Title: PSI 2019/2020 - Lie Groups and Lie Algebras - Lecture 2

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Collection: PSI 2019/2020 - Lie Groups and Lie Algebras (Ali)

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Refs:

1 Groups, Representations and Physics – H.F. Jones

2 Group Theory in a Nutshell for Physicists – A. Zee

Group homomorphisms

Group Homomorphisms

A group homo f between two gps G_1 & G_2 is a map

$$f: G_1 \longrightarrow G_2$$

s.t. $\forall$ elements a & b in G_1

$$f(ab) = f(a)\,f(b)$$

GPS

Example:

$\mathbb{R}^+$ ~ under arithmetic multiplication this is a group

$\mathbb{R}^+ \cong \mathbb{R}$

$a, b \in \mathbb{R}^+$

$\log(ab) = \log(a) + \log(b)$

$\ln = \log$

Symmetric Groups: S_n

n objects can n! configurations

Defn S_n is the group of all permutations of n objects.

$O(S_n) = n!$

1- $S_2 = \{(), (12)\}$

Abelian $S_2 \cong \mathbb{Z}_2$

$\mathbb{Z}_2 = \{1, -1\}^{\times}$

2. S_3 has $3! = 6$ elements.

(): Do nothing. Identity element

(12): Exchange objects in position 1 & 2

(23): " " " " 2 & 3

(31): " " " " 3 & 1

(123): $\begin{matrix} abc \\ 123 \end{matrix} \longrightarrow \begin{matrix} cab \\ 123 \end{matrix}$

(321) $\begin{matrix} abc \\ 123 \end{matrix} \longrightarrow \begin{matrix} bca \\ 123 \end{matrix}$

S_3 has $3! = 6$ elements.

nothing. Identity element

change objects in position 1 & 2

" " " " 2 & 3

" " " " 3 & 1

abc → cab
123 123

abc → bca
123 123

$(123) = (12)(23)$

$(321) = (32)(21)$
$= (23)(12)$

$(12)(23) \neq (23)(12)$

S_3 is non-Abelian.

Permutation Group, P_n

Group of permutations (not necessarily all possible ones) of n objects.

Ex S_4 $4! = 24$

$\{(\,), (12), (34), (1234)\}$

$$\begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}$$

A Theorem about Permutation Groups

$\cdot$	a	b	c	$\cdots$
a	a^2	ab	ac	$\cdots$
b	ba	b^2	bc	$\cdots$
c	ca	cb	c^2	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

