Title: Wilson line impurities, flows and entanglement entropy

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Collection: Boundaries and Defects in Quantum Field Theory

Date: August 06, 2019 - 4:15 PM

URL: http://pirsa.org/19080064

Wilson line impurities, flows and entanglement

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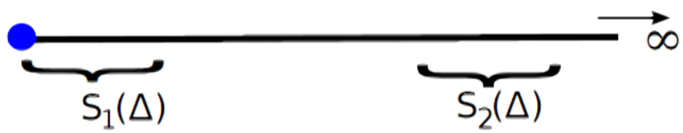
August 6, 2019,
Boundaries and Defects in QFT, Perimeter Institute

Based on: Work in progress w/ D. Gutiez, C. Hoyos, M. Jarvinen
& 1711.01554 (with D. Silvani)

Introduction and Motivation

- Quantum Impurities:
  Gapless quantum bulk d.o.f. interacting with a point-like impurity carrying quantum d.o.f. E.g. Kondo effect.
- Heavy quark probes in CFTs e.g. $\mathcal{N}=4$ SYM are such impurites; study their QM and deformations.
- Kondo effect $\leftrightarrow$ CFT_2 with boundary: for flows triggered on boundary (and critical bulk), a g-theorem applies. [Affleck-Ludwig; Friedan-Konechny]
- Entanglement Entropy (EE) can be used to compute g [Takayanagi; Casini-Landea-Torroba]:



$$S^{(1)}_{\rm EE}(\Delta) - S^{(2)}_{\rm EE}(\Delta) = \ln g\,.$$

Introduction and Motivation

- Higher dimensions [Fujita-Takayanagi-Tonni; Estes-Jensen-O'Bannon-Tsatis-Wrase]: EE contribution to S^{d-2}, from codimension one defect in CFT_d $\leftrightarrow$ g-function

- Heavy quark "impurities" in gauge theories compute Wilson/Polyakov loops. For generic representations these are computed holographically by objects with nontrivial dynamics.

  Is there a version of the g-theorem via EE for these ?

Outline

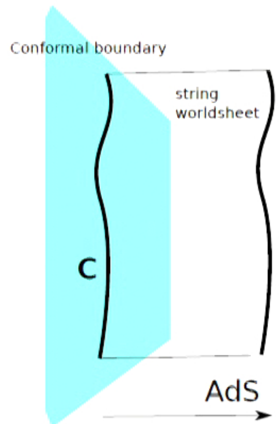
- Quantum mechanics of Wilson line impurities
- Saddle points and relation to probe D-branes
- D3-/D5-brane non-conformal embeddings
- Entanglement entropies

Wilson lines/heavy quarks

- Wilson loops in representation R and contour C:

$$W[C] = \frac{1}{\dim[R]} \mathrm{Tr}_R \mathcal{P} \exp\left(i \oint_C A_\mu \dot{x}^\mu \, ds \right)$$

- In large-N gauge theories with string duals, computed by (multiple) string worldsheets [Maldacena (2001); Rey-Theisen-Yee (2001)]



Wilson lines/heavy quarks

- Wilson line $\leftrightarrow$ Phase associated to heavy quark worldline.
- Heavy quark "impurity" interacts with the "ambient" Yang-Mills degrees of freedom:

$$\mathcal{L} = \mathcal{L}_{\rm imp} + \mathcal{L}_{\rm int} + \mathcal{L}_{\rm YM}$$

- Integrate out Yang-Mills d.o.f. to obtain a defect action. [E.g. C.Hoyos (2018)]

BPS Wilson lines

- SUSY YM theories: natural to consider BPS Wilson lines, e.g. in $\mathcal{N} = 4$ SUSY Yang-Mills:

$$W[C] = \frac{1}{\dim[R]} \mathrm{Tr}_R \mathcal{P} \exp\left[i \oint_C \left(A_\mu \dot{x}^\mu + \Phi_J n^J(s) |\dot{x}| \right) ds \right]$$

