Title: PSI 2019/2020 - Lie Groups and Lie Algebras - Lecture 5

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Collection: PSI 2019/2020 - Lie Groups and Lie Algebras (Ali)

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Last Time: Time Inversion $\hat{T}$

$$[\hat{T}, \hat{H}] = 0 \quad \text{— (0)}$$

$\hat{H}$ ↳ Hamiltonian

$$|\psi(t)\rangle = e^{-i\hat{H}t}|\psi(0)\rangle \quad \text{— (1)}$$

$$\hat{T}|\psi(t)\rangle = |\psi(-t)\rangle$$

$$= \hat{T} e^{-i\hat{H}t} \hat{T}^{-1}\hat{T}|\psi(0)\rangle$$

$\hat{T}^{-1}\hat{T} = \mathbb{1}$

$$= e^{+i\hat{H}t}|\psi(0)\rangle$$

$t \to -t$ in (1)

$$|\psi(-t)\rangle = e^{+i\hat{H}t}|\psi(0)\rangle$$

So if we assume in addition to (0) the following

$$\hat{T}_i = -i\hat{T}$$

$$\hat{T}_i \hat{T}^{\dagger} = -i$$

Shannon entropy

Claim: Degenerate subspaces form an irrep of the symmetry group G.

$$[H, D(g)] = 0 \quad \forall g \in G$$

$D(g)$ acts on abstract Hilbert space $\mathcal{H}$.

If $D(g)$ act non-trivially on some $|\psi\rangle \xrightarrow{D(g)} |\psi'\rangle$

$$D(g)|\psi\rangle \quad \forall g \in G$$

$= e^{\frac{i}{\hbar}Ht} |\psi(0)\rangle$

$\hat{T}_i \hat{T}^{-1} = -i$

What is the dimension of the subspace generated by $D(g)|\psi\rangle \ \forall g \in G$.

Ans: K is some positive integer larger than 1.

Choose basis: $\{|\psi_1\rangle, |\psi_2\rangle \cdots, |\psi_K\rangle\}$

Because $[D(g), H] = 0$

$\Rightarrow |\psi_i\rangle$ have the same energy Eigenvalue

Degeneracy in QM

Claim: Degenerate subspaces form rep of the symmetry group G.

$$[H, D(g)] = 0 \quad \forall\, g \in G$$

$D(g)$ acts on abstract Hilbert space $\mathcal{H}$.

If $D(g)$ act non-trivially on some $|\psi\rangle \xrightarrow{D(g)} |\psi'\rangle$

$$D(g)\,|\psi\rangle \quad \forall\, g \in G$$

$\hat{T}_i \hat{T}^{-1} = -i$

$D(g)|\psi_i\rangle \quad \forall g \in G$

ubspace

ter than 1.

y Eigenvalue

$$D(g)|\psi_i\rangle = \sum_{j=1}^{k} |\psi_j\rangle\langle\psi_j|D(g)|\psi_i\rangle$$

$$= \sum_{j=1}^{k} |\psi_j\rangle D(g)_{ji}$$

$D(g)_{ji} \rightarrow$ A rep of G

Unitary matrices

$D(g)|\psi\rangle \quad \forall g \in G$

$$\sum_{j=1}^{k} |\psi_j\rangle\langle\psi_j| D(g) |\psi_i\rangle$$

$$\sum_{j=1}^{k} |\psi_j\rangle D(g)_{ji}$$

rep of G

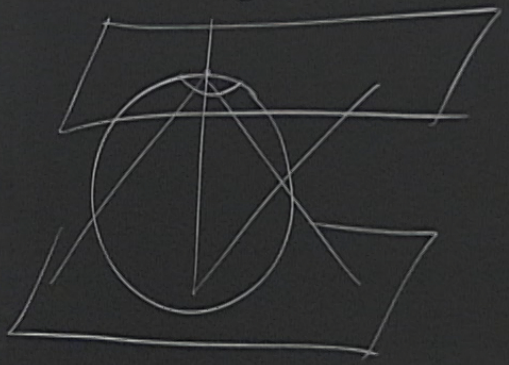
unitary matrices

Schur's Lemma:

$$H = \begin{pmatrix} \lambda \mathbb{1}_k & 0 & 0 \\ 0 & \mu \mathbb{1}_m & 0 \\ 0 & 0 & \sigma \mathbb{1}_p \end{pmatrix}$$

Lie Groups

A Lie group G is a group which is also a manifold.

