For construction of models it is convenient to use perturbative methods: pAQFT [Epstein-Glaser 73, Dütsch-Fredenhagen 00, Brunetti-Fredenhagen 00, Brunetti-Dütsch-Fredenhagen 09, Hollands 08, Fredenhagen-KR 11,...]

Algebraic QFT on causal sets

- One can replace the smooth manifold with a discrete set of points equipped with the causal order relation and the main ideas of pAQFT carry over.
- This has been done in our recent work: *Algebraic Classical and Quantum Field Theory on Causal Sets*, Edmund Dable-Heath, Christopher J. Fewster, KR, Nick Woods, [arXiv:1908.01973].
- Let $(\mathcal{C}, \preceq)$ be a discrete set of points $\mathcal{C}$ with a relation $\preceq$:

$$\begin{aligned} x \preceq y \preceq z &\implies x \preceq z, && \textit{transitivity} \\ x \preceq y \text{ and } y \preceq x &\implies x = y, && \textit{acyclicity} \\ |I(x,y)| &< \infty, && \textit{local finiteness} \end{aligned}$$

where

$$I(x,y) = \{z \in \mathcal{C} \mid x \preceq z \preceq y\}$$

and we write $x \prec y$ if $x \preceq y$ and $x \neq y$.

1. Algebraic QFT and its generalizations

2. Scalar fields on causal sets
   - Classical theory
   - Free quantum theory
   - Interaction

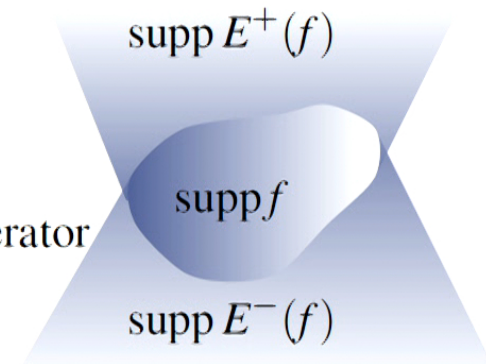
Physical input

- A fixed causal set $(\mathcal{C}, \preceq)$.
- Scalar field configuration space $\mathcal{E}$: choice of objects we want to study in our theory, $\mathcal{E}$ consists of maps $\phi : \mathcal{C} \to \mathbb{R}$,
- For a finite causal set of cardinality N, $\mathcal{E} = \mathbb{R}^N$.
- Observables: $\mathcal{F} = \mathcal{C}^\infty(\mathcal{E}, \mathbb{C})$, i.e. functionals on the configuration space.
- Free dynamics: a discretized retarded Green function E^+ (ideally coming from a discretization of some normally hyperbolic operator).
- Interaction: we use a modification of the Lagrangian formalism (fully covariant), where the choice of interaction is realized as the choice of some $V \in \mathcal{F}$.

Green operators in the continuum

- Let M be a globally hyperbolic spacetime, and $\mathcal{E}(M) = \mathcal{C}^\infty(M, \mathbb{R})$.
- For the free scalar field the equation of motion is $P\varphi = 0$, where $P = -(\Box + m^2)$ is (minus) the Klein-Gordon operator.
- For globally hyperbolic M, P admits retarded and advanced Green's operators E^+, E^-. They satisfy: $P \circ E^\pm = \mathrm{id}_{\mathcal{D}(M)}$, $E^\pm \circ (P\big|_{\mathcal{D}(M)}) = \mathrm{id}_{\mathcal{D}(M)}$ and

$$\mathrm{supp}(E^+) \subset \{(x,y) \in M^2 | x \in J^+(y)\},$$
$$\mathrm{supp}(E^-) \subset \{(x,y) \in M^2 | x \in J^-(y)\}.$$



- Their difference is the Pauli-Jordan operator

$$E \doteq E^+ - E^-.$$

Discretized Green operators I

- We focus on finite causal sets (neglecting edge effects for the moment) and study equations taking the form
$$P\phi = Kf\,,$$
where $f, \phi \in \mathbb{R}^N$ are the source and solution respectively.
- The map K provides additional freedom to determine the way in which a continuum source is discretized.
- In agreement with [Aslanbeigi-Saravani-Sorkin 2014], we require P to be a retarded operator, i.e. $(P\phi)_p$ is a linear combination of ϕ_q with $q \preceq p$. Similarly for K.
- We require both P and K to be real.
- In the natural labeling, P is lower triangular and its diagonal entries are all nonvanishing. Consequently, P is invertible.
- The retarded Green operator is then
$$E^+ \doteq P^{-1}K\,.$$

Discretized Green operators II

- We define the advanced Green operator to be

$$E^- \doteq (E^+)^T ,$$

and the Pauli-Jordan operator is the anti-symmetric matrix

$$E = E^- - E^+ = (E^+)^T - E^+ .$$

- As a specific example of P, we recall the discretized d'Alembertian proposed in [Sorkin 09]:

$$(P_S)_{pq} = \begin{cases} 1, & p = q \\ -2, 4, -2, \ p \neq q, & n(p,q) = 1, 2, 3 \text{ respectively} \\ 0, & \text{otherwise,} \end{cases}$$

where $n(p,q) = |I(q,p)| - 1$.
- Another proposal (using some extra structure called "preferred past") can be found in our paper.

