

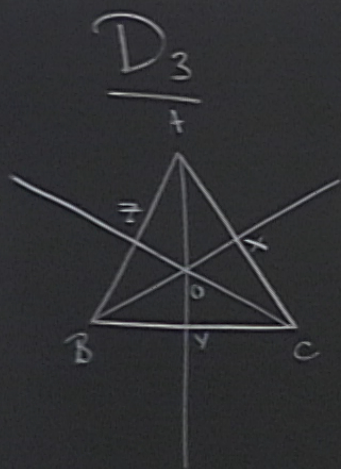
Title: PSI 2019/2020 - Lie Groups and Lie Algebras - Lecture 3

Speakers: Tibra Ali

Collection: PSI 2019/2020 - Lie Groups and Lie Algebras (Ali)

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$$D_3 = C_3 \cup \{b_1, b_2, b_3\}$$

b_1 : Reflection about BOX

b_2 : " " AOY

b_3 : " " COZ

$$C_3 = \{ \alpha, \alpha^2, e \}, \alpha: \text{cw rotation by } 120^\circ$$

Subgroups: $C_3 \nsubseteq 3 C_2s: \{e, b_i\}$

$$H = \{e, b_1\}$$

$$eH = \{e, b_1\} = H$$

$$\alpha H = \{\alpha, \alpha b_1\} =$$

$$\alpha^2 H = \{\alpha^2, \alpha^2 b_1\} =$$

120°

For H to be a normal subgroup

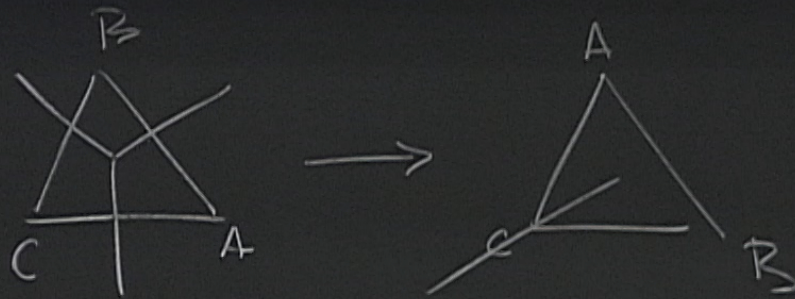
$$gHg^{-1} = H$$

$$gH = Hg$$

Left coset = Right coset.

$$He = \{ e, b_1 \}$$

$$H\alpha = \{ \alpha, b_1\alpha \}$$



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$$gHg^{-1} = H$$

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$$H\alpha = \{\alpha, b_1\alpha\} = \{\alpha, b_3\}$$

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$$gHg^{-1} = H$$

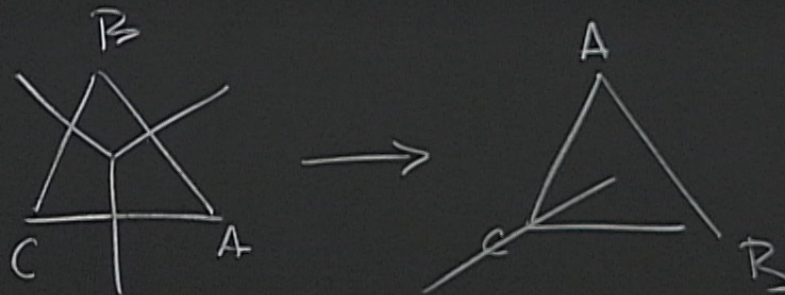
$$gH = Hg$$

Left coset = Right coset.

$$He = \{e, b_1\}$$

$$Ha = \{a, b_1 a\} = \{a, b_2\}$$

H is not a normal subgroup.