$\{\Phi_J\}_{J=1,\ldots,6}$: $\mathcal{N} = 4$ scalars and $n^J n_J = 1$

- Defect theories for $\frac{1}{2}$-BPS Wilson lines (straight and circular):

[Gomis-Passerini (2006)]

Rank k **symmetric** rep. $\leftrightarrow$ Bosonic QM

Rank k **antisymmetric** rep. $\leftrightarrow$ Fermionic QM

Defect theory

$$\langle W(C)\rangle = \int \mathcal{D}A_\mu \mathcal{D}\Phi\, [d\chi]\, [d\chi^\dagger][da_\tau]\, e^{i\left(S_{\rm imp}+S_{\rm int}+S_{\mathcal{N}=4}\right)}$$

$$S_{\rm imp} + S_{\rm int} = \int d\tau \left(i\chi^\dagger_m \left[\partial_\tau + ia_\tau + i\left(A_\tau + n_J\phi^J\right)\right]\chi^m + ka_\tau\right)$$

- $\{\chi^m\}$: Bosons/fermions in fundamental of $SU(N)$
- a_τ: $U(1)$ gauge field/Lagrange multiplier for particle number (rank):

$$\boxed{\chi^\dagger_m\chi^m = k}$$

- Integrate out $\tilde{A} = (A_\tau + n_J\Phi^J)$

Defect theory

- $\langle \tilde{A}(\tau)\tilde{A}(\tau')\rangle = D(\tau - \tau')$
- $D(\tau - \tau')$ vanishes for the straight line and constant for circular loop [Erickson-Semenoff-Zarembo(2000)]:

$$D_\circ(\tau - \tau') = \frac{\lambda}{4N(2\pi R)^2} \qquad \lambda \equiv g^2_{\rm YM} N\,.$$

$$S_{\rm eff} = \int_0^{2\pi R} d\tau \left(i\chi^\dagger_m \partial_\tau \chi^m - \tfrac{i}{2} D_\circ \int_0^{2\pi R} \chi^\dagger_m(\tau)\chi^n(\tau)\,\chi^\dagger_n(\tau')\,\chi^m(\tau')\,d\tau' + a_\tau(\chi^\dagger_m\chi^m - k)\right)\,.$$

Defect theory

- **Two routes**:

  (i) Define $M^n_m \sim \int d\tau \chi^\dagger_m(\tau)\chi^n(\tau)$; integrate out $\{\chi_m\}$

  $\Longrightarrow \langle \mathrm{Tr}_k M \rangle$ in **Gaussian matrix model for M.**

  (ii) Define bilocal field $\Sigma(\tau,\tau') \sim \frac{1}{N}\chi^m(\tau)\chi^\dagger_m(\tau')$; integrate out $\{\chi_m\}$ $\Longrightarrow$ Effective action and **large-N saddle for Σ.**

- Routes (i) and (ii) agree for fermionic QM / **Antisymmetric** and with strong coupling wrapped D5-brane $\sim AdS_2 \times S^4$:

  [Yamaguchi (2006); Hartnoll-SPK (2006);]

$$\langle W_{\mathcal{A}_k} \rangle = \exp\left[\frac{2N}{3\pi}\sqrt{\lambda}\sin^3\theta_k\right], \qquad \frac{k}{N} = \text{fixed}, \qquad \lambda \gg 1,$$

$$\theta_k - \sin\theta_k \cos\theta_k = \pi k/N$$

Defect theory

- $\langle \tilde{A}(\tau)\tilde{A}(\tau')\rangle \;=\; D(\tau-\tau')$
- $D(\tau-\tau')$ vanishes for the straight line and constant for circular loop [Erickson-Semenoff-Zarembo(2000)]:

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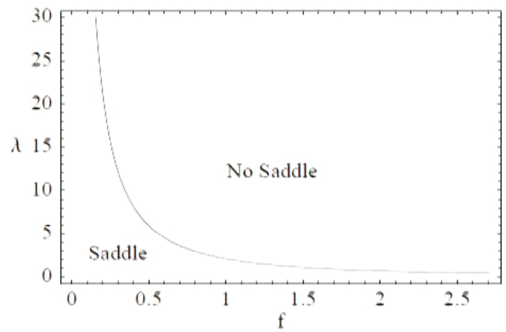
$$S_{\rm eff} \;=\; \int_0^{2\pi R} d\tau\,\Big(i\chi_m^\dagger\partial_\tau\chi^m \;-\; \tfrac{i}{2}D_\circ\int_0^{2\pi R}\chi_m^\dagger(\tau)\chi^n(\tau)\,\chi_n^\dagger(\tau')\,\chi^m(\tau')\,d\tau' + a_\tau(\chi_m^\dagger\chi^m \;-\; k)\Big)\,.$$

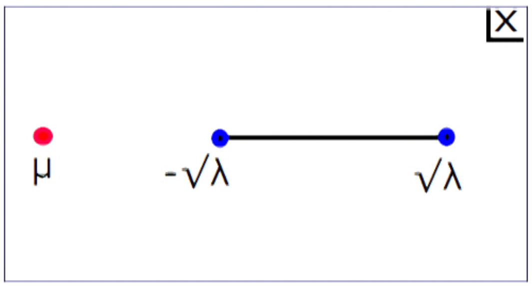
Defect theory: Symmetric rep

- For **symmetric rep** $\frac{k}{N}$ fixed, large-N saddle pt in Gaussian MM:

$$f = \frac{k}{N} = \frac{2}{\lambda}\int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \frac{dx}{\pi} \frac{\sqrt{\lambda - x^2}}{e^{(x-\mu)} - 1}$$

Weak coupling saddle disappears at critical value of λ (μ hits branch point)





- For large λ, a different saddle with one eigenvalue "pulled out" matches supergravity result from wrapped D3-brane $\sim AdS_2 \times S^2$. [Yamaguchi '07]

Bosonic defect theory (Symmetric rep. loop)

- Separate out zero modes: $\chi^m = v^m + \delta\chi^m(\tau)$
- "Radial": $\rho = \frac{1}{N}\sum v_m^\dagger v^m$ Bilocal: $\Sigma = \frac{1}{N}\delta\chi^m(\tau)\delta\chi_m^\dagger(\tau')$

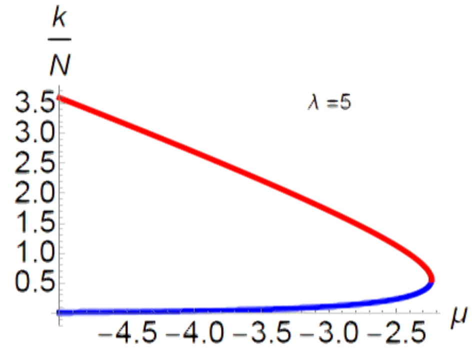
  Measure: $\prod_m dv^m\, dv_m^\dagger \sim d\rho\,\rho^{(2N-1)/2}$
- Integrate out $\{\delta\chi^m\} \longrightarrow S_{\rm eff}[\rho,\Sigma] \sim \mathcal{O}(N)$
- **Two Saddle points**: $\rho_\pm = \frac{2}{\lambda}(\mu \pm \sqrt{\mu^2-\lambda})$

$$\mathbf{S}_- : \qquad \frac{k}{N} = \frac{2}{\lambda}\int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \frac{dx}{\pi}\, \frac{\sqrt{\lambda - x^2}}{e^{(x-\mu)} - 1}\,.$$

$$\mathbf{S}_+ : \qquad \frac{k}{N} = \frac{4}{\lambda}\sqrt{\mu^2-\lambda} + \frac{2}{\lambda}\int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \frac{dx}{\pi}\, \frac{\sqrt{\lambda - x^2}}{e^{(x-\mu)} - 1}\,.$$

