Title: Quantum Work of an Optical Lattice and Boundary Field Theory

Speakers: Natan Andrei

Collection: Boundaries and Defects in Quantum Field Theory

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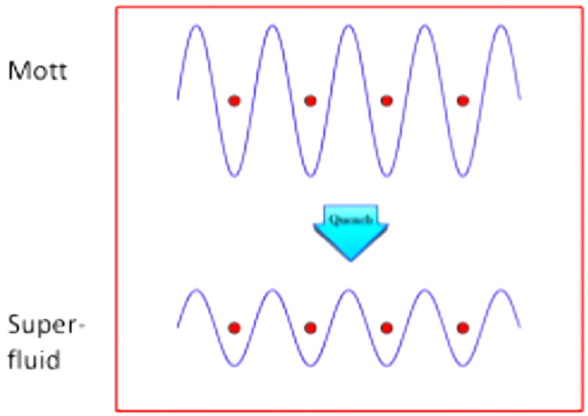
Abstract: We study the quantum work associated with the nonequilibrium quench of an optical lattice as it evolves from initial Mott type states with large potential barriers under the Sine-Gordon Hamiltonian that describes the dynamics of the system when the barriers are suddenly lowered. The calculations are carried out by means of the Boundary Bethe Ansatz approach where the initial and final states of the quench are applied as boundary states on the evolving system. We calculate exactly the Loschmidt amplitude, the fidelity and work distribution characterizing the quenches for different values of the interaction strength.

Quantum Work of an Optical Lattice
and
Boundaries in Field Theory

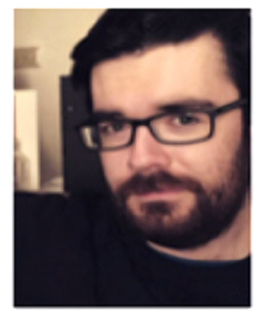
Quantum quench of an optical lattice



The Sine-Gordon model

Natan Andrei
Rutgers university

RUTGERS



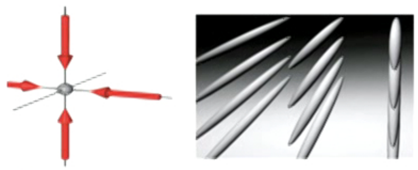
Colin Rylands
Maryland University

Boundaries and defects in Quantum Field Theory
Perimeter Institute - Aug, 2019

Non equilibrium dynamics of quantum systems - Quantum Quench of Optical Lattices

- Nonequilibrium – the new frontier : Old and new questions
  - Many experiments: cold atom systems, nano-devices, molecular electronics
  - Isolated systems – effects not washed out by coupling to environment
  - Fine control of parameters
  - Many systems described by integrable Hamiltonians

- Nonequilibrium Quench protocol: release bosons in optical traps



Bloch et al
2008

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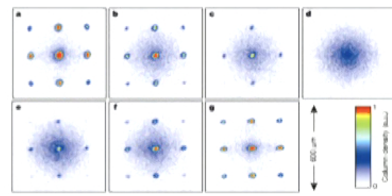
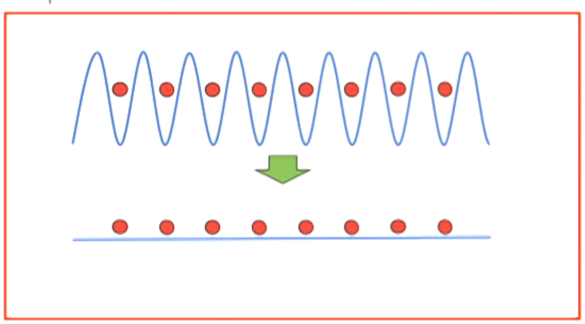
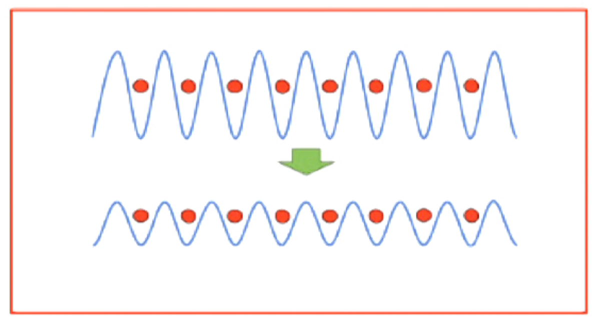


Mott



The Lieb-Liniger model

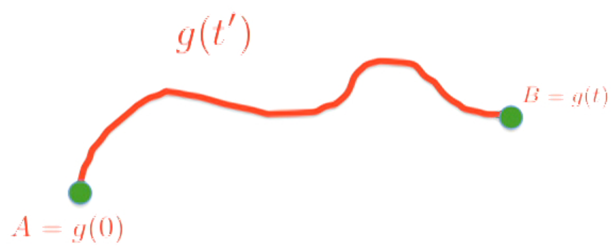


The sine-Gordon model

Time evolution - Quench protocol in isolated systems

Quench protocol

- **Isolated system** with Hamiltonian $H(t) = H[g(t)]$ depends on "work parameter" $g(t)$
- Initial state, $|\Phi_i\rangle$, typically ground state of $H(0) = H[A]$
- Evolve initial state under $H(t') = H[g(t')]$ from $t' = 0$ to $t' = t$ where



$$|\Phi_i, t\rangle = Te^{-i\int_0^t H(t')dt'}|\Phi_i\rangle \Longrightarrow |\Phi_i, t\rangle = e^{-iHt}|\Phi_i\rangle$$

Sudden quench

- Process depends on initial state and on Hamiltonian
- Local characteristics: calculate evolution of local observables, correlations ..
- Global characteristics: **quantum work,** spread of entanglement ..

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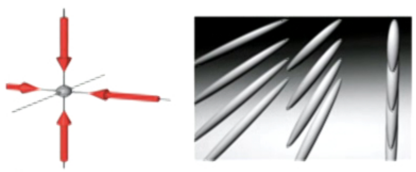

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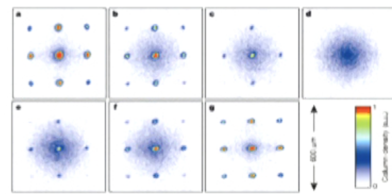

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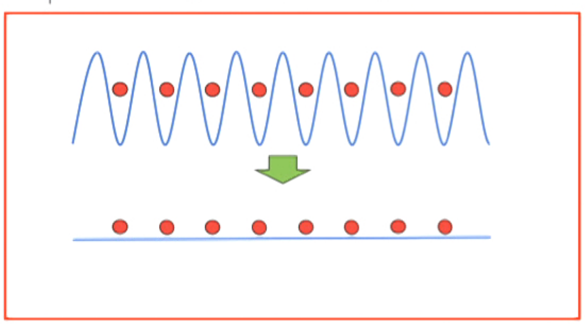

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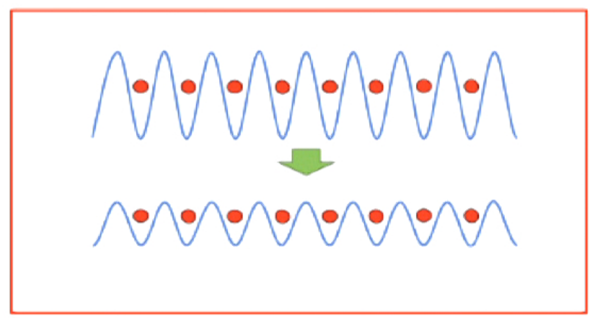

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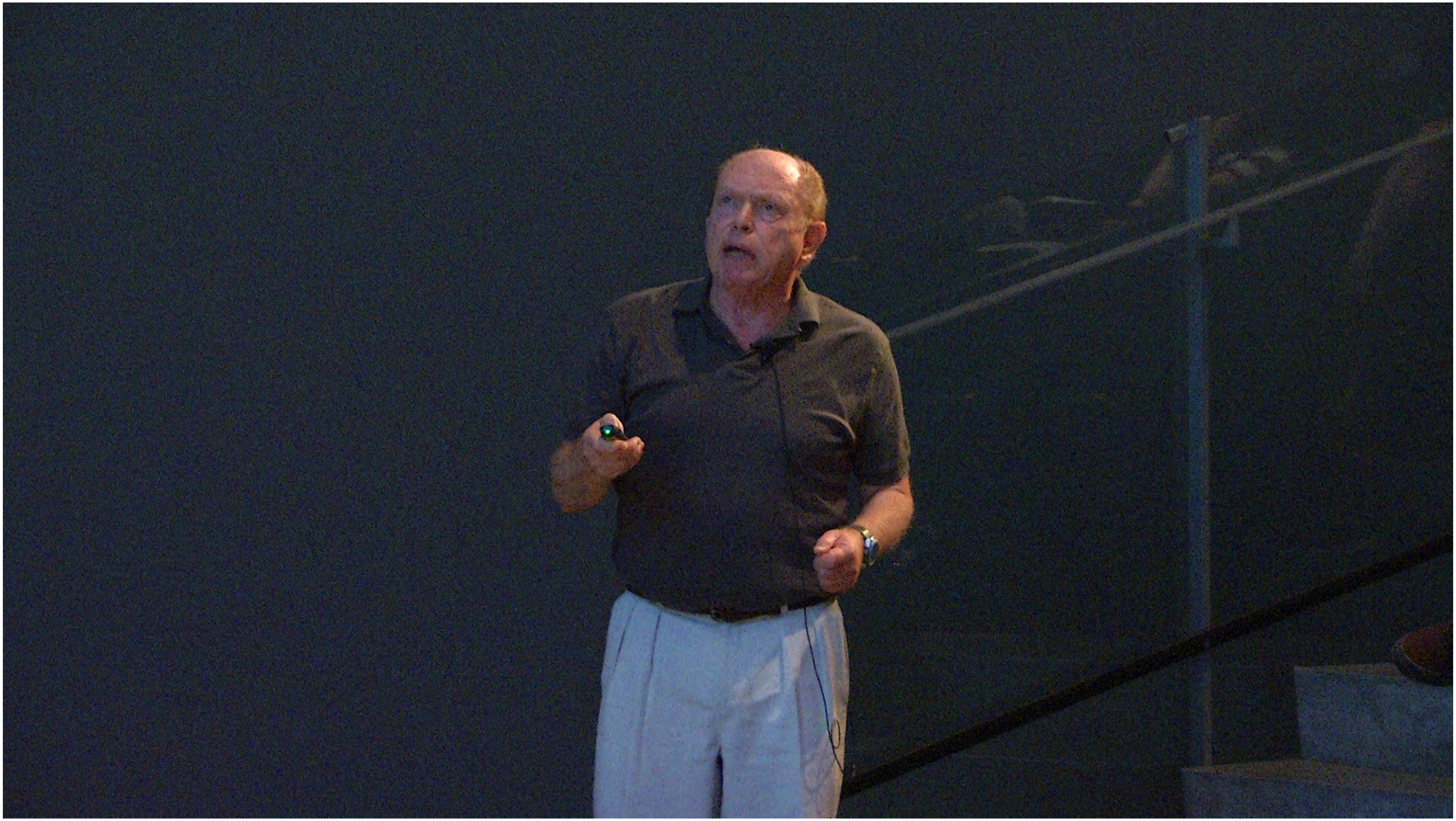

