Title: PSI 2019/2020 - Math for QFT - Lecture 5

Speakers: Dan Wohns

Collection: PSI 2019/2020 - Math for QFT (Wohns)

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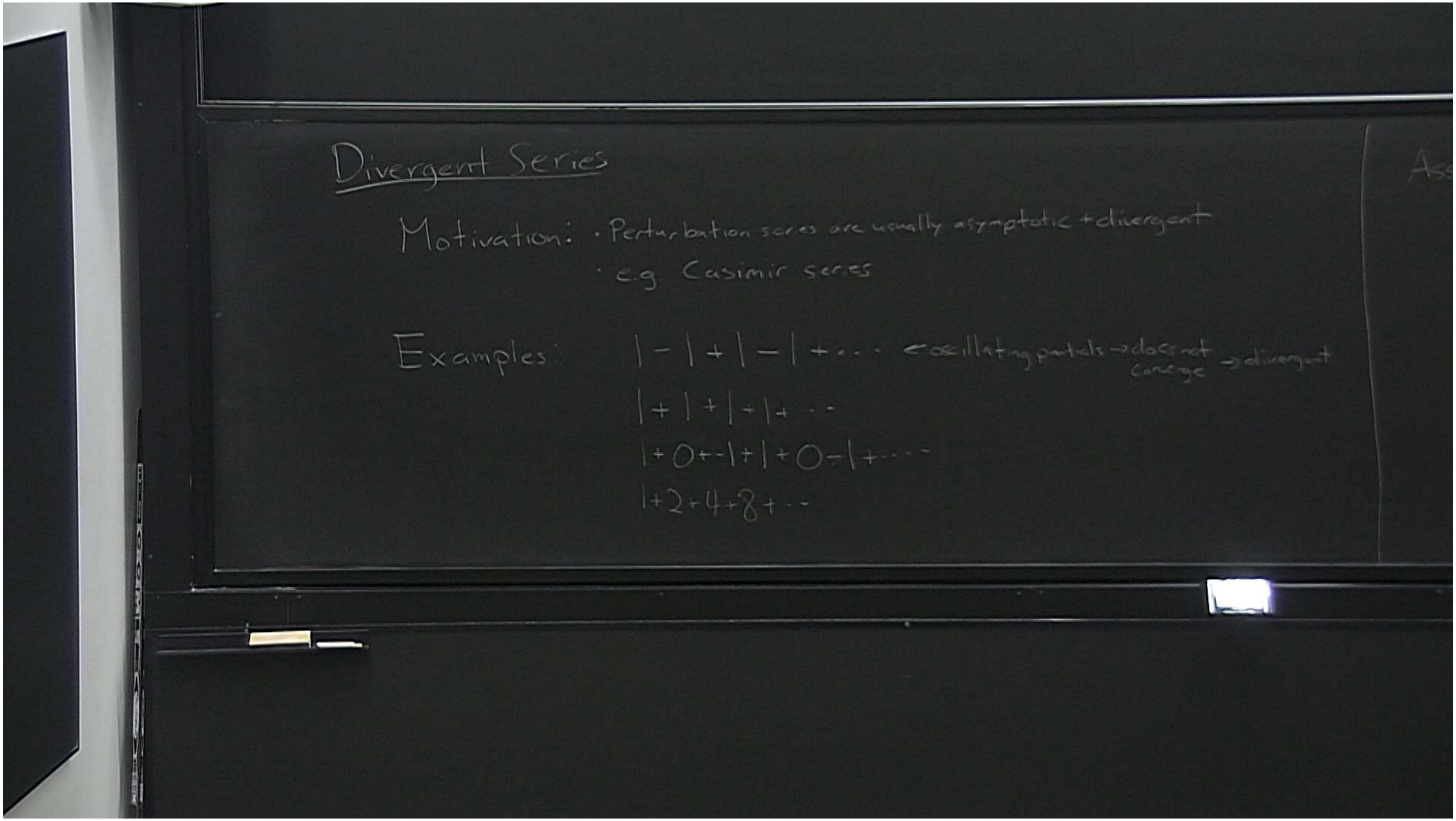


Divergent Series
Motivation: · Perturbation series are usually asymptotic + divergent
· e.g. Casimir series
Examples: 1 − 1 + 1 − 1 + . . . ← oscillating partials → does not converge → divergent
1 + 1 + 1 + 1 + . . .
1 + 0 + −1 + 1 + 0 − 1 + . . .
1 + 2 + 4 + 8 + . . .

