Title: Wilson loops and defect CFT

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Abstract: I will overview recent results on the defect CFT corresponding to Wilson loop operators in N=4 SYM theory. In particular, I will review the calculation of defect correlators at strong coupling using the AdS2 string worldsheet, and I will present exact results for correlation functions in a subsector of the defect CFT using localization. I will also discuss a defect RG flow from the BPS to the ordinary Wilson loop, which can be used to provide a test of the "defect F-theorem" for one-dimensional defects.

Wilson Loops and Defect CFT

Simone Giombi

Princeton University

Boundaries and Defects in Quantum Field Theory
Perimeter Institute, Aug. 8, 2019

Based mainly on SG, Roiban, Tseytlin arXiv: 1706.00756
SG, Komatsu arXiv: 1802.05201, arXiv:1811.02369
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Outline

- Half-BPS Wilson loop in $\mathcal{N}$=4 SYM as conformal defect
- Defect correlators at strong coupling from AdS_2 string worldsheet
- Exact results from localization
- Non-supersymmetric Wilson loop, defect RG flow and a test of “defect F-theorem”

Wilson loops in *N*=4 SYM

- In *N*=4 SYM, it is natural to study Wilson loop operators that include couplings to the six adjoint scalars Φ^I ("Maldacena-Wilson" loop)

$$W = \mathrm{tr} P e^{\oint dt \left(i \dot{x}^\mu A_\mu + |\dot{x}| \theta^I \Phi^I \right)}$$

where $x^\mu(t)$ is a loop in spacetime and $\theta^I(t)$ a unit 6-vector.

- Special choices of (x^μ, θ^I) lead to families of Wilson loop operators preserving various fractions of the superconformal symmetry

(Zarembo '02; Drukker, SG, Ricci, Trancanelli '07)

Half-BPS Wilson loop

- The most supersymmetric case is the 1/2-BPS Wilson loop:
  - $x^\mu(t)$: a circle, or infinite straight line in R^4 (related by conf. transf.)
  - θ^I : a constant unit 6-vector

- E.g. take the line x^0=t, and $\theta^I = \delta^{I6}$

$$W = \mathrm{tr} P e^{\int dt \left(i A_t + \Phi^6 \right)}$$

- This preserves 8 Q's and 8 S's (superconformal charges): 1/2-BPS. (Similarly for a circle, but it preserves 16 lin. combinations of Q and S)

Half-BPS Wilson line as conformal defect

- Let us recall the symmetries preserved by the 1/2-BPS Wilson line. The bosonic symmetries are
  - SO(3): rotations around the line (i=1,2,3)
  - SO(5): R-symmetry rotations of the five scalars Φ^a, a=1,…,5 that do not couple to the Wilson loop operator
  - *SL(2,R):* dilatation, translations and special conformal transformation on the line. **1d conformal symmetry** *(Kapustin '05; Drukker, Kawamoto '06)*

- Together with the 16 supercharges, these combine into the 1d superconformal group OSp(4*|4) $\supset$ SL(2,R)xSO(3)xSO(5)
- Since it preserves a 1d conformal subgroup of the 4d conformal symmetry, the 1/2-BPS Wilson loop can be viewed as a *conformal defect* of the 4d theory

Correlators on the defect

- As usual in defect CFT, we can study correlators of operators on the defect, of bulk operators, or mixed bulk/defect ones. We will mainly focus on correlators of operators on the defect
- Given some local operators *$O_i(t)$* in the *adjoint* of the gauge group, consider

$$\langle\langle O_1(t_1)O_2(t_2)\cdots O_n(t_n)\rangle\rangle \equiv \frac{\langle \mathrm{tr}P\left[O_1(t_1)\ e^{\int dt(iA_t+\Phi^6)}\ O_2(t_2)\ e^{\int dt(iA_t+\Phi^6)}\cdots\ O_n(t_n)\ e^{\int dt(iA_t+\Phi^6)}\right]\rangle}{\langle W\rangle}$$

$$\equiv \frac{\langle \mathrm{tr}P\left[O_1(t_1)O_2(t_2)\cdots O_n(t_n)e^{\int dt(iA_t+\Phi^6)}\right]\rangle}{\langle W\rangle}$$

- Such defect correlators arise naturally when we consider small deformations of the Wilson loop (*Polyakov, Rychkov '00 ; Drukker, Kawamoto '06*)
- They encode information on expectation value of Wilson loops of more general shapes and scalar coupling

Correlators on the defect

- These defect correlators are constrained by the SL(2,R) 1d conformal symmetry as in general CFT_d
- Defect operator can be organized in primaries and descendants. Primaries are labelled by their scaling dimension Δ and SO(3)xSO(5) representation
- Correlation functions constrained by conf. symmetry:

$$\langle O_\Delta(t_1) O_\Delta(t_2)\rangle = \frac{1}{t_{12}^{2\Delta}} \qquad \langle O_1(t_1) O_2(t_2) O_3(t_3)\rangle = \frac{c_{123}}{t_{12}^{\Delta_1+\Delta_2-\Delta_3} t_{23}^{\Delta_2+\Delta_3-\Delta_1} t_{31}^{\Delta_3+\Delta_1-\Delta_2}}$$

$$\langle O_\Delta(t_1) O_\Delta(t_2) O_\Delta(t_3) O_\Delta(t_4)\rangle = \frac{1}{(t_{12}t_{34})^{2\Delta}} \mathcal{G}(\chi) \qquad \chi = \frac{t_{12}t_{34}}{t_{13}t_{24}}$$

