Title: PSI 2019/2020 - Lie Groups and Lie Algebras - Lecture 4

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Collection: PSI 2019/2020 - Lie Groups and Lie Algebras (Ali)

Date: August 13, 2019 - 10:45 AM

URL: http://pirsa.org/19080048


THEORETICAL
PHYSICIST

$K \rightarrow$ normal subgroup.

Representations:

$SO(3)$, $O(3)$, $SO(1,3)$, $O(1,3)$

$GL(n, \mathbb{C})$: The group $n \times n$ matrices with complex entries and non-zero determinant.

Closure:

$M_1 M_2 = M_3$

$\det(M_1 M_2) = \det M_3$

$\det M_1 \det M_2 = \det M_3$

$\dim GL(n, \mathbb{R}) = n^2$

$\det M \neq 1$.

...33)

...atrices with

...terminat.

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Def^n: Rep

Closure:

$M_1 M_2 = M_3$

$\det(M_1 M_2) = \det M_3$

$\det M_1 \det M_2 = \det M_3$

$\mathrm{Dim}\, GL(n,\mathbb{R}) = 2n^2$

$\det M \neq 0$

3)

trices with

terminant.



Defn: Representation

A representation of the gp G is defined as the homomorphism $D: G \to GL(n, \mathbb{C})$ such that the group structure is preserved, i.e.,

$$D(g_1 g_2) = D(g_1) D(g_2)$$

Comments:

1. $D(e) = \mathbb{1}$
2. $D(g^{-1}) = [D(g)]^{-1}$

Examples: $SO(3) \nsubseteq SO(1,3)$

$\cap$ $\cap$

$GL(3,\mathbb{R})$ $GL(4,\mathbb{R})$

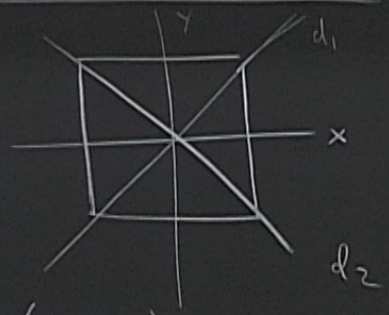
$SO(3)$ was defined by $O^T O = \mathbb{1}$ with O a 3×3 real matrix

$\det O = 1$

Such matrix rep. is known the defining representation.

The Dihedra Group D_4

$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $R^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $R^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$



$r_x = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $r_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$d_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $d_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Representations can be faithful or unfaithful.

Defn: Faithful Rep.

If D is a rep of G that maps distinct elements of $g \in G$ to distinct elements $D(G) = \{D(g) \mid g \in G\}$ then we say the rep is faithful. Otherwise the rep is called an unfaithful rep.

Ex: Trivial rep $D: G \longrightarrow 1$.

If $g \in G$

then we say

an unfaithful

Def: Equivalent Representations

Two reps D & D' of G are said to be equivalent if $D'(g) = S D(g) S^{-1}$ $\forall g$.

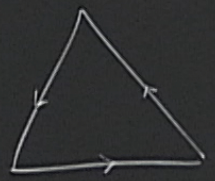
Note that the matrix S is independent of g.

Comment

1. We think of $D(g)$ & $D'(g)$ being the 'same' but represented in different bases.

C_3

$$D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$D(\alpha) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D(\alpha^2) = \begin{pmatrix} -1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D = \left(\begin{array}{cc|c} 2\times 2 & & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

Reducibility of

D

A rep^n of d

reducible if

$-\sqrt{3}/2 \quad 0$
$-1/2 \quad 0$
$0 \quad 1$
$\sqrt{3}/2 \quad 0$
$-1/2$
$0 \quad 1$

Reducibility of Reps

A rep^n D of dim $n+m$ is said to be reducible if $D(g)$ takes the form

$$D(g) = \begin{pmatrix} A(g) & C(g) \\ 0 & B(g) \end{pmatrix} \quad \forall g \in G$$