Is C_3 a word

Is C_3 a normal subgroup? $H = C_3$

$$\left. \begin{aligned} eH &= H = \{e, \alpha, \alpha^2\} \\ b_1H &= \{b_1, b_2, b_3\} \end{aligned} \right\} \begin{array}{l} \text{Left} \\ \text{coset} \end{array}$$

$$He = \{e, \alpha, \alpha^2\}$$

$$Hb_1 = \{b_1, b_3, b_2\}$$

C_3 is a normal subgroup

D_3/C_3 is a coset group

$$E \equiv \{e, \alpha, \alpha^2\}$$

$$B \equiv \{b_1, b_2, b_3\}$$

$$E^2 = (eH)(eH) = e^2H = eH = E$$

$$EB = eH b_1H = eb_1H = b_1H = B$$

$$BE = B$$

$$B^2 = E$$

C_2

$$C_2 \cong D_3 / C_3$$

$$= eH = E$$

$$= b_1H = B$$

Rotation in 3D

$$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow r' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} r$$

$\uparrow$
 O

$$r^T r = r'^T r'$$

$$O^T = O^{-1}$$

$$O^T O = \mathbb{1}$$

$$(\det O)^2 = 1$$

$$\det O = +1$$

$$\det O = -1$$

$$O = \exp [i \Theta \cdot T]$$

$T_a \rightarrow$ generators

$$T_a^\dagger = T_a$$

Rotation in 3D

$$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow r' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} r$$

