

Title: PSI 2019/2020 - Lie Groups and Lie Algebras - Lecture 6

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Collection: PSI 2019/2020 - Lie Groups and Lie Algebras (Ali)

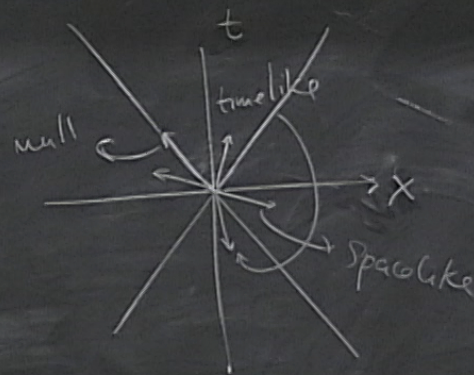
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$V^\mu \rightarrow$ contravariant

$\alpha_\mu \rightarrow$ covariant

$$V_M^2 \begin{matrix} > 0 \\ < 0 \\ = 0 \end{matrix}$$



Metric tensor $\eta_{\mu\nu}$

$$V_M^2 = V^\mu V^\nu \eta_{\mu\nu}$$

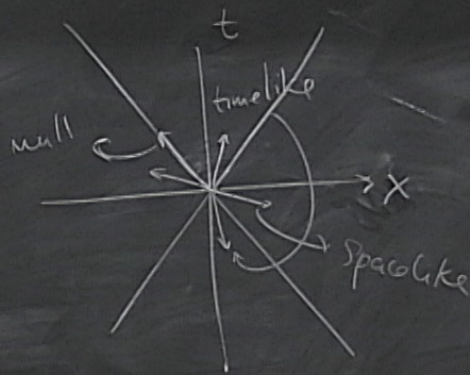
$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$V_E^2 = V^\mu V^\nu \delta_{\mu\nu} \text{ - Euclidean}$$

$V^\mu \rightarrow$ contravariant

$\alpha_\mu \rightarrow$ covariant

$$V_M^2 \begin{cases} > 0 \\ < 0 \\ = 0 \end{cases}$$



$$V_M^2 = V_M V^M$$

$$V_M = \eta_{M\mu} V^\mu$$

Metric tensor $\eta_{\mu\nu}$

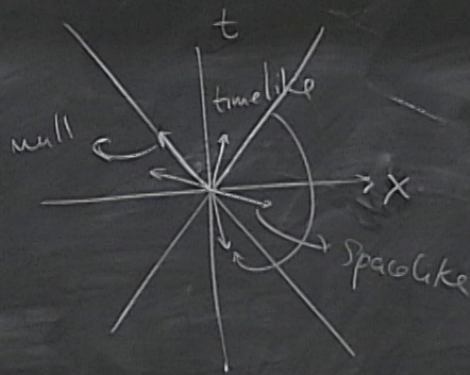
$$V_M^2 = V^\mu V^\nu \eta_{\mu\nu} \quad \eta_{\mu\nu} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$V_E^2 = V^\mu V^\nu \delta_{\mu\nu} \text{ - Euclidean}$$

$V^\mu \rightarrow$ contravariant

$\alpha_\mu \rightarrow$ covariant

$$V_M^2 \begin{cases} > 0 \\ < 0 \\ = 0 \end{cases}$$



$$V_M^2 = V_\mu V^\mu$$

$$\checkmark V_\mu = \eta_{\mu\nu} V^\nu$$

$$\eta^{\mu\nu} \eta_{\nu\rho} = \delta^\mu_\rho$$

$$\checkmark V^\mu = \eta^{\mu\nu} V_\nu$$

Metric tensor $\eta_{\mu\nu}$

$$V_\mu^2 = V^\mu V^\nu \eta_{\mu\nu} \quad \eta_{\mu\nu} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$V_E^2 = V^\mu V^\nu \delta_{\mu\nu} \text{ - Euclidean}$$

$$D(g)|\psi\rangle \quad \forall g \in G$$

$$H|\psi\rangle = E|\psi\rangle$$

$$D_{ij}(g)$$

$$\left(\begin{array}{c|c} D_{(1)} & \\ \hline & D_{(2)} \end{array} \right) H = \left(\begin{array}{c|c} \lambda \mathbb{1}_K & \\ \hline & \mu \mathbb{1}_P \end{array} \right) \left(\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right)$$

Properties of Lie Alg & Lie Brackets.

1. $[X, Y] = -[Y, X]$

2. $T_a \rightarrow$ generators

$\{T_a\}$ as vector space over $\mathbb{R}$ or $\mathbb{C}$.

3. $[T_a, T_b] = i f_{ab}^c T_c$

$f_{ab}^c \rightarrow$ real. when $\{T_a\}$ vector space over $\mathbb{R}$.

