Title: Symmetries and Dualities of Abelian TQFTs

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Symmetries and Dualities of Abelian TQFTs

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D. Delmastro and J.G., arXiv:1904.12884

In Memoriam

Before starting I would like to take a moment to remember our colleagues who have tragically passed doing what they loved

Ann Nelson

Steven Gubser

Introduction

- In this talk we answer the following question:

what are the symmetries of abelian TQFTs?

e.g: what are the symmetries of $U(1)_k$ Chern-Simons theory?

This foundational physics question is connected with number theory

Symmetries

- Symmetries play a pivotal role in our description of nature

- In Quantum Mechanics symmetries are implemented in $\mathcal{H}$ either by
  - unitary
  - anti-unitary $\Longrightarrow$ time-reversal transformations

- Symmetries are realized by invertible topological defects

- 't Hooft anomalies for global symmetries serve as lamposts for non-perturbative dynamics. They are renormalization group invariants

$$\mathcal{A}_{UV} = \mathcal{A}_{IR}$$

Symmetries in QFT

- A sufficient condition for a transformation g to be a symmetry is that

$$g \cdot S = S$$

- If S is invariant, quantum theory obeys Ward identities

$$\langle g \cdot \mathcal{O}_1 \dots g \cdot \mathcal{O}_m \rangle = \begin{cases} \langle \mathcal{O}_1 \dots \mathcal{O}_m \rangle & g \text{ unitary} \\ \langle \mathcal{O}_1 \dots \mathcal{O}_m \rangle^* & g \text{ anti-unitary} \end{cases}$$

- Quantum systems can be endowed by symmetries not visible classically

$$g \cdot S \neq S$$

which nevertheless obey Ward identities

- Such inherently "quantum" symmetries play a prominent role in this talk

Motivation

- TQFTs in 2+1d play a central role in physics and mathematics
  - Emergent description of gapped quantum phases of matter, e.g.:
    - Integer Quantum Hall Effect
    - Fractional Quantum Hall Effect
    - Topological insulators and superconductors
  - Describe the nonperturbative infrared dynamics of gauge theories:
    - CFT
    - Gapped phase: TQFT

    E.g: J.G, Komargodski, Seiberg

    $$SU(2) \text{ Yang-Mills } + \psi \text{ in adjoint} \Longrightarrow U(1)_2 + \text{massless fermion}$$
  - Supported on domain walls/defects $3+1$-dimensional gauge theories

    Witten, ...
  - Applications in mathematics: knot invariants, representation theory, ...

- Study of phases of matter with symmetries have lead to a classification schemes of topological phases for matter

- At long wavelength symmetries are realized in the emergent TQFT

  - Symmetries in TQFTs describing trivial gapped phases: SPT phases

  - Symmetries in TQFTs describing nontrivial gapped phases: SET phases

- Very little is known about the symmetries of TQFTs

  - What are the symmetries of a TQFT?

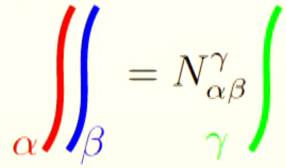
  - What are their 't Hooft anomalies?

- QFTs are subject to non-trivial dualities

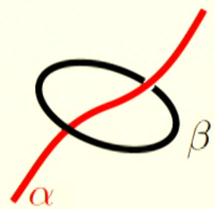
  - What are the dualities of TQFTs?

TQFT

- A TQFT specified by Moore-Seiberg data, a MTC $\mathcal{C}$. This includes:
  - Fusion of anyons: $\alpha \times \beta = N_{\alpha\beta}^{\gamma}\, \gamma$, $\qquad N_{\alpha\beta}^{\gamma} \in \mathbb{Z}$



    Fusion algebra $\mathcal{A}$
  - Anyon topological spins: $\theta(\alpha)$
  - Braiding matrix: $B(\alpha, \beta)$



  - $[F]$ and $[R]$ symbols, subject to local isomorphisms/gauge redundancy
  - S and T matrices
  - Data obeys groupoid relations: i.e. hexagon and pentagon identities

Symmetries of a TQFT

- The group of symmetries of a TQFT is the automorphism group $\text{Aut}(\mathcal{C})$ of the associated MTC

  - A symmetry $g \in \text{Aut}(\mathcal{C})$ acts as a permutation on the anyons: $\alpha \to g(\alpha)$

  - Preserves fusion: $g(\alpha \times \beta) = g(\alpha) \times g(\beta)$

  - If symmetry is unitary, g preserves MTC data, e.g. spin and braiding

$$\theta(g(\alpha)) = \theta(\alpha)\,, \qquad B(g(\alpha), g(\beta)) = B(\alpha, \beta)\,, \ \ldots$$