$\uparrow$
 O

$$r^T r = r'^T r'$$

$$O^T = O^{-1}$$

$$O^T O = \mathbb{1}$$

$$(\det O)^2 = 1$$

$$\det O = +1 \text{ --- } SO(3)$$

$$\det O = -1$$

$$O = \exp [i \theta_a T_a] \quad \epsilon_{123}$$

$T_a \rightarrow$ generators

$$T_a^\dagger = T_a$$

$$\left. \frac{\partial O}{\partial \theta_a} \right|_{\theta_a=0} = i T_a$$

$$[T_a, T_b] = i \epsilon_{abc} T_c$$

$[\theta \cdot T]$

os

$$\epsilon_{123} = +1$$

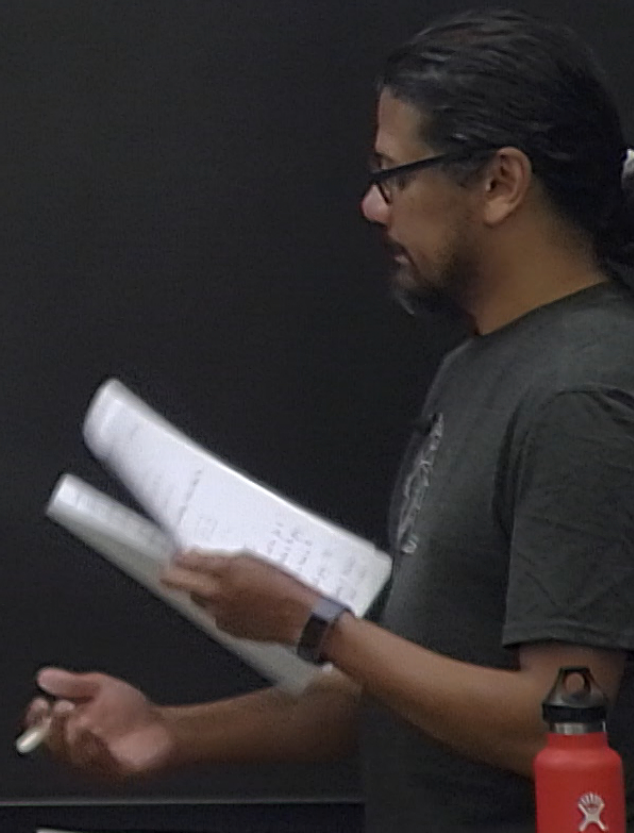
$$\epsilon_{132} = -1$$

$\underline{\theta \cdot I}$

$$\epsilon_{123} = +1$$

$$\epsilon_{132} = -1$$

$$O = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \equiv P$$



$$O = \exp \left[\frac{i \theta \cdot T}{M} \right]$$

$T_a \rightarrow$ generators

$$T_a^\dagger = T_a$$

$$\frac{\partial O}{\partial \theta_a} \bigg|_{\theta_a=0} = i T_a$$

$$[T_a, T_b] = i \epsilon_{abc} T_c$$

$$\epsilon_{123} = +1$$

$$\epsilon_{132} = -1$$

$$O = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \equiv P$$

$$O \simeq 1 + i \theta \cdot T$$

$$O = e^M$$

$$O^T =$$

$$O = e^M \quad O^{-1} = e^{-M}$$

$$O^T = e^{M^T}$$

$$O^T = O^{-1}$$

$$e^{M^T} = e^{-M}$$

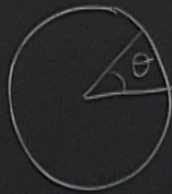
$$M^T = -M$$

$$T_a^T = -T_a$$

Rotations in 2D

Rotations in a plane form $SO(2)$

$$O = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$



$SO(2) \rightarrow$ Abelian group.

$$\dim SO(2) = 1$$

$$SO(2) \cong S^1$$

$SO(2)$ is isomorphic to S^1

Conjugacy Classes

Defn: Conjugate elements:

Two elements $a \neq b$ of a gp G are conjugate to each other if there exist some element g (not necessarily unique) st

$$a = gbg^{-1}$$

And g is called the

Conjugacy Classes

Defn: Conjugate elements:

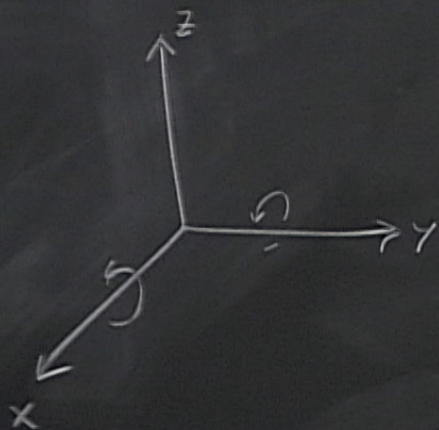
Two elements a & b of a gp G are
conjugate to each other if there exist some
element g (not necessarily unique) st
$$a = gbg^{-1} \quad (*)$$

And g ...
We write
an equation

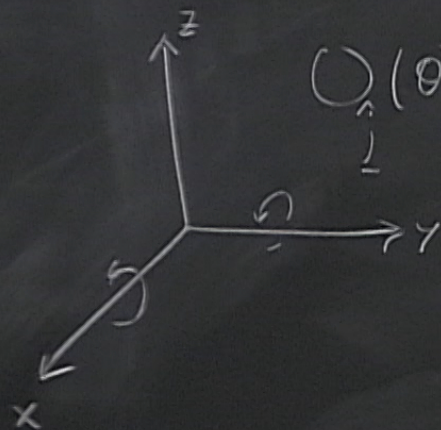
And g is called the conjugating element.

We write $a \sim b$ because $(*)$ is
an equivalence relationship.

$$O_{\hat{z}}(\theta) r = r'$$



$$O_{\hat{z}}(\theta) \mathbf{r} = \mathbf{r}'$$



$$O_{\hat{z}}(\theta) \tilde{\mathbf{r}} = \tilde{\mathbf{r}}'$$

$$O_{\hat{k}}\left(\frac{\pi}{2}\right) \mathbf{r}$$

$$O_{\underline{j}}(\theta) O_{\underline{k}}(\frac{\pi}{2}) r = O_{\underline{k}}(\frac{\pi}{2}) r'$$

$$O_{\underline{k}}(\frac{\pi}{2}) O_{\underline{j}}(\theta) O_{\underline{k}}(-\frac{\pi}{2}) O_{\underline{k}}(\frac{\pi}{2})$$