The Peierls bracket

- We define the Poisson bracket of the free theory as

$$\{F,G\} \doteq \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\delta F}{\delta \phi_i} E^{ij} \frac{\delta G}{\delta \phi_j} \equiv F_{,i} E^{ij} G_{,j} \equiv (F^{(1)})^T E G^{(1)} ,$$

where we used the Euclidean inner product to raise one of the indices in $E_i^{\ j}$.

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  where we used the Euclidean inner product to raise one of the indices in $E_i^{\ j}$.
- We have shown that this agrees with the idea of Peierls to define the canonical bracket as the difference between the retarded and advanced response.

Future and past inifinity

- Following [Sorkin 09], we introduce layers:

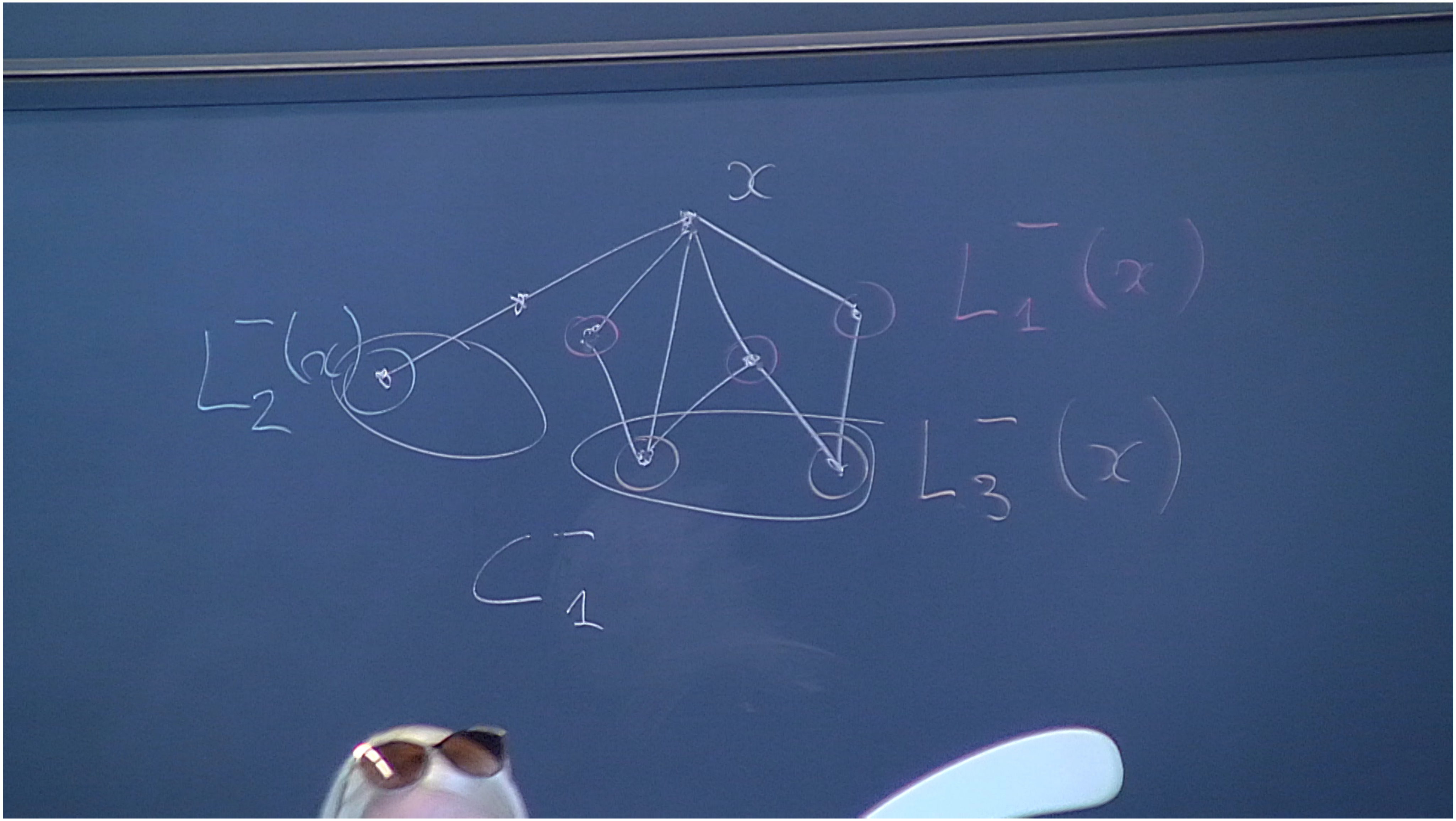
$$L_i^-(x) \doteq \{y \in \mathcal{C} \mid y \prec x,\ n(x,y) = i\},$$

and dual layers using the reversed order:

$$L_i^+(x) \doteq \{y \in \mathcal{C} \mid y \succ x,\ n(y,x) = i\}.$$

- The n-layer *past infinity* C_n^- is defined by

$$C_n^- \doteq \{x \in \mathcal{C} \mid L_i^-(x) = \emptyset, \forall i \geq n\}.$$



x
$L_1^-(x)$
$L_2^-(x)$
$L_3^-(x)$
C_1^-

Edge effects

- Define the projection to future/past infinity as

$$(S_n^{\pm})_{xx} = \begin{cases} 1, & \text{if } x \in C_n^{\pm} \\ 0, & \text{otherwise}. \end{cases}$$

- For a discretized d'Alembertian that uses the data from n layer-past, the edge effects in a finite causal set are taken into account by using:

$$P\phi = Kf + \phi^- ,$$

where $\phi^- = S_n^- \phi$ is the projection of ϕ onto past infinity and the source f vanishes in C_n^-.