Merging saddles for symmetric loop

- Fixed λ and increasing k/N the saddles $\mathbf{S}_\pm$ merge **smoothly**



- Fixed k/N, free energy is smooth in λ. Merger point for$\lambda \gg 1$: $\boxed{\dfrac{k}{N} \simeq \sqrt{\dfrac{2}{\pi}}\dfrac{\zeta(3/2)}{\lambda_{\rm crit}^{3/4}}}$

- Saddle $\mathbf{S}_+$ matches D3-brane Wilson loop when $\lambda \gg 1$:

$$W_{\mathcal{S}_k} = \exp\left[2N\left(\sinh^{-1}\kappa + \kappa\sqrt{1+\kappa^2}\right)\right] \qquad \kappa = \frac{k\sqrt{\lambda}}{N}$$

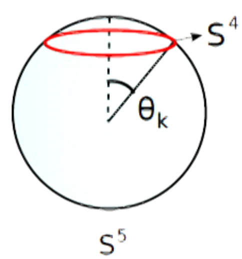
[Drukker-Fiol]

Dual D-brane probes

- When $\mathrm{rank}(R) = k$ with $\frac{k}{N}$ = fixed as $N \to \infty$

  k strings $\longrightarrow$ wrapped D-brane.

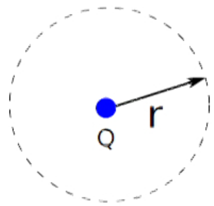
- Rank k antisymm. $(\mathcal{A}_k)$: **D5-brane** - $AdS_2 \times S^4 \subset AdS_5 \times S^5$
  Strong coupling $\Longrightarrow$ Fermi levels in impurity theory filled;
  $\mu = -\sqrt{\lambda}\cos\theta_k$



[Camino-Paredes-Ramallo (2001); Hartnoll-SPK (2006); Yamaguchi (2006)]

- Rank k symm. $(\mathcal{S}_k)$: **D3-brane** -$AdS_2 \times S^2 \subset AdS_5$ [Drukker-Fiol]
  Strong coupling $\Longrightarrow$ Zero modes/Bose condensate in impurity theory

Probing the defect



- Probe this with:
  - (i) $\langle \mathcal{O}_{\rm YM} \rangle (r)$: Gauge-invariant Yang-Mills operators
  - (ii) $S_{\rm EE}(r)$: Entanglement Entropy across sphere
  - (iii) Deformations of $\mathcal{L}_{\rm imp}$

Yang-Mills VEVs induced by heavy quarks

- Static quark sources Yang-Mills fields: $\langle \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} \rangle$, $\langle T_{\mu\nu} \rangle \ldots$
- VEVs $\leftrightarrow$ Normalizable modes of AdS_5 fields
- $S_{\rm bulk} \sim N^2 S_{\rm sugra}[\phi, g_{\mu\nu} \ldots] + \int J_{\rm source}$

  $J_{\rm F1} \sim \sqrt{\lambda}$ $\qquad$ $J_{\rm D3,D5} \sim N\sqrt{\lambda}$

- Dilaton ϕ $\leftrightarrow$ Scalar "glueball" $\sim \frac{1}{N} \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} + \ldots$

  Metric $g_{\mu\nu}$ $\leftrightarrow$ Stress tensor $T_{\mu\nu}$

Yang-Mills VEVs

E.g. Extracting $\langle \mathcal{O} \rangle = \langle \frac{1}{N} \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} + \ldots \rangle$:

- Solve linearized e.o.m. of dilaton in presence of source in AdS_5

$$\phi(\vec{x}, z) = \frac{1}{N} \int d^5x'\, G_{\mathrm{AdS}}(x^\mu, z; x'^\mu, z')\, J_{\mathrm{source}}(x', z')$$

[Danielsson, Kesko-Vakkuri,Kruczenski ('98)]

- Expand RHS near AdS-boundary $z \to 0$ and find coefficient of z^4, given $\Delta_{\mathcal{O}} = 4$.