...ptotic + divergent

...ng particles → does not converge → divergent

Associativity: $(1-1)+(1-1)+(1-1)+\cdots = 0$

$1+(-1+1)+(-1+1)+\cdots = 1$

$\boxed{\text{Addition is } \underline{\text{not}}\ \infty \text{ associative or commutative}}$

Commutativity: $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges

$1 + \frac{1}{3} + \frac{1}{5} + \cdots$ and $-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} + \cdots$ diverges

$\pi = \underbrace{1 + \frac{1}{3} + \frac{1}{5} + \cdots}_{>\pi} - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} \cdots \underbrace{+++}_{>\pi} \underbrace{---}_{<\pi}$

ough $\left(a_n \sim n^k\right)$

$$f(x) = 1 - x f(x) \rightarrow f(x) = \frac{1}{1+x}$$

$E(\varphi) = \frac{1}{2}$ $\leftarrow$ not unreasonable but may seem arbitrary?

Borel summation

$$n! = \Gamma(n+1) = \int_0^\infty dt\, e^{-t} t^n$$

$$1 = \int_0^\infty dt\, e^{-t} \frac{t^n}{n!}$$

$$B(\varphi) \equiv \int_0^\infty dt\, e^{-t} \sum_{n=0}^{\infty} \frac{a_n t^n}{n!}$$

$(B(\varphi)(1))$

Exercise: Borel sum of Grandi

Borel summation

$$n! = \Gamma(n+1) = \int_0^\infty dt\, e^{-t} t^n$$

$$1 = \int_0^\infty dt\, e^{-t} \frac{t^n}{n!}$$

$$B(\varphi) \equiv \int_0^\infty dt\, e^{-t} \sum_{n=0}^{\infty} \frac{a_n t^n}{n!}$$