  - If symmetry is anti-unitary, g preserves MTC data up to $*$-conjugation

$$\theta(g(\alpha)) = (\theta(\alpha))^*\,, \qquad B(g(\alpha), g(\beta)) = (B(\alpha, \beta))^*\,, \ \ldots$$

- Very little is concretely known about the symmetries of TQFTs

Abelian TQFTs

- In an abelian TQFT all anyons are invertible $\leftrightarrow$ fusion rules of $\mathcal{C}$ are abelian

$$\forall\alpha \;\; \exists \;\; \alpha^{-1} \;\; \text{s.t.} \;\; \alpha \times \alpha^{-1} = \mathbf{1}$$

- In an abelian TQFT the entire data is uniquely determined by $(\mathcal{A}, \theta(\alpha))$
  E.g:

$$B(\alpha, \beta) = \frac{\theta(\alpha \times \beta)}{\theta(\alpha)\theta(\beta)}$$

- In an abelian TQFT the fusion algebra $\mathcal{A}$ is a finite abelian group

- In an abelian TQFT a symmetry is an element of $Aut(\mathcal{A})$ that preserves spin

- Any abelian TQFT admits a description as abelian Chern-Simons theory

$$\mathcal{L}_K = \frac{1}{4\pi} a^T K da$$

  - K is an integral, symmetric matrix

$U(1)_k$ Chern-Simons

- The classical action is

$$\mathcal{L} = \frac{k}{4\pi} a \wedge da$$

  - $k \in \mathbb{Z}$
  - $Z[T^2] = |k|$
  - Wilson line $W_\alpha = e^{i\alpha \oint a}$ describes the worldline of anyon with spin

$$h_\alpha = \frac{\alpha^2}{2k}$$

  - The braiding phase of anyons α and β is

$$B(\alpha, \beta) = e^{\frac{2\pi i \alpha\beta}{k}}$$

    - $B(\alpha, \alpha) = \theta(\alpha)^2$
    - $\theta(\alpha) = e^{2\pi i h_\alpha}$ is the topological spin of anyon α
    - spin of anyon h_α defined mod 1

- For $k \in$ even, there are k distinct anyons: $\alpha \in \{0, 1, \ldots, k-1\}$

$$\alpha \times \beta = \alpha + \beta \mod k$$

  - Describes bosonic FQH state: $\alpha = k$ line realizes the microscopic boson
  - Bosonic TQFT
  - $\mathcal{A} = \mathbb{Z}_k$

- For $k \in$ odd, there are $2k$ distinct anyons: $\alpha \in \{0, 1, \ldots, 2k-1\}$

$$\alpha \times \beta = \alpha + \beta \mod 2k$$

  - The line $\alpha = k$ is a transparent fermion
  - $\nu = 1/k$ Laughlin FQH state: $\alpha = k$ line realizes the microscopic electron
  - Spin TQFT. Requires a choice of a spin structure
  - $\mathcal{A} = \mathbb{Z}_{2k}$

Classical symmetries of $U(1)_k$

- $\forall\, k > 2$, there is a classical $\mathbb{Z}_2$ charge conjugation symmetry C
  - Under $a \to -a$, the action is invariant $S \to S$
  - C acts as by permuting the anyons

$$C : \alpha \to -\alpha = \begin{cases} k - \alpha & \text{for } k \text{ even} \\ 2k - \alpha & \text{for } k \text{ odd} \end{cases}$$

- Under the action of time-reversal $T : \begin{cases} a_0 \to a_0 \\ a_i \to -a_i \end{cases}$, $S \to -S$.

  $U(1)_k$ does not admit a “classical” anti-unitary symmetry

Questions

- Does $U(1)_k$ admit “quantum” T-reversal symmetries?
- What is the group of unitary symmetries of $U(1)_k$?