Defect fermion/boson description

- There is no local 1d lagrangian describing these CFT_1 correlators, but there is a description in terms of N 1d fermions χ^i (or bosons) coupled to the bulk *N*=4 SYM fields (Gomis, Passerini '06)

$$\mathrm{tr}_{A_k} P e^{\int dt(iA+\phi)} = \int D\bar{\chi} D\chi Da\, e^{-S_\chi} \qquad A_k: \text{ rank-}k \text{ antisymmetric repr.}$$

$$S_\chi = \int dt\, [\bar{\chi}(\partial_t - ia)\chi + i\bar{\chi}(iA+\phi)\chi] + ik \int dt\, a$$

- Similarly, 1d bosons give rank-*k* symmetric representation
- Defect local operators are the gauge invariant objects like $\bar{\chi}_i O^i_j \chi^j$ inserted on the line. General defect correlators are obtained by inserting such operators and computing path integral over 4d and 1d fields with the defect-CFT action

$$S_{\text{defect}-\text{CFT}} = S_{\mathcal{N}=4} + S_\chi$$

Correlators on the defect

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$$\equiv \frac{\langle \mathrm{tr} P\left[O_1(t_1)O_2(t_2)\cdots O_n(t_n)e^{\int dt(iA_t+\Phi^6)}\right]\rangle}{\langle W\rangle}$$

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The "super-displacement" multiplet

- Among the possible defect primaries, a special role is played by a set of 8_B+8_F "elementary insertions" forming a short multiplet of Osp(4*|4).
- The 8 bosonic insertions are
  - The 5 scalars not coupled to the loop $\Phi^a, \quad a = 1, \ldots, 5 \qquad \Delta = 1$
  - The "displacement operator" $\mathbb{F}_{ti} \equiv iF_{ti} + D_i\Phi^6, \quad i = 1, 2, 3 \qquad \Delta = 2$
- These operators have protected scaling dimensions, due to being in a short multiplet

The displacement operator, which is related to deformations of the defect in the transverse directions, has in fact, more generally, protected scaling dimension Δ=2 for any line defect, independently from supersymmetry

Two-point functions

- Because they have protected scaling dimensions, their exact 2-point functions take the form

$$\langle\langle \Phi^a(t_1)\Phi^b(t_2)\rangle\rangle = \delta^{ab}\frac{C_\Phi(\lambda)}{t_{12}^2}\,, \qquad \langle\langle \mathbb{F}_{ti}(t_1)\mathbb{F}_{tj}(t_2)\rangle\rangle = \delta_{ij}\frac{C_{\mathbb{F}}(\lambda)}{t_{12}^4}$$

- The normalization factors are related to the so-called "Brehmsstrahlung function" *(Correa, Maldacena, Sever '12)* , and can be determined exactly using supersymmetric localization. In the planar limit:

$$C_\Phi(\lambda) = 2B(\lambda)\,, \qquad C_{\mathbb{F}}(\lambda) = 12B(\lambda)$$

$$B(\lambda) = \frac{\sqrt{\lambda}\, I_2(\sqrt{\lambda})}{4\pi^2\, I_1(\sqrt{\lambda})}$$

Defect chiral primaries

- A more general class of protected defect operators is given by the products (inserted inside WL trace as usual):

$$\Phi^{(a_1} \ldots \Phi^{a_J)}$$

in the symmetric traceless of SO(5). They are in short multiplets and have protected scaling dimension Δ=J

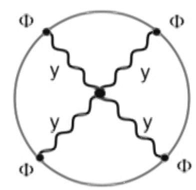
- Analogous to the familiar single-trace chiral primaries tr(Z^J)
- Exact results for correlation functions of these operators can be obtained from localization *(SG, Komatsu '18)*

Defect correlators at strong coupling

- In general correlation functions of operator insertions on the Wilson loop are non-trivial functions of position and of the coupling constant

- At weak coupling, they can be computed in perturbation theory
(Cooke, Dekel, Drukker, '17; Kyriu, Komatsu '18)

- At strong coupling, they can be computed from string theory using the AdS_2 worldsheet dual to the Wilson loop *(SG, Roiban, Tseytlin '17)*

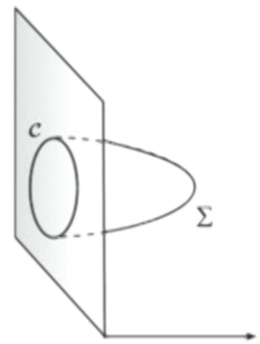


Wilson loop from string theory

- In AdS/CFT dictionary, the Wilson loop operator is dual to a minimal string surface ending on the contour defining the operator at the boundary

$$\langle W \rangle = \mathcal{Z}_{\text{string}} = \int_{X^M|_{\partial\Sigma}=(x^r(t),\theta^I(t))} \mathcal{D}X^M \mathcal{D}\psi \, e^{-S_{\text{string}}}$$

$$\langle W \rangle \overset{\lambda\to\infty}{\sim} e^{-S_{\text{class.}}} = e^{-\frac{\sqrt{\lambda}}{2\pi} A_{\text{reg}}}$$



- The bosonic part of the $AdS_5 x S^5$ string action reads, taking Poincare coordinates and using Nambu-Goto form (we omit fermions):

$$S_B = \frac{\sqrt{\lambda}}{2\pi} \int d^2\sigma \sqrt{\det\left[\frac{1}{z^2}\left(\partial_\mu x^r \partial_\nu x^r + \partial_\mu z \partial_\nu z\right) + \frac{\partial_\mu y^a \partial_\nu y^a}{(1+\frac{1}{4}y^2)^2}\right]}$$

where σ^μ=(t,s) are worldsheet coordinates, r=(0,i), i=1,2,3 label the coordinates of the (Euclidean) boundary, and a=1,…,5 are S^5 directions