$(B(\varphi)(1))$

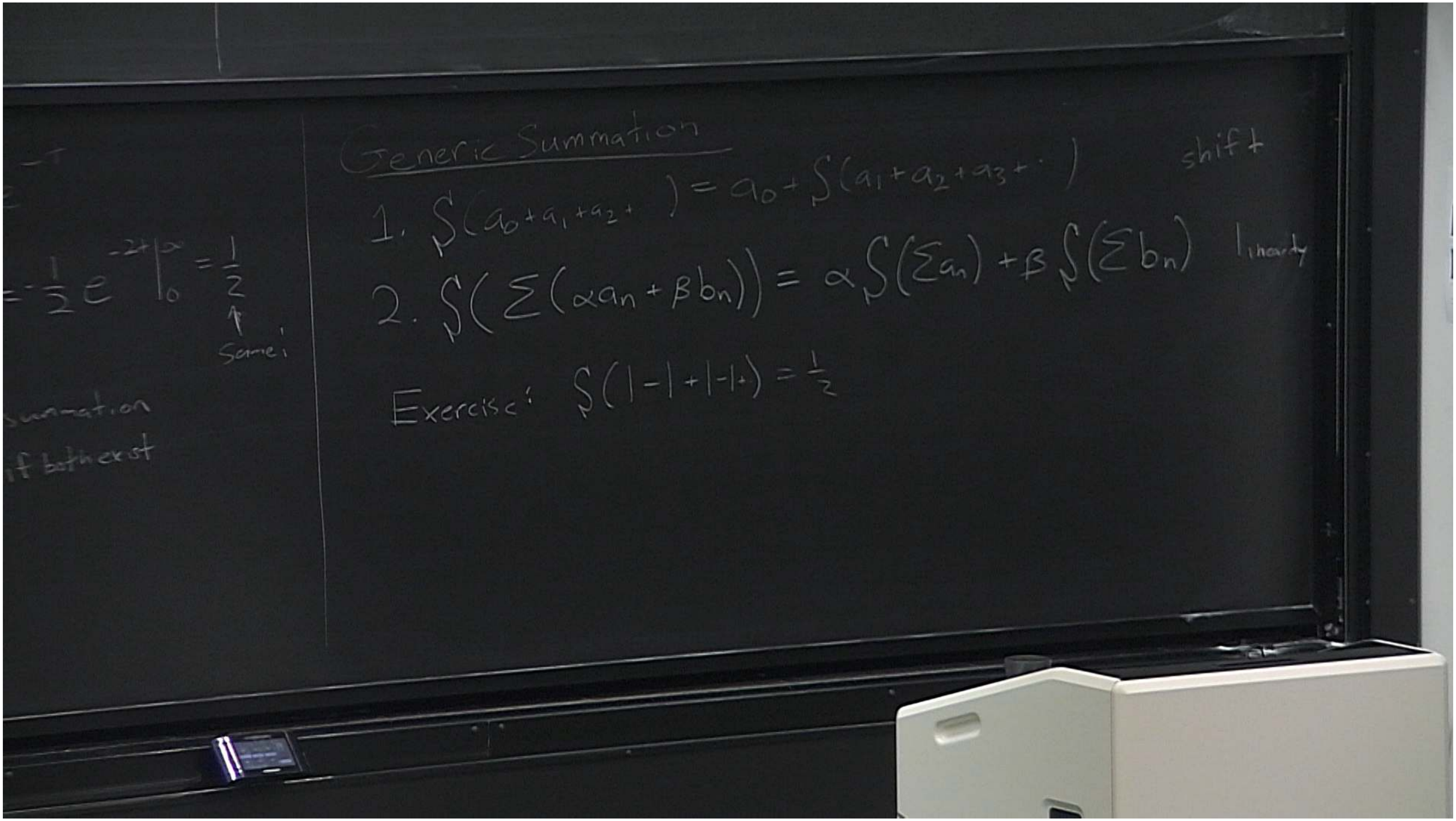
Exercise: Borel sum of Grandi

$$\sum_{n=0}^{\infty} \frac{(-t)^n}{n!} = e^{-t}$$

$$\int_0^\infty e^{-2t} dt = -\frac{1}{2} e^{-2t}\Big|_0^\infty = \frac{1}{2}$$

Same!

Euler + Borel summation always agree if both exist


Generic Summation
1. $S(a_0+a_1+a_2+\ )=a_0+S(a_1+a_2+a_3+\cdots)$
shift
2. $S(\sum(\alpha a_n+\beta b_n))=\alpha S(\sum a_n)+\beta S(\sum b_n)$
linearity
Exercise: $S(1-1+1-1+)=\frac{1}{2}$
$=-\frac{1}{2}e^{-2t}\Big|_0^\infty=\frac{1}{2}$
Same!
summation
if both exist

$= \frac{1}{2}$

$\uparrow$

Same!

Generic Summation

1. $S(a_0 + a_1 + a_2 + \cdots) = a_0 + S(a_1 + a_2 + a_3 + \cdots)$ shift

2. $S\left(\sum(\alpha a_n + \beta b_n)\right) = \alpha S\left(\sum a_n\right) + \beta S\left(\sum b_n\right)$ linearity

Exercise: $S(1 - 1 + 1 - 1 + \cdots) = \frac{1}{2}$

$$
\begin{aligned}
S &= 1 + S(-1 + 1 - 1) \quad \#1 \\
&= 1 - S(1 - 1 + 1) \quad \#2 \\
&= 1 - S \rightarrow \boxed{S = \tfrac{1}{2}}
\end{aligned}
$$