Non equilibrium dynamics of quantum systems - Quantum Quench of Optical Lattices

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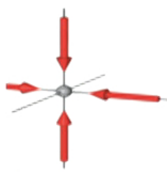

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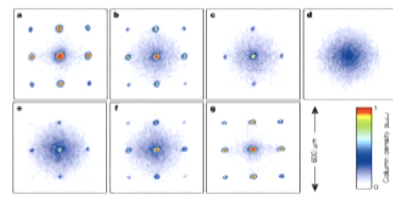

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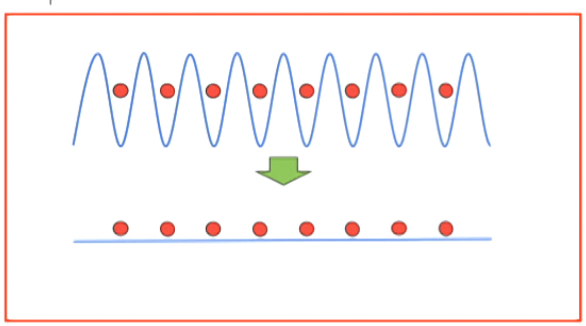

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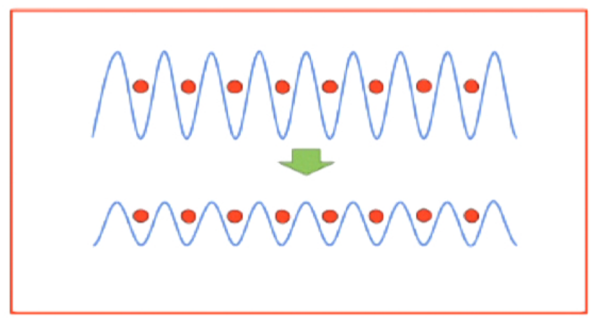

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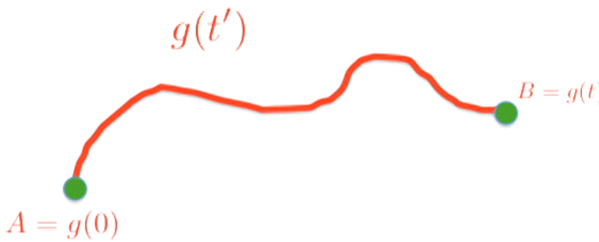

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Sudden quench

- Process depends on initial state and on Hamiltonian
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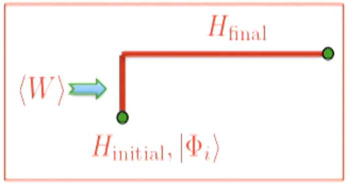
Nonequilibrium Thermodynamics -Work done in a quench

- During the quench energy is pumped into the system - $W = E_{\rm final} - E_{\rm initial}$: **work** is done

First law of thermo' $dE = dQ + dW$ but system isolated: $dQ = 0$, so $dE = dW$



Work – *random variable* : involves two measurements - initial and final energies,
- at initial time - yielding $|\Phi_i\rangle$ and ϵ_i with probability $P_i = 1$
- at final time - yielding $|\Psi_n\rangle$ and E_n with probability $P_{i\to n} = |\langle \Psi_n|\Phi_i\rangle|^2$

- For a sudden quench the work distribution: *(Talkner et al '07 "Not an observable")*

$$P(W) = \sum_n \delta\left(W - (E_n - \epsilon_i)\right) |\langle \Psi_n|\Phi_i\rangle|^2$$

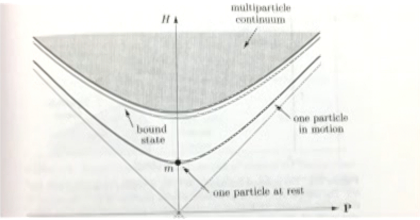
- The average is:

$$\langle W \rangle = Tr(H_{\rm f}\rho_{\rm f}) - Tr(H_{\rm i}\rho_{\rm i})$$

Nonequilibrium Thermodynamics -Work done in a quench

- $P(W)$ **has the form of the spectral function:**

Silva, Gambassi, Palmai, Sotiriadis, Mussardo, Calabrese, Goold .... : '08 –'18



- Defined for $W \geq \delta E = E_0 - \epsilon_i$

- A delta function at threshold $W = \delta E$
 weighted by the fidelity $\mathcal{F} = |\langle \Psi_0 | \Phi_i \rangle|^2$ transition from the initial state to the ground state of H_f

- If H_f is gapped and translation invariant - continuum of excited states into which $|\Phi_i\rangle$ can transition with $W = 2m + \delta E$ lowest threshold for the continuum.

- Power like behavior at threshold $P(W) \sim \theta(W - \delta E - 2m)(W - \delta E - 2m)^\alpha$

- Similarly the four-particle emission continuum threshold $W = 4m + \delta E$

- It there are bound states m_b then delta function may appear at $W = m_b + \delta E$

- **Work distribution directly measureable**

- ***Reversible and irreversible processes, entropy production, spread of entanglement***

- ***Fluctuation theorems*** : *Jarzynski '97* $\langle e^{-\beta W} \rangle = e^{-\beta \Delta F}$ *Cohen-Galavotti '95* $\frac{P(S)}{P(-S)} = e^{\frac{S}{k_B}}$
 Crooks '99 $\frac{P_F(W)}{P_B(-W)} = e^{\frac{W - \Delta F}{T}}$ *and more.*

 - relate equilibrium and nonequilibrium, relate forward and backward evolution

Nonequilibrium Dynamics – Loschmidt amplitude

- Claim: Work distribution is related to the Loschmidt Amplitude (*Talkner et al 2007*)

$$\mathcal{G}(t) = \langle \Phi_i | e^{-iHt} | \Phi_i \rangle$$

$$\Rightarrow \quad P(W) = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi} e^{iWt + i\epsilon_i t} \mathcal{G}(t)$$

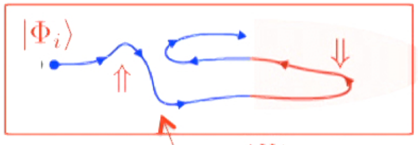
Related concepts:

Return Probability
Survival probability
Persistence
Scrambling

- The Loschmidt amplitude probes the full Hilbert space of states

$$\mathcal{G}(t) = \langle \Phi_i | e^{-iHt} | \Phi_i \rangle = \sum_n e^{-iE_n t} |\langle \Psi_n | \Phi_i \rangle|^2$$

starting from and weighted by overlaps with initial state $|\Phi_i\rangle$

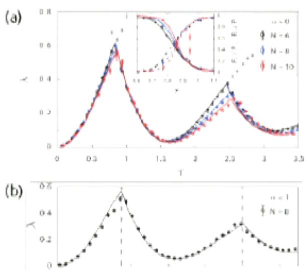


- May exhibit singularities Dynamical Quantum Phase Transitions DQPT *(Heyl '18)* some similarities to thermodynamic phase transitions, Fisher &Lee-Yang zeroes

$$\lambda(t) = \lim_{N\to\infty} \frac{1}{N} \log |\mathcal{G}(t)|^2$$

DQPT depends on initial state, not only on the Hamiltonian

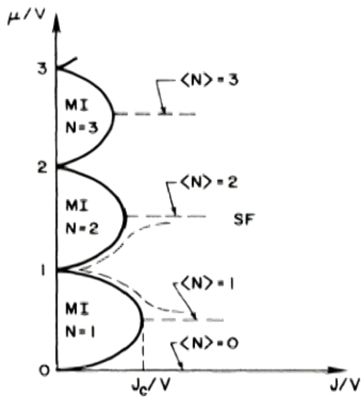
Observation of DQPT in Ising model



Jurevic et al '17

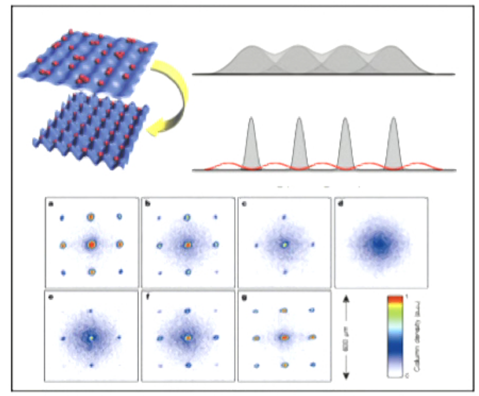
Quench of an optical lattice: *gapped scenario*