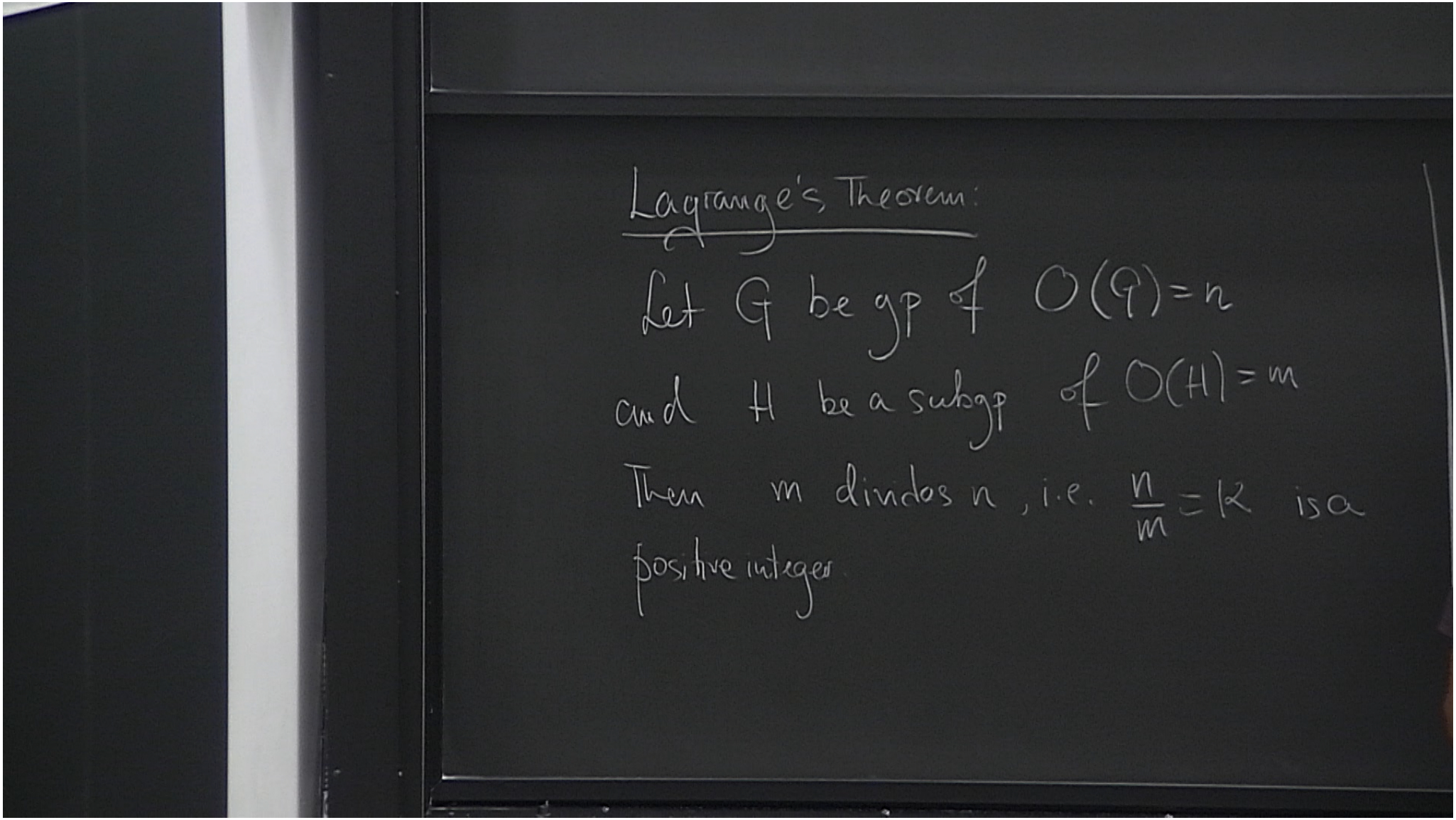
Each element occurs once & only once in each row (column).

$bc = dc$

$b = d$

Cayley's Theorem

Any finite group is isomorphic to a permutation group (a subgroup of S_n).


Lagrange's Theorem:
Let G be gp of O(G) = n
and H be a subgp of O(H) = m
Then m divides n, i.e. n/m = k is a
positive integer

Def'n: Cosets

Let G be gp of which $H = \{h_1, h_2, \ldots h_r\}$ is a subgp. Let $g \in G$ which is not in H

Then the (left) coset of g, $_{\text{of } g}$ is defined to be

$$\{gh_1, gh_2, \ldots, gh_r\} \equiv gH$$

If $O(G)/O(H) = k$

$$G = H + g_1 H + g_2 H + \cdots + g_{k-1} H$$

This is the coset decomposition of G wrt H.

Coset Space

The coset space G/H is defined to be the set $\{H, g_1 H, g_2 H, \ldots, g_{k-1} H\}$.

A coset space is <u>not</u> necessarily a group.

However, if H satisfies $H = gHg^{-1}$ $\forall g$ then G/H can be given a group structure.

H when it satisfies $gHg^{-1} = H$

is called a normal subgroup.

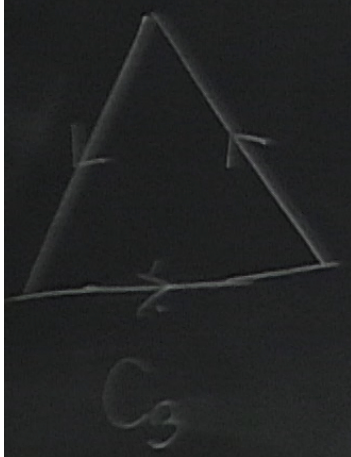
Defn. Centre:

The centre Z of a gp G is the subset of G such that

$$\forall z \in Z \ \& \ \forall g \in G$$

$$zg = gz$$

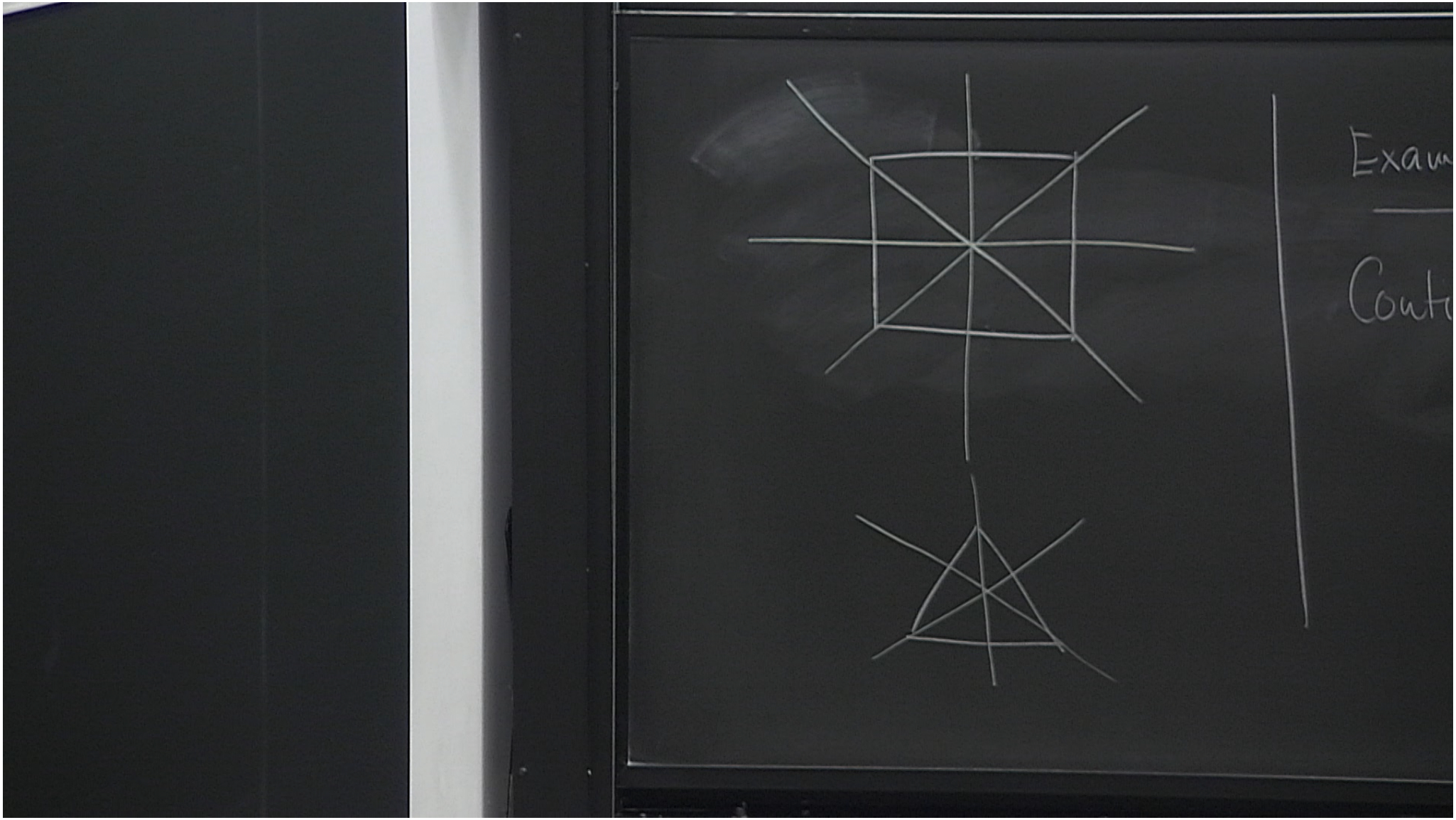
$a^n \equiv e$



C_3

Dihedral Group D_n

Dihedral gp D_n is the symmetry gp of an n-sided regular polygon without directed sides.


Exam
Cont

Examples of continuous groups

Continuous gps are gps whose elements are labelled by a set of continuous parameters. For Lie groups the number continuous parameters is finite

Generalizations to infinite dim lie algebras — Affine lie algebras

Rotations in 3D

Position vector in 3D

$$\underline{r} = x\hat{\underline{i}} + y\hat{\underline{j}} + z\hat{\underline{k}}$$

$\underline{r}$, $\vec{r}$ $\qquad r = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Consider the linear transformations on r st Euclidean norm

$$r^2 = r^T r = x^2 + y^2 + z^2$$

remains invariant.

$r \longrightarrow r' = Or$, O : 3×3 matrix

$r'^T r' = r^T r$

$(Or)^T (Or) = r^T r$

$r^T O^T O r = r^T r$

$O^T O = \mathbb{1}$

$O O^T = \mathbb{1}$

$$O^{-1} = O^T$$

$\mathbb{1} \longrightarrow e$

Three dimensional

Orthogonal group – $O(3)$.