Perturbation Theory

$$f(g) \underset{g\to 0}{\sim} \sum_{n=0}^{\infty} a_n g^n \equiv \varphi(g)$$

pert theory

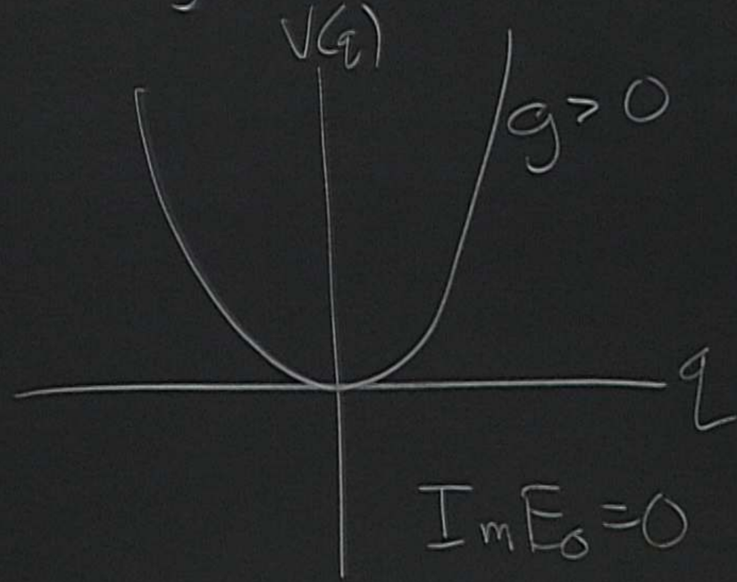
exact quantity of interest

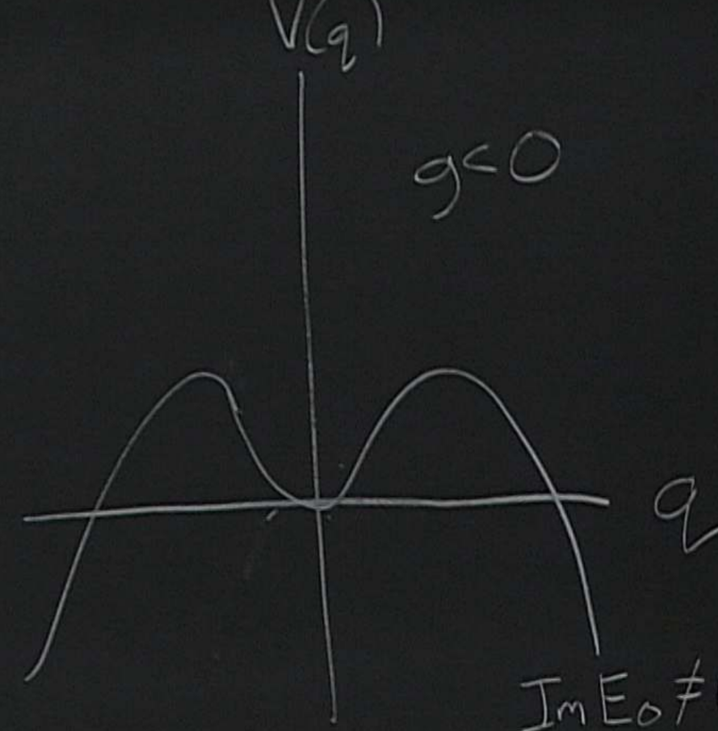
? - sometimes through summation of divergent series