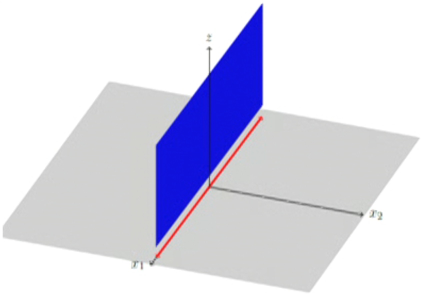
AdS_2 minimal surface

- The minimal surface dual to the 1/2-BPS Wilson line is given by

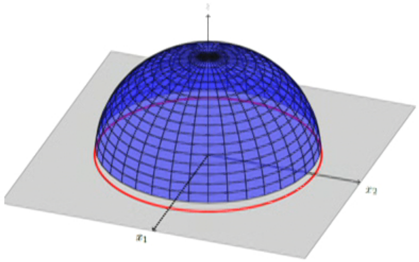
$$z = s \ , \qquad x^0 = t \ , \qquad x^i = 0 \ , \qquad y^a = 0$$

- The induced metric is just that of AdS_2 in Poincare coordinates

$$ds_2^2 = \frac{1}{s^2}\left(dt^2 + ds^2\right)$$



- Similarly, one can describe the minimal surface for the circular Wilson loop, which is given by AdS_2 with the hyperbolic disk coordinates



$$ds_2^2 = \frac{d\sigma^2 + d\tau^2}{\sinh^2 \sigma}$$

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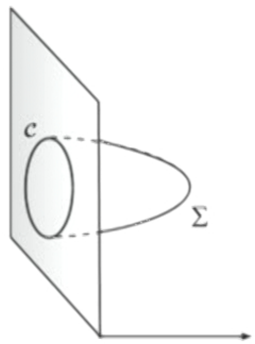

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where σ^μ=(t,s) are worldsheet coordinates, r=(0,i), i=1,2,3 label the coordinates of the (Euclidean) boundary, and a=1,…,5 are S^5 directions

AdS_2 minimal surface

- So the minimal surface dual to 1/2-BPS Wilson loop is an AdS_2 worldsheet embedded in AdS_5, and sitting at a point on S^5
- It preserves the same superconformal symmetry OSp(4*|4) as the dual Wilson loop operator
- SL(2,R) is just realized as the isometry of AdS_2
- The SO(3)xSO(5) correspond to rotations of the transverse coordinates $x^i(t,s)$ (i=1,2,3) and $y^a(t,s)$ (a=1,…,5)
- By expanding the string sigma model around this minimal surface, we can study the dynamics of small fluctuations of the worldsheet

Worldsheet fluctuations as fields in AdS_2

- It is convenient to adopt a *static gauge* where x^0 and z (which are identified with the AdS_2 worldsheet coordinates) do not fluctuate
- Then we get a Lagrangian for the 8 transverse fluctuations $x^i(t,s)$ and $y^a(t,s)$, which can be viewed as fields propagating in a rigid AdS_2

$$S_B = \frac{\sqrt{\lambda}}{2\pi} \int d^2\sigma \sqrt{g} L_B$$

$$L_B = 1 + L_2 + L_4 + \dots$$

$$L_2 = \frac{1}{2} g^{\mu\nu} \partial_\mu x^i \partial_\nu x^i + x^i x^i + \frac{1}{2} g^{\mu\nu} \partial_\mu y^a \partial_\nu y^a$$

$$L_{4y} = y^2 (\partial y)^2 + (\partial y)^4 + \dots$$

etc.

Worldsheet fluctuations as fields in AdS_2

- From the quadratic Lagrangian $L_2 = \frac{1}{2} g^{\mu\nu}\partial_\mu x^i \partial_\nu x^i + x^i x^i + \frac{1}{2} g^{\mu\nu}\partial_\mu y^a \partial_\nu y^a$ we find
  - 5 massless scalars y^a
  - 3 scalars x^i with $m^2=2$
- Since these may be viewed as scalar fields in AdS_2, they should be dual to operators inserted at the d=1 boundary, with dimension given by

$$\Delta(\Delta - 1) = m^2$$

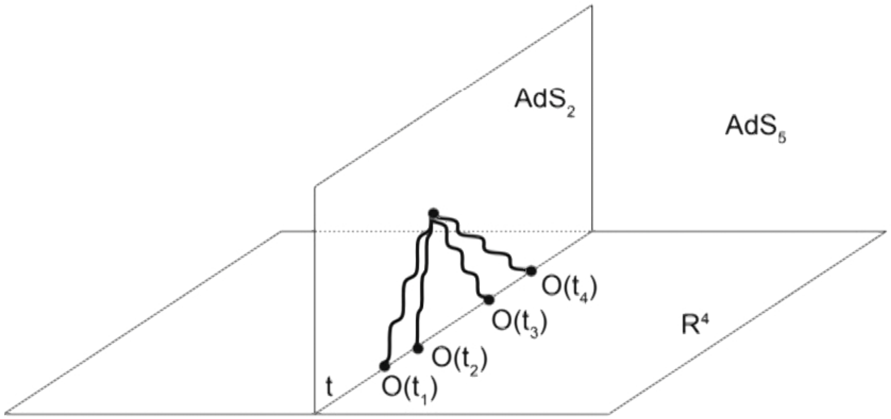
- So we recover the 8 bosonic operators in the super-displacement multiplet

$$y^a \quad \leftrightarrow \quad \Phi^a \quad \Delta = 1$$

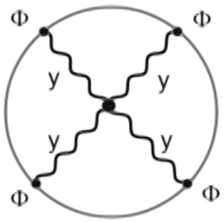
$$x^i \quad \leftrightarrow \quad \mathbb{F}_{ti} \quad \Delta = 2$$

Four-point functions

- The four-point functions of the dual operators at strong coupling can then be obtained from familiar AdS/CFT techniques by computing Witten diagrams in AdS_2



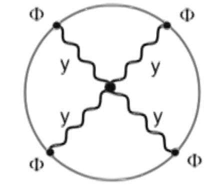
- E.g., the leading tree level connected term just involves contact 4-point interactions, with Witten diagram