Rotations are a subgroup of $O(3)$.

Rotations consists Os st

$$\det O = +1$$

Note that

$$\det O = \pm 1$$

$O(3)$

of O(3).

What is the number real parameters?

O have $3 \times 3 = 9$ real parameters

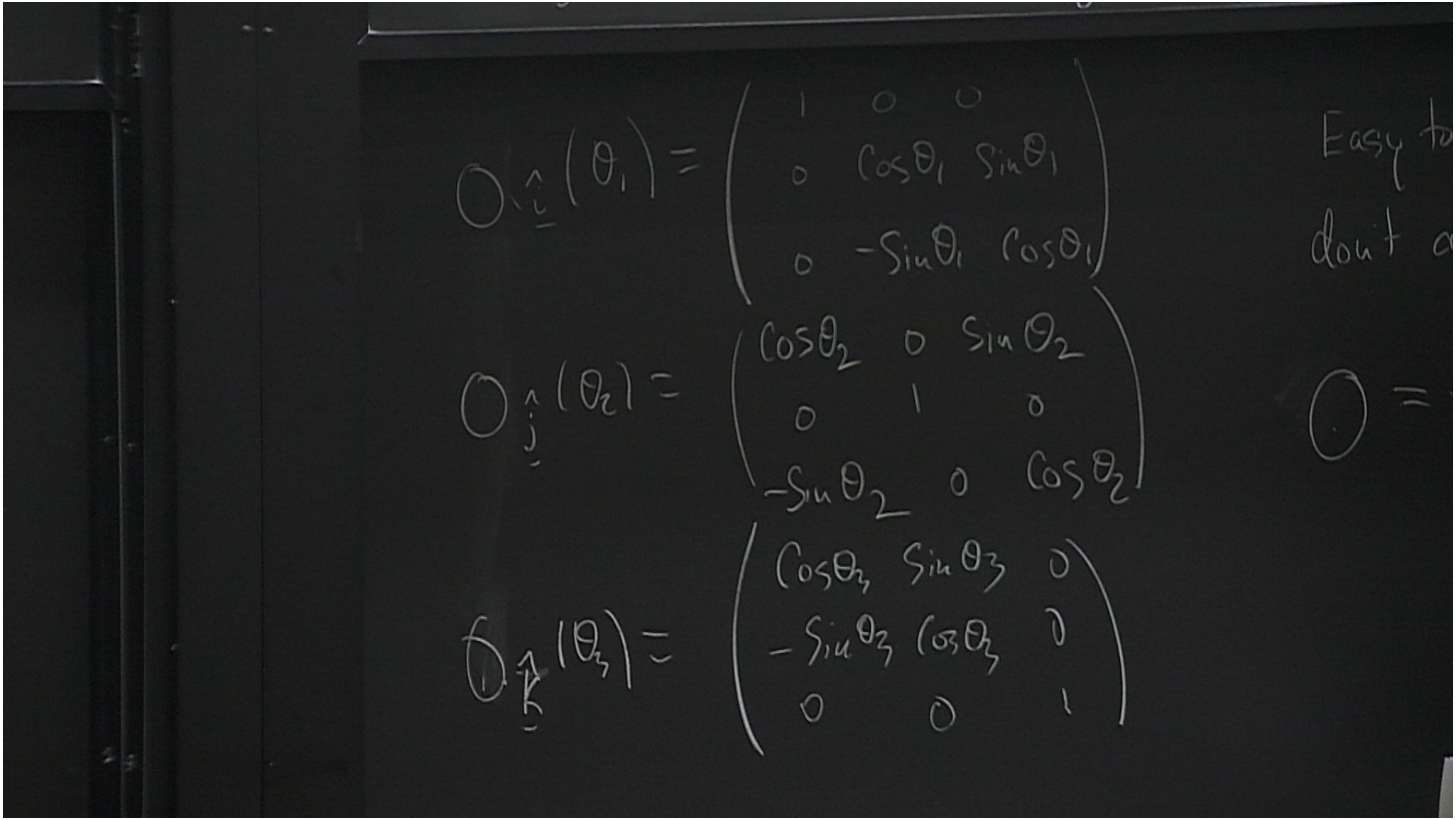
-6 Conditions.