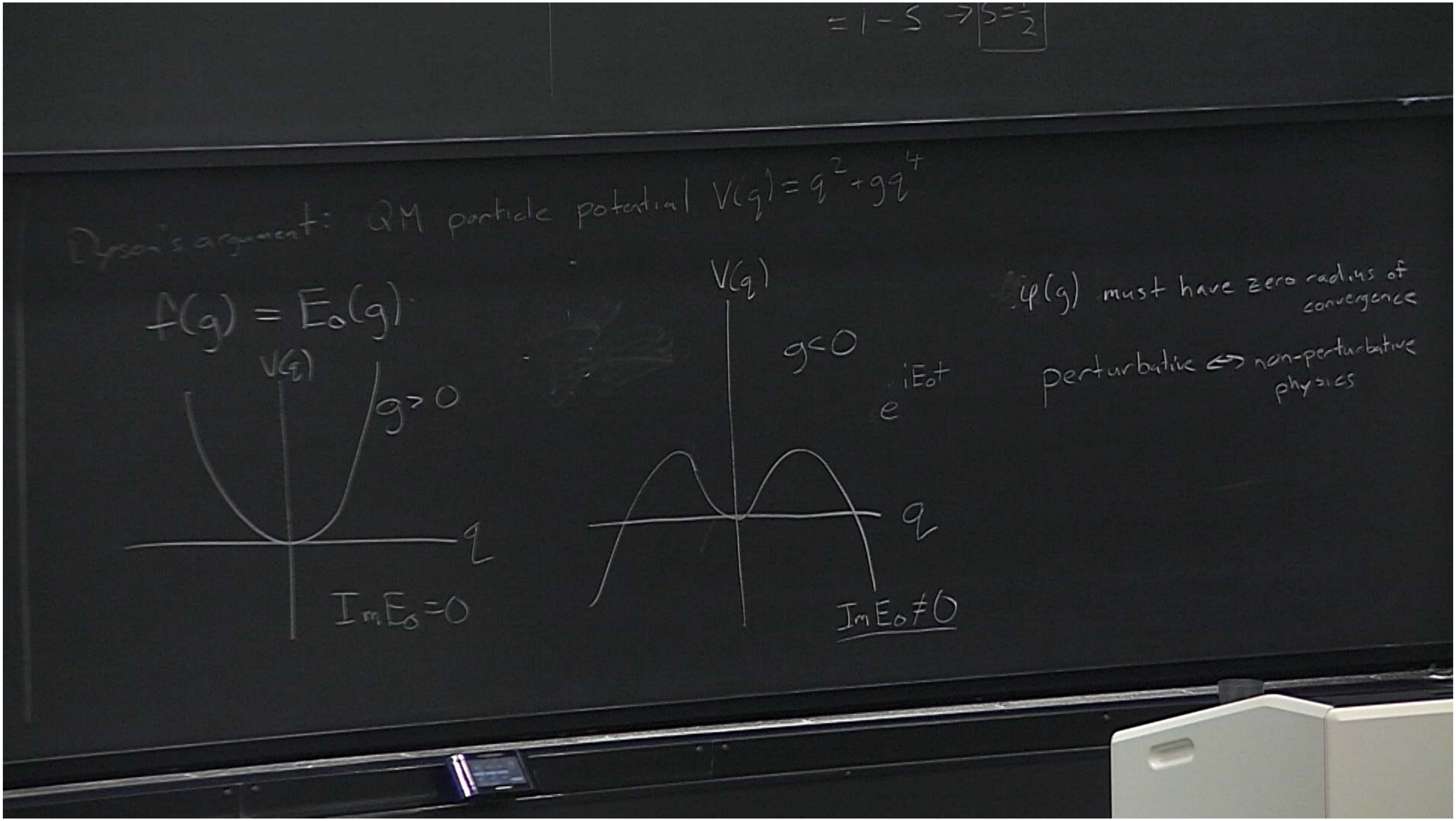
Dyson's

Dyson's argument: QM particle potential $V(q) = q^2 + g q^4$

$f(g) = E_0(g)$






$= 1 - S \to \boxed{S = \frac{1}{2}}$
Dyson's argument: QM particle potential $V(q) = q^2 + g q^4$
$f(g) = E_0(g)$
$V(q)$
$g > 0$
q
$\mathrm{Im}\, E_0 = 0$
$V(q)$
$g < 0$
$e^{iE_0 t}$
q
$\mathrm{Im}\, E_0 \neq 0$
$\varphi(g)$ must have zero radius of convergence
perturbative $\Leftrightarrow$ non-perturbative physics