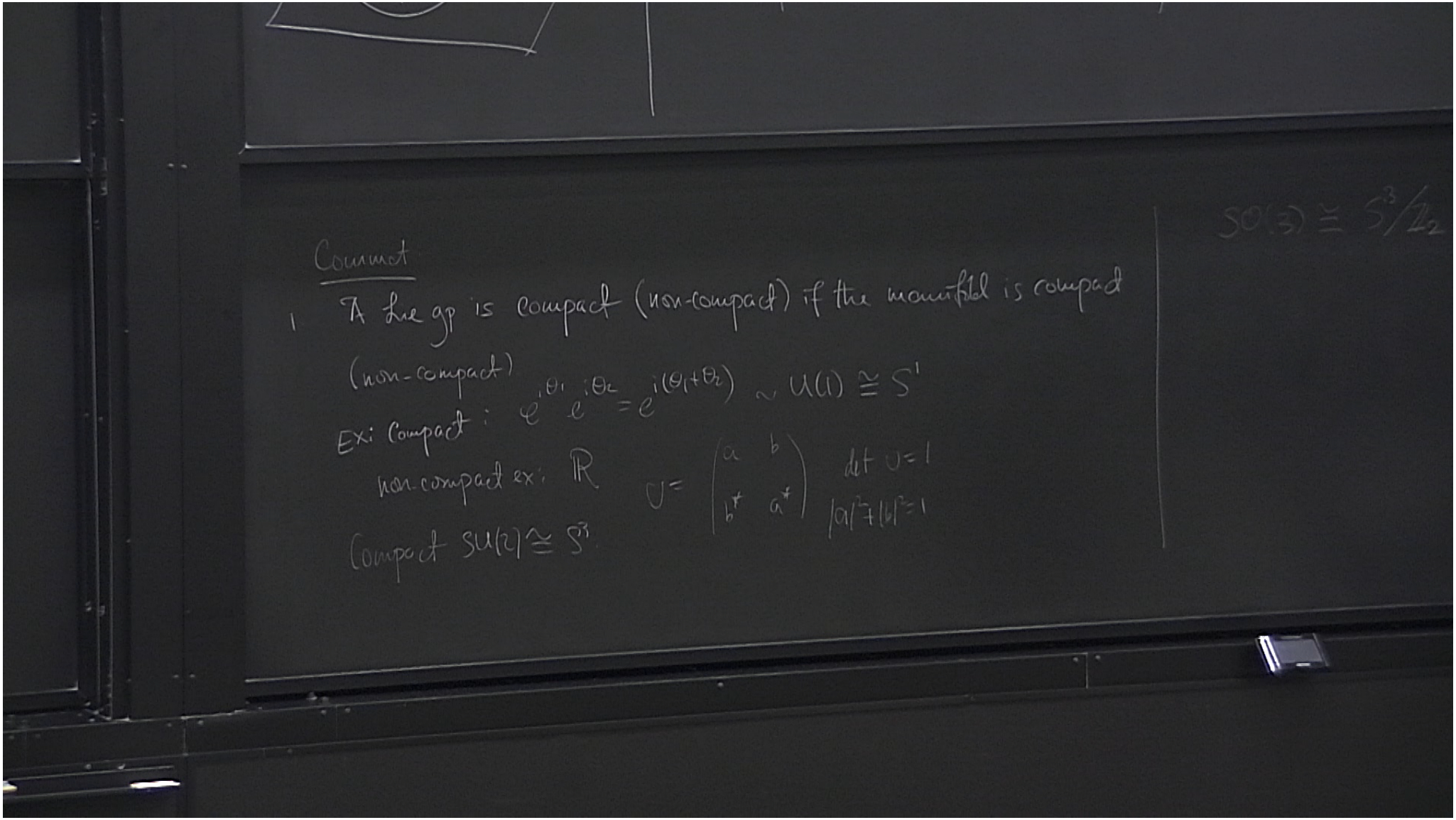


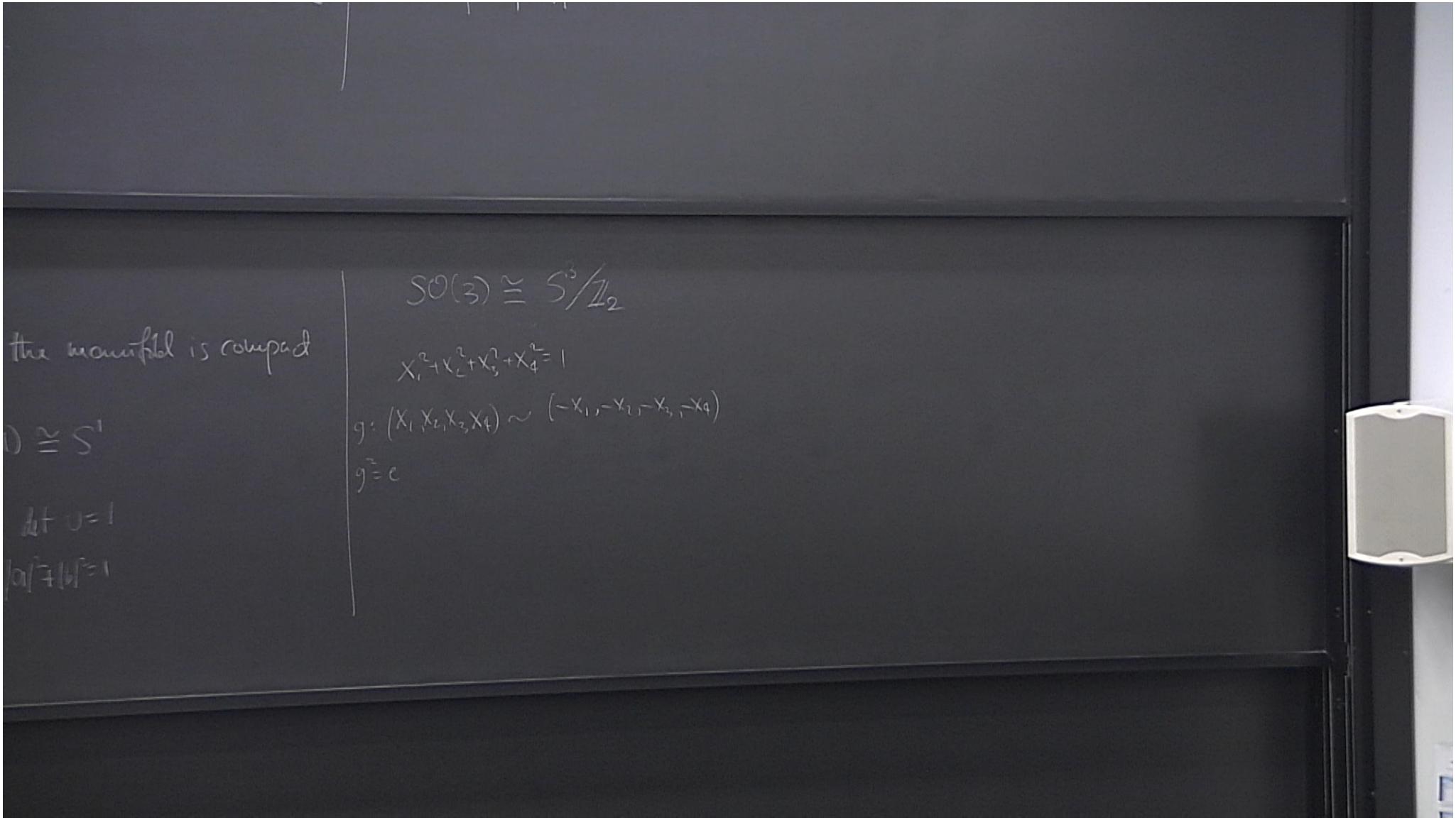
There are continuous parameters labelling the group elements and these parameters are coordinates on some curved space.

which is also a manifold.

re continuous parameters
g the group elements and
rameters are coordinates on some
l space.

In addition to the group structure $G \times G \to G$
a Lie group also inherits all the additional
structure of a manifold such as differentiability
continuity, topology


Connect
1 A Lie gp is compact (non-compact) if the manifold is compact
(non-compact)
Ex: Compact : $e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)} \sim U(1) \cong S^1$
non-compact ex: $\mathbb{R}$
Compact $SU(2) \cong S^3$
$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$ $\det U = 1$
$|a|^2 + |b|^2 = 1$
$SO(3) \cong S^3/\mathbb{Z}_2$


the manifold is compact
$\cong S^1$
$SO(3) \cong S^3/\mathbb{Z}_2$
$x_1^2+x_2^2+x_3^2+x_4^2=1$
$g: (x_1,x_2,x_3,x_4) \sim (-x_1,-x_2,-x_3,-x_4)$
$g^2=e$