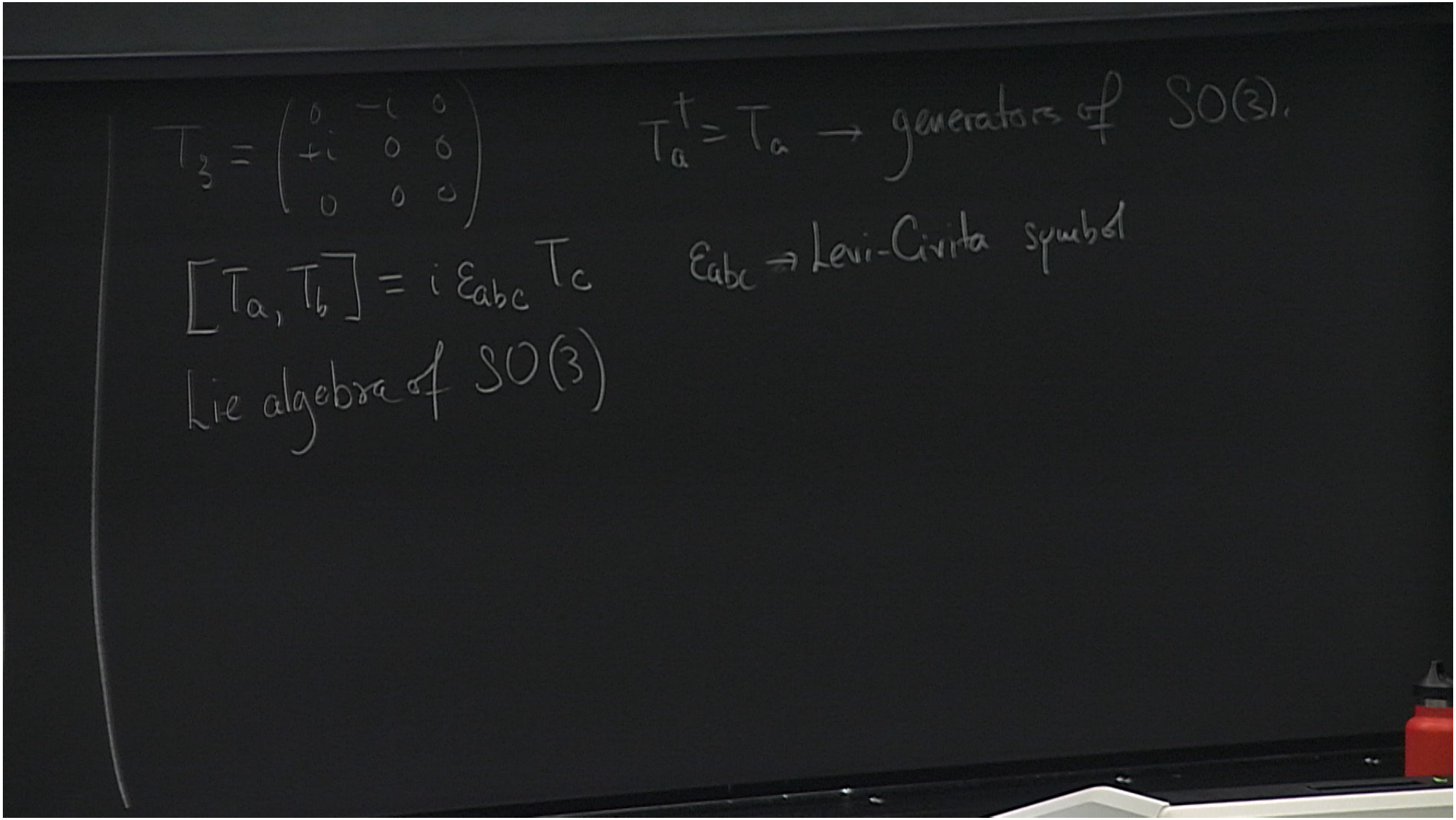
3 real parameter.

+1

O(3) has infinite number of elements.

Dimension of O(3): # of real parameters needed to specify an element which is 3.