$U(1)_k$ T-reversal symmetries

- We determine the T-reversal symmetries of $U(1)_k$ by solving the equations

$$T(\alpha \times \beta) = T(\alpha) \times T(\beta)$$

$$\theta(T(\alpha)) = (\theta(\alpha))^* \Longleftrightarrow h_{T(\alpha)} = -h_\alpha \mod 1$$

$\Longrightarrow$ entire abelian MTC admits an anti-unitary automorphism

bosonic $U(1)_k$: $\nexists$ a T-reversal symmetry

- A necessary condition is that $\forall \alpha$, $\exists$ an β with $h_\alpha = -h_\beta \mod 1$
- Imposing this on the generating line $\alpha = 1$, and that k is even, we find that

$$\frac{1+\beta^2}{2k} \neq \mathbb{Z} \qquad \text{QED}$$

spin $U(1)_k$:

- k odd: $\mathcal{A} = \mathbb{Z}_{2k}$. Anyons labeled by $\alpha \in \{0, 1, \ldots, 2k-1\}$
- $U(1)_k \times \{1, \psi\}$ for k even: $\mathcal{A} = \mathbb{Z}_k \times \mathbb{Z}_2$. Anyons labeled by pair (α, β):

$$\alpha \in \{0, 1, \ldots, k-1\}$$

$$\beta \in \{0, 1\}$$

with $\psi = (0, 1)$

- Action of fusion homomorphism $T \in Aut(\mathcal{A})$ fixed by action on generators

$$T : \alpha \to q\,\alpha$$

$$T : (\alpha, \beta) \to \begin{cases} (q\,\alpha, \beta) \\ (q\,\alpha, \alpha + \beta) \end{cases}$$

where $q \in \mathbb{Z}$

- Imposing $h_{T(\alpha)} = -h_\alpha$ mod 1 for the generators of fusion algebra requires that

$$pk - q^2 = 1 \Longleftrightarrow \begin{cases} 2pk - q^2 = 1\,, & k \text{ odd} \\ (2p-1)k - q^2 = 1\,, & k \text{ even} \end{cases} \tag{1}$$

for $p, q \in \mathbb{Z}$

- If the generators have a T-reversal image, all anyons do

$$h_{T(\alpha)} = \frac{q^2\alpha^2}{2k} = \frac{(2pk-1)\alpha^2}{2k} = -\frac{\alpha^2}{2k} \text{ mod } 1 = -h_\alpha \text{ mod } 1$$

$$h_{T(\alpha,\beta)} = \frac{q^2\alpha^2}{2k} + \frac{1}{2}(\alpha+\beta)^2 = -\frac{\alpha^2}{2k} + \frac{\beta^2}{2} \text{ mod } 1 = -h_{(\alpha,\beta)} \text{ mod } 1$$

$\Longrightarrow$

$U(1)_k$ admits a T-reversal transformation iff $kp - q^2 = 1$

$k = 1,\, 2,\, 5,\, 10,\, 13,\, 17,\, 25,\, 26,\, 29,\, 34,\, 37,\, 41,\, 50,\, 53,\, 58,\, 61,\, 65,\, 73,\, 74,\, 82,\, \ldots$

Summary

- Spin $U(1)_k$ Chern-Simons is T-invariant if and only if $k \in \mathbb{T}$

$$\mathbb{T} = \{k \in \mathbb{Z} | kp - q^2 = 1 \text{ for } p, q \in \mathbb{Z}\}$$

  - $k \in \mathbb{T}$ if and only if $k = a^2 + b^2$ for relatively prime $a, b \in \mathbb{Z}$
  - Given the prime factorization of k

$$k = 2^a \left[\prod_{\pi \equiv 1 \mod 4} \pi^\alpha \right] \left[\prod_{\pi \equiv 3 \mod 4} \pi^\beta \right]$$

  $k \in \mathbb{T}$ iff $\beta = 0$ and $a \in \{0, 1\}$. k has only Pythagorean prime factors

  - $\mathbb{T} \supset \mathbb{P}$, where $\mathbb{P}$ are the solutions of the Pell equation $kp^2 - q^2 = 1$. Witten showed that when $k \in \mathbb{P}$, then $U(1)_k$ is T-invariant
  - T-invariance for $k \in \mathbb{T}$ can also be proven by a path integral argument

- There are $2^{\varpi(k)}$ solutions to $kp - q^2 = 1 \Longrightarrow$ T-reversal transformations

$$\varpi(k) = \begin{cases} k \text{ odd : number of distinct prime factors of } k \\ k \text{ even : number of distinct prime factors of } k/2 \end{cases}$$

  - For $k \in \mathbb{T} > 2$ the time-reversal algebra is $\mathbb{Z}_4$: $T_i^2 = (-1)^F C$
  - For $k \in \{1, 2\}$ the time-reversal algebra is $\mathbb{Z}_2$: $T^2 = (-1)^F$

$U(1)_k$ unitary symmetries

- $\forall k > 2$ there is charge conjugation symmetry
- We determine the unitary symmetries of $U(1)_k$ by solving the equations

$$U(\alpha \times \beta) = U(\alpha) \times U(\beta)$$

$$\theta(U(\alpha)) = (\theta(\alpha)) \Longleftrightarrow h_{U(\alpha)} = h_\alpha \mod 1$$