4. Jacobi identity

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$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

$$5. f_{ab}^d f_{dc}^e + f_{ca}^d f_{db}^e + f_{bc}^d f_{da}^e = 0$$

or $\mathbb{C}$.

space over $\mathbb{R}$.

4. Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

$$5. f_{ab}^d f_{dc}^e + f_{ca}^d f_{db}^e + f_{bc}^d f_{da}^e = 0$$

$$\begin{array}{l} f_{12}^3 = 5 \\ f_{31}^2 = 6 \end{array}$$

5. Using the adjoint rep define a metric
on $\{T^a\}$ known the Cartan metric

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on $\{T^a\}$ known the Cartan metric

$$g_{ab} = \text{Tr}_{\text{adj}}(T_a T_b)$$

$$f_{abc} := g_{cd} f^{cd}{}_b$$

If gp is compact $f_{abc} = f[abc]$

$$f_{abc} = -f_{bac}$$

$$f_{abc} = -f_{acb}$$

$$f_{abc} = -f_{bac}$$

$$f_{abc} = -f_{acb}$$

G-module

The vector V on which a rep D of G acts is called a G -module.

$G \rightarrow \text{Lie group}$

$\mathfrak{g} \leftarrow \mathfrak{g} \rightarrow \text{Lie alg of } G$

Rep of Lie algs

A rep ρ of a Lie alg $\mathfrak{g}$ is map

$$\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(n, \mathbb{C})$$

st. the Lie bracket is preserved:

$$\rho([X, Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X).$$

The adjoint rep of

The adjoint rep of Lie alg.

The Adjoint action of $g \in G$
on $T_a \in \tilde{\mathfrak{g}}$ is given by

$$\underline{\text{Adj}}_g(T_a) = g T_a g^{-1}$$

This action forms a rep of Lie alg.

$$g T_a g^{-1} = \tilde{T}_a$$

$$[\tilde{T}_a, \tilde{T}_b] = i f_{abc} \tilde{T}_c$$

$$\tilde{T}_c = C_{cb} T_a$$

$$\text{Let take } g \simeq 1 + i \theta_c T_c$$

$$C_{ab} \simeq \delta_{ab} - i \theta_c F_{abc}$$

$$\boxed{\underline{\theta} \cdot \underline{T} = \theta_a T_a}$$

$$g T_a g^{-1} = \hat{T}_a$$

$$(1 + i \underline{\theta} \cdot \underline{T}) T_a (1 - i \underline{\theta} \cdot \underline{T}) = T_a - i \theta_c F_{abc} T_b + \dots$$

$$[T_c, T_a] = -F_{abc} T_b$$

$$\text{adj}_{T_c}(T_a) = [T_c, T_a]$$

$$= -F_{abc} T_b$$

$$(F_{ab})_{ab} =$$

$$(F_c)_{ab} = -F_{abc}$$

$T_b + \dots$

$$Ad_{T_c}(T_a) = (F_c)_{ab} T_b$$

$$(F_c)_{ab} = -F_{abc}$$

$$\text{Ad}_{T_c}(T_a) = (F_c)_{ab} T_b$$

$$(F_c)_{ab} = -if_{ca}{}^b$$

$$[F_a, F_b] =$$

$$\delta q = \epsilon(\Delta q)$$

$$[F_a, F_b] = i f_{ab}^c F_c$$

adj action generates a rep of the Lie alg.

a, b, T_b

$$\delta q = \epsilon (\Delta q)$$

$$[F_a, F_b] = i f_{ab}^c F_c$$

adj action generates a rep of the Lie alg.

Comment Suppose $\dim \mathfrak{g} = n$

$$T^a \quad a = 1, \dots, n$$

$SU(2)$ - 2×2 unitary matrices with unit determinant.

$$\dim(SU(2)) = 3$$

T_a

$$\boxed{\Theta \cdot \mathbb{I} = \Theta_a T_a}$$

$$g T_a g^{-1} = \hat{T}_a$$

$$(1 + i \Theta \cdot \mathbb{I}) T_a (1 - i \Theta \cdot \mathbb{I}) = T_a - i \Theta_c F_{abc} T_b$$

$$\checkmark [T_c, T_a] = -F_{abc} T_b$$

$$\checkmark \boxed{\text{adj}_{T_c}(T_a) = [T_c, T_a]} \text{ Defn} \\ = -F_{abc} T_b$$

Dim of the matrices in the adj rep is the
Same as the dim of the Lie alg (or the group)

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Examples of matrix Lie group & Lie algebras