Compact $SU(2) \cong S^3$

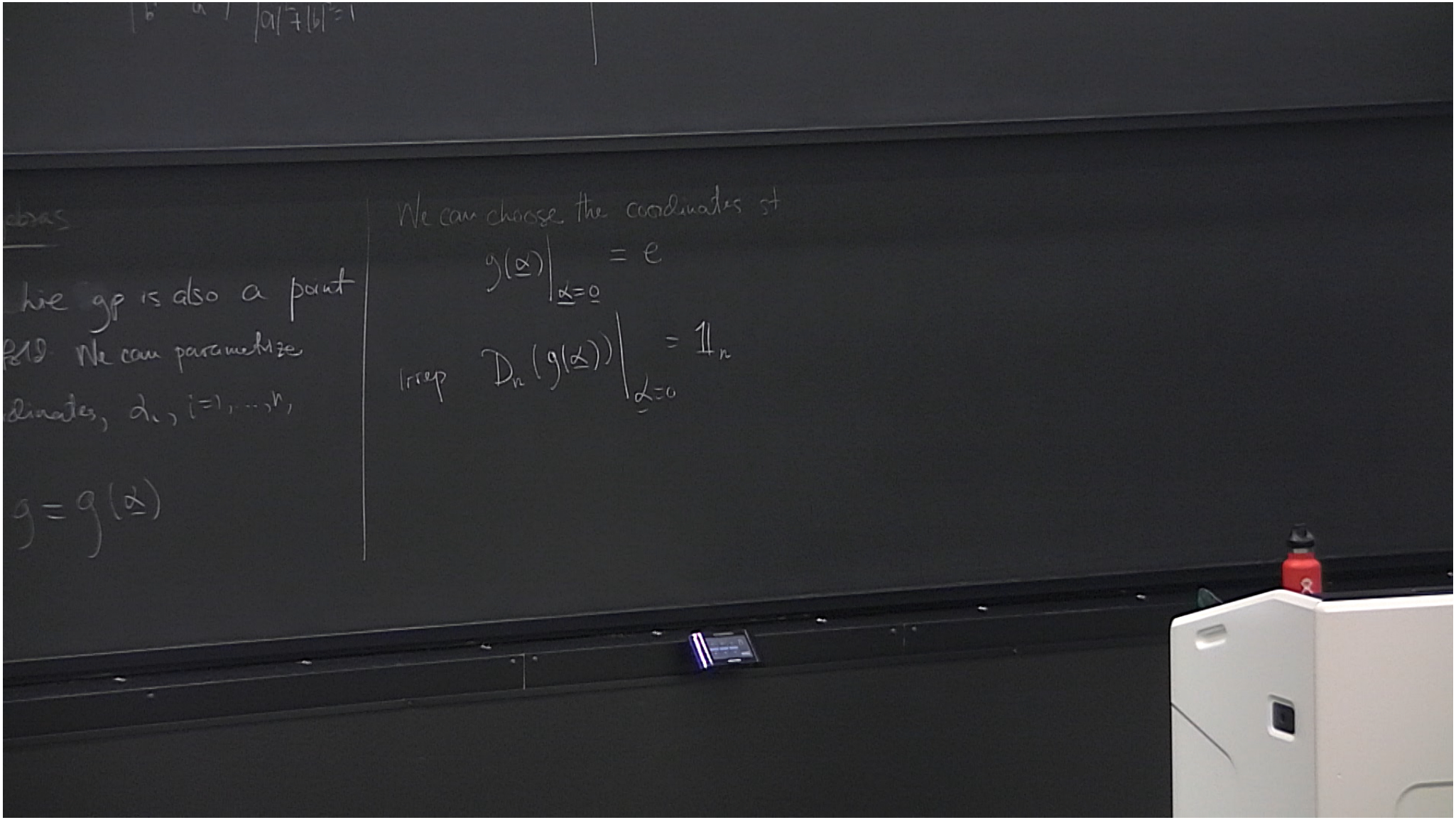
$|a|^2 + |b|^2 = 1$

Lie groups & Lie Algebras

An element of a lie gp is also a point on the some manifold. We can parametrize elements by the coordinates, α_i, $i = 1, \ldots, n$, on the manifold

$$g = g(\vec{\alpha})$$

We can choose


|a|²+|b|²=1
ebras
Lie gp is also a part
812 We can parametrize
dinates, α_i, $i=1,\dots,n$,
$g = g(\underline{\alpha})$
We can choose the coordinates st
$g(\underline{\alpha})\big|_{\underline{\alpha}=\underline{0}} = e$
Irrep $D_n(g(\underline{\alpha}))\big|_{\underline{\alpha}=0} = \mathbb{1}_n$