$\underbrace{\hspace{10em}}$
 $\underline{1}$

$$O_{\hat{j}}(0) O_{\hat{k}}(\frac{\pi}{2}) r = O_{\hat{k}}(\frac{\pi}{2}) r'$$

$$O_{\hat{k}}(-\frac{\pi}{2}) O_{\hat{j}}(0) O_{\hat{k}}(\frac{\pi}{2}) r = r'$$

$$O_{\hat{j}}(\theta) O_{\hat{k}}\left(\frac{\pi}{2}\right) r = O_{\hat{k}}\left(\frac{\pi}{2}\right) r'$$

$$\underbrace{O_{\hat{k}}\left(-\frac{\pi}{2}\right) O_{\hat{j}}(\theta) O_{\hat{k}}\left(\frac{\pi}{2}\right)}_{O_{\hat{i}}(\theta)} r = r'$$

$$O_{\hat{i}}(\theta) =$$

$$O_{\hat{i}}(\theta) = g O_{\hat{j}}(\theta) g^{-1}$$

Def 4. Simple Group

The identity element e & G are automatically normal subgroups. If e & G are the only normal subgroups of G then we say G is a simple group.

Defⁿ: Direct Product of groups

A group G is said to be the direct product of two subgroups

A & B if

(i) All elements of A commute with all elements of B

(ii) Every element of

(ii) Every element of Q can be written in a unique way as $g = ab$ with $a \in A, b \in B$.

(ii) Every element of G can be written in a unique way as $g = ab$ with $a \in A, b \in B$.

Comments:

$$\begin{aligned} \text{Consider } g_1 g_2 &= (a_1 b_1)(a_2 b_2) \\ &= (a_1 a_2)(b_1 b_2) \end{aligned}$$

Multiplying elements of G is multiplying elements of A & B separately.

2 Both A & B are normal subgps.

Therefore $G/A \cong B$, $G/B \cong A$

$$SU(2)_L \times SU(2)_R$$

The Lorentz Group

Lorentz transformations of Special Relativity
also form a group. Unit $c=1$

$$\Delta S^2 = -(t_1 - t_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

$$\Delta S^2 \begin{matrix} > 0 \\ = 0 \\ < 0 \end{matrix}$$

Coordinate on Space-time

$$X_i^M = (t_i^M, x_i^M, y_i^M, z_i^M)$$

Coordinate on Space-time

$$X_i^M = (t_i, x_i, y_i, z_i)$$

$$X_i^0 = t, \quad X_i^3 = z_i$$

$$X_2^M = (0, 0, 0, 0)$$

$$X_i^M \equiv X^M$$

$$\Delta S^2 = X^\mu X^\nu \eta_{\mu\nu}$$

$$\Delta S^2 = X^\mu X^\nu \eta_{\mu\nu} = \sum_{\mu, \nu}^3 X^\mu X^\nu \eta_{\mu\nu}$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}$$

$\eta_{\mu\nu} \rightarrow$ Metric tensor

In matrix language

$$\Delta S^2 = X^T \eta X$$

$$X^\mu \rightarrow X'^\mu = \Lambda^\mu{}_\nu X^\nu$$

$$\Lambda^T \eta \Lambda = \eta$$

$\Lambda \rightarrow 4 \times 4$ matrix

$\Lambda \rightarrow$ Lorentz if

$$X'^\mu \eta_{\mu\nu} X'^\nu = X^\mu \eta_{\mu\nu} X^\nu$$

$$(\Lambda^\mu{}_\sigma X^\sigma) \eta_{\mu\nu} (\Lambda^\nu{}_\rho X^\rho) = X^\sigma \eta_{\sigma\rho} X^\rho$$

$$\boxed{\Lambda^\mu{}_\sigma \eta_{\mu\nu} \Lambda^\nu{}_\rho = \eta_{\sigma\rho}}$$

$$\Delta S^2 \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases}$$

$$\Delta S^2 = X^\mu X^\nu \eta_{\mu\nu} = \sum_{\mu, \nu} X^\mu X^\nu \eta_{\mu\nu}$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}$$