- Quench system from an initial Mott state, the ground state of very high barriers (BH model)



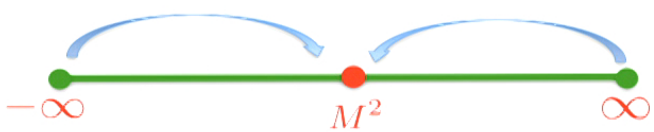
Fisher et al, 1989

Greiner et al '02 Bloch et al '08

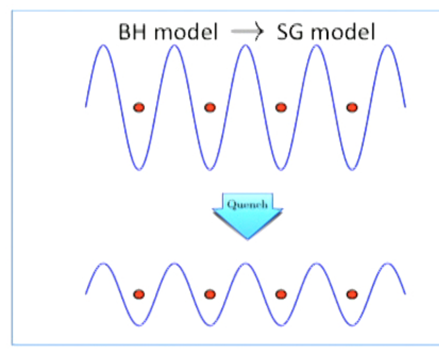


- Quench (*e.g.* SG as effective low-E theory of BHub)

$$H_{SG}(M^2 = \pm\infty, \beta) \to H_{SG}(M^2, \beta)$$



$$H_{SG}(M^2, \beta) = \frac{1}{2} \int \mathrm{d}x \left\{ \Pi^2(x) + [\partial\phi(x)]^2 - M^2 \cos[\beta\phi(x)] \right\}$$



Mott

SF

- Another realization: pair of coupled one-dimensional condensates of interacting atoms *(Gritsev et al '07)*

The Sine-Gordon / Massive Thirring model

- The Sine-Gordon model: **Quantum Integrable** (*Zamolodchikov^2 '1978)*

$$H_{SG}(M,\beta) = \frac{1}{2}\int \mathrm{d}x \left\{\Pi^2(x) + [\partial\phi(x)]^2 - M^2\cos[\beta\phi(x)]\right\}$$

- Low-energy effective action of many classical and quantum systems (e.g *Giamarchi book*)

Bose Hubbard model
Spin chains
Interacting bosons
Quantum impurity models
KT phase transitions

- It is equivalent to the massive Thirring model *(S. Coleman '78)*

$$H_{MTM}(m_0,g) = -i\int \left(\psi_+^\dagger(x)\partial_x\psi_+(x) - \psi_-^\dagger(x)\partial_x\psi_-(x)\right) + \\ + m_0\int\left(\psi_+^\dagger(x)\psi_-(x) + \psi_-^\dagger(x)\psi_+(x)\right) + 4g\int \psi_+^\dagger(x)\psi_-^\dagger(x)\psi_-(x)\psi_+(x)$$

Parameters are related in a non-universal way (depend on reno' scheme) $M,\beta \leftrightarrow m_0, g$

- Hamiltonian is **Quantum Integrable** (*Bergknoff and Thacker 1978*)

spectrum: solitons, anti-solitons of mass m for repulsive interaction $4\pi < \beta^2 < 8\pi$

solitons, anti-solitons and breather bound states for attractive interactions $0 < \beta^2 < 4\pi$

- Non interacting limit: $g = 0,\ \beta^2 = 4\pi$ Luther-Emery point

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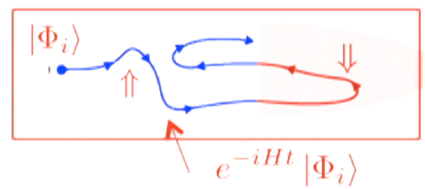

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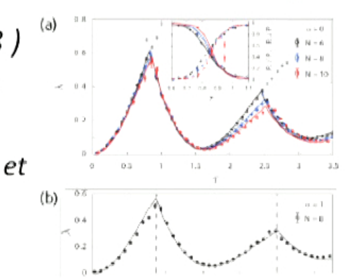
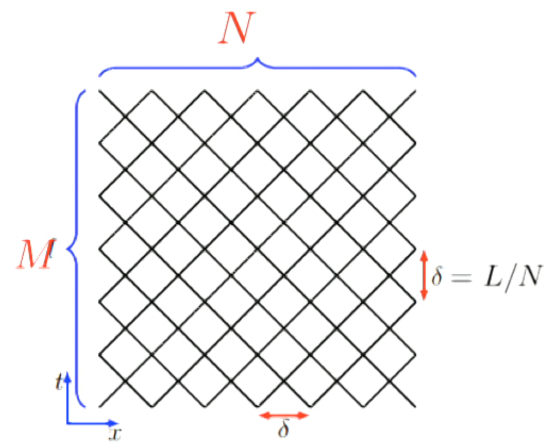


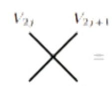
Jurevic et al '17

The MTM model in space time lattice

- Represent the massive Thirring model on the light-cone lattice in space-time in terms of spins while preserving integrability (*Destri - De Vega '87)*
- Left and right moving bare fermions evolve in a discretized Minkowski space time



- To each link emanating a vector space $V = \mathbb{C}^2$ with V_{2j} associated with left movers, V_{2j-1} right movers of j^{th} intersection.
- Jordan-Wigner $\begin{array}{l} \sigma^+_{2j-1} \to \psi^\dagger_+(j\delta) \\ \sigma^+_{2j} \to \psi^\dagger_-(j\delta) \end{array}$
- Each vertex associate matrix



$$= R_{2j,2j+2}(2\Theta) = \begin{pmatrix} \sinh(2\Theta+\eta) & 0 & 0 & 0 \\ 0 & \sinh(2\Theta) & \sinh(\eta) & 0 \\ 0 & \sinh(\eta) & \sinh(2\Theta) & 0 \\ 0 & 0 & 0 & \sinh(2\Theta+\eta) \end{pmatrix}$$

microscopic transition amplitudes

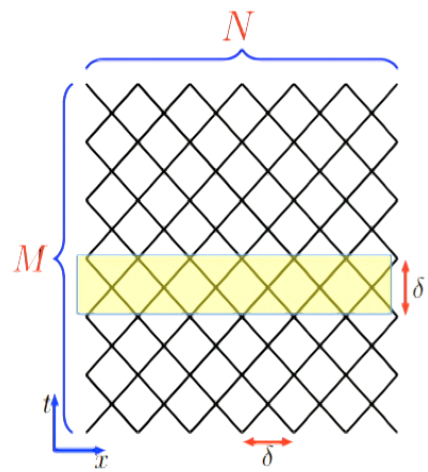
$\eta = i\gamma$ with $\gamma \sim \pi/2 + g$ or $\frac{\gamma}{\pi} = 1 - \frac{\beta^2}{8\pi}$

Θ rapidity cutoff for bare particles, related to bare mass $m_0 = \frac{4N}{L}\sin\gamma\, e^{-2\Theta}$, mass $m = \frac{4N}{L} e^{-\frac{\pi}{\gamma}\Theta}$

- Thermo limit $N, L \to \infty$ with $\delta = L/N$ fixed, continuum limit $\delta \to 0, \Theta \to \infty$, with m fixed

Time Evolution on the Lattice

- Time evolution by δ consist of a move to the left followed by a move to the right (*DDV '87, '89, '92*)



$$e^{-iH\delta} = \left[\frac{\sinh(2\Theta+\eta)}{\sinh(2\Theta-\eta)}\right]^N \tau(-\Theta)\tau^{-1}(\Theta)$$

where

$$\tau(u) = \mathrm{Tr}_k\left[R_{k1}(u-\Theta)R_{k2}(u+\Theta)\ldots R_{k2N-1}(u-\Theta)R_{k2N}(u+\Theta)\right]$$

- The time evolution over interval $t = M\delta$

$$e^{-iHt} = \lim_{N\to\infty}\left[\frac{\sinh(2\Theta+\eta)}{\sinh(2\Theta-\eta)}\right]^{NM}\left[\tau(-\Theta)\tau^{-1}(\Theta)\right]^M$$

- The states in this regularization scheme take the form *(Thacker '81)*

$$|\Psi\rangle = \sum_{\{j\}} a_{\{j\}} \prod_{l=1}^{N} \sigma^+_{j_l}|\Downarrow\rangle \xrightarrow[\text{continuum limit}]{} = \sum_{\{j\}} a_{\{j\}} \prod_{l=1}^{N} \psi^+_{a_{j_l}}(j_l\delta)|0\rangle$$