$= \hat{T} e^{-iHt} \hat{T}^{-1} \hat{T} |\psi(0)\rangle$

$= e^{+iHt} |\psi(0)\rangle$

$\hat{T}^{-1}\hat{T} = \mathbb{1}$

So if we assume in addition to (0) the following

$\hat{T}_i = -i\hat{T}$ ✓

$\hat{T}_i \hat{T}^{\dagger} = -i$

If $D(g)$

For small $\delta\alpha_i$

$$D_n(g(\underline{\delta\alpha})) \simeq \mathbb{1}_n + \delta\alpha_i \left.\frac{\partial D_n(g(\underline{\alpha}))}{\partial \alpha_i}\right|_{\underline{\alpha}=0}$$

$$X_i \equiv -i \left.\frac{\partial D_n}{\partial \alpha_i}\right|_{\underline{\alpha}=0} \rightsquigarrow \text{Generators}$$

$$D_n(g(\delta\underline{\alpha})) \simeq \mathbb{1} + i\delta\alpha_i X_i$$

addition to (0) the following

$\hat{T}_i = -i\hat{T}_i$

$\hat{T}_i \hat{T}_i^{\dagger} = -i$

If $D(g)$ act non-trivially on some $|\psi\rangle \xrightarrow{D(g)} |\psi'\rangle$

$D(g)\,|\psi\rangle \;\forall g \in G$

$\left.\frac{\partial D_n(g(\alpha))}{\partial \alpha_i}\right|_{\underline{\alpha}=0}$

generators

Lets take $\alpha_i = N\,\delta\alpha_i$

$$\lim_{N\to\infty} \left(\mathbb{1} + i\,\delta\alpha_i X_i\right)^N = \lim_{N\to\infty} \left(\mathbb{1} + i\,\frac{\alpha_i}{N} X_i\right)^N$$

$$= e^{i\alpha_i X_i}$$

The form of a finite element of the lie group

is: $e^{i\alpha_i X_i}$

g) act non-trivially on some $|\psi\rangle \xrightarrow{D(g)} |\psi'\rangle$

$D(g)|\psi\rangle \quad \forall g \in G$

$N \in \alpha_i$

$\left(\;\right)^N = \lim_{N\to\infty}\left(\mathbb{1} + i\frac{\alpha_i}{N}X_i\right)^N$

$i\alpha_i X_i$

element of the Lie group

This is the exponential map $\alpha_i X_i \longrightarrow e^{i\alpha_i X_i}$

Now we will derive the form of the algebra satisfied by $\{X_i\}$

$$e^{i\alpha_i X_i}\, e^{i\beta_j X_j} = e^{i\gamma_k X_k}$$

If $[\alpha_i X_i, \beta_j X_j] = 0$

$$\gamma_k X_k = (\alpha_i + \beta_i) X_i$$

Lie gr

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elem

on

$$i\gamma_k X_k = \log\left[e^{i\alpha_i X_i}\, e^{i\beta_j X_j}\right]$$

$$\log(1+X) = X - \frac{X^2}{2} + \cdots$$

BCH Formula

↳ Baker-Campbell-Hausdorff

$$X = e^{i\underline{\alpha}\cdot\underline{X}}\, e^{i\underline{\beta}\cdot\underline{X}} - 1$$

$$X - \frac{X^2}{2} = i(\alpha_i X_i + \beta_i X_i) - \frac{1}{2}[\alpha_i X_i, \beta_j X_j]$$

$e^{i\alpha_i X_i}\, e^{i\beta_j}$

$e^{i\beta_j X_j}]$

BCH Formula

↳ Baker-Campbell-Hausdorff

$-\frac{1}{2}[\alpha_i X_i, \beta_j X_j]$

$$e^{i\alpha_i X_i} e^{i\beta_j X_j} \simeq e^{i(\alpha_i X_i + \beta_j X_j) - \frac{1}{2}[\alpha_i X_i, \beta_j X_j]}$$

$e^{i\theta_i X_i}$

$$\boxed{[X_i, X_j] = i f_{ij}^{\ k} X_k} \sim \text{Lie Algebra}$$

$f_{ij}^{\ k} \rightarrow$ structure constant.