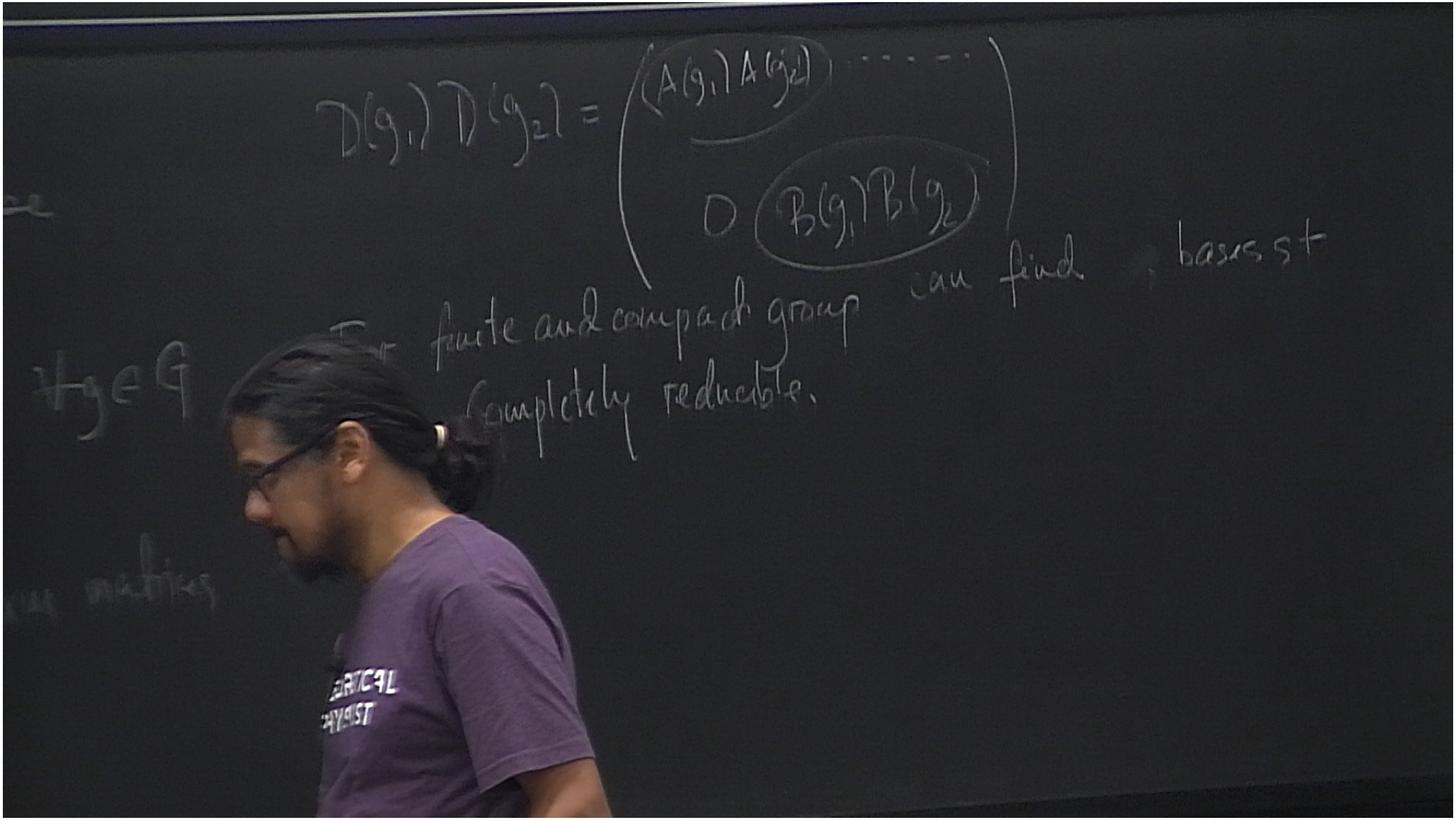
where A, C, B are $n \times n$, $n \times m$ & $m \times m$ matrices respectively.

$$D(g_1)D(g_2) = \begin{pmatrix} A(g_1)A(g_2) & \cdots \\ 0 & B(g_1)B(g_2) \end{pmatrix}$$

For finite and compact group

$\forall g \in G$

matrices


$D(g_1)D(g_2) = \begin{pmatrix} A(g_1)A(g_2) & \cdots \\ 0 & B(g_1)B(g_2) \end{pmatrix}$
finite and compact group
can find
; bases s.t
completely reducible.

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For finite and compact group G can find a basis s.t. $C=0$. Completely reducible.

$$D(g_1)D(g_2) = \begin{pmatrix} A(g_1)A(g_2) & \cdots \\ 0 & B(g_1)B(g_2) \end{pmatrix}$$

For finite and compact group can find basis s.t

$C=0$. Completely reducible.

If a rep is not reducible then the rep is an irreducible rep (irrep).