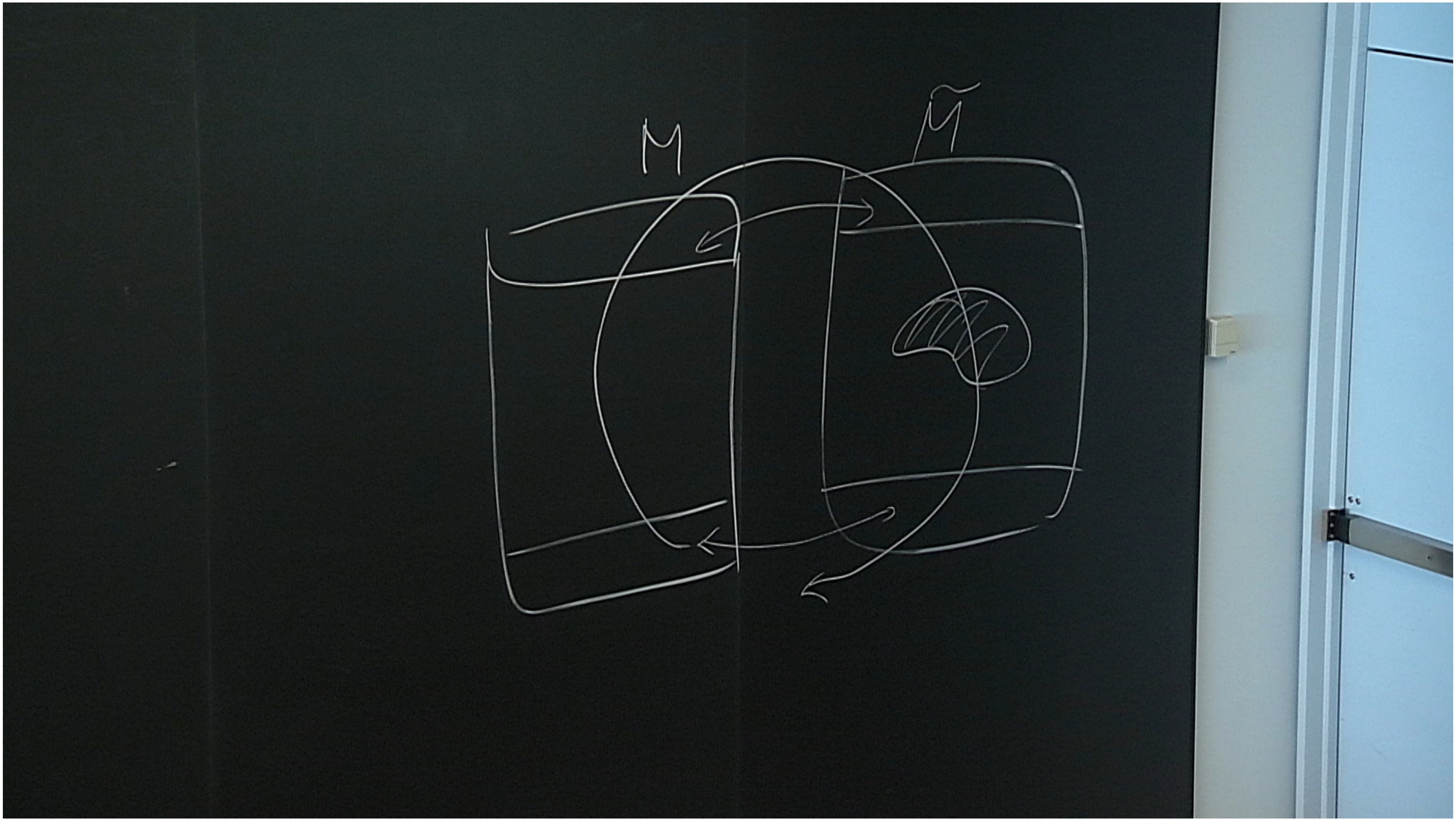
- For $f = 0$, we have the source-free equation with Cauchy data ϕ^- and the solution is $\phi = E^+ \phi^-$.
- The solution space is $\mathcal{E}^+_{Sol} \doteq E^+ \mathcal{E}(C_n^-)$.

Relative Cauchy evolution

- A particularly simple situation occurs if the solutions are also in bijection with data on future infinity.
- In this case $\alpha^+ = S_n^+ E^+|_{\mathcal{E}(C_n^-)}$ is an iso $\alpha^+ : \mathcal{E}(C_n^-) \to \mathcal{E}(C_n^+)$, called the Cauchy evolution.
- For the existence of Cauchy evolution, we need in particular that $C_n^\pm$ have equal cardinality.
- We can use the Cauchy evolution to compare the dynamics of the theory on two causal sets $\mathcal{C}$ and $\widetilde{\mathcal{C}}$, provided we have isomorphisms: $\iota^\pm : \mathcal{E}(C_n^\pm) \to \mathcal{E}(\widetilde{C}_n^\pm)$.
- The relative Cauchy evolution is a linear isomorphism on the solution space $\mathcal{E}^+_{Sol}(\mathcal{C})$ defined by

$$\mathtt{rce}(\phi) \doteq E^+(\iota^-)^{-1}(\widetilde{\alpha}^+)^{-1}\iota^+ S_k^+ \phi\,,$$

- In the continuum, `rce` allows one to characterize the dynamics and reconstruct the stress-energy tensor.