Some comments

- These calculations are technically very similar to Witten diagram calculations in SUGRA in $AdS_5 x S^5$, but the interpretation is a bit different
- In the SUGRA case, one computes correlation functions of single trace local operators like, trZ^J, dual to closed string states. The expansion parameter is $G_N \sim 1/N^2$
- In our case, we compute correlators of insertions inside the Wilson loop trace (it is an expectation value of a *single trace,* non-local operator), dual to open string fluctuations. The expansion parameter is the worldsheet sigma-model coupling, i.e. string tension or $1/\sqrt{\lambda}$

Summary of 4-point function result from string theory



- Let us consider just the 4-point function of the S^5 fluctuations y^a , dual to the Δ=1 operator insertions Φ^a on the line defect.
- The 4pt function is specified by a function of cross ratio (multiplied by the fixed prefactor $1/(t_{12}\, t_{34})^2$). We can decompose it in the singlet (S), symmetric traceless (T) and antisymmetric (A) channels of SO(5). Calculation of AdS_2 Witten diagrams gives

$$G_S^{(1)}(\chi) = -\frac{2\left(\chi^4-4\chi^3+9\chi^2-10\chi+5\right)}{5(\chi-1)^2}+\frac{\chi^2\left(2\chi^4-11\chi^3+21\chi^2-20\chi+10\right)}{5(\chi-1)^3}\log|\chi| - \frac{2\chi^4-5\chi^3-5\chi+10}{5\chi}\log|1-\chi|\ ,$$

$$\chi=\frac{t_{12}t_{34}}{t_{13}t_{24}}$$

$$G_T^{(1)}(\chi) = -\frac{\chi^2\left(2\chi^2-3\chi+3\right)}{2(\chi-1)^2}+\frac{\chi^4\left(\chi^2-3\chi+3\right)}{(\chi-1)^3}\log|\chi|-\chi^3\log|1-\chi|\ ,$$

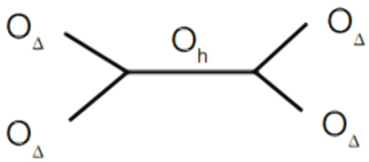
$$G_A^{(1)}(\chi) = \frac{\chi\left(-2\chi^3+5\chi^2-3\chi+2\right)}{2(\chi-1)^2}+\frac{\chi^3\left(\chi^3-4\chi^2+6\chi-4\right)}{(\chi-1)^3}\log|\chi|-(\chi^3-\chi^2-1)\log|1-\chi|$$

(SG, Roiban, Tseytlin '17)

Extracting OPE data

- From the small χ expansion we can read off the anomalous dimensions and OPE coefficients of "two-particle" operators appearing in the OPE

$$\mathcal{G}(\chi) = \sum_h c_{\Delta,\Delta;h}\, \chi^h\ {}_2F_1(h,h,2h,\chi)$$



- The lowest-lying unprotected operator is the singlet "2-particle" bound state $y^a\, y^a$, whose dimension turns out to be

$$\Delta_{y^a y^a} = 2 - \frac{5}{\sqrt{\lambda}} + \dots .$$

The dimension of the Φ^6 insertion

- At weak coupling, the lowest dimension singlet, unprotected operator in the defect primary spectrum is Φ^6: this is the insertion of the scalar that appears in the Wilson loop exponent
- Its dimension is known to 1-loop order *(Alday, Maldacena '07; Polchinski,Sully '11)*

$$\Delta_{\Phi^6} = 1 + \frac{\lambda}{4\pi^2} + \dots$$

- It is natural to expect that this operator smoothly goes to the lowest unprotected singlet at strong coupling, i.e .the "2-particle" operator $y^a\, y^a$. So we expect at strong coupling

$$\Delta_{\Phi^6} = 2 - \frac{5}{\sqrt{\lambda}} + \dots$$

Exact results from localization

- It turns out to be possible to derive a number of exact results for the correlators of a special type of protected insertions on the Wilson loop
- To use localization, we consider the 1/2-BPS circular loop rather than straight line. Correlators on the circle are related to those on the line by a conformal transformation, e.g.

$$\langle\langle O_\Delta(t_1)O_\Delta(t_2)\rangle\rangle_{\text{line}} = \frac{C_O}{t_{12}^{2\Delta}} \quad \rightarrow \quad \langle\langle O_\Delta(\tau_1)O_\Delta(\tau_2)\rangle\rangle_{\text{circle}} = \frac{C_O}{\left(2\sin\frac{\tau_{12}}{2}\right)^{2\Delta}}$$

and similarly for 4-point functions, with cross ratio now given by

$$\chi = \frac{\sin\frac{\tau_{12}}{2}\sin\frac{\tau_{34}}{2}}{\sin\frac{\tau_{13}}{2}\sin\frac{\tau_{24}}{2}}$$

Exact results from localization

- Recall that the expectation value of the circular loop is given exactly by the Gaussian matrix model (Erickson, Semenoff, Zarembo '00; Drukker, Gross '00; Pestun '09)

$$\langle W_{\mathrm{circle}} \rangle = \int DM \, \frac{1}{N} \mathrm{tr} e^{M} e^{-\frac{2N}{\lambda} \mathrm{tr} M^2} \overset{N \to \infty}{=} \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})$$

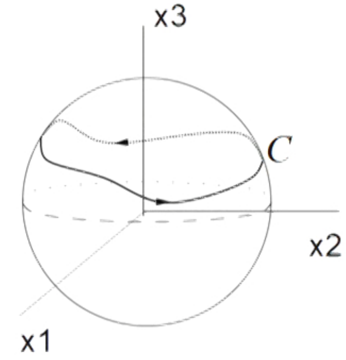
- To derive exact results for correlators of insertions on the Wilson loop, we will need to consider a more general family of 1/8-BPS Wilson loops constructed in *Drukker, SG, Ricci, Trancanelli '07*