Conformal impurity with representation $\mathcal{A}_k$

- The straight (BPS) Wilson line in $\mathcal{N} = 4$ SYM has $\langle W \rangle = 1$ for any representation.
- For representation $\mathcal{A}_k$: D5-brane embedding $\simeq AdS_2 \times S^4$

$$\underline{AdS_5}: \quad ds^2 = \frac{dz^2}{z^2} + \frac{1}{z^2}\left(-dt^2 + d\vec{x}^2\right)$$

$$S^5: \quad ds^2 = d\theta^2 + \sin^2\theta \, d\Omega_4^2$$

D5-brane: (t, z, Ω_4) $\quad \vec{x} = 0$

$$\theta = \theta(z)$$

- Preserves $SO(5)$ R-symmetry

D5-brane embedding

$$S_{\rm D5} = N\frac{\sqrt{\lambda}}{8\pi^2}\left[\int dt\,dr\,d\Omega_4\sqrt{*g+2\pi\alpha' F} - \int 2\pi\alpha' F\wedge *C_4\right]$$

$$\frac{\delta S}{\delta F} = -k$$

- World-volume electric field fixes number of units of string charge.
- E.o.m. yields a constant solution: $\theta(z) = \theta_k$

$$\frac{\pi k}{N} = \theta_k - \sin\theta_k\cos\theta_k$$

- World-volume $\simeq AdS_2\times S^4 \implies$ Conformal quantum mechanics on impurity

Conformal impurity with representation $\mathcal{A}_k$

- Lightest (linearized) fluctuation of D5-brane embedding: $\delta\theta(z)$ ("slipping" mode)

$$m^2 = 12\,, \qquad \Delta = \tfrac{1}{2} + \sqrt{\tfrac{1}{4} + m^2} = 4$$

Irrelevant operator in CFT_1

- For conformal impurity $\langle \mathrm{Tr} F^2 \rangle \sim r^{-4}$ where $(r = |\vec{x}|)$:

$$\tfrac{1}{N}\langle \mathrm{Tr} F^2 \rangle_{\mathcal{A}_k} = \frac{\sqrt{2}}{24\pi^2}\, \sqrt{\lambda}\, \sin^3\theta_k\, \frac{1}{r^4}$$

- Same factor appears as action for circular Wilson loop and Polyakov loop in rep. $\mathcal{A}_k$

"Collapsed" solution

- The wrapped D5-brane permits a distinct finite action solution with $\theta = \pi$.
- AdS_2 factor with collapsed S^4; $\mathrm{Tension} = \frac{k\sqrt{\lambda}}{2\pi}$ $(\sim \int {}^*C_4 \wedge F_{0r})$
- Lightest fluctuation $\delta\theta(z)$: $m^2 = 0 \leftrightarrow \Delta = 1$ in CFT_1
- $$\frac{1}{N}\langle \mathrm{Tr} F^2 \rangle_{\mathrm{collapsed}} = \frac{\sqrt{2}}{24\pi^2}\sqrt{\lambda}\,\frac{3\pi k}{2N}\,\frac{1}{r^4}$$

  Strength $\propto$ k coincident strings/quarks

Interpolating solution

- Exact interpolating solution (BPS "D5-spike"):

$$\frac{1}{z(\theta)} = \frac{A}{\sin\theta}\left(\theta - \sin\theta\cos\theta - \frac{\pi k}{N}\right)^{1/3}$$

[Callan-Guijosa-Savvidy (1998)]

- k fundamental reps. in UV (small z) $\longrightarrow \mathcal{A}_k$ in IR (large z)