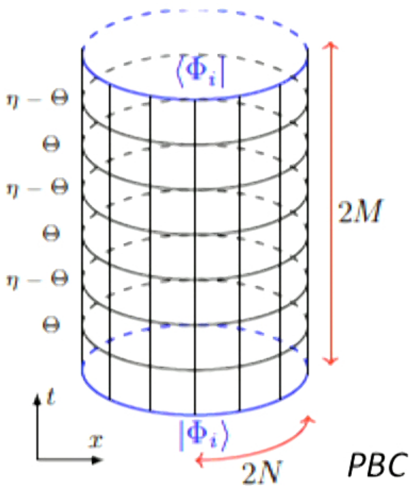
The Loschmidt Amplitude on the Lattice

- The survival amplitude to end up in state $|\Phi_i\rangle$ having propagated for time $t = M\delta$ on lattice of length $L = N\delta \to \infty$:

$$\mathcal{G}(t = M\delta) = \lim_{L=N\delta\to\infty} \left[\frac{\sinh(2\Theta+\eta)}{\sinh(2\Theta-\eta)}\right]^{NM} \langle\Phi_i| \left[\tau(-\Theta)\tau^{-1}(\Theta)\right]^M |\Phi_i\rangle$$

$$= \lim_{L\to\infty} \frac{\langle\Phi_i| \left[\tau(-\Theta)\tau(\Theta-\eta)\right]^M |\Phi_i\rangle}{\sinh^{2NM}(2\Theta-\eta)\sinh^{2NM}(\eta)}$$

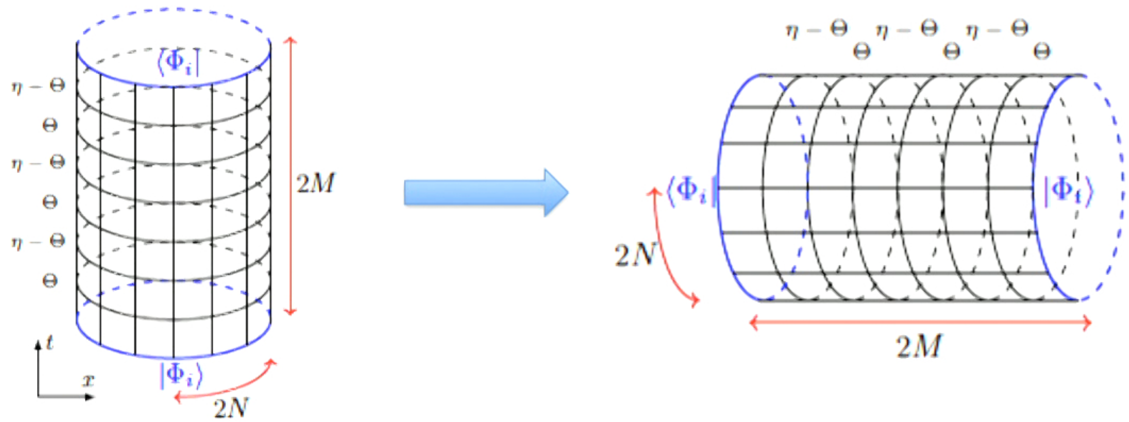
used identity $\tau^{-1}(\Theta) \propto \tau(\Theta-\eta)$ *(Wang et al. 2017)*



- Calculating the Loschmidt amplitude became a classical 2d lattice problem, with $\tau(-\Theta)\,\tau(\Theta-\eta)$ the vertical transfer matrix of an inhomogeneous 6-vertex model on a $2N \times 2M$ lattice, with periodic BC in the horizontal direction and $|\Phi_i\rangle$ initial and final boundary states.

Rotating space and time

- Rotating to a more convenient configuration (modular transformation)



The initial state becomes boundary conditions in the horizontal direction, the periodic boundary conditions becoming a trace in the time direction and parameters associated to vertical lines being exchanged with those associated to horizontal lines.

$$\mathcal{G}(t) = \lim_{L=N\delta\to\infty}\left[\frac{1}{\sinh(2\Theta-\eta)\sinh(\eta)}\right]^{2NM} \mathrm{Tr}\,\left[\langle\Phi_i|\otimes_{j=1}^{N}\left[T_{2j-1}(-\Theta)\times T_{2j}(\Theta)\right]|\Phi_i\rangle\right]$$

The trace being due to horizontal PBC

with the "quantum transfer matrix" *(Pozsgay '13)*

$$T_j(u) = R_{j1}(u-\Theta)R_{j2}(u-(-\Theta+\eta))\dots R_{j2M-1}(u-\Theta)R_{j2M}(u-(-\Theta+\eta))$$

acting in the horizontal space.

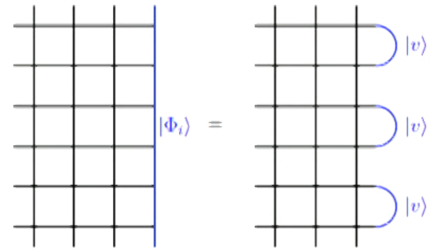
Some interesting initial states

- Consider initial states that can be written as the product of **two site states**

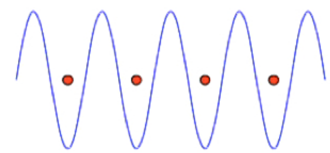
$$|\Phi_i\rangle = \otimes_{j=1}^{N} \frac{1}{\sqrt{\langle v_j|v_j\rangle}}|v_j\rangle$$

with

$$|v_j\rangle = c_1|\uparrow\downarrow\rangle + c_2|\downarrow\uparrow\rangle$$



The states $c_1 = \pm c_2$ correspond to the ground state and highest excited state of the Hamiltonian when the initial mass m_i is very large. In bosonic language the coefficient of the $\cos(\beta\phi(x))$ term is very large with the initial states $|\Phi_i\rangle$ corresponding to those which minimize or maximize this particular term.



- \- In a low energy description of the superfluid-Mott transition these initial states correspond to having a very deep optical lattice with either positive of negative on site interaction

  \- In a low energy description of a pair of coupled condensates these states correspond to an initial phase difference of 0 or π .

- Note: Quench process involves all eigenstates of the post-quench Hamiltonian which may result in the Sine-Gordon description being invalid.

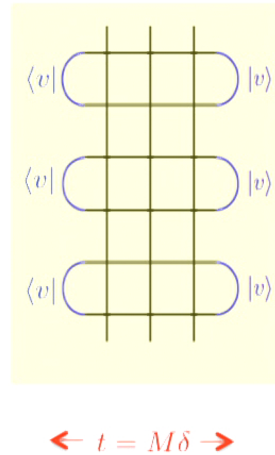
Loschmidt amplitude

- The Loschmidt amplitude takes the form

$$\mathcal{G}(t) = \lim_{N\to\infty}\left[\frac{1}{\sinh(2\Theta-\eta)\sinh(\eta)}\right]^{2NM}\mathrm{Tr}\left[\frac{\langle v|T(-\Theta)\otimes T(\Theta)|v\rangle}{\langle v|v\rangle}\right]^{N}$$

$$= \lim_{N\to\infty}\left[\frac{1}{\sinh(2\Theta-\eta)\sinh(\eta)}\right]^{2NM}\sum_j\left[\frac{\Lambda_j}{\langle v|v\rangle}\right]^{N}$$

$$= \lim_{N\to\infty}\left[\frac{1}{\sinh(2\Theta-\eta)\sinh(\eta)}\right]^{2NM}\left(\frac{\Lambda_{\max}}{\langle v|v\rangle}\right)^{N}$$



- Maximal eigenvalue dominates in the infinite volume limit $N \to \infty$

- If a level crossing occurs at some time t_c then a DQPT takes place.

Maximal Eigenvalue

- Need the maximal eigenvalue of the quantum transfer matrix $\langle v|\, T(u) \otimes T(-u)\, |v\rangle$ evaluated at $u = -\Theta$

Equivalent to finding the ground state energy of the XXZ model with open boundary conditions *(Pozsgay '13, Piroli, Pozsgay, Vernier '17)*:

- Claim: For an initial state $|v(\xi)\rangle$ specified in the continuum limit by $c_1 = e^{2\xi} c_2$ (i.e. $\xi = i\pi/2$ is ground state, and $\xi = 0$ highest excited state), the unique maximal eigenvalue is given by: $\Lambda_{\max} = \Lambda_\xi(-\Theta)$ **Book: Off Diagonal BA** Wang, Yang, Cao, Shi '17

$$\Lambda_\xi(u) = \frac{\sinh(2u+\eta)}{\sinh(2u)} \sinh^2(u + (\xi - \eta/2)) \left[\sinh(u+\Theta)\sinh(u-\Theta+\eta)\right]^{2M} \prod_k^M \frac{\sinh(u-\lambda_k-\eta)}{\sinh(u-\lambda_k)} \frac{\sinh(u+\lambda_k-\eta)}{\sinh(u+\lambda_k)}$$
$$+ \frac{\sinh(2u-\eta)}{\sinh(2u)} \sinh^2(u - (\xi - \eta/2)) \left[\sinh(u-\Theta)\sinh(u+\Theta-\eta)\right]^{2M} \prod_k^M \frac{\sinh(u-\lambda_k+\eta)}{\sinh(u-\lambda_k)} \frac{\sinh(u+\lambda_k+\eta)}{\sinh(u+\lambda_k)}$$

with the Bethe parameters $\{\lambda_j,\ j = 1 \ldots M = t/\delta\}$ satisfying (for initial state $|v(\xi)\rangle$)

$$\frac{\sinh(2\lambda_j - \eta)}{\sinh(2\lambda_j + \eta)} \frac{\sinh^2(\lambda_j - (\xi - \eta/2))}{\sinh^2(\lambda_j + (\xi - \eta/2))} \left[\frac{\sinh(\lambda_j - \Theta)\sinh(\lambda_j + \Theta - \eta)}{\sinh(\lambda_j + \Theta)\sinh(\lambda_j - \Theta + \eta)}\right]^{2M} = -\prod_k^M \frac{\sinh(\lambda_j - \lambda_k - \eta)}{\sinh(\lambda_j - \lambda_k + \eta)} \frac{\sinh(\lambda_j + \lambda_k - \eta)}{\sinh(\lambda_j + \lambda_k + \eta)}$$