$O_{\hat{i}}(\theta_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix}$
$O_{\hat{j}}(\theta_2) = \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}$
$O_{\hat{k}}(\theta_3) = \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
Easy to
dont c
O =


$T_3 = \begin{pmatrix} 0 & -i & 0 \\ +i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$T_a^\dagger = T_a \rightarrow$ generators of $SO(3)$.
$[T_a, T_b] = i\,\varepsilon_{abc}\,T_c$
$\varepsilon_{abc} \Rightarrow$ Levi-Civita symbol
Lie algebra of $SO(3)$

A general rotation / element of $SO(3)$

$$O(\underline{\theta}) = \exp\left[i\,\underline{\theta}\cdot\underline{T}\right]$$

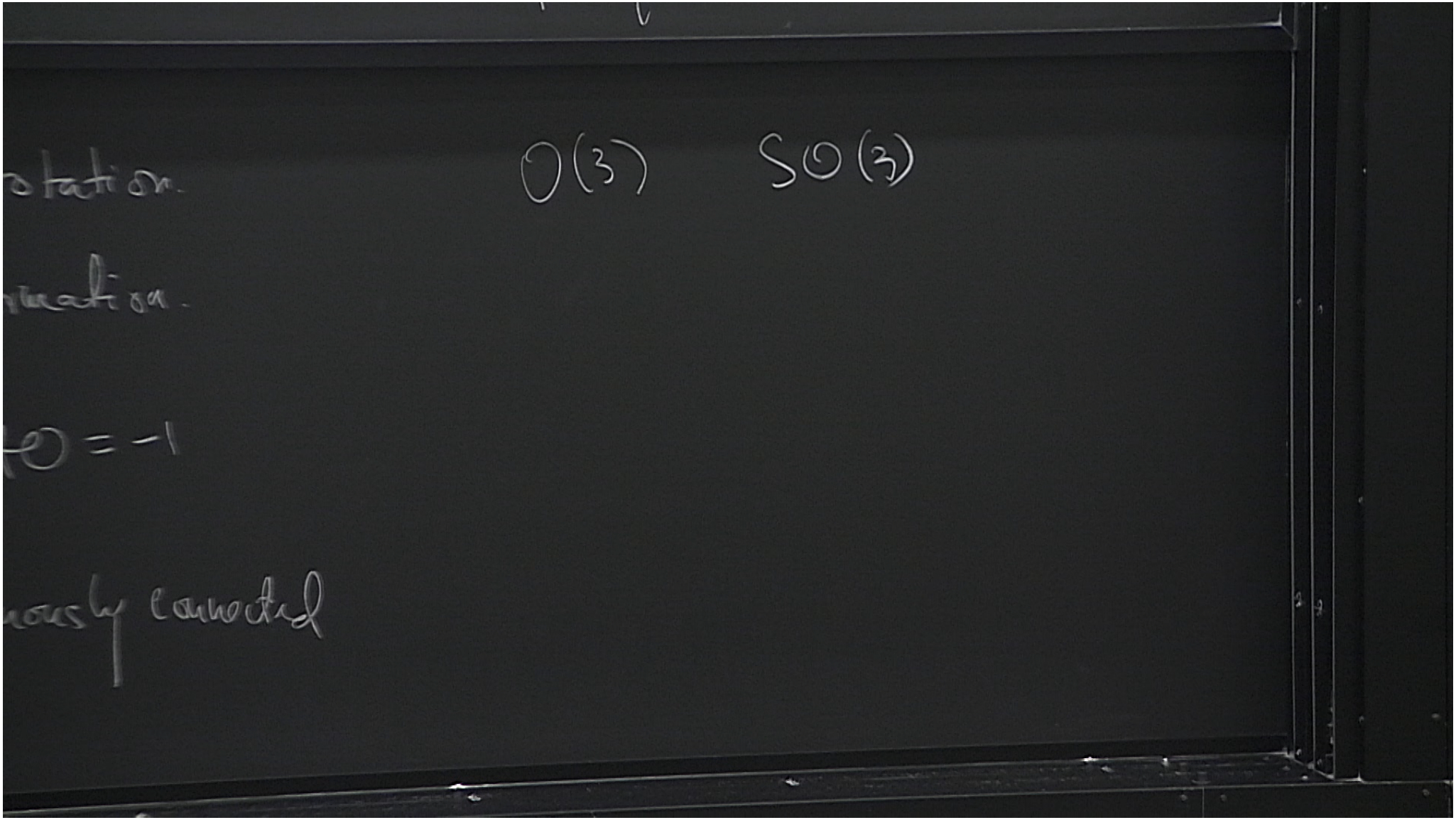
What about those elements for which $\det O = -1$

$$O \equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \equiv P \sim \text{parity}$$

P is a combination of reflection + rotation.

P is an example of a discrete transformation.

No real parameter labelling the det O = −1 elements. These elements are <u>not</u> continuously connected to the identity.


O(3)
SO(3)