- These Wilson loops are defined on generic contours on an S^2 subspace of R^4 (or S^4), and couple to three of the six scalar fields, say Φ^1, Φ^2, Φ^3

The 1/8-BPS Wilson loops

- Explicitly, take an S^2 given by $x_1^2 + x_2^2 + x_3^2 = 1$ in Cartesian coordinates, and define the Wilson loop operator

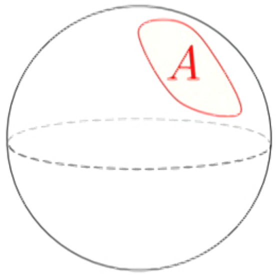
$$\mathcal{W} \equiv \frac{1}{N} \mathrm{tr}\, \mathrm{P} \left[e^{\oint_C \left(i A_j + \epsilon_{kjl} x^k \Phi^l \right) dx^j} \right]$$



- This preserves 1/8 of the superconformal symmetries for generic contour
- The 1/2-BPS circle is a special case: it corresponds to the contour being a great circle of S^2
- It was conjectured in *Drukker et al '07*, and essentially proved in *Pestun '09* by localization, that the expectation value, as well as correlators of any number of Wilson loops on the S^2, is captured by 2d YM theory (more precisely its truncation to the *"zero-instanton"* sector)

The 1/8-BPS Wilson loops

- This in particular implies that the expectation value only depends on the area singled out by the loop on S^2



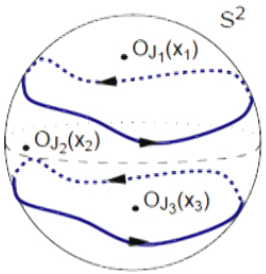
- The expectation value is given by the same function as for the 1/2-BPS circular loop, but with a rescaled coupling constant. E.g. in the planar limit:

$$\langle W(A)\rangle = \frac{2}{\sqrt{\lambda'}} I_1(\sqrt{\lambda'})\,, \qquad \lambda' \equiv \frac{A(4\pi - A)}{4\pi^2}\lambda$$

with $A=2\pi$ being the 1/2-BPS case

1/8-BPS Wilson loops and local operators

- More generally, localization applies to general correlation functions of Wilson loops and local operators *(SG, Pestun '09-'12)*



$$\langle W_{R_1}(\mathcal{C}_1) W_{R_2}(\mathcal{C}_2) \cdots O_{J_1}(x_1) O_{J_2}(x_2) \cdots \rangle_{4d}$$
$$= \langle W^{2d}_{R_1}(\mathcal{C}_1) W^{2d}_{R_2}(\mathcal{C}_2) \cdots \mathrm{tr}\, F^{J_1}_{2d}(x_1)\, \mathrm{tr}\, F^{J_2}_{2d}(x_2) \cdots \rangle_{\text{2d YM}}$$

- The relevant local operators may be inserted inside the Wilson loop trace, or in the "bulk", and they involve the position-dependent combination of scalars

$$(x_1\Phi_1 + x_2\Phi_2 + x_3\Phi_3 + i\Phi_4)^J \equiv \tilde{\Phi}^J, \qquad x_1^2 + x_2^2 + x_3^2 = 1$$

1/8-BPS Wilson loops and local operators

- These are just chiral primaries of the form $(Y \cdot \Phi)^J$, with *Y* a null vector which is taken to be position dependent. They were first studied in *Drukker, Plefka '09*
- A crucial property is that their correlation functions are *position independent.*
- In the localization approach, they are mapped to insertions of powers of the Hodge dual of the 2d YM field strength:

$$\tilde{\Phi} \quad \Leftrightarrow \quad i * F_{2d}$$

Correlators on the Wilson loop

- Focusing on our problem of defect correlators on the circular loop, this means that localization allows us to study correlators of the operators

$$\tilde{\Phi}^J = (Y_i(\tau_i) \cdot \Phi(\tau_i))^J \qquad\qquad Y_i = (\cos\tau_i, \sin\tau_i, 0, i, 0, 0)$$

- These operators form a *"topological subsector"* of the defect CFT, since their *n*-point correlation functions

$$\langle \tilde{\Phi}^{L_1}(\tau_1)\tilde{\Phi}^{L_2}(\tau_2)\cdots\tilde{\Phi}^{L_n}(\tau_n)\rangle_{\text{circle}}$$

are completely position independent

Defect CFT data from topological correlators

- Note that the 2-point and 3-point functions of the general defect chiral primaries are completely fixed by symmetries up to overall functions of the coupling

$$\langle\langle (Y_1 \cdot \vec{\Phi})^{L_1}(\tau_1) \ \ (Y_2 \cdot \vec{\Phi})^{L_2}(\tau_2) \rangle\rangle_{\text{circle}} = n_{L_1}(\lambda, N) \times \frac{\delta_{L_1,L_2}(Y_1 \cdot Y_2)^{L_1}}{}$$

$$\langle\langle (Y_1 \cdot \vec{\Phi})^{L_1}(\tau_1) \ \ (Y_2 \cdot \vec{\Phi})^{L_2}(\tau_2) \ \ (Y_3 \cdot \Phi)^{L_3}(\tau_3) \rangle\rangle_{\text{circle}} = c_{L_1,L_2,L_3}(\lambda, N) \times \frac{(Y_1 \cdot Y_2)^{L_{12|3}}(Y_2 \cdot Y_3)^{L_{23|1}}(Y_3 \cdot Y_1)^{L_{31|2}}}{\left(2\sin\frac{\tau_{12}}{2}\right)^{2L_{12|3}}\left(2\sin\frac{\tau_{23}}{2}\right)^{2L_{23|1}}\left(2\sin\frac{\tau_{31}}{2}\right)^{2L_{31|2}}}$$

So we can use localization for the "topological correlators" to find the exact 2-point normalization and structure constants of the general chiral primaries on the defect.