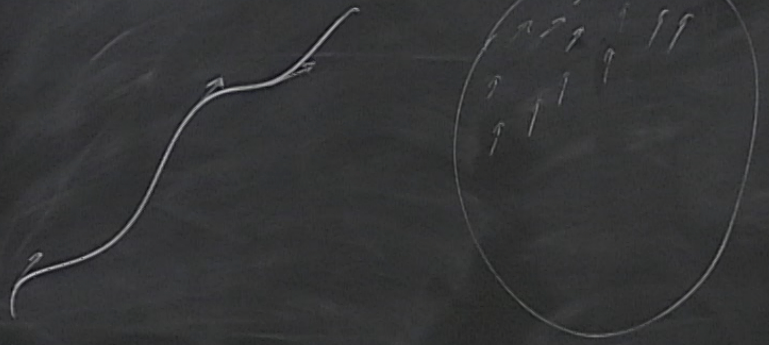
$-\frac{1}{2}[\alpha_i x_i, \beta_j x_j]$

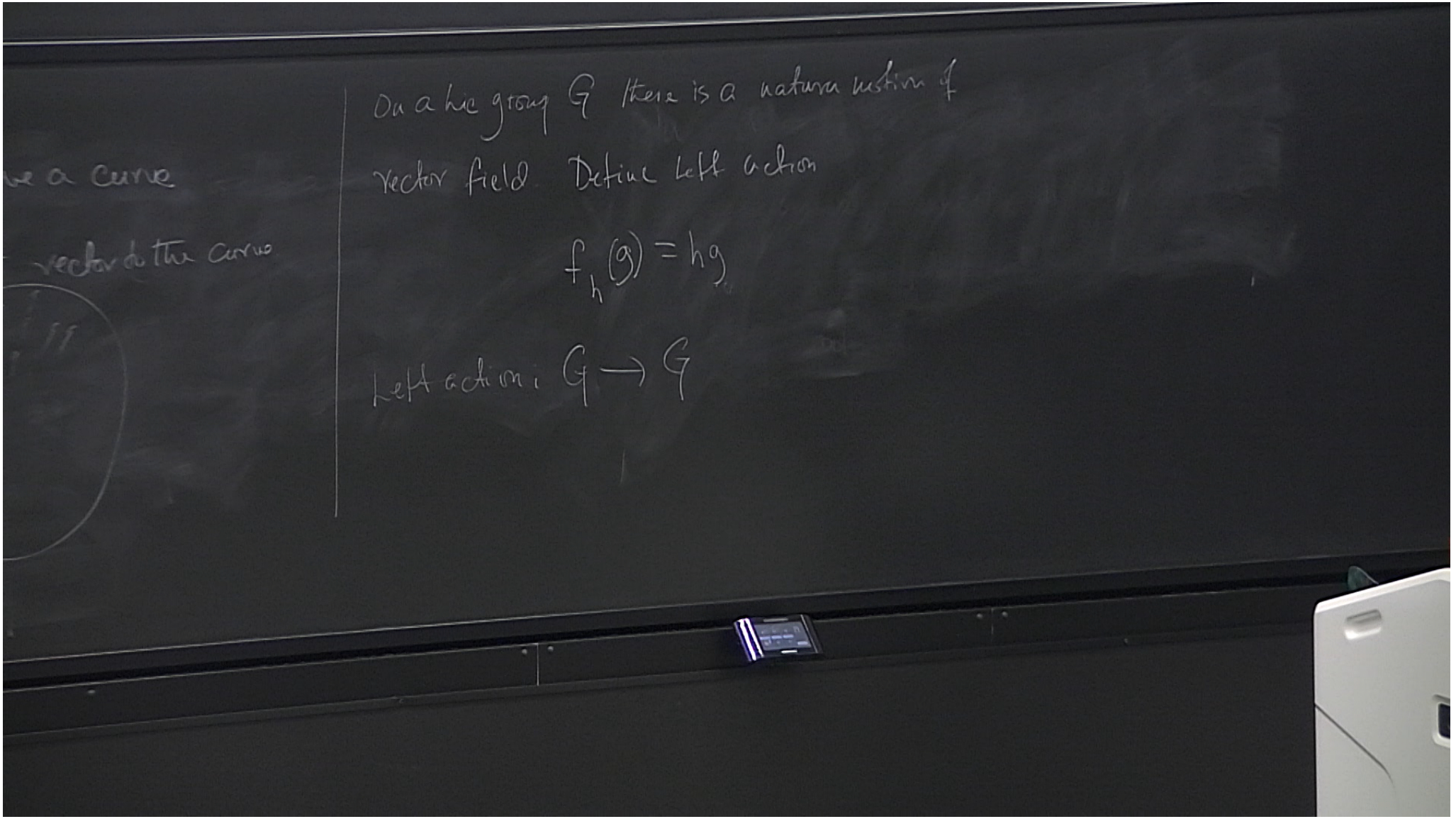
Algebra

The Lie algebra captures the non-commutative nature of the Lie group multiplication in the neighbourhood of e.