A nonlinear integral equation (NLIE) for an Auxiliary Function

- It is convenient to rewrite the BA equations in terms of an auxiliary function $\mathfrak{a}_{t,\xi}(u)$:

$$\mathfrak{a}_{t,\xi}(u) = K(u,\xi)\left[\frac{\sinh(u-\Theta)\sinh(u+\Theta-\eta)}{\sinh(u+\Theta)\sinh(u-\Theta+\eta)}\right]^{2M}\prod_{k}^{M}\frac{\sinh(u-\lambda_k+\eta)}{\sinh(u-\lambda_k-\eta)}\frac{\sinh(u+\lambda_k+\eta)}{\sinh(u+\lambda_k-\eta)}$$

with the boundary term $K(u,\xi) = \frac{\sinh(2u-\eta)\sinh^2(u-(\xi-\eta/2))}{\sinh(2u+\eta)\sinh^2(u+(\xi-\eta/2))}$ encoding initial state

$\xi = i\pi/2$ - quench from ground state

$\xi = 0$ - quench from max excited state

The BA equations for eigenvalues are: $\mathfrak{a}_{t,\xi}(u) = -1$

Eqn depends on its eigenvalues: turn it into a Non Linear Integral Equation - NLIE *(DDV '92)*

The Loschmidt Amplitude

- In terms of the auxiliary function the Loschmidt amplitude is given by:

$$\log \mathcal{G}(t) = -iE_0 t + \log \mathcal{F} - i\frac{mL}{4\gamma}\int_{-\infty}^{\infty} d\mu\, e^{\frac{\pi}{\gamma}(\mu+i\zeta)} \log\left[1 + \mathfrak{a}_{l,\xi}(\mu + i\zeta)\right] + i\frac{mL}{4\gamma}\int_{-\infty}^{\infty} d\mu\, e^{\frac{\pi}{\gamma}(\mu-i\zeta)} \log\left[1 + \mathfrak{a}_{l,\xi}^{-1}(\mu - i\zeta)\right]$$

where:

- $E_0 = -L\int_{-\infty}^{\infty} \frac{d\mu}{\pi} m\, m_0 \cosh(2\mu)\cosh\left(\frac{\pi}{\gamma}\mu\right)$ ground state energy

- $\log \mathcal{F} = -i\frac{mL}{4\gamma}\int_{-\infty}^{\infty} d\mu\, e^{\frac{\pi}{\gamma}(\mu-i\zeta)} \log\left[1 + K^{-1}(\mu - i\zeta)\right] + i\frac{mL}{4\gamma}\int_{-\infty}^{\infty} d\mu\, e^{\frac{\pi}{\gamma}(\mu+i\zeta)} \log\left[1 + K(\mu + i\zeta)\right]$

$\mathcal{F}$ - the Fidelity $\mathcal{F} = |\langle\Psi_0|\Phi_i\rangle|^2$

It will appear in the work distribution as the weight of the delta function peak.

It is the boundary contribution to the ground state of on a finite interval *(LeClair, Mussardo, Saleur, Skorik '95)*

- Compare

$$\log \mathcal{G}(t) = \log \sum_n e^{-iE_n t} |C_n|^2$$

with $C_n = \langle\Psi_n|\Phi_i\rangle$ the overlaps from initial state

$$= -iE_0 t + \ln|C_0|^2 + \log\left[1 + \sum_n \frac{|C_n|^2}{|C_0|^2} e^{-i(E_n - E_0)t}\right]$$

Maximal Eigenvalue

- Need the maximal eigenvalue of the quantum transfer matrix $\langle v| \, T(u) \otimes T(-u) \, |v\rangle$ evaluated at $u = -\Theta$

Equivalent to finding the ground state energy of the XXZ model with open boundary conditions *(Pozsgay '13, Piroli, Pozsgay, Vernier '17)*:

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Wang, Yang, Cao, Shi '17

$$\Lambda_\xi(u) = \frac{\sinh(2u+\eta)}{\sinh(2u)} \sinh^2(u + (\xi - \eta/2)) \left[\sinh(u+\Theta)\sinh(u-\Theta+\eta)\right]^{2M} \prod_k^M \frac{\sinh(u-\lambda_k-\eta)}{\sinh(u-\lambda_k)} \frac{\sinh(u+\lambda_k-\eta)}{\sinh(u+\lambda_k)}$$
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with the Bethe parameters $\{\lambda_j, \; j = 1 \ldots M = t/\delta\}$ satisfying (for initial state $|v(\xi)\rangle$)

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The Loschmidt Amplitude

- In terms of the auxiliary function the Loschmidt amplitude is given by:

$$\log \mathcal{G}(t) = -iE_0 t + \log \mathcal{F} - i\frac{mL}{4\gamma}\int_{-\infty}^{\infty} d\mu\, e^{\frac{\pi}{\gamma}(\mu+i\zeta)} \log\left[1+\mathfrak{a}_{l,\xi}(\mu+i\zeta)\right] + i\frac{mL}{4\gamma}\int_{-\infty}^{\infty} d\mu\, e^{\frac{\pi}{\gamma}(\mu-i\zeta)} \log\left[1+\mathfrak{a}_{l,\xi}^{-1}(\mu-i\zeta)\right]$$

where:

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- $\log \mathcal{F} = -i\frac{mL}{4\gamma}\int_{-\infty}^{\infty} d\mu\, e^{\frac{\pi}{\gamma}(\mu-i\zeta)} \log\left[1+K^{-1}(\mu-i\zeta)\right] + i\frac{mL}{4\gamma}\int_{-\infty}^{\infty} d\mu\, e^{\frac{\pi}{\gamma}(\mu+i\zeta)} \log\left[1+K(\mu+i\zeta)\right]$

$\mathcal{F}$ - the Fidelity $\mathcal{F} = |\langle\Psi_0|\Phi_i\rangle|^2$

It will appear in the work distribution as the weight of the delta function peak.

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- Compare

$$\log \mathcal{G}(t) = \log \sum_n e^{-iE_n t}|C_n|^2$$

with $C_n = \langle\Psi_n|\Phi_i\rangle$ the overlaps from initial state

$$= -iE_0 t + \ln|C_0|^2 + \log\left[1+\sum_n \frac{|C_n|^2}{|C_0|^2} e^{-i(E_n-E_0)t}\right]$$

A nonlinear integral equation (NLIE) for an Auxiliary Function

- One can write a non linear integral equation for $\mathfrak{a}(u)$ *(DDV '92, Pozsgay '13)*

$$\log \mathfrak{a}_{t,\xi}(u) = -2mt \sinh\left(\frac{\pi}{\gamma}u\right) + \log \mathbb{K}_\xi(u) + \int_{-\infty}^{\infty} \mathrm{d}\mu\, G(u-\mu-i\zeta,\gamma)\log\left[1+\mathfrak{a}_{t,\xi}(\mu+i\zeta)\right] - \int_{-\infty}^{\infty} \mathrm{d}\mu\, G(u-\mu+i\zeta,\gamma)\log\left[1+\mathfrak{a}_{t,\xi}^{-1}(\mu-i\zeta)\right]$$

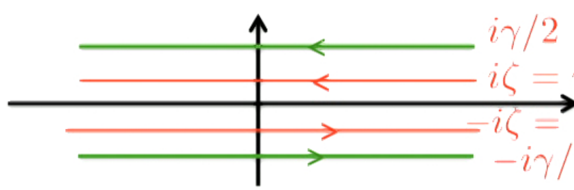
where:

$$\log \mathbb{K}_\xi(u) = J * \log\left[\frac{\sinh(u-i\gamma)}{\sinh(u+i\gamma)}K(u,\xi)\right]$$

$$J(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} e^{i\omega x} \frac{\sinh(\pi\omega/2)}{2\cosh(\gamma\omega/2)\sinh[(\pi-\gamma)\omega/2]}$$

$$G(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} e^{i\omega x} \frac{\sinh[(\pi-2\gamma)\omega/2]}{2\cosh(\gamma\omega/2)\sinh[(\pi-\gamma)\omega/2]}$$

The integral run over contours $\mu \pm i\zeta$ in the complex plane with $2\zeta < \min(\gamma, \pi-\gamma)$ to reproduce the BA eqns. The form of the NLIE depends on whether the interaction repulsive or attractive and reflects the existence of bound states



Generalities

- Non-equilibrium strong-weak duality

  - Under the replacement, $\gamma \to \pi - \gamma$ and $\xi \to i\pi/2 - \xi$

    both the BA eqns and the Loschmidt amplitude are modified by $t \to -t$

    $$\mathcal{G}(t)|_{\xi,\gamma} = \mathcal{G}(-t)|_{i\frac{\pi}{2}-\xi,\pi-\gamma}$$

    relates the non-equilibrium behavior of the system in the repulsive regime to that of the attractive regime.

  - Free theory is self dual

- Dynamical quantum phase transition - DQPT

- No DQPT appears for the two quenches we studied. It occurs when there is a crossing of Λ_ξ^{max} with next level, leading to non-analyticity of the auxiliary function $\mathfrak{a}_\xi(u)$.