M
M̃

Interacting equations of motion

- Let $\mathcal{C}$ be finite with $|\mathcal{C}| = N$. Take $V \in \mathcal{F} = \mathcal{C}^\infty(\mathcal{E}, \mathbb{C})$ and let λ be the coupling constant.
- We work perturbatively, so the space of observables is now extended to include formal power series in the coupling constant λ, i.e. it becomes $\mathcal{F}[[\lambda]]$.
- The interacting field equations are given by

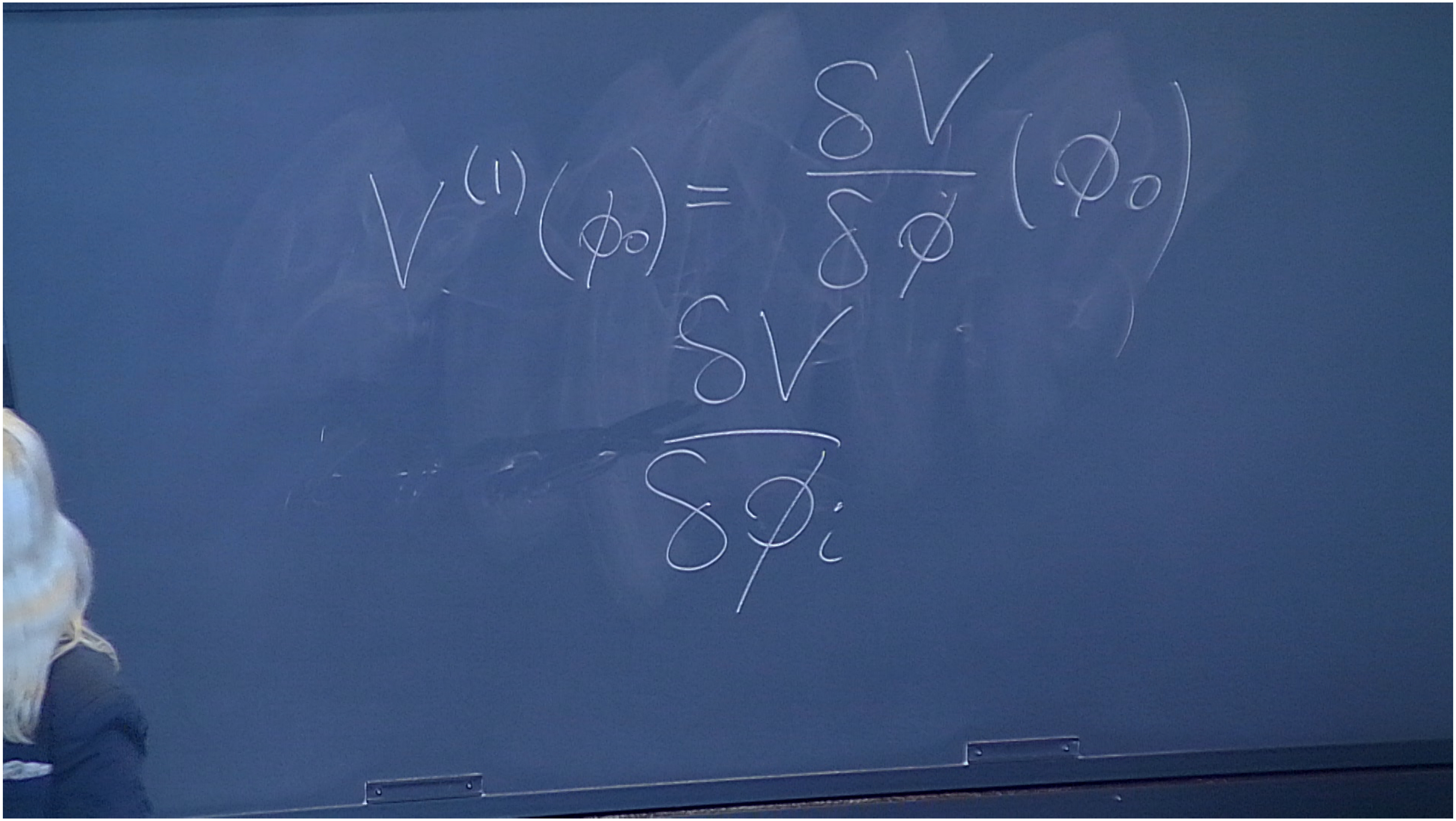
$$P\phi + \lambda K(V^{(1)}(\phi)) = \phi^- \,.$$

- The interacting field equations linearized about $\phi \in \mathcal{E}$, are

$$P\psi + \lambda K V^{(2)}(\phi)\psi = \psi^- \,,$$

where $V^{(2)}(\phi)$ is an $N \times N$ matrix with components

$$(V^{(2)}(\phi))_{ij} = V_{,ij}(\phi) \,.$$


$V^{(1)}(\phi_0) = \frac{\delta V}{\delta \phi}(\phi_0)$
$\frac{\delta V}{\delta \phi_i}$