- For small z (UV)

$$\theta(z) \simeq \pi - A z \ldots$$

Interpreted as a VEV for the $\Delta = 1$ operator in CFT_1

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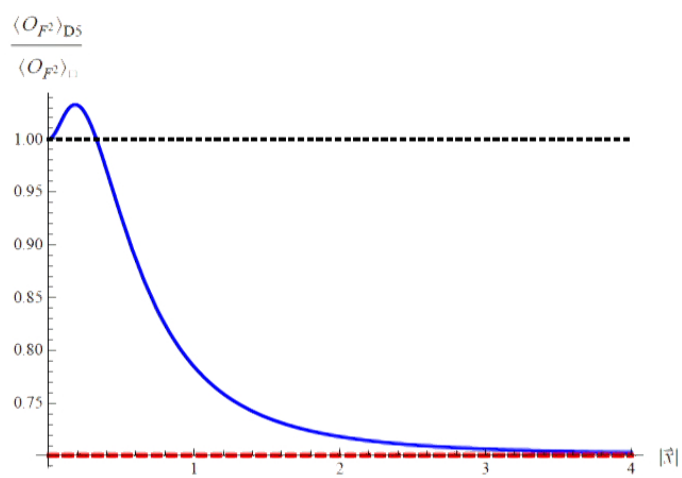
Interpolating $\frac{1}{N}\langle \mathrm{Tr} F^2 \rangle$

$$\langle \mathcal{O}_{F^2} \rangle = \frac{5\sqrt{2\lambda}}{16\pi^2} \int_0^\infty \frac{dv}{v^4} \frac{\sqrt{1+v^2\theta'(v)^2}\sqrt{D(\theta)^2+\sin^8\theta}}{(v^{-2}+r^2)^{7/2}}$$



- Interpretation: k fundamental sources screened to antisymmetric representation.[SPK-Silvani 1611.06033]

Symmetric represention $\mathcal{S}_k$ and deformation

- D3-brane embedding for rep. $\mathcal{S}_k$: $AdS_2 \times S^2$

$$AdS_5: \quad ds^2 = \frac{dz^2}{z^2} + \frac{1}{z^2}\left(-dt^2 + d\rho^2 + r^2 d\Omega_2^2\right)$$

$$D3: \quad (t, z, r, \Omega_2) \qquad \boxed{r = \kappa\, z} \qquad \kappa \equiv \frac{\sqrt{\lambda}k}{4N}$$

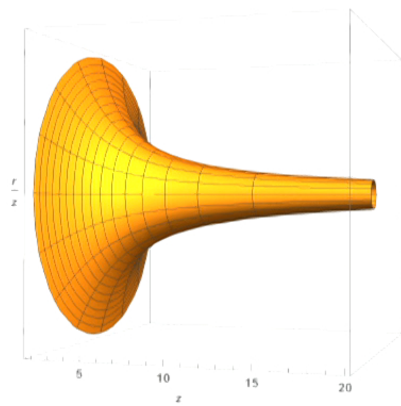
- "Breathing mode" of S^2 has $m^2 = 0$: $\Delta = 1$ in CFT_1

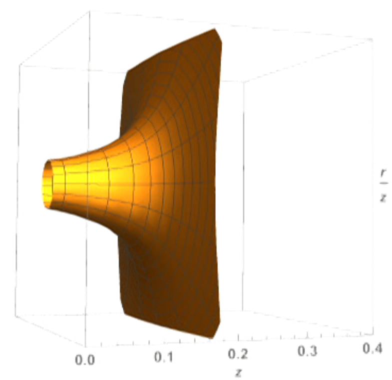
- Exact non-conformal (BPS) embedding: $\boxed{r = \dfrac{\kappa\, z}{1 + a\kappa\, z}}$

[J. H. Schwarz (2014)]

Two types of non-conformal solutions

- **D3-spike** ($a > 0$):
  Source in rep. $\mathcal{S}_k$ in UV ($az \ll 1$) $\longrightarrow$ k fundamental quarks in IR





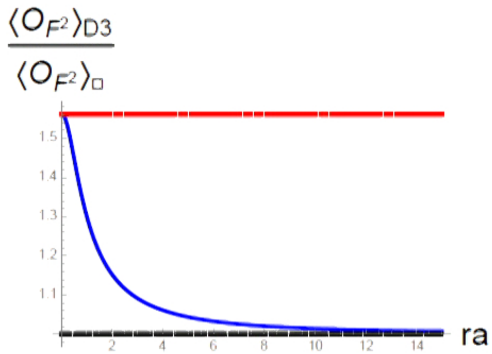
- **Coulomb branch Wilson line** ($a < 0$): Rep. $\mathcal{S}_k$ heavy quark in a Coulomb phase with $SU(N) \to U(1) \times SU(N-1)$