$g \in G$

matrices

$D(\alpha^2) = \begin{pmatrix} -\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$D = \left(\begin{array}{c|c} 2\times 2 & \begin{matrix}0\\0\end{matrix} \\ \hline 0\;0 & 1 \end{array}\right)$

where A, C, B are $n\times n$, $n\times m$ & $m\times m$ matrices respectively.

Comment:

1. For finite Abelian groups the complex and unitary representations ($U^\dagger = U^{-1}$) are 1 dimensional. Take C_3 then its non-trivial complex irreducible reps are:

$$\{1, e^{i2\pi/3}, e^{i4\pi/3}\}$$

$$\{1, e^{-i2\pi/3}, e^{-i4\pi/3}\}$$

$$\{1, 1, 1\}$$

inequivalent reps. They are also irreps

$$D(a^2) = \begin{pmatrix} -1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & \\ 0 & 0 & 1 \end{pmatrix}$$

$$D = \left(\begin{array}{c|c} 2\times 2 & \begin{matrix}0\\0\end{matrix} \\ \hline \begin{matrix}0 & 0\end{matrix} & 1 \end{array}\right)$$

where A, C, B are $n\times n$, $n\times m$ & $m\times m$ matrices respectively.

If a rep is not ... rep (irrep)

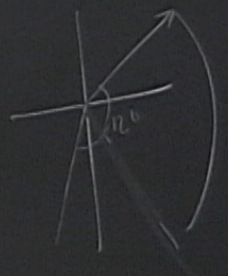
Comment:

1. For finite Abelian groups the complex and unitary representations ($U^\dagger = U^{-1}$) are 1 dimensional. Take C_3 then its non-trivial complex irreducible reps are:

Faithful $\left\{ \{1, e^{i2\pi/3}, e^{i4\pi/3}\} \right.$

$\left. \{1, e^{-i2\pi/3}, e^{-i4\pi/3}\} \right\}$ inequivalent reps. They are also irreps

unfaithful $\leftarrow \{1, 1, 1\}$



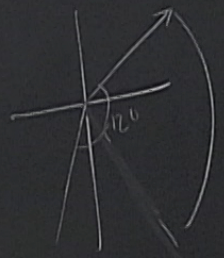
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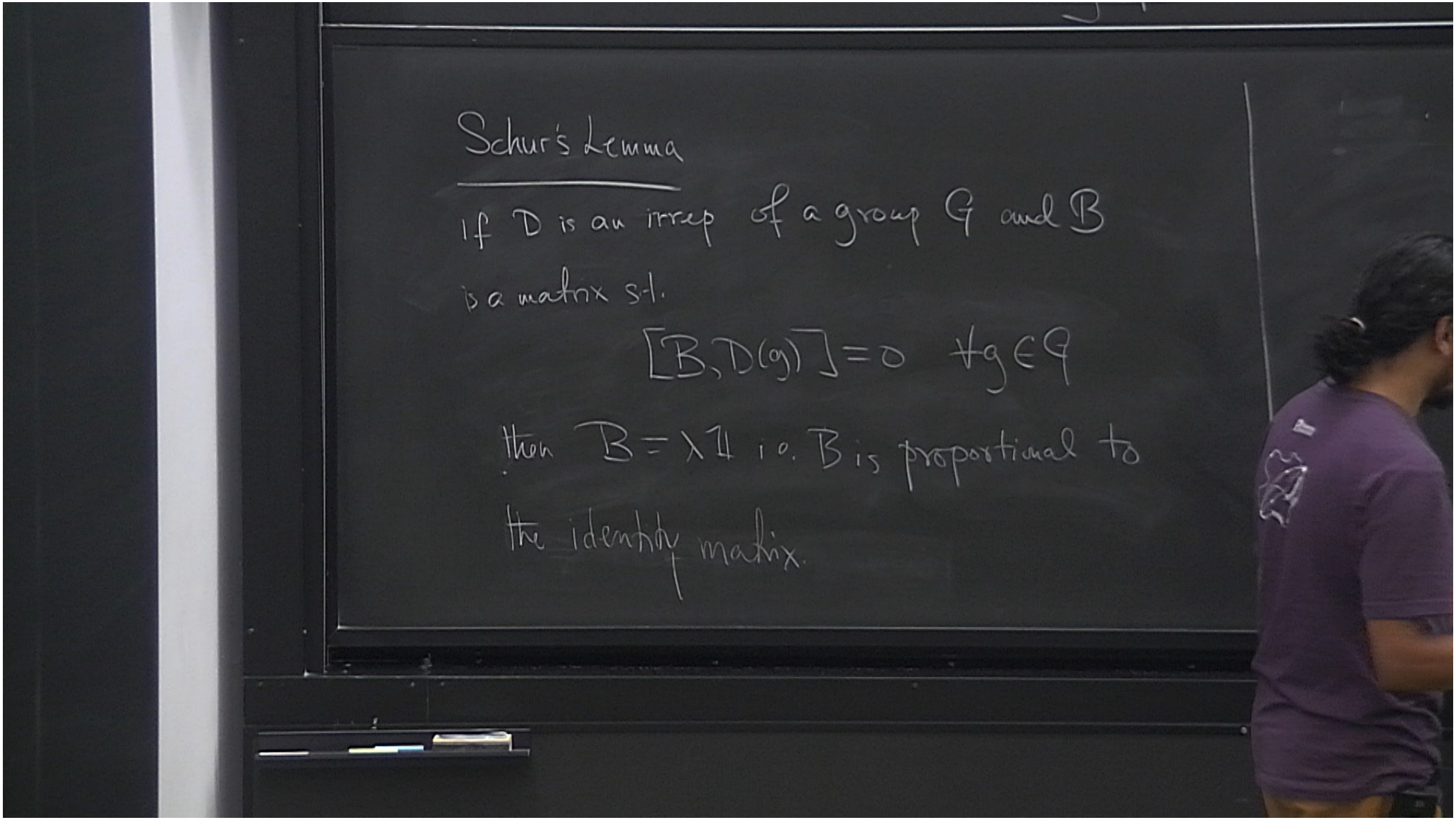
inequivalent reps