  Need: $1 + \mathfrak{a}(\mu + i\zeta) = 0$ or $1 + \mathfrak{a}^{-1}(\mu - i\zeta) = 0$ but BAE require real solutions

- DQPT will appear in finite mass quenches -

- $\log[1 + \mathfrak{a}_\xi(u)]$ is related to a non-equilibrium distribution function, therefore non-analyticity in the Loschmidt amplitude is reflected also in quenched observables.

The free-Fermion point - I

Set $\gamma = \frac{\pi}{2}$ namely $\beta^2 = 4\pi$, then $G(x) = 0, \; J(x) = \delta(x)$

- The NLIE yields the auxiliary function:

$$\log \mathfrak{a}_{t,\xi}(u) = -2mt\sinh(2u) + \log\left[-\frac{\sinh^2(u - (\xi - i\pi/4))}{\sinh^2(u + (\xi - i\pi/4))}\right].$$

- The Loschmidt amplitude:

$$\log \mathcal{G}(t) = -iE_0 t + \ln \mathcal{F} + L\int_{-\infty}^{\infty} \frac{d\mu}{\pi} m_0 \cosh(2\mu) \; \log\left\{1 + \left[\frac{\cosh(\mu - \xi)}{\sinh(\mu + \xi)}\right]^2 e^{-2im_0 t\cosh(2\mu)}\right\}$$

- Fidelity:

$$\mathcal{F} = -L\int_{-\infty}^{\infty} \frac{d\mu}{\pi} m_0 \cosh(2\mu) \log\left\{1 + \left[\frac{\cosh(\mu - \xi)}{\sinh(\mu + \xi)}\right]^2\right\}$$

Same result can be obtained by Bogoliubov rotation (*Silva '08)*:

- Note: under $\xi \to i\frac{\pi}{2} - \xi$ the two initial states are mapped into each other.

We have $K(u) \to K^{-1}(u)$ and so: $\mathcal{G}(t)|_{\xi} = \mathcal{G}(-t)|_{i\frac{\pi}{2}-\xi}$ i.e. related by time reversal

- Loschmidt echo is the same for both quenches - initial states are related by a particle-hole transformation which is preserved by the quench

The free-Fermion point - II

- The work distribution for the quench from the initial ground state :

$$P(W) = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{\pi} e^{iWt + i\epsilon_i t} \mathcal{G}(t)$$

Silva '08, Smacchia, Silva '13

$$= \mathcal{F} \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{\pi} e^{iWt - i\delta E t} \left[1 + L \int_{-\infty}^{\infty} \frac{\mathrm{d}\mu}{\pi} m_0 \cosh(2\mu) \log \left\{ 1 + \tanh^2(\mu) e^{-2im_0 t \cosh(2\mu)} \right\} + \dots \right]$$

$$= \mathcal{F}\delta(W - \delta E) + m_0 \frac{\mathcal{F}L}{2\pi} \theta(W - \delta E - 2m_0) \sqrt{\frac{W - \delta E - 2m_0}{(W - \delta E + 2m_0)^3}} (W - \delta E) + \dots$$

- The term $\delta(W - \delta E)$ weighted by the fidelity $\mathcal{F}$ comes from the transition to the ground state.
- There is an edge singularity at $W = 2m_0 + \delta E$ with exponent 1/2.
- Keeping further terms in the expansion of the exponential to further edges at $W = 2n\, m_0 + \delta E$.

- The work distribution for quench from the initial state of maximal energy: $W - \delta E \to W$

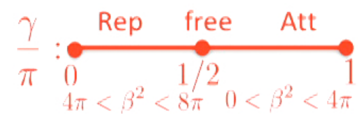
$$P(W) = \mathcal{F}\delta(W) + 2m_0 \mathcal{F} L \delta(W - m_0) + 2\mathcal{F}^2 m_0^2 L^2 \delta(W - 2m_0)$$
$$+ m_0 \frac{\mathcal{F}L}{2\pi} \theta(W - 2m_0) W \sqrt{\frac{W + 2m_0}{(W - 2m_0)^3}} + \dots$$

(Gambassi, Silva '11
Smacchia, Silva '13
Palmai '15)

Additional delta functions at $W = m_0 + \delta E$ and at the edge $W = 2m_0 + \delta E$. the Hilbert space of the free fermion model splits into sectors with even and odd particle number. The previous initial state is contained in the even sector whereas the max excited state has overlap with both sectors. It can transition to a single particle excited state with zero momentum resulting in the additional delta function. Critical exponent at threshold = - 3/2

Interactions - the repulsive regime

Turn on repulsive interaction: $\gamma < \pi/2$, or $4\pi < \beta^2 < 8\pi$:



- Work with the auxiliary function:

$$\mathfrak{y}_{t,\xi}(u) = \log \mathfrak{a}_{t,\xi}(u + i\gamma/2 - i\epsilon) + 2mit\cosh\left(\frac{\pi}{\gamma}u\right) - \log \mathcal{K}_{t,\xi}(u)$$

where $\mathcal{K}(u) = \mathbb{K}(u + i\gamma/2)$ is the shifted boundary phase shift

- The auxiliary function satisfies

$$\mathfrak{y}_{t,\xi}(u) = \int_{-\infty}^{\infty} \mathrm{d}\mu\, G(u-\mu,\gamma)\log\left[1 + \mathcal{K}_{t,\xi}(\mu)e^{\mathfrak{y}_{t,\xi}(\mu) - 2imt\cosh\left(\frac{\pi}{\gamma}\mu\right)}\right]$$
$$- \int_{-\infty}^{\infty} \mathrm{d}\mu\, G(u-\mu+i\gamma-i\epsilon,\gamma)\log\left[1 + \mathcal{K}_{t,\xi}(-\mu)e^{\mathfrak{y}_{t,\xi}(-\mu) - 2imt\cosh\left(\frac{\pi}{\gamma}\mu\right)}\right]$$

- And the Loschmidt amplitude

$$\log \mathcal{G}_{\xi}(t) = -iE_0 t + \log\mathcal{F} + \frac{mL}{4\gamma}\int_{-\infty}^{\infty}\mathrm{d}\mu\,\cosh\left(\frac{\pi}{\gamma}\mu\right)\left\{\log\left[1 + \mathcal{K}_{t,\xi}(\mu)e^{\mathfrak{y}_{t,\xi}(\mu) - 2imt\cosh\left(\frac{\pi}{\gamma}u\right)}\right]\right.$$
$$\left. + \log\left[1 + \mathcal{K}(-\mu)e^{\mathfrak{y}_{t,\xi}(-\mu) - 2imt\cosh\left(\frac{\pi}{\gamma}\mu\right)}\right]\right\}$$

Interactions - the repulsive regime

- For the quench from the ground state:

$$P(W) = \mathcal{F}\delta(W-\delta E) + \frac{\mathcal{F}L}{4\pi}\theta(W-\delta E-2m)\frac{\mathrm{Re}\left\{\mathcal{K}_{i\frac{\pi}{2}}\left(\frac{\gamma}{\pi}\mathrm{arcosh}\left[\frac{W-\delta E}{2m}\right]\right)\right\}}{\sqrt{(W-\delta E)^2/4m^2-1}}(W-\delta E)+\ldots$$

- Resembles the non interaction case: after expanding $\mathcal{K}$ about the edge singularity $W-\delta E \sim 2m$ we get that the edge exponent is 1/2.

- The effect of interactions on $P(W)$ near the threshold is negligible, since in this region the quench process is governed by transitions to excited states containing only two quasi-particles moving away from each other.

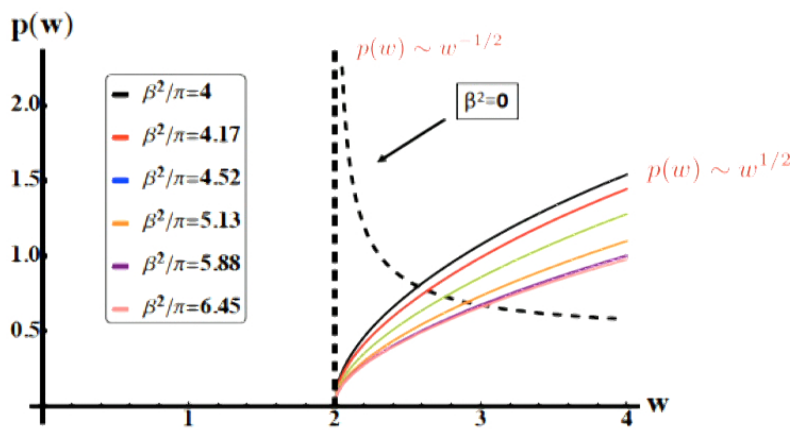
- For the quench from the max excited state:

$$P(W) = \mathcal{F}\delta(W-\delta E) + m\mathcal{F}L\frac{g_0^2}{2}\delta(W-\delta E-m) + \frac{1}{2}\left[m\mathcal{F}L\frac{g_0^2}{2}\right]^2\delta(W-\delta E-2m)$$
$$+\frac{\mathcal{F}L}{4\pi}\theta(W-\delta E-2m)\frac{\mathrm{Re}\left\{\mathcal{K}_0\left(\frac{\gamma}{\pi}\mathrm{arcosh}\left[\frac{W-\delta E}{2m}\right]\right)\right\}}{\sqrt{(W-\delta E)^2/4m^2-1}}(W-\delta E)+\ldots$$

as $\mathcal{K}_{\xi=0}(x)$ diverges $1/x$ we have again an edge exponent of -3/2.