$\eta_{\mu\nu} \rightarrow$ Metric tensor

In matrix language

$$\Delta S^2 = X^T \eta X$$

$$X^\mu \rightarrow X'^\mu = \Lambda^\mu{}_\nu X^\nu$$

$\Lambda \rightarrow 4 \times 4$ matrix

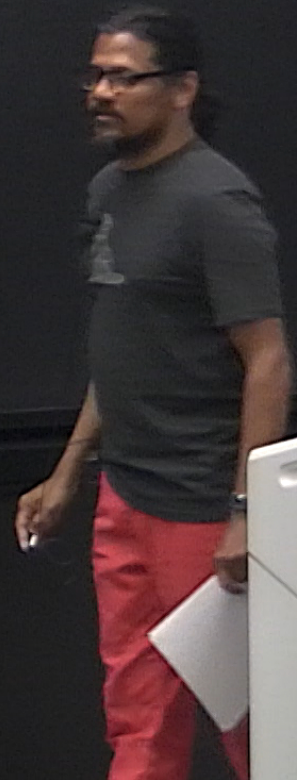
$\Lambda \rightarrow$ Lorentz if

$$X'^\mu \eta_{\mu\nu} X'^\nu = X^\mu \eta_{\mu\nu} X^\nu$$

$$(\Lambda^\mu{}_\sigma X^\sigma) \eta_{\mu\nu} (\Lambda^\nu{}_\rho X^\rho) = X^\sigma \eta_{\sigma\rho} X^\rho$$

$$\Lambda^\mu{}_\sigma \eta_{\mu\nu} \Lambda^\nu{}_\rho = \eta_{\sigma\rho}$$

$$\Lambda^T \eta \Lambda = \eta$$



$$\boxed{\Lambda^T \eta \Lambda = \eta} \longrightarrow O(1,3)$$

$$\boxed{O^T \mathbb{1} O = \mathbb{1}}$$

$$p_\mu x^\mu = x^\sigma \eta_{\sigma\rho} x^\rho$$

$$\boxed{p_\rho}$$

$$\boxed{\Lambda^T \eta \Lambda = \eta}$$

$$\boxed{O^T \mathbb{1} O = \mathbb{1}}$$

$\longrightarrow O(1,3)$ preserves $\eta_{\mu\nu}$
Lorentz gp

$\eta_{\mu\nu} x^\mu$

1. What is the dim of $O(1,3)$?

$$\Lambda^T \eta \Lambda = \eta$$

$\Lambda \rightarrow 16$ real components

$4 + 6 = 10$ conditions

6 real independent parameters

3 $\rightarrow$ rotations

3 $\rightarrow$ boosts

$$\dim(O(1,3)) = 6$$

$SO(1,3)$

components
conditions

$$SO(1,3)$$

$$\Lambda^0_0 > 0$$

$$P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

The Translation Group

$$X^M \rightarrow X^M + a^M$$

This is an Abelian Group : $\mathbb{R}^4$

The Poincaré Group

The Poincaré Group

The group of symmetries of d dim
Minkowski Space time.

Includes Lorentz gp $\hat{=}$ the translation group.

$$a_i \in \mathbb{R}^4$$

$$\Lambda_i \in SO(1,3)$$

The composition law:

$$(a_1, \Lambda_1)(a_2, \Lambda_2) = (a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2)$$

$$a_i \in \mathbb{R}^4$$

$$\Lambda_i \in SO(1,3)$$

The composition law:

$$(a_1, \Lambda_1)(a_2, \Lambda_2) = (a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2)$$

Semidirect product of $SO(1,3) \ltimes \mathbb{R}^4$.

$3 \rightarrow$ boosts,

unitary

Representation Theory

Vectors $\longleftrightarrow$ state of a system

Linear Op acting on vector $\longleftrightarrow$ Symmetry trans. acting states

Linear op $\rightarrow$ Represented by linear differential operators

$\rightarrow$ " " Finite dim (sometimes infinite dim) matrices

We are interested in groups as represented by
finite dimensional matrices

$$GL(n, \mathbb{C})$$