- Of course, for higher-point functions, one cannot fully reconstruct the general correlators from the topological ones.

Correlators from localization

- Using the localization correspondence

$$\tilde{\Phi} \quad \Leftrightarrow \quad i * F_{2d}$$

and area-preserving invariance in 2d YM, one can obtain correlators of *L*-point scalar insertions by taking multiple area-derivatives of the Wilson loop VEV

$$\langle \underbrace{\tilde{\Phi} \cdots \tilde{\Phi}}_{L} \rangle|_{\text{circle}} = \left. \frac{\partial^L \langle \mathcal{W} \rangle}{(\partial A)^L} \right|_{A=2\pi}$$

- E.g. for the 2-point function, we get for the normalized correlator

$$\langle\langle \tilde{\Phi}(\tau_1) \tilde{\Phi}(\tau_2) \rangle\rangle = \frac{\partial^2}{\partial A^2} \log \langle W(A) \rangle \Big|_{A=2\pi} = -\frac{\sqrt{\lambda} I_2(\sqrt{\lambda})}{4\pi^2 I_1(\sqrt{\lambda})}$$

which gives the Bremsstrahlung function B(λ)

The general composite operators

- For general operators $\tilde{\Phi}^J = (Y_i(\tau_i)\cdot\Phi(\tau_i))^J$, one needs to define the properly normal-ordered composite operators
- This can be accomplished systematically by a Gram-Schmidt orthogonalization procedure *(SG, Komatsu '18)*
- In the planar limit, one gets the explicit form of the charge-*L* operators as a determinant

$$:\tilde{\Phi}^L: = \frac{1}{D_L}\begin{vmatrix} \langle\mathcal{W}\rangle & \langle\mathcal{W}\rangle^{(1)} & \cdots & \langle\mathcal{W}\rangle^{(L)} \\ \langle\mathcal{W}\rangle^{(1)} & \langle\mathcal{W}\rangle^{(2)} & \cdots & \langle\mathcal{W}\rangle^{(L+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \langle\mathcal{W}\rangle^{(L-1)} & \langle\mathcal{W}\rangle^{(L)} & \cdots & \langle\mathcal{W}\rangle^{(2L-1)} \\ \mathbf{1} & \tilde{\Phi} & \cdots & \underbrace{\tilde{\Phi}\cdots\tilde{\Phi}}_{L} \end{vmatrix} \qquad D_L = \begin{vmatrix} \langle\mathcal{W}\rangle & \langle\mathcal{W}\rangle^{(1)} & \cdots & \langle\mathcal{W}\rangle^{(L-1)} \\ \langle\mathcal{W}\rangle^{(1)} & \langle\mathcal{W}\rangle^{(2)} & \cdots & \langle\mathcal{W}\rangle^{(L)} \\ \vdots & \vdots & \ddots & \vdots \\ \langle\mathcal{W}\rangle^{(L-1)} & \langle\mathcal{W}\rangle^{(L)} & \cdots & \langle\mathcal{W}\rangle^{(2L-2)} \end{vmatrix}$$

with $\langle\mathcal{W}\rangle^{(k)} \equiv (\partial_A)^k \langle\mathcal{W}\rangle$

- This allows to obtain exact results for all correlation functions in the topological subsector

Some explicit 3-point function results

$$\langle\langle :\tilde{\Phi}^2::\tilde{\Phi}::\tilde{\Phi}:\rangle\rangle = \langle\langle :\tilde{\Phi}^2::\tilde{\Phi}^2:\rangle\rangle$$

$$\langle\langle :\tilde{\Phi}^2::\tilde{\Phi}^2::\tilde{\Phi}^2:\rangle\rangle = -\frac{\lambda^{3/2} I_0\left(\sqrt{\lambda}\right)^3}{32\pi^6 I_1\left(\sqrt{\lambda}\right)^3} + \frac{51\lambda}{32\pi^6} - \frac{3\lambda I_0\left(\sqrt{\lambda}\right)^2}{8\pi^6 I_1\left(\sqrt{\lambda}\right)^2} - \frac{3\sqrt{\lambda}(\lambda+40) I_0\left(\sqrt{\lambda}\right)}{32\pi^6 I_1\left(\sqrt{\lambda}\right)} + \frac{37}{4\pi^6}$$

$$\langle\langle :\tilde{\Phi}^3::\tilde{\Phi}^2::\tilde{\Phi}^1:\rangle\rangle = \langle\langle :\tilde{\Phi}^3::\tilde{\Phi}^3:\rangle\rangle$$

$$\langle\langle :\tilde{\Phi}^3::\tilde{\Phi}^3::\tilde{\Phi}^2:\rangle\rangle = -\frac{3\lambda(5\lambda+72) I_0\left(\sqrt{\lambda}\right)^4}{256\pi^8 I_1\left(\sqrt{\lambda}\right)^2 I_2\left(\sqrt{\lambda}\right)^2} - \frac{3\sqrt{\lambda}(127\lambda+1920) I_0\left(\sqrt{\lambda}\right)^3}{128\pi^8 I_1\left(\sqrt{\lambda}\right) I_2\left(\sqrt{\lambda}\right)^2} + \frac{3(\lambda(2\lambda+579)+6192) I_0\left(\sqrt{\lambda}\right)^2}{64\pi^8 I_2\left(\sqrt{\lambda}\right)^2}$$
$$+ \frac{3(\lambda(5\lambda-757)-6336) I_1\left(\sqrt{\lambda}\right) I_0\left(\sqrt{\lambda}\right)}{32\pi^8\sqrt{\lambda}\left(I_0\left(\sqrt{\lambda}\right) - \frac{2I_1\left(\sqrt{\lambda}\right)}{\sqrt{\lambda}}\right)^2} + \frac{3(\lambda(\lambda(9\lambda-112)+4960)+34176) I_1\left(\sqrt{\lambda}\right)^2}{256\pi^8 \lambda I_2\left(\sqrt{\lambda}\right)^2}.$$