Work distribution at threshold: quench from ground state

Repulsive regime $4\pi < \beta^2 < 8\pi$



- The edge of the rescaled work distribution $p(w) = 4\pi P(W)/(mL\mathcal{F})$ for quench in the Sine-Gordon model, measured in units of mass from the ground state energy $w = (W - \delta E)/m$. The dashed line shows same quantity for non interacting bosons while solid lines are for different values of interaction strength. The solid black line is for a quench of free fermions.
- The interaction modifies the exponent from $-1/2$ to $1/2$.

Interactions – solve NLIE by iteration

- With respect to the non interacting case, besides mass reno' $m_0 \to m$, rapidity reno' $2\mu \to \pi\mu/\gamma$ and boundary scattering reno' $K_\xi \to \mathcal{K}_\xi$ we have the interactions encoded in $\mathfrak{y}(\mu)$

- To solve NLIE expand exponential and log, keeping lowest orders *(Palmai, Sotiriadis '14)*

The NLIE (for ground state quench)

$$\mathfrak{y}(u) = \sum_{n=1,l=0}^{\infty} \frac{(-n)^l}{nl!} \left[\int_{-\infty}^{\infty} \mathrm{d}\mu\, G(u-\mu,\gamma) \mathcal{K}^n_{i\frac{\pi}{2}}(\mu) \mathfrak{y}^l(\mu) e^{-2imnt\cosh\left(\frac{\pi}{\gamma}\mu\right)} - \int_{-\infty}^{\infty} \mathrm{d}\mu\, G(u-\mu+i\gamma-i\epsilon,\gamma) \bar{\mathcal{K}}^n_{i\frac{\pi}{2}}(\mu) \bar{\mathfrak{y}}^l(\mu) e^{-2imnt\cosh\left(\frac{\pi}{\gamma}\mu\right)} \right]$$

The echo

$$\log \mathcal{G}(t) = -iE_0 t + \mathcal{F} + \frac{mL}{4\gamma} \sum_{n=l,l=0}^{\infty} \frac{(-n)^l}{nl!} \left[\int_{-\infty}^{\infty} \mathrm{d}\mu\, e^{\frac{\pi}{\gamma}\mu} \left\{ \mathcal{K}^n_{i\frac{\pi}{2}}(\mu) \mathfrak{y}^l(\mu) e^{-2imnt\cosh\left(\frac{\pi}{\gamma}\mu\right)} + \bar{\mathcal{K}}^n_{i\frac{\pi}{2}}(\mu) \bar{\mathfrak{y}}^l(\mu) e^{-2imnt\cosh\left(\frac{\pi}{\gamma}\mu\right)} \right\} \right]$$

- Start iteration with non interacting value: $\mathfrak{y}_{[0]} = 0$ only $l = 0$ enters and each n term comes with $e^{-2imnt\cosh\frac{\pi}{\gamma}\mu}$, contributing to $P(W)$ for $W > 2mn$. We also obtain:

$$\mathfrak{y}_{[1]}(u) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}\mu}{n} e^{-2imnt\cosh\left(\frac{\pi}{\gamma}\mu\right)} \left[G(u-\mu,\gamma)\mathcal{K}^n_{i\frac{\pi}{2}}(\mu) - G(u-\mu+i\gamma-i\epsilon,\gamma)\bar{\mathcal{K}}^n_{i\frac{\pi}{2}}(\mu) \right]$$

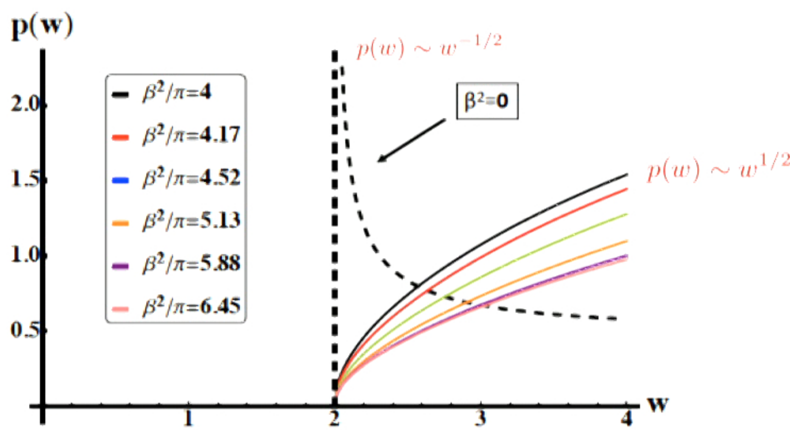
Inserting into amplitude, we note that only $n=1, l=0$ term has factor $e^{-2imt\cosh\frac{\pi}{\gamma}\mu}$ so only it can contribute in the region $0 < W < 4m$ (we absorb δE into the work $W - \delta E \to W$).

- Iterating further gives the exact expression for a larger window. After the $(n-2)^{\mathrm{th}}$ iteration,

  we get an exact expression for $P(W)$ with $W < 2mn$

Work distribution at threshold: quench from ground state

Repulsive regime $4\pi < \beta^2 < 8\pi$

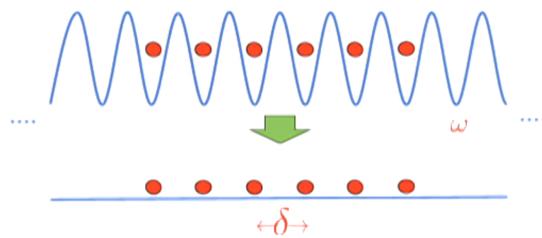


- The edge of the rescaled work distribution $p(w) = 4\pi P(W)/(mL\mathcal{F})$ for quench in the Sine-Gordon model, measured in units of mass from the ground state energy $w = (W - \delta E)/m$. The dashed line shows same quantity for non interacting bosons while solid lines are for different values of interaction strength. The solid black line is for a quench of free fermions.
- The interaction modifies the exponent from $-1/2$ to $1/2$.

Quench of an optical lattice: *gapless scenario*

- **Quench protocol**: Release N neutral bosons from a deep trap $\bar{x}_j = j\delta,\ j = 1 \cdots N$

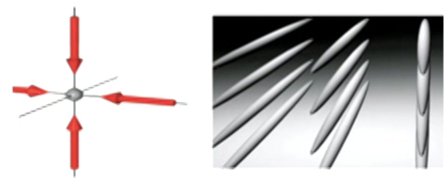
$$|\Psi_i\rangle = \int \mathrm{d}^N x \prod_{j=1}^{N} \left[\frac{m\omega}{\pi}\right]^{\frac{1}{4}} e^{-\frac{m\omega}{2}(x_j - \bar{x}_j)^2} b^\dagger(x_j)|0\rangle$$



- Bosons well described by the Lieb-Liniger Hamiltonian

$$H_{LL} = \frac{1}{2m} \int \mathrm{d}x \left[-b^\dagger(x)\partial_x^2 b(x) + c\, b^\dagger(x) b(x) b^\dagger(x) b(x) \right]$$

I. Bloch et al



Coupling: $c > 0$ repulsive, $c < 0$ attractive

- Allow them to evolve $|\Psi(t)\rangle = e^{-itH_{LL}}|\Psi_i\rangle$ **Localized peaks broaden and bosons begin to collide**

- How to calculate evolution? Use partition of the unity?

$$|\Psi_i(t)\rangle = e^{-itH_{LL}}|\Psi_i\rangle = e^{-itH_{LL}} \sum_n |n\rangle\langle n|\Psi_i\rangle = \sum_n e^{-iE_n t}|n\rangle\langle n|\Psi_i\rangle$$

- But what are the eigenstates $|n\rangle$ and how to calculate ***overlaps***?

Eigenstates and Yudson states

- The exact eigenstates of the Lieb-Liniger model *(Lieb-Liniger '63)*

$$|\{k_j\}\rangle = \int \mathrm{d}^N x \prod_{\substack{i,j=1\\ i<j}}^{N} \frac{k_i - k_j - ic\,\mathrm{sgn}(x_i - x_j)}{k_i - k_j - ic} \prod_{l=1}^{N} e^{ik_l x_l} b^\dagger(x_l)|0\rangle$$

For PBC the momenta satisfy $k_j = \frac{2\pi}{L} n_j - \frac{1}{L}\sum_{l\neq j} \varphi(k_j - k_l)$ with $\varphi(x) = 2\arctan(x/c)$

The integer quantum numbers $\{n_j\}$ determine the momenta $\{k_j\}$, denote eigenstates $|\{n_j\}\rangle$

- Claim *(Yudson '82, Goldstein, NA '12)* : Can rewrite the standard partition of the unity

$$I_N = \sum_{n_1<\cdots<n_N} \frac{|\{n_j\}\rangle\langle\{n_j\}|}{\mathcal{N}(\{n_j\})}$$ as $$I_N = \sum_{n_1,\ldots,n_N} \frac{|\{n_j\}\rangle(\{n_j\}|}{\mathcal{N}(\{n_j\})}$$

normalization

$$\mathcal{N}(\{n\}) = \det\left[\delta_{jk}\left(L + \sum_{l=1}^{N} \varphi'(k_j - k_l)\right) - \varphi'(k_j - k_k)\right]$$

in terms of Yudson states $|\{n\}) = \int \mathrm{d}^N x\, \theta(x_1 > \cdots > x_N) \prod_l^N e^{ik_l x_l}\, b^\dagger(x_l)\, |0\rangle$

- This allows the computation of overlaps and therefore of time evolution of $|\Psi_i\rangle$:

$$|\Psi_i(t)\rangle = \left[\frac{4\pi\hbar}{m\omega}\right]^{\frac{N}{4}} \sum_{n_1,\ldots,n_N} \frac{e^{-\sum_{j=1}^{N}\left[\frac{\hbar}{2m\omega}(1+i\hbar\omega t)k_j^2 + ik_j\bar{x}_j\right]}}{\mathcal{N}(\{n_j\})} |\{n_j\}\rangle$$