- $U(1)_k$ has a unitary symmetry, which must act as $U : \alpha \to q\,\alpha$, iff

$$2kp + q^2 = 1$$

- The group of unitary symmetries of $U(1)_k$ is $(\mathbb{Z}_2)^{\varpi(k)}$

  e.g: $U(1)_{12}$ has a quantum symmetry acting as $\alpha \to \pm 5\alpha$
- The time-reversal transformations we found earlier can be written: $T_i = TU_i$
- Spin $U(1)_k$ has extra $\mathbb{Z}_2$ symmetry compared to bosonic $U(1)_k$ when $k = 8n$

Symmetries of K-matrix Chern-Simons

- The classical action is

$$\mathcal{L} = \frac{1}{4\pi} a^T K \wedge da$$

  - K is an integral, symmetric matrix
  - $Z[T^2] = |\det(K)|$
  - Charges of anyons take values in the lattice $\mathbb{Z}^n/(K\mathbb{Z})^n$

$$h_\alpha = \frac{1}{2}\alpha^T K^{-1} \alpha$$

  - Anyon fusion: $\alpha \times \beta = \alpha + \beta \text{ mod } K$
  - The braiding phase of anyons α and β is

$$B(\alpha, \beta) = e^{2\pi i \alpha^T K^{-1} \beta}$$

  - Theory is bosonic for K even and spin for K odd

Symmetries

- The most general fusion homomorphism is

$$T : \alpha \to Q\,\alpha$$

where Q is an $n \times n$ integral matrix

- This homomorphism is a symmetry of TQFT iff
  - There are solutions to the equation

    $$P \pm Q^T K^{-1} Q = K^{-1}$$

    for Q, P integral matrices
  - Q is invertible modulo K over the integers
    - Solutions with $P = 0$ are classical symmetries
    - Solutions with $P \neq 0$ are "quantum" symmetries
    - Equations have a rich arithmetic

Examples

- $(\mathbb{Z}_{k_1})_{k_2}$ twisted gauge theory. Described by

$$K = \begin{pmatrix} 0 & k_1 \\ k_1 & k_2 \end{pmatrix}$$

- $(\mathbb{Z}_{k_1})_{k_2}$ is T-invariant iff

$$k_2 \in \mu(k_1)\mathbb{Z}$$

where

$$\mu(a) = \frac{a}{\text{Pythagorean prime factors of a}}$$

- Symmetry groups

k	$Aut((\mathbb{Z}_k)_0)$	$Aut_U((\mathbb{Z}_k)_0)$	$Aut((\mathbb{Z}_k)_{\mu(k)})$	$Aut_U((\mathbb{Z}_k)_{\mu(k)})$
2	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
3	D_8	$\mathbb{Z}_2^2$	D_8	$\mathbb{Z}_2^2$
4	D_8	$\mathbb{Z}_2^2$	D_8	$\mathbb{Z}_2^2$
5	$\mathbb{Z}_4 \circ D_8$	D_8	$\mathbb{Z}_4$	$\mathbb{Z}_2$
6	$\mathbb{Z}_2 \times D_8$	$\mathbb{Z}_2^3$	D_8	$\mathbb{Z}_2^2$
7	$\mathbb{Z}_3 \rtimes D_8$	D_{12}	$\mathbb{Z}_3 \rtimes D_8$	D_{12}
8	$\mathbb{Z}_2 \times D_8$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2 \times D_8$	$\mathbb{Z}_2^3$

Dualities

- We developed a criterion to determine dual TQFT's, extending celebrated level/rank dualities

- $\mathrm{CS}[K_1] \longleftrightarrow \mathrm{CS}[K_2]$ if and only if

$$P + Q^T K_1^{-1} Q = K_2^{-1}$$

for Q, P integral matrices

  - Solutions with $P \neq 0$ yield nontrivial dualities

- Examples:

$$\begin{aligned}(\mathbb{Z}_7)_2 &\quad\leftrightarrow\quad (\mathbb{Z}_7)_4 \\ (\mathbb{Z}_7)_3 &\quad\leftrightarrow\quad (\mathbb{Z}_7)_5\end{aligned}$$

Conclusions

- We have determined the unitary and anti-unitary symmetries of TQFTs

Plethora of “quantum” symmetries

- Found many new TQFT dualities
- Interesting problems for the future include
  - Classification of associated ’t Hooft anomalies
  - Constructing anomaly indicators
  - ...

Still leaning new and elementary facts about TQFTs!