Interacting Poisson bracket

- The interacting Poisson bracket is given by

$$\{G,H\}_{\lambda V} \doteq G_{,i} E_{\lambda V}(\phi)^{ij} H_{,j}\,,$$

where $E_{\lambda V}(\phi) = E^{+}_{\lambda V}(\phi) - (E^{+}_{\lambda V}(\phi))^{T}$, and $E^{+}_{\lambda V}(\phi)$ is the retarded Green function for the linearized field equations.
- Starting from E^{+}, we construct the interacting one using:

$$E^{+}_{\lambda V} = E^{+} + \sum_{n=1}^{\infty} (-\lambda)^{n} E^{+} \left(V^{(2)} E^{+}\right)^{n}\,.$$

- We introduce the retarded classical Møller map:

$$\mathtt{r}_{\lambda V}(\phi) = \phi - \lambda E^{+} V^{(1)}(\mathtt{r}_{\lambda V}(\phi))\,,$$

- which induces $(r_{\lambda V} F)(\phi) \doteq F \circ \mathtt{r}_{\lambda V}(\phi)$ on the observables.
- Analogously to the continuum case, the Peierls bracket satisfies:

$$\{F,G\}_{\lambda V} = r_{\lambda V}^{-1}\{r_{\lambda V}F, r_{\lambda V}G\}\,.$$

The Moyal–Weyl $\star$-product

- Restrict to the subspace of $\mathcal{F}$ that consists of smooth functionals F such that $F^{(n)}(\varphi) = 0$ for all $n > N$, $\varphi \in \mathcal{E}$. Denote it by $\mathcal{F}_{\mathrm{pol}}$.
- Define the Moyal–Weyl product as

$$F \star G \doteq m \circ e^{\frac{1}{2} i\hbar D_E} (F \otimes G) \,,$$

where $F, G \in \mathcal{F}_{\mathrm{pol}}$, m is the pointwise multiplication and for a given $N \times N$ matrix K:

$$D_K \doteq K_{ij} \frac{\delta}{\delta \phi_i} \frac{\delta}{\delta \phi_j} \equiv \left\langle K, \tfrac{\delta}{\delta \phi} \otimes \tfrac{\delta}{\delta \phi} \right\rangle \,,$$

- We obtain a non-commutative algebra $\mathfrak{A} \doteq (\mathcal{F}_{\mathrm{pol}}, \star)$, which is the analogue of the continuum off-shell algebra.

Digression: Commutation relations in continuum

- **Smeared fields**: Let $\mathcal{D}(M) = \mathcal{C}_c^\infty(M, \mathbb{R})$ and $f, f' \in \mathcal{D}(M)$.

$$\Phi_f(\varphi) \doteq \int f(x)\varphi(x)d\mu_g(x), \quad \Phi_{f'}(\varphi) \doteq \int f'(x)\varphi(x)d\mu_g(x)$$

- $[\Phi_f, \Phi_{f'}]_\star = \Phi_f \star \Phi_{f'} - \Phi_{f'} \star \Phi_f = i\hbar \langle \Delta, f \otimes f' \rangle$.
- Formally, we can consider $\Phi_x \doteq \Phi(\delta_x)$, where δ_x is the Dirac delta supported at some $x \in M$.
- for $M = \mathbb{M}$ (Minkowski spacetime):
  $[\Phi_{(0,\mathbf{x})}, \Phi_{(0,\mathbf{y})}]_\star = \Delta(0, \mathbf{x}; 0, \mathbf{y}) = 0$.
  $[\Phi_{(0,\mathbf{x})}, \dot{\Phi}_{(0,\mathbf{y})}]_\star = \partial_{y^0}\Delta(0, \mathbf{x}; 0, \mathbf{y}) = i\hbar\delta(\mathbf{x} - \mathbf{y})$, where dot denotes the time derivative.