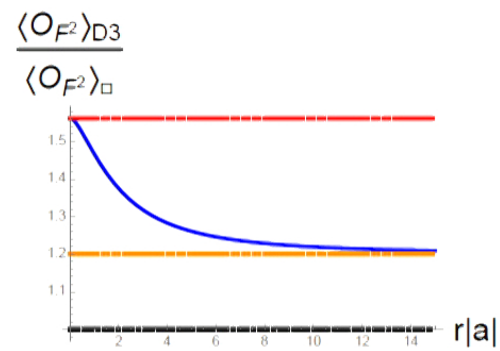
  Indicates scalar VEV: $\langle\Phi\rangle(r) = \kappa|a| + \frac{\kappa}{r}$

Interpolating solution

- Small z: $\frac{r}{z} = \kappa - a\kappa^2 z \ldots$

  Flow triggered by VEV of $\Delta = 1$ operator in UV





- **UV**: $\frac{1}{N}\langle \mathrm{Tr}\, F^2 \rangle \rightarrow \frac{\sqrt{2}}{4\pi}\,\kappa\sqrt{1+\kappa^2}\, r^{-4}$ (Fiol, Garolera, Lewkowycz (2012))

  IR: $(a > 0) \quad \rightarrow \frac{\sqrt{2}}{4\pi}\,\kappa\,\rho^{-4}$ $\qquad a < 0: \quad \rightarrow \frac{\sqrt{2}}{4\pi}\,\kappa^2\,\rho^{-4}$

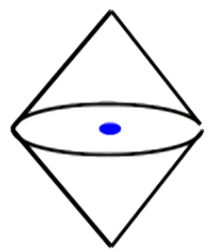
- $(a > 0)$ soln: Symmetric rep. source "dissociating" into fundamental quarks

Brief Summary

- All non-conformal cases $\leftrightarrow$ flows triggered by VEV for a $\Delta = 1$ mode in CFT_1
- All accompanied by a decrease of the strength of source towards the IR $\rightarrow$ "thinning out" of degrees of freedom
- We turn to an alternative measure: The contribution from the impurity to the Entanglement Entropy of a spherical region enclosing it

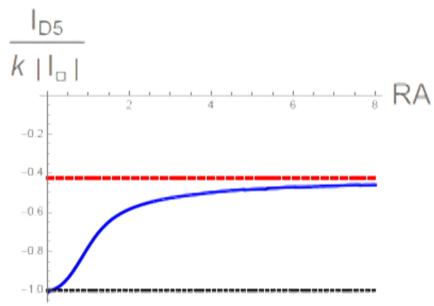
Generalized gravitational entropy

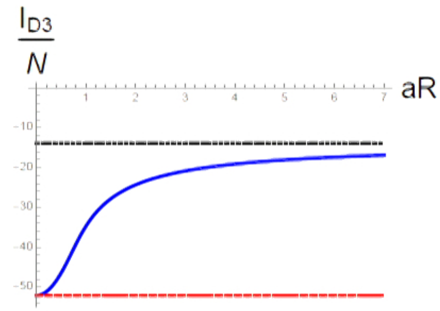
- Implement replica trick in the bulk. [Lewkowycz-Maldacena (2013)]
- We are interested in the EE excess due to impurity for a spherical region $\mathcal{B}$ of radius r centred around the impurity.



- Map (CHM) the causal development $\mathcal{D}$ of the $\mathcal{B}$ to the Rindler wedge, which is conformal to hyperbolic space H_3 at temperature $\beta = \frac{1}{2\pi}$ [Casini-Huerta-Myers (2011)]
- The bulk dual of $\mathcal{D}$ is hyperbolically sliced AdS_5 with horizon and Hawking temperature β.