They are also irreps

Dim of a rep D is the dim of the matrices D

$U(g_1)U(g_2) = e^{i\phi(g_1,g_2)} U(g_1 g_2)$

$\alpha \; UL^{-1}$


Schur's Lemma
If D is an irrep of a group G and B is a matrix s.t.
$[B, D(g)] = 0 \quad \forall g \in G$
then $B = \lambda \mathbb{1}$ i.e. B is proportional to the identity matrix.

Characters:

Characters are a convenient way of distinguishing different reps.

Defn: Character:

The character of a representation D of a gp G is the set $\chi = \{\chi(g) \mid g \in G\}$

where $\chi(g)$ is the trace of the rep matrix $D(g)$:

Day

reps

ation D of

$= \{\chi(g) \mid g \in G\}$

the rep matrix $D(g)$:

$\chi(g) = \mathrm{Tr}\, D(g)$. $D(g) = S\, D'(g)\, S^{-1}$ $\forall g \in G$

Comments:

1. $\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB)$
$= \mathrm{Tr}(BCA)$

$\chi(g) = \mathrm{Tr}(D(g)) = \mathrm{Tr}(S D'(g) S^{-1})$
$= \mathrm{Tr}(D'(g)) = \chi(g)$

2. Character is a function of conjugacy class:

$g_1 \sim g_2 \qquad g_1 = g g_2 g^{-1}$

$\chi(g_1) = \mathrm{tr}(g g_2 g^{-1})$

$= \mathrm{tr}(g_2) = \chi(g_2)$

3. Important theorem:

$\chi^{(\mu)} \rightarrow$ characters of a rep μ

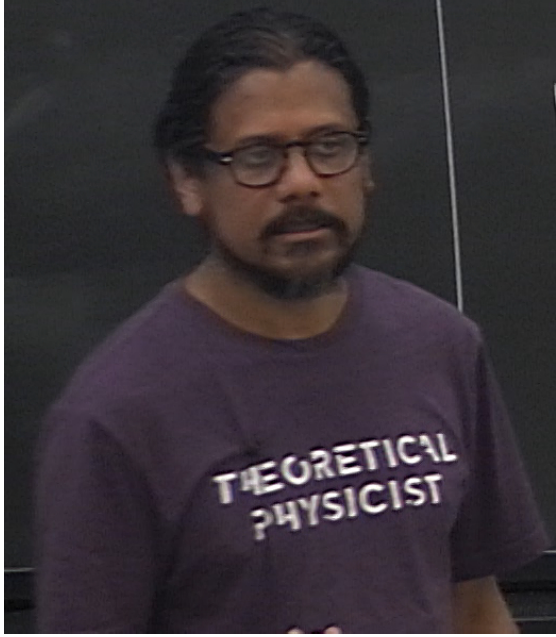
where $\chi(g)$ is the trace of the rep matrix $D(g)$:

$$\frac{1}{[g]} \sum_{g \in G} \chi^{(\mu)}(g)\, \chi^{(\nu)}(g^{-1}) = \delta^{\mu\nu}$$