Weak and Strong coupling checks

- One may test the exact results for the correlators

$$\langle :\tilde{\Phi}^{L_1}::\tilde{\Phi}^{L_2}: \cdots :\tilde{\Phi}^{L_m}:\rangle = \oint d\mu \prod_{k=1}^{m} Q_{L_k}(x)$$

in the weak and strong coupling limit (where $Q_L(x)$ turn out to be related to Chebyshev and Hermite polynomials respectively)

- We have checked that the first two orders in perturbation theory both at weak and strong coupling indeed precisely agree with the exact predictions

- On the string theory side, we essentially need to compute correlation functions of products of S^5 fluctuations $(Y(\tau)\cdot y)^L = \tilde{y}^L$, bringing the insertion points of these operators to the boundary of AdS_2

Non-supersymmetric Wilson loop

- It is natural to also study the ordinary Wilson loop operator

$$\mathrm{tr} P e^{\oint iA}$$

- For smooth contours, there are no logarithmic divergences in its expectation value
- When the contour is a line or circle, the operator preserves again an SL(2,R) conformal symmetry
- Correlation functions of operator insertions should then define a non-supersymmetric defect CFT with SL(2,R)xSO(3)xSO(6) symmetry

A defect RG flow

- It is useful to consider a more general operator interpolating between the ordinary Wilson loop and the Maldacena-Wilson loop

$$W^{(\zeta)} = \mathrm{tr} P e^{\oint iA + \zeta \Phi^6}$$

- For generic ζ, there are logarithmic divergences and ζ develops a non-trivial beta function. At weak coupling it can be computed to be *(Alday, Maldacena '07; Polchinski,Sully '11)*

$$\beta_\zeta = -\frac{\lambda}{8\pi^2}\zeta(1-\zeta^2) + O(\lambda^2)$$

- $\zeta=0$ and $\zeta=1$ are conformal fixed points corresponding to the ordinary and 1/2-BPS Wilson loop

A defect RG flow

- The dimension of Φ^6 at the fixed points at weak coupling can be found to be

$$\Delta(1) = 1 + \frac{\lambda}{4\pi^2} + \ldots \,, \qquad \Delta(0) = 1 - \frac{\lambda}{8\pi^2} + \ldots$$

- Since Φ^6 is a slightly relevant defect operator at $\zeta = 0$, running of ζ can be interpreted as a defect RG flow between the non-susy Wilson loop in the UV and the 1/2-BPS Maldacena-Wilson loop in the IR
- It is natural to expect that F = log<W> plays the role of a d=1 "defect free energy", that should satisfy

$$F_{UV} > F_{IR}$$

(see *Kobayashi, Nishioka, Sato, Watanabe '18 and Nishioka's talk*)

- A direct perturbative calculation (Beccaria, SG, Tseytlin '17) gives

$$\langle W^{(\zeta)} \rangle = 1 + \frac{1}{8}\lambda + \left[\frac{1}{192} + \frac{1}{128\,\pi^2}(1-\zeta^2)^2\right]\lambda^2 + \mathcal{O}(\lambda^3)$$

which indeed satisfies

$$\log\langle W^{(0)} \rangle \;>\; \log\langle W^{(1)} \rangle$$

The inequality can be also shown to hold at strong coupling, where one finds

$$\langle W^{(0)} \rangle \sim \sqrt{\lambda} e^{\sqrt{\lambda}} \qquad \langle W^{(1)} \rangle \sim \lambda^{-3/4} e^{\sqrt{\lambda}} \qquad \lambda \gg 1$$

Ordinary Wilson loop at strong coupling

- The dual of the ordinary Wilson loop $\mathrm{tr} P e^{\oint iA}$ at strong coupling should be a string worldsheet with *Neumann*, rather than Dirichlet, boundary conditions on S^5 *(Alday, Maldacena '07; Polchinski,Sully '11)*
- For circular/straight Wilson loop, the worldsheet is still AdS_2, but it does not sit at a point on S^5, rather we should integrate over 5 zero modes representing the position on S^5
- More explicitly, using S^5 embedding coordinates Y^A ($Y^A Y^A=1$), we may write

$$Y^A = \sqrt{1-\zeta^2}\, n^A + \zeta^A \qquad\qquad n^A \zeta^A = 0$$

with ζ^A (σ) fluctuations obeying Neumann conditions, and integration over the constant unit vector n^A restores SO(6) symmetry

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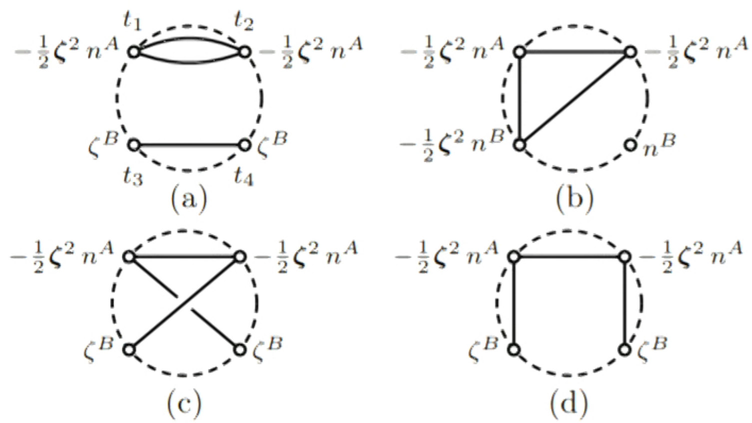
Correlation functions at strong coupling