On a manifold if you have a curve

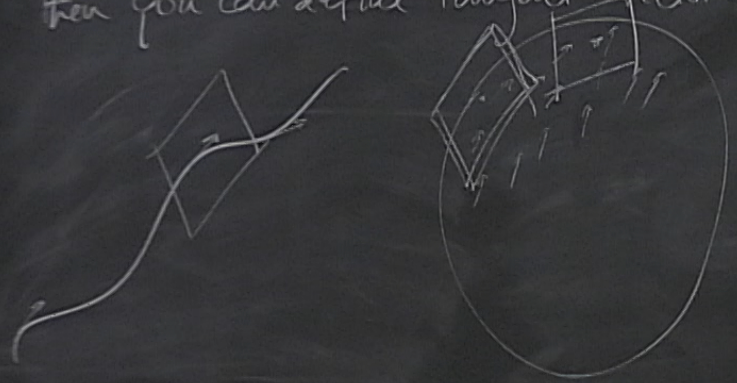
then you can define tangent vector to the curve




e a curve
vector to the curve
On a lie group G there is a natural action of
vector field. Define left action
$f_h(g) = hg$
Left action: $G \to G$

On a manifold if you have a curve

then you can define tangent vector to the curve



On a lie group G there is a natural notion of vector field Define left action

$$f_h(g) = hg$$

Left action: $G \to G$

$$T(G) \to T(G)$$

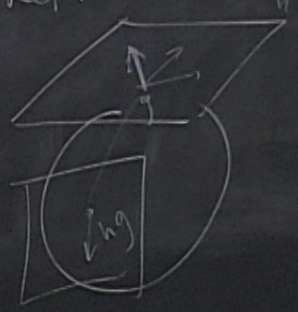
On a lie group G there is a natural notion of vector field. Define left action

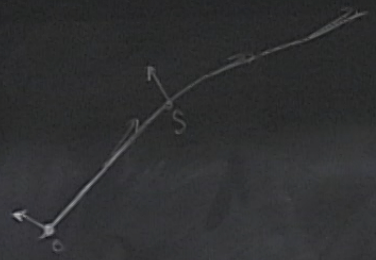
$$f_h(g) = hg$$

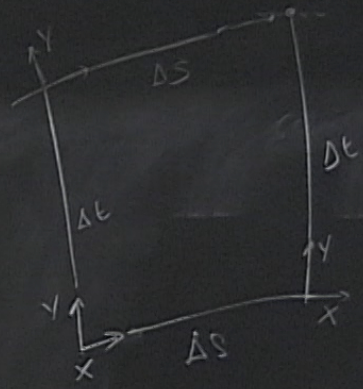
Left action: $G \to G$

$T(G) \to T(G)$

Left-invariant vector field. Take one point on G







Lie derivative. A vector field allows one

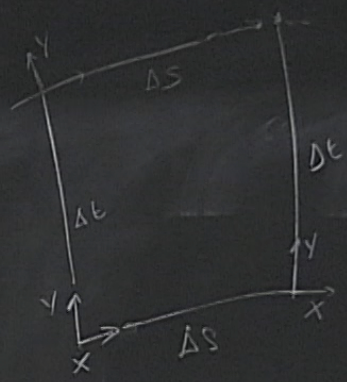
to define such a derivative.

$$\mathcal{L}_X Y = [X, Y] = \sum_\alpha c_\alpha T_\alpha$$

measures the failure of small rectangels to close







Lie derivative. A vector field allows one to define such a derivative.