$[g]$ → Order of the group

R_i elements in the conjugacy class K_i.

then $\frac{1}{[g]} \sum_i R_i\, \chi^{\mu}_i\, \chi^{\nu}_i = \delta^{\mu\nu}$



$$\frac{1}{[g]} \sum_{g \in G} \chi^{(\mu)}(g) \chi^{(\nu)}(g^{-1}) = \delta^{\mu\nu}$$

$[g] \rightarrow$ Order of the group.

k_i elements in the conjugacy class K_i.

then $\frac{1}{[g]} \sum_i k_i \chi^{\mu}_i \chi^{\nu}_i = \delta^{\mu\nu}$

Has generalizations for compact continuous groups.

For finite groups the number of inequivalent irreps, r, is equal to the total number of conjugacy classes, k.

$r = k$

Character table of C_3:

$\omega = e^{2\pi i/3}$

C_3	e	a	a^2
$D^{(1)}$	1	1	1
$D^{(2)}$	1	ω	ω^2
$D^{(3)}$	1	ω^2	ω

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Rep Theory & Quantum Mech

Theorem (Wigner - 1931)

The group of symmetries of a quantum mech system is represented by linear, unitary or anti unitary (unitary + antilinear) operators action on a Hilbert space.

Under some symmetry transformation (rotations, translations) is given by $g \in G$

$$|\psi_1\rangle \to U(g)|\psi_1\rangle$$

$$|\psi_2\rangle \to U(g)|\psi_2\rangle$$

$$|\langle\psi_2|U^\dagger(g)U(g)|\psi_1\rangle|^2 = |\langle\psi_1|\psi_2\rangle|^2$$

Under some symmetry transformation (rotations, translations) is given by $g \in G$

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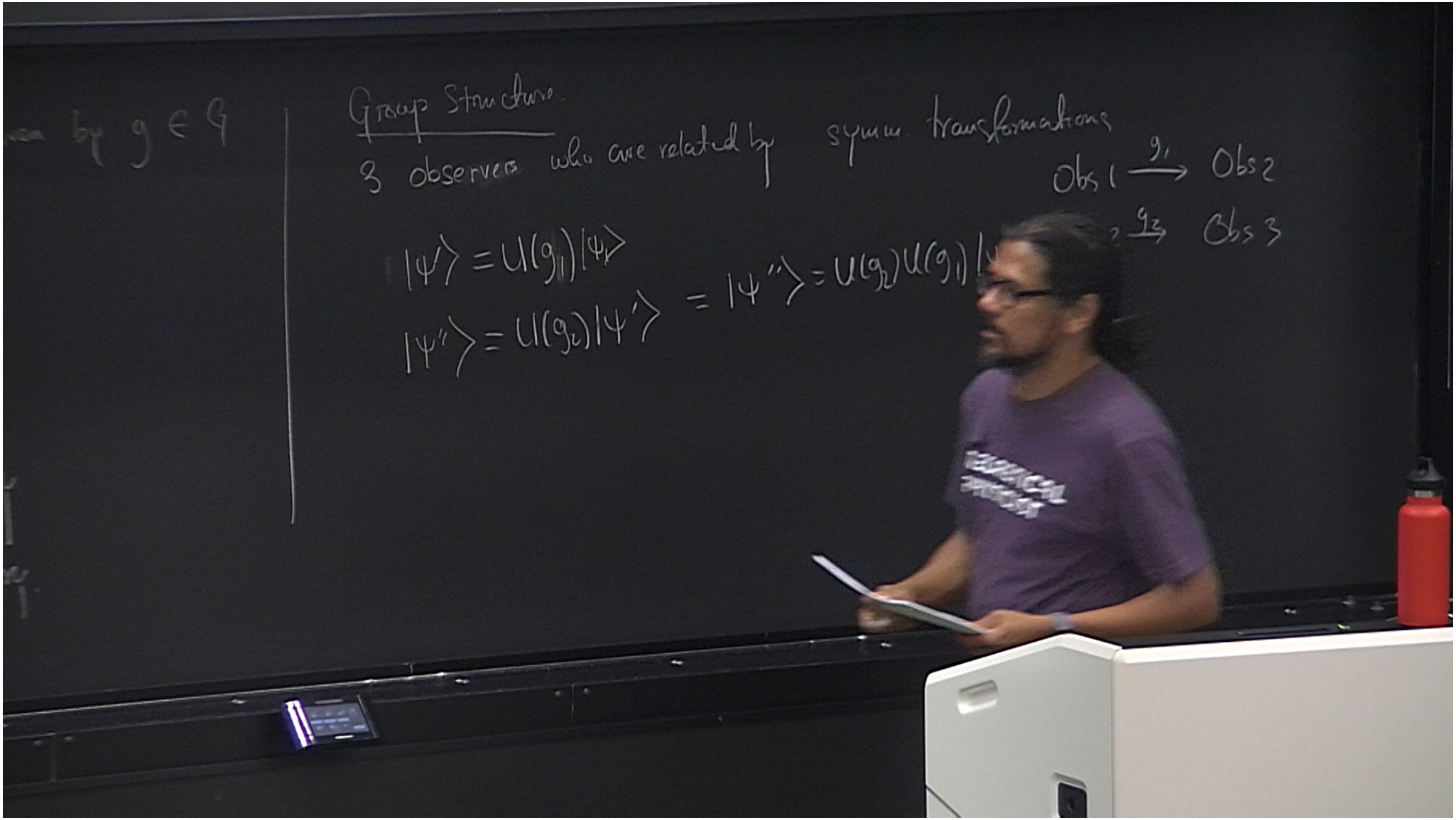
$$|\psi_2\rangle \to U(g)|\psi_2\rangle$$