$O(n)$: Consists of real $n \times n$ matrices O st

$$O^T O = I$$

$$O_{ij} O_{ik} = \delta_{jk} \quad (+)$$

Dim $O(n)$

$O \rightarrow n^2$ real components

Constraints: n constraints from $j=k$

$$\frac{n^2 - n}{2} \quad " \quad " \quad j \neq k$$

$$\begin{aligned} \# \text{ of independent parameters} &= n^2 - \left(n + \frac{n^2 - n}{2} \right) \\ &= \frac{n(n-1)}{2} \end{aligned}$$

$$\text{Dim } O(n) = \frac{1}{2} n(n-1)$$

$$\text{Dim } (O(2)) = \frac{1}{2} 2 \cdot 1 = 1$$

$$\text{Dim } (O(3)) = \frac{1}{2} 3 \cdot 2 = 3$$

$$\text{Dim } (O(4)) = 6$$

$=k$

$\neq k$

$$+ \frac{n^2 - n}{2}$$

Dim of the matrices in the adj rep is the
Same as the dim of the Lie alg (or the group)

Examples of matrix Lie group & Lie algebras

$O(n)$: Consists of real $n \times n$ matrices O st

$$O^T O = I$$

$$\Rightarrow \det O = \pm 1$$

$$\det O = +1$$

$$SO(n)$$

Generators: $O \simeq \mathbb{1} + \theta_i T_i$

$e^{\theta_i T_i}$

$$O^T O = \mathbb{1}$$

$$T_i^T = -T_i$$

Labelling generators of $O(n)$:

$$T^{ab} = -T^{ba}$$

$$a, b = 1, \dots, n$$

$$T^{12} = -T^{21} \quad T_{ii} = 0$$

There are only $\frac{1}{2}n(n-1)$ linearly independent
generators.

$SO(3), O(3)$

$T_{12} \quad T_{23} \quad T_{31}$

There are only $\frac{1}{2}n(n-1)$ linearly independent generators.

$$\underline{SO(3), O(3)}$$

$$T_{12} \quad T_{23} \quad T_{31}$$

$$J_a = \frac{1}{2} \epsilon_{abc} T_{bc}$$

$$[J_a, J_b] = \epsilon_{abc} J_c$$

dependent

$$(T^{ab})^T = -T^{ab}$$

$$(T^{ab})_{ij} = \delta_i^a \delta_j^b - \delta_j^a \delta_i^b$$

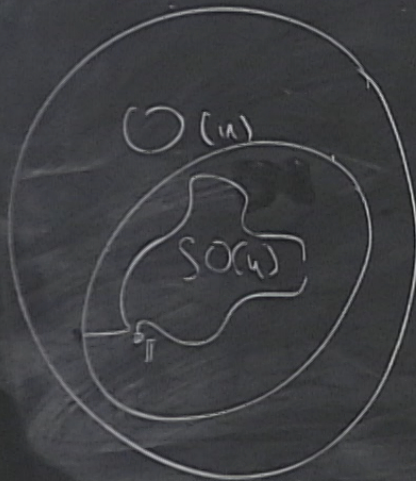
Generators of
 $SO(n), O(n)$

Lie alg of $SO(n)$

$$[T^{ab}, T^{cd}] = \delta^{ac} T^{bd} - \delta^{bc} T^{ad} - \delta^{ad} T^{bc} + \delta^{bd} T^{ac}$$

Lie alg of $SO(n)$

$$[T^{ab}, T^{cd}] = \delta^{ac} T^{bd} - \delta^{bc} T^{ad} - \delta^{ad} T^{bc} + \delta^{bd} T^{ac}$$



$$\det O = -1$$

The maximal set of commuting generators of Lie algs

plays an important role both in classification of Lie algs and rep theory. This set is called the

Cartan subalg (CSA).

$$\text{Lie} \rightarrow \{T^{ab}\} \supset \{H^{ab}\} \quad [H^{ab}, H^{cd}] = 0$$

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$SO(1,3)$

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Cartan subalgebra (CS)

$$\{H^{ab}\} \quad [H^{ab}, H^{cd}] = 0$$

$SO(1,3)$

$$\begin{matrix} T^{01} & T^{02} & T^{03} \\ T^{12} & T^{23} & T^{31} \end{matrix}$$

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Cartan subalg (CSA).

$$\text{Lie} \rightarrow \{T^{ab}\} \supset \{H^{ab}\} \quad [H^{ab}, H^{cd}] = 0$$

$SO(1,3)$

$$T^{01} \quad T^{02} \quad T^{03}$$

$$T^{12} \quad T^{23} \quad T^{31}$$

$$\omega_{\mu\nu} = -\omega_{\nu\mu} \quad \exp[i \omega_{\mu\nu} T^{\mu\nu}]$$

$$\omega_{0i} \rightarrow \text{boost}$$

$$| \quad \omega_{ij}$$

T^{ab} have a subset $\{H^{ab}\}$ which have the prop

$$[H^{ab}, H^{cd}] = 0$$

then that ^{sub} set is the CSA.