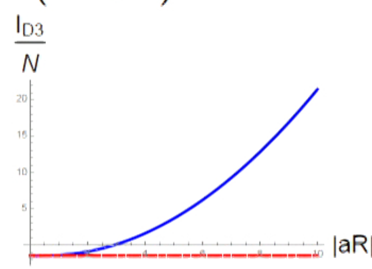
Action in entanglement wedge: $I_{\rm D3,D5} = -\ln Z_{\mathcal{H}}$





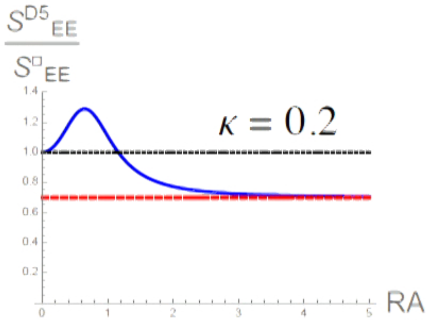
- Both monotonic, interpolating between result for k fundamentals and symmetric/antisymmetric reps.

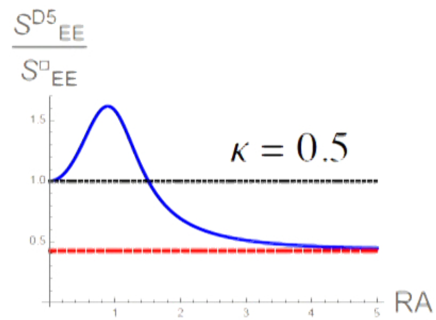
- Coulomb branch soln. $(a < 0)$:



- $I_{\rm D3}|_{|ar|\gg 1} \simeq N\left(\frac{1}{2}(\kappa a r)^2 - \ln(2\kappa|a|r)\right)$, coefficient of log universal [Casini-Huerta-Myers[(2011)]

Impurity EE for screening to $\mathcal{A}_k$

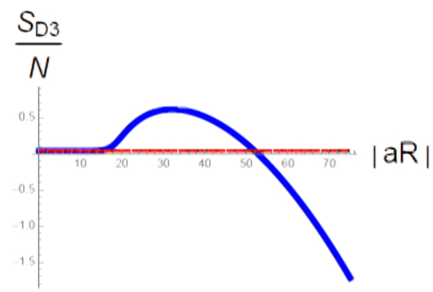




$$\left.\frac{S_{\rm D5}}{k\,S_{\Box}}\right|_{Ar\to 0} = 1\,, \qquad \left.\frac{S_{\rm D5}}{k\,S_{\Box}}\right|_{Ar\to\infty} = \frac{2N}{3\pi k}\,\sin^3\theta_k\,.$$

- Non-monotonic, settling at lower value in deep IR, and qualitatively tracking $\langle {\rm Tr} F^2\rangle$

Coulomb Branch $\mathcal{S}_k$



$$S_{\mathrm{D3}}\Big|_{ar\gg 1} = N\left[-\frac{1}{3}(|a|r\tilde{\kappa})^2 - c(\kappa)|a|r + \frac{2}{3}\ln\left(2|a|r\tilde{\kappa}\right)\right] .$$

- Not purely a Coulomb branch effect: linear term appears to have nontrivial dependence on κ

Conclusions, further questions

- D3/D5 Wilson line embeddings allow interesting examples of non-conformal defects.
- Analogous embeddings in $AdS_7 \times S^4$ recently found for M5-brane Wilson surfaces. [R. Rodgers (2018); Estes-Krym-O'Bannon-Robinson-Rodgers (2018)]
- All the flows originate from a "VEV" of marginal operator in the UV of the impurity theory. E.g. for D3-brane, bosonic QM, $\mathcal{O} = \chi^{\dagger}_{m} \Phi^{J} n_{J} \chi^{m}$. Study deformations of the impurity QM.
- While impurity contribution to EE is not monotonic, the defect contribution to the free energy restricted to entanglement wedge is a candidate g-function.

  [Kobayashi-Nishioka-Sato-Watanabe (2018)]