- Similarly to the case of the 1/2-BPS loop, one can compute 4-point functions of scalar insertions from the AdS_2 worldsheet theory. Considerably more complicated due to Neumann boundary conditions and logarithmic propagators
- Explicit result for scalar 4-point function

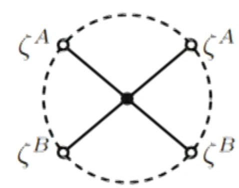
(Beccaria, SG, Tseytlin '19)



$$\langle Y^A(t_1) Y^A(t_2) Y^B(t_3) Y^B(t_4)\rangle = \frac{1}{|t_{12}t_{34}|^{2\Delta}}\, G_S$$

$$G_S = 1 + \frac{10}{(\sqrt{\lambda})^2}\log^2(1-\chi) + \frac{1}{(\sqrt{\lambda})^3} G_S^{(3)} + \mathcal{O}\big(\tfrac{1}{(\sqrt{\lambda})^4}\big)\,,$$

$$G_S^{(3)} = 80\Big[\mathrm{Li}_3(\chi) + \mathrm{Li}_3\big(\tfrac{\chi}{\chi-1}\big) - \mathrm{Li}_2(\chi)\,\log(1-\chi)\Big] + 40\,\log\frac{\chi}{1-\chi}\,\log^2(1-\chi)$$
$$- 10\,\frac{\chi^2}{1-\chi}\,\log\chi + 5\,\big(5 - \tfrac{10}{\chi} - 2\,\chi\big)\,\log(1-\chi) - 50 + 4\,d_2\,\log^2(1-\chi)$$

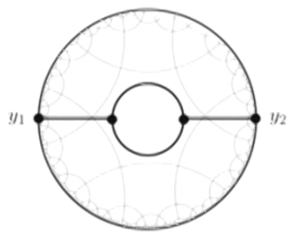


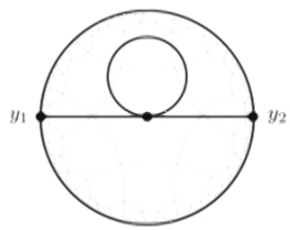
Conclusion

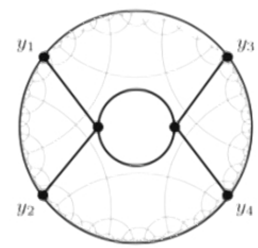
- Correlation functions on the straight/circular Wilson loop have the structure of a d=1 conformal system living on the defect
- Correlator of operator insertions on the loop are dual to AdS_2 amplitudes for the fluctuations of the open string worldsheet
  - Is there a manifestation of integrability in the AdS_2 amplitudes? AdS_2 analog of S-matrix factorization? (Mellin amplitudes for 1d CFT?)
- For the defect CFT on the 1/2-BPS Wilson loop, exact results may be obtained in a "topological subsector" of special operator insertions
- Perhaps combining information from localization, integrability and bootstrap techniques *(Liendo, Meneghelli '16; Liendo, Meneghelli, Mitev '18. Also Mazac, Paulos '18-'19...)* one may be able to solve this defect CFT
- The non-supersymmetric Wilson loop in $\mathcal{N}$=4 SYM define another interesting, non-supersymmetric defect CFT.
  - Integrability? *(Correa, Leoni, Luque '18)*

Conclusion

- Some other directions
  - Higher-point functions (localization checks; signs of integrability?...)
  - Loops in the AdS_2 worldsheet theory ($1/\sqrt{\lambda}$ corrections to defect CFT_1 data)



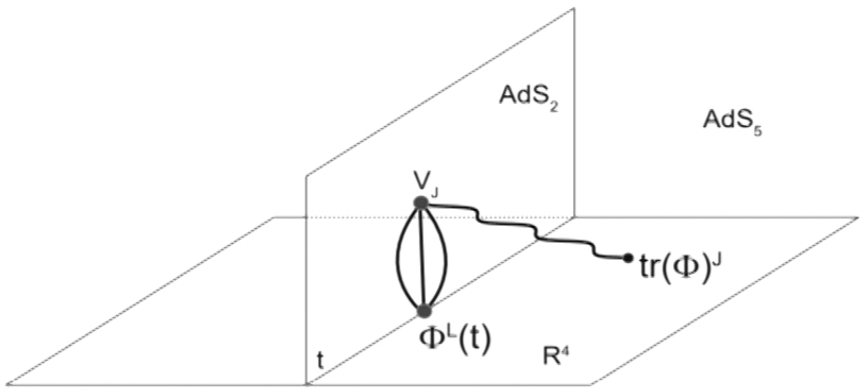




  - Wilson loops in more general representations
    - "Giant Wilson loops": rank k~N symmetric/antisymmetric representations dual to D3/D5 branes with AdS_2xS^2 and AdS_2xS^4 worldvolumes
    - "Bubbling Geometries"

Bulk-defect correlators

- So far I focused mainly on correlators of defect operators, which are captured by the AdS_2 open string worldsheet theory
- But one more generally can also consider also "bulk-defect" correlators: correlation functions of the Wilson and single-trace operators inserted away from the loop, e.g. <W trZ^J>, <W[O(t)] trZ^J>...
- This correspond to an "open-closed" string amplitude of the schematic form (to leading order):



Localization: *(SG, Komatsu '18)*

$$\langle \mathcal{W}[\prod_{k=1}^{n} \tilde{\Phi}^{L_k}]\mathrm{tr}[\tilde{\Phi}^J]\rangle \sim \oint d\mu \; B_J(x) \prod_{k=1}^{n} Q_{L_k}(x)$$

$$B_J(x) = \frac{4\pi g x^{J+1}}{1+x^2}$$