The Wick product

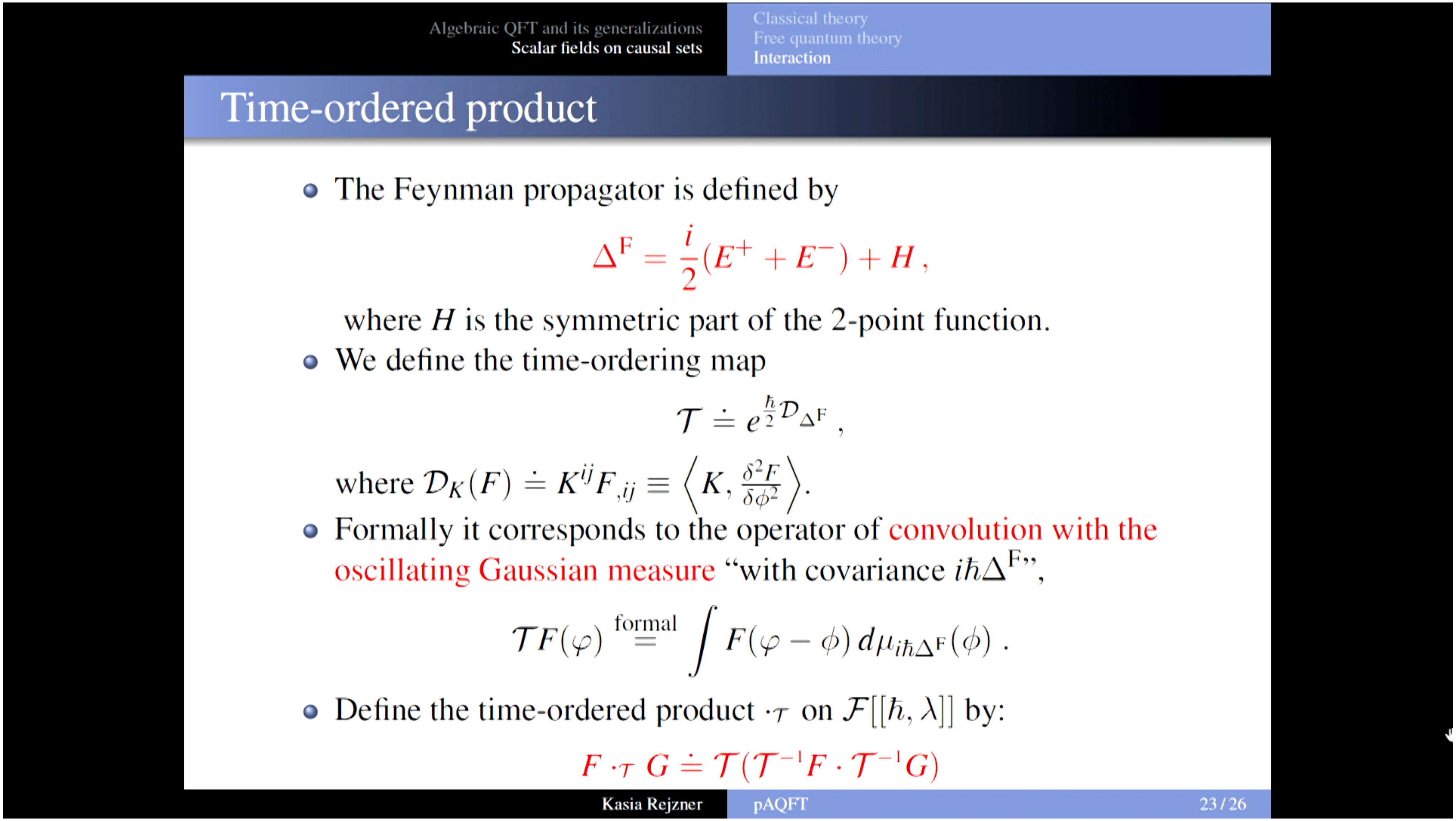
- The Wick product is defined by

$$F \star_H G \doteq m \circ e^{\hbar D_W}(F \otimes G)\,,$$

  where $W = \frac{i}{2}E + H$ is a complex hermitian matrix, interpreted as the two-point function of a quasifree state on $\mathfrak{A}$.
- Denote $\mathfrak{A}_H \doteq (\mathcal{F}_{\mathrm{pol}}, \star_H)$.
- We require W to have the following properties
  - **W1** $E = 2\,\mathrm{Im}\,W$, i.e., $H = \mathrm{Re}\,H$ (recall that E is real by definition).
  - **W2** W is a positive definite matrix, meaning that $f^\dagger Wf \geq 0$, where $f^\dagger$ is the hermitian conjugate of $f \in \mathbb{C}^N$.
  - **W3** $\ker W \subset \ker E$ (a proxy for W solving the equations of motion)
- Physically, passing from $\star$ to $\star_H$ corresponds to normal-ordering with respect to the quasifree state determined by W.

SJ axioms

- In order to find a specific choice of W, we will follow the ideas of [Johnston 09, Afshordi-Aslanbeigi-Sorkin 12] and take W as the Sorkin-Johnston (SJ) two-point function.
- Sorkin's W is the unique $N \times N$ matrix satisfying the following properties:
  - SJ 1 $W - \overline{W} = iE$, where bar denotes the complex conjugation,
  - SJ 2 $W \geq 0$,
  - SJ 3 $\overline{W}W = 0$.
- It was shown by Sorkin that the unique W satisfying the axioms above is given by $W = \frac{1}{2}(iE + \sqrt{-E^2})$, where the square root is the unique positive semi-definite square root of the positive semi-definite matrix $(iE)^2 = (iE)(iE)^\dagger$.


Algebraic QFT and its generalizations
Scalar fields on causal sets
Classical theory
Free quantum theory
Interaction
Time-ordered product
The Feynman propagator is defined by
$$\Delta^{\mathrm{F}} = \frac{i}{2}(E^+ + E^-) + H\,,$$
where H is the symmetric part of the 2-point function.
We define the time-ordering map
$$\mathcal{T} \doteq e^{\frac{\hbar}{2}\mathcal{D}_{\Delta^{\mathrm{F}}}}\,,$$
where $\mathcal{D}_K(F) \doteq K^{ij}F_{,ij} \equiv \left\langle K, \frac{\delta^2 F}{\delta\phi^2}\right\rangle$.
Formally it corresponds to the operator of convolution with the oscillating Gaussian measure “with covariance $i\hbar\Delta^{\mathrm{F}}$”,
$$\mathcal{T}F(\varphi) \stackrel{\text{formal}}{=} \int F(\varphi - \phi)\, d\mu_{i\hbar\Delta^{\mathrm{F}}}(\phi)\,.$$
Define the time-ordered product $\cdot_{\mathcal{T}}$ on $\mathcal{F}[[\hbar,\lambda]]$ by:
$$F \cdot_{\mathcal{T}} G \doteq \mathcal{T}(\mathcal{T}^{-1}F \cdot \mathcal{T}^{-1}G)$$
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Formal S-matrix