Optimal truncation

Typically in QM + QFT

$$a_n \underset{n\to\infty}{\sim} A^{-n}\, n!$$

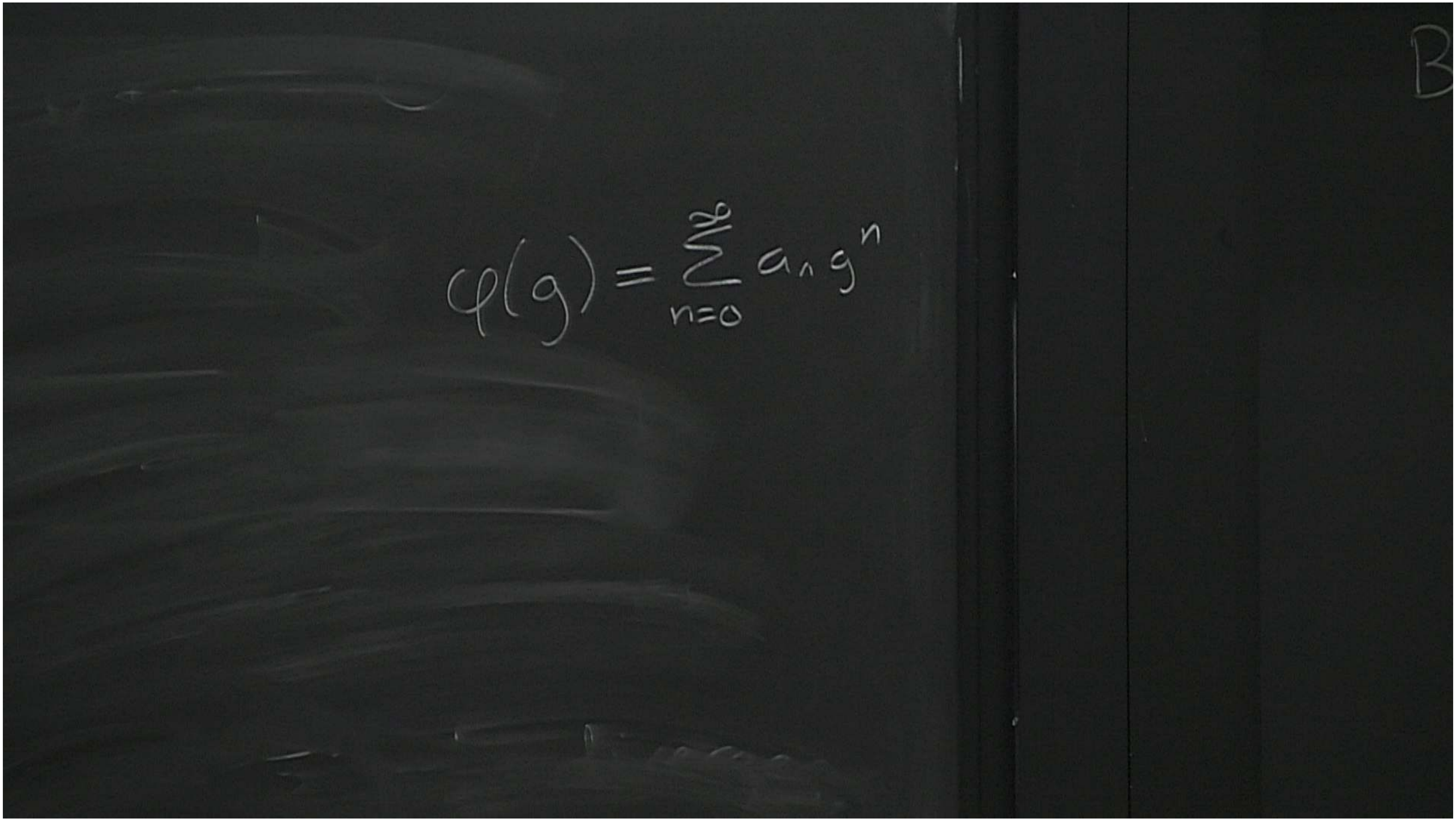
typically: optimal truncation is to keep up to the smallest term

$$N_* \text{ minimizes } |a_N g^N| = c N! \left(\frac{g}{A}\right)^N$$

$$= c \exp\left[N\left(\log N - 1 - \log\left|\frac{A}{g}\right|\right)\right]$$

$$N_* \simeq \left|\frac{A}{g}\right|$$

error $\simeq$ next term $\simeq \exp[-|A/g|]$ - exponentially accurate


$\varphi(g) = \sum_{n=0}^{\infty} a_n g^n$

Borel transform: $\hat{\varphi}(t) = \sum_n \frac{a_n}{n!} t^n$

Borel sum: $B(\varphi)(z) = \int_0^\infty e^{-t}\, \hat{\varphi}(zt)\, dt$

$$= \int_0^\infty e^{-t} \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n t^n \, dt$$

$$\underset{z \to 0}{\sim} \sum_{n=0}^{\infty} \frac{a_n}{n!} \Gamma(n+1) z^n = \varphi(z)$$

$$\begin{tikzcd} f(g) \arrow[r, bend left, "\text{perturbation}"] & \varphi(g) \arrow[l, bend left, "\text{Borel sum}"] \end{tikzcd}$$

(sometimes need non-perturbative info)