$$\mathcal{L}_X Y = [X, Y] = \sum_a c_a T_a$$

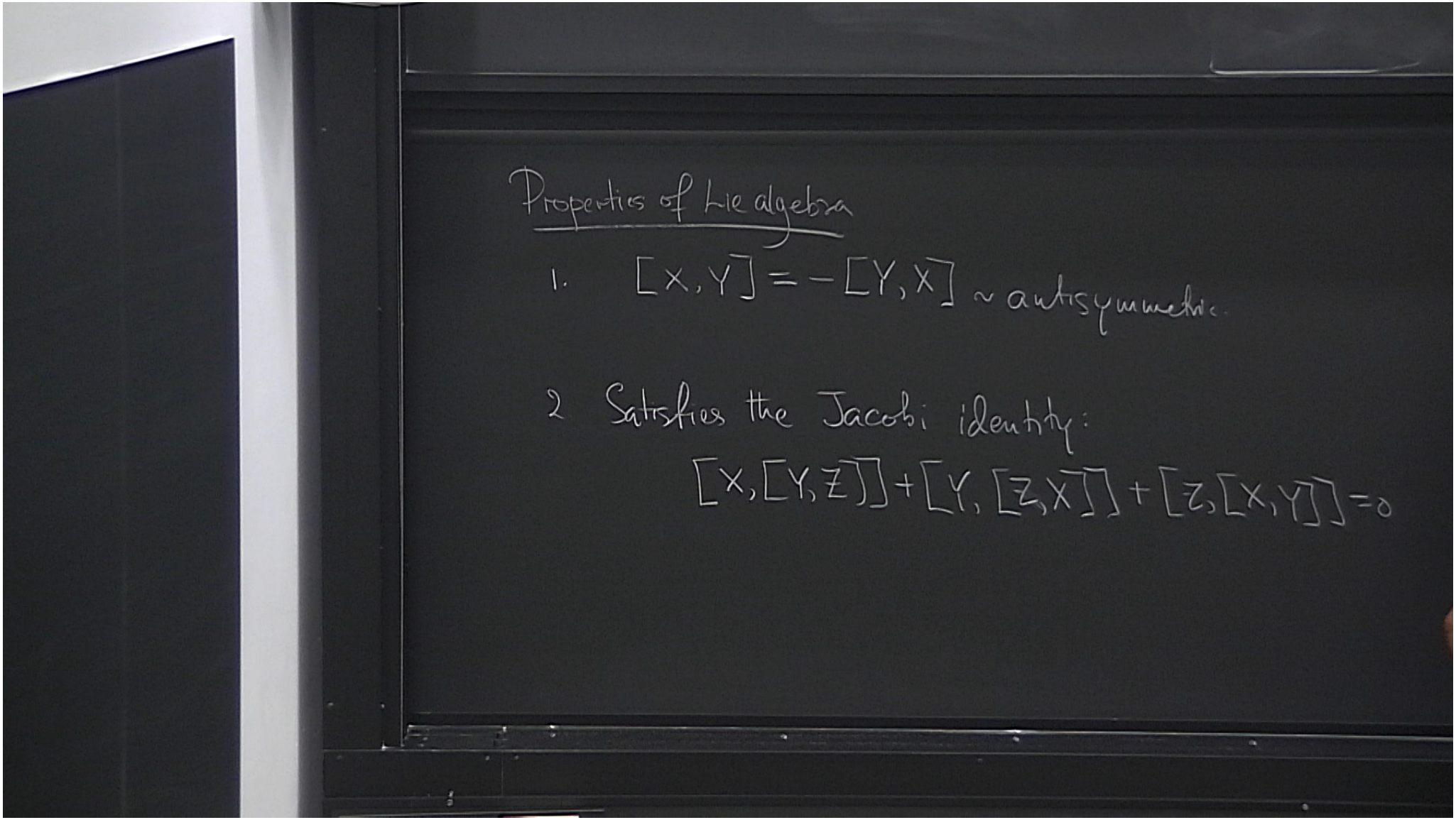
measures the failure of small rectangels to close

n independent vector fields

$$[T_a, T_b] = i f_{ab}{}^c T_c$$

Lie algebra

e.g. $\alpha_a T_a \rightarrow \exp(i\alpha_a T_a)$


Properties of Lie algebra
1. $[X,Y] = -[Y,X]$ ~ antisymmetric.
2 Satisfies the Jacobi identity:
$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$

3. Choose basis so that

$$[T_a, T_b] = i f_{ab}{}^{c} T_c$$

4. Jacobi identity

$$f_{ab}{}^{d} f_{dc}{}^{e} + f_{ca}{}^{d} f_{db}{}^{e} + f_{bc}{}^{d} f_{da}{}^{e} = 0$$

5. (

th

5. On compact group define metric on the Lie alg.

$$g_{ab} = \lambda \, \mathrm{Tr}(T_a T_b)$$ ~ Cartan metric (Killing form)

Using g_{ab}

$$f_{abc} = g_{cd} f_{ab}{}^{d}$$

$f_{abc} = f_{[abc]}$ ~ Completely antisymm