$$|\langle\psi_2|U^\dagger(g)U(g)|\psi_1\rangle|^2 = |\langle\psi_1|\psi_2\rangle|^2$$

Has two solutions:

1. $\langle\psi_1|U^\dagger U|\psi_2\rangle = \langle\psi_1|\psi_2\rangle \to U$ is unitary

2. $\langle\psi_1|U^\dagger U|\psi_2\rangle = \langle\psi_1|\psi_2\rangle^* \to U$ is antiunitary.


by g ∈ G
Group Structure
3 observers who are related by symm. transformations
Obs 1 → Obs 2
Obs 3
|ψ⟩ = U(g1)|ψ⟩
|ψ'⟩ = U(g2)|ψ⟩ = |ψ''⟩ = U(g2)U(g1)

Group Structure:

3 observers who are related by symm. transformations

Obs 1 $\xrightarrow{g_1}$ Obs 2

Obs 2 $\xrightarrow{g_2}$ Obs 3

$|\psi'\rangle = U(g_1)|\psi\rangle$

$|\psi''\rangle = U(g_2)|\psi'\rangle = |\psi''\rangle = U(g_2)U(g_1)|\psi\rangle$

$|\psi'\rangle = U(g_3)|\psi\rangle$

$U(g_3) = U(g_2)\,U(g_1)$

Comments:

1. In QM we are allowed

$$U(g_3) = e^{i\phi(g_1,g_2)}\,U(g_2)\,U(g_1)$$

→ Projective Reps.

$= \mathrm{Tr}(D'(g)) = \chi(g)$

where $\chi(g)$ is the trace of the rep matrix $D(g)$:

2 Time reversal op, $\hat{T}$, in QM is an antiunitary op.

$$\boxed{[\hat{T}, \hat{H}] = 0} \quad -(*)$$

$$|\psi(t)\rangle = e^{-i\hat{H}t}|\psi(0)\rangle$$

$$\hat{T}|\psi(t)\rangle = |\psi(-t)\rangle = \hat{T}e^{-i\hat{H}t}\hat{T}^{-1}\hat{T}|\psi(0)\rangle = \hat{T}e^{-i\hat{H}t}\hat{T}^{-1}|\psi(0)\rangle = e^{+i\hat{H}t}|\psi(0)\rangle$$

$$|\psi(-t)\rangle = e^{i\hat{H}t}|\psi(0)\rangle$$

$(*)$ only $\Rightarrow$ $\hat{T}e^{-i\hat{H}t}\hat{T}^{-1} = e^{-i\hat{H}t}$

$$\boxed{\hat{T}\, i\, \hat{T}^{-1} = -i}$$