Other approach:
Quench action
Perfetto *et al.* 1904.06259

The Loschmidt Amplitude

- The Loschmidt Amplitude:

$$\mathcal{G}(t) = \langle \Phi_i | \, e^{-iHt} \, | \Phi_i \rangle = \langle \Phi_i | \Phi_i(t) \rangle = \left[\frac{4\pi}{m\omega} \right]^{\frac{N}{2}} \sum_{n_1, \dots, n_N} e^{-\frac{1}{m\omega}\left[1 + i\frac{\omega}{2}t\right] \sum_{j=1}^{N} k_j^2} \frac{G(\{n\})}{\mathcal{N}(\{n\})}$$

with

$$G(\{n\}) = \det\left[e^{-ik_j(\bar{x}_j - \bar{x}_k) - i\theta(j-k)\varphi(k_j - k_k)} \right]$$

$$\mathcal{N}(\{n\}) = \det\left[\delta_{jk} \left(L + \sum_{l=1}^{N} \varphi'(k_j - k_l) \right) - \varphi'(k_j - k_k) \right]$$

- Exact for any c, N, L.
- Displays **recurrences** : $\tau = (1 + 2\rho/c)^2 \, L^2/\pi\omega$

with $\rho = N/L$ and ω characterizing the initial state

The Loschmidt Amplitude (infinite volume)

Open system (partially filled lattice) $\rho = N/L \ll 1/\delta$

- ***Repulsive interactions*** Simplifies for strong repulsion $c \gg m\omega$

$$\mathcal{G}(t) = \frac{1}{\left[1 + i\frac{\omega}{2}t\right]^{\frac{N}{2}}} \sum_P (-1)^P e^{-\frac{\omega \alpha_P^2}{4\left(1+i\frac{\omega}{2}t\right)}}$$

with $\alpha_P^2 = m\delta_{\text{eff}}^2 \|P\|^2/2$

$\delta_{\text{eff}} = \left[1 + \frac{2}{c\delta}\right]\delta$ Effective increased distance due to repulsion

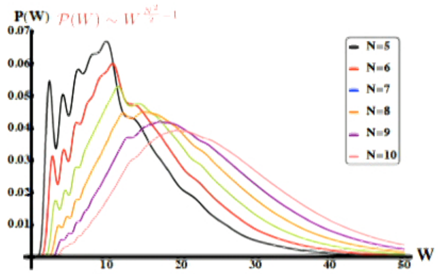
$\|P\|^2 = \sum_j^N (j - P(j))^2$ Measures how many particles were exchanged

Sum over permutations corresponds to a sum over particles exchanging positions: e.g $\|P\|^2 = 2$ corresponds to a neighboring pair exchanging positions, $\|P\|^2 = 8$ could be 4 nearest neighbor exchanges or 1 next nearest neighbor exchange.

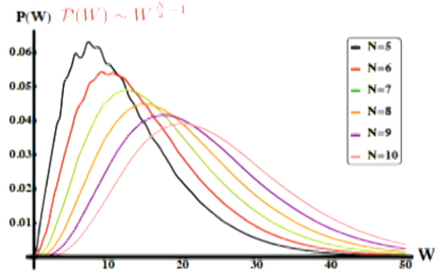
The Work Distribution (repulsive interactions, infinite volume)

- The work distribution is obtained from the Loschmidt amplitude

$$P(W) = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi} e^{iWt} \mathcal{G}(t) = \frac{e^{-\frac{2W}{\omega}}}{W} \left[\frac{2W}{\omega}\right]^{\frac{N}{2}} \sum_P (-1)^P \frac{J_{\frac{N-2}{2}}(2\sqrt{\alpha_P^2 W})}{[\alpha_P^2 W]^{\frac{N-2}{4}}}$$



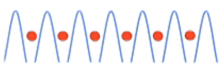
Work distribution for repulsive bosons



Work distribution for free bosons

$5 \leq N \leq 10$

$\delta/m = 2, \ \omega = 10$

- For large value of W (short times) no dependence on interaction (no overlaps initially) 

- For small values of W strong effect of interactions: $\mathcal{P}(W) \sim W^{\frac{N^2}{2}-1}$ as opposed to: $\mathcal{P}(W) \sim W^{\frac{N}{2}-1}$
Indeed, $|\mathcal{G}(t)|^2 \to 1/t^{N^2}$ vs. $|\mathcal{G}(t)|^2 \to 1/t^N$ as $t \to \infty$. In 1-d even weak interaction have strong effect.

- Average work $\langle W \rangle = \int \mathrm{d}W \, W \mathcal{P}(W) = N\omega/4 = \langle \Psi_i | H | \Psi_i \rangle$

- Exponentiated work (yields all moments) $\langle e^{-\beta W} \rangle = \left(\frac{1}{1+\frac{\omega\beta}{2}} \right)^{\frac{N}{2}} \left[1 + \sum_{P \neq I} (-1)^P \frac{e^{-\frac{\omega \alpha_P^2}{2+\omega\beta}}}{\Gamma(N/2)} \right]$

cf Jarzynski equality

$e^{-\beta \Delta F} = \langle e^{-\beta W} \rangle$

The Work distribution (attractive interactions, infinite volume)

For $c < 0$ there are bound states of n -bosons, $n = 2, \cdots, N$
(momenta k form complex n- strings: $k_j = k + ijc/2, \; j = 1 \cdots n$

- Using a generalization of Yudson representation, we find

$$\mathcal{P}_{c<0}(W) = \mathcal{P}_{\text{unbound}}(W) + \mathcal{P}_{\text{bound}}(W)$$

- $\mathcal{P}_{\text{unbound}}(W)$ same expression as before with $c \to -c$, so $\delta_{\text{eff}} < \delta$ due to attraction.

  Similar to super Tonks-Girardeau experiment, where one prepares system with large $c > 0$ and then suddenly quenches to large $c < 0$. Note, our expression valid for any c .

- $\mathcal{P}_{\text{bound}}(W)$ is due to the strings, n- string contributing $\mathcal{P}_{n-\text{bound}}(W) \propto |c|^{n-1} e^{-n|c|\delta}$

  Transitions to states containing bound states are highly suppressed and in the true super-TG limit vanish entirely.

- For finite $c < 0$ new effects: bound states lower the energy and work distribution becomes non vanishing at negative values of W.

- Indeed, for a 2- string $\mathcal{P}_{\text{bound}}(W) \approx N \sqrt{\frac{2\pi\omega}{m}} \frac{e^{-|c|\delta - \frac{2W}{\omega}}}{\Gamma\left(\frac{N}{2} - 1\right)} \left[\frac{2(W + \frac{|c|^2}{4m})}{\omega} \right]^{\frac{N}{2} - 2}$, non vanishing for $-|c|^2/4m < W$

- There is a non zero probability that work can be extracted from the system. This does not violate the 2nd law of thermodynamics since $\langle W \rangle > 0$ *Jarzynski '11*

The Work distribution (repulsive interactions, finite volume)

Start from a **fully occupied lattice** – boundary effects important

As before:

$$P(W) = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi} e^{iWt} \mathcal{G}(t) = \frac{e^{-\frac{2W}{\omega}}}{W} \left[\frac{2W}{\omega}\right]^{\frac{N}{2}} \sum_P (-1)^P \frac{J_{\frac{N-2}{2}}(2\sqrt{\alpha_P^2 W})}{[\alpha_P^2 W]^{\frac{N-2}{4}}}$$

But boundary conditions enter when calculating $\alpha_P^2 = m\delta_{\text{eff}}^2 ||P||^2/2$

For instance when $\rho \ll \delta^{-1}$ the permutation $P = (23...N1)$ gives $\alpha_P^2 = m\delta_{\text{eff}}^2 N(N-1)/2$
however with PBC it gives $\alpha_P^2 = m\delta_{\text{eff}}^2 N/2$

- Region $W \sim \langle W \rangle$ not affected by BC, dominated by few exchanges of particles average work is as before.

- Region $W \ll \langle W \rangle$ strongly affected, all permutation contribute

We have $\mathcal{P}(W) \sim W^{\frac{N}{2}}$ corresponding to $|\mathcal{G}(t)|^2 \to 1/t^{N+2}$

As opposed to $\mathcal{P}(W) \sim W^{\frac{N^2}{2}-1}$ and $|\mathcal{G}(t)|^2 \to 1/t^{N^2}$ as before.
The strongly interacting particles have no space to expand into unlike previously, resulting in slower decay of the echo.

Conclusions

- Quenches realizable in cold atoms experiments, work distribution measurable

- Calculated work distribution of a quench in a strongly interacting, gapless system.

- Studied bound state contributions to the work distribution.
  Showed they dramatically change the distribution and allow for negative values of work

- Showed that interactions strongly affect the universal edge exponents of the work distribution and also the long time decay of the Loschmidt echo

- Calculated work statistics for some quenches of the SG Hamiltonian – describes a sudden lowering of periodic potential

- Calculated for attractive and repulsive interactions, determined critical exponents at threshold

To do

- Connect to nonequilibrium thermodynamics: entropy production, fluctuation theorems
- Small and large systems - increase role of fluctuations (fluctuation theorems)
- Quench across critical points, defect production, Kibble-Zurek dynamics, scaling and universality
- Time dependent quenches: slow drives, Floquet