- More explicitly:

$$F \cdot_{\mathcal{T}} G \doteq m \circ e^{\frac{i\hbar}{2} D_{\Delta^{\mathrm{F}}}} (F \otimes G) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} F_{,i_1 \dots i_n} (\Delta^{\mathrm{F}})^{i_1 j_1} \dots (\Delta^{\mathrm{F}})^{i_n j_n} G_{,j_1 \dots j_n} \,.$$

- The formal S-matrix for the interaction V and coupling constant λ is now given by

$$\mathcal{S}(\lambda V) \doteq e_{\mathcal{T}}^{\frac{i}{\hbar}\lambda V} = \sum_{n=0}^{\infty} \frac{\lambda^n i^n}{\hbar^n n!} \underbrace{V \cdot_{\mathcal{T}} \dots \cdot_{\mathcal{T}} V}_{n} \,.$$

- For a functional $F \in \mathcal{F}$, the corresponding quantum interacting field is given by $R_{\lambda V}(F)$, where $R_{\lambda V}$ is the retarded quantum Møller operator defined by

$$R_{\lambda V}(F) \doteq -i\hbar \frac{d}{d\mu} \mathcal{S}(\lambda V)^{-1} \star_H \mathcal{S}(\lambda V + \mu F)\big|_{\mu=0} \,.$$

Interacting star product

- We can also use the Møller operator to deform the free star product and obtain the interacting one, using the formula:

$$F \star_{H,\mathrm{int}} G \doteq R_{\lambda V}^{-1}(R_{\lambda V}(F) \star_H R_{\lambda V}(G))\,.$$

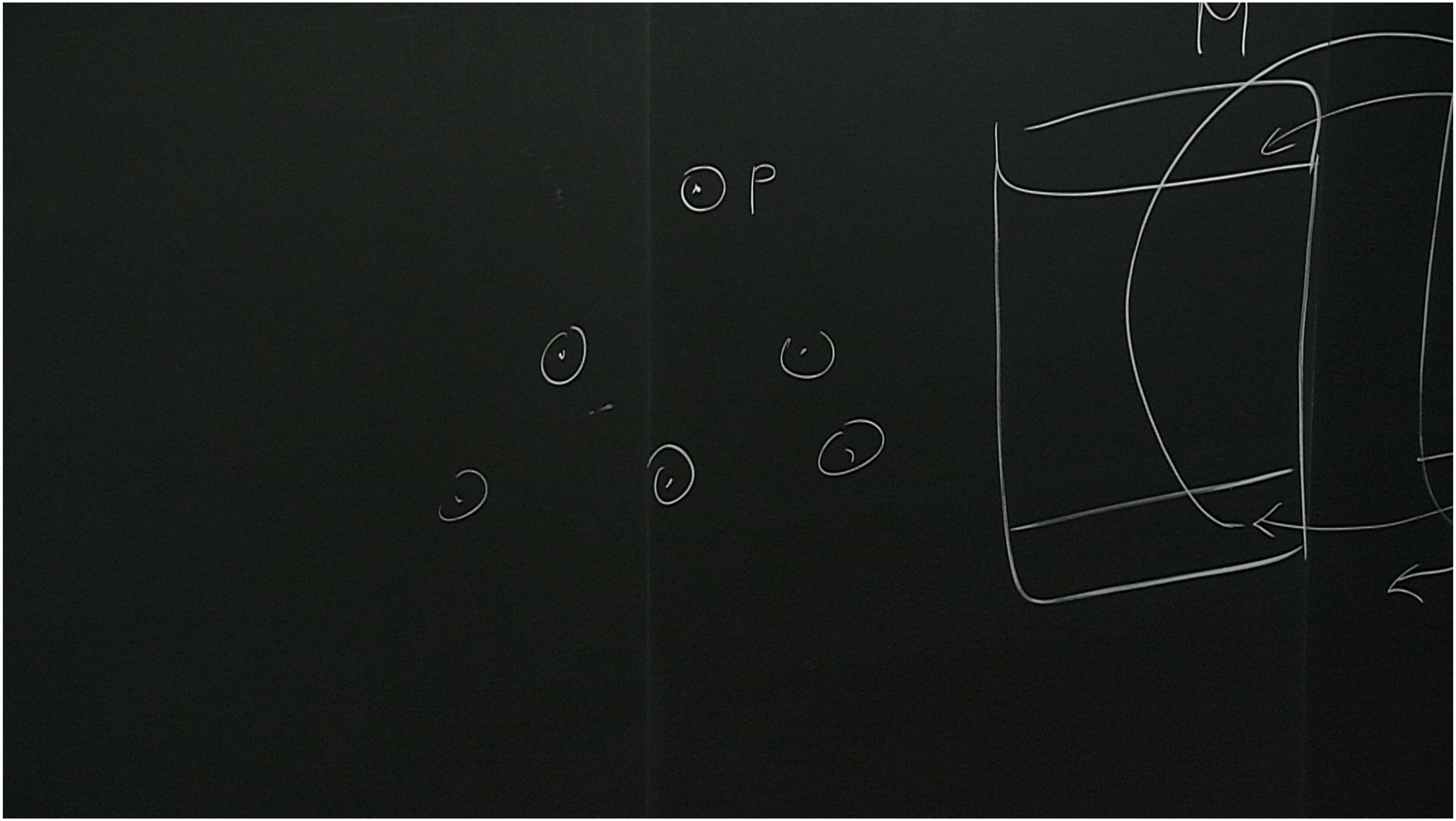
- This way we obtain the interacting algebra $\mathfrak{A}_H^{\mathrm{int}} \doteq (\mathcal{F}[[\hbar, \lambda]], \star_{H,\mathrm{int}})$. Given a state ω on the free algebra, we can construct the state ω_{int} on $\mathfrak{A}_H^{\mathrm{int}}$ using the pullback:

$$\omega_{\mathrm{int}}(F) \doteq \omega \circ R_{\lambda V}(F)\,.$$

- The n-point correlation function of smeared interacting fields is given by:

$$\omega_{\mathrm{int}}(\Phi_{g_1} \star_{H,\mathrm{int}} \cdots {}_{H,\mathrm{int}} \Phi_{g_n}) = \omega(R_{\lambda V}(\Phi_{g_1}) \star_H \cdots \star_H R_{\lambda V}(\Phi_{g_n})) \\ = (R_{\lambda V}(\Phi_{g_1}) \star_H \cdots \star_H R_{\lambda V}(\Phi_{g_n}))(0)\,.$$

Thank you very much for your attention!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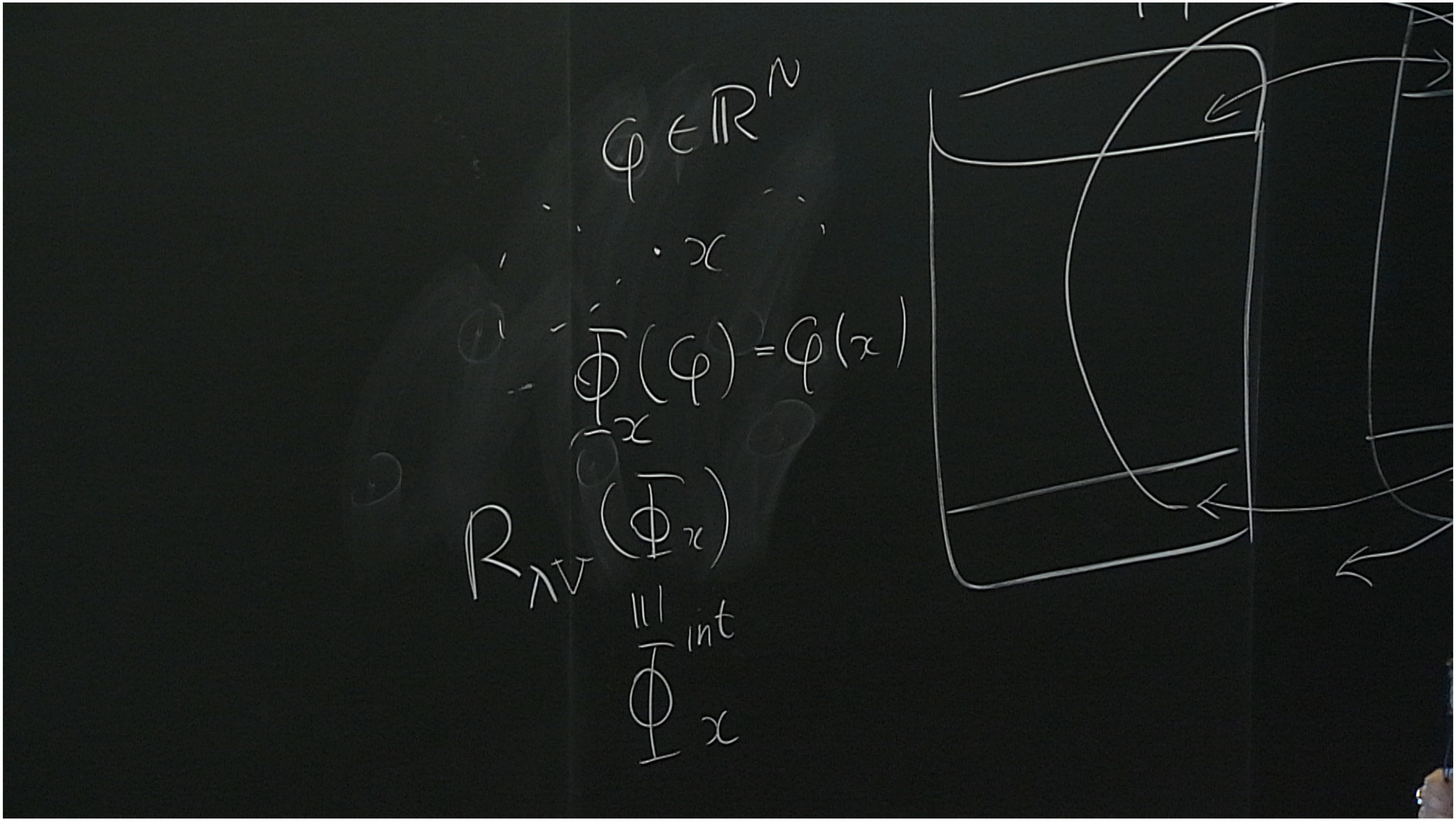

$\varphi \in \mathbb{R}^N$
$\cdot x$
$\Phi_x(\varphi) = \varphi(x)$
$R_{\Lambda V}(\Phi_x)$
$\cong_{int}$
Φ_x