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$$\underline{n = 2k}$$

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 $[H^{ab}, H^{cd}] = 0$

then that ^{sub} set is the CSA.

$$n = 2k$$

k is the dim CSA = Rank of the Lie alg or the Lie group.

$$n = 2k + 1$$

$$SO(2k)$$

$SO(2k) \rightarrow \text{rank is } k$

$SO(2k+1) \rightarrow \text{rank is } k$

of the Lie group.

$$\underline{O(n, m)}$$

$$O^T \eta O = \eta$$

$SO(n, m)$ subgroup

$SO(1, 3) \rightarrow$ Lorentz =

$SO(2, n) \rightarrow$ Conformal group

$$\eta = \begin{pmatrix} \underbrace{-1 \dots -1}_n & & \\ & \underbrace{+1 \dots +1}_m & \\ & & \end{pmatrix}$$

$$\delta \leftrightarrow \eta$$

SU(n)

Consists of U st $U^\dagger = U^{-1}$ ✓, U are $n \times n$ matrices
and $\det U = +1$ ✓ with complex entries

$$U^\dagger U = I$$

U has $2n^2$ real parameters.

$$n + n(n-1)$$

$$n + n(n-1) + 1 \quad \text{Constraints}$$

$$\dim SU(n) = 2n^2 - [n + n(n-1) + 1] \\ = n^2 - 1$$

$$U = e^{i\theta_a T_a} \quad \theta_a \rightarrow \text{real } n^2 - 1 \text{ parameters}$$

$$U^\dagger = U^{-1}$$

$$e^{-i\theta_a T_a^\dagger} = e^{-i\theta_a T_a}$$

Ge

$$\delta q = \epsilon (\Delta q)$$

constraints

$$(n-1)+1]$$

and n^2-1 parameters

Generators T_a

$$\boxed{T_a^\dagger = T_a} \sim \text{hermitian}$$

$$\text{Rank of } SU(n) = n-1$$

$$\delta q = \epsilon (\Delta q)$$

Constraints

$$[n + n(n-1) + 1]$$

$a \rightarrow$ real $n^2 - 1$ parameters

Generators T_a

$$\boxed{T_a^\dagger = T_a} \sim \text{hermitian}$$

Rank of $SU(n) = n-1$

$$\det M = e^{\text{Tr} \log M}$$

$$\det U = 1$$

$$\delta q = \epsilon (\Delta q)$$

Constraints

$$[n + n(n-1)/2]$$

$a \rightarrow$ real $n^2 - 1$ parameters

Generators T_a

$$\boxed{T_a^\dagger = T_a} \sim \text{hermitian}$$

$$\text{Rank of } SU(n) = n-1 \quad e^{i \epsilon_a T_a}$$

$$\det M = e^{\text{Tr} \log M} \Rightarrow \text{Tr } T_a = 0$$

$$\det U = 1$$

a) $n-1$ linearly independent mutually commuting matrices

which the CSA of $SU(n)$

$$H_1 = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 & \\ & & & & -1 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}$$

$$H_{n-1} = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & +1 & \\ & & & & -1 \end{pmatrix}$$

a) $n-1$ linearly independent mutually commuting matrices

which form the CSA of $SU(n)$

$$H_1 = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 & \\ & & & & -1 \end{pmatrix}, H_2 = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}$$

$$H_{n-1} = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

$$\text{Rank } SU(n) = n-1$$

b) $\frac{n(n-1)}{2}$

$$[T^{ab}]_{ij} =$$

c) $\frac{n(n-1)}{2}$ (un)

$$[\tilde{T}^{ab}]_{ij}$$

os) b) $\frac{n(n-1)}{2}$ real, symmetric matrices

$$[T^{ab}]_{ij} = \frac{1}{2}(\delta_i^a \delta_j^b + \delta_j^a \delta_i^b), \quad (? \quad T^{aa} = 0)$$

c) $\frac{n(n-1)}{2}$ imaginary and antisymmetric

$$[\tilde{T}^{ab}]_{ij} = \frac{i}{2}(\delta_i^a \delta_j^b - \delta_j^a \delta_i^b)$$

as b) $\frac{n(n-1)}{2}$ real, symmetric matrices

$$[T^{ab}]_{ij} = \frac{1}{2}(\delta_i^a \delta_j^b + \delta_j^a \delta_i^b), \quad a < b$$

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