Title: PSI 2019/2020 - Math for QFT - Lecture 4

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Collection: PSI 2019/2020 - Math for QFT (Wohns)

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Saddle-Point Method

$$I(z) = \int_C g(w)\, e^{z f(w)}\, dw \qquad \text{for } z \to \infty \qquad z \in \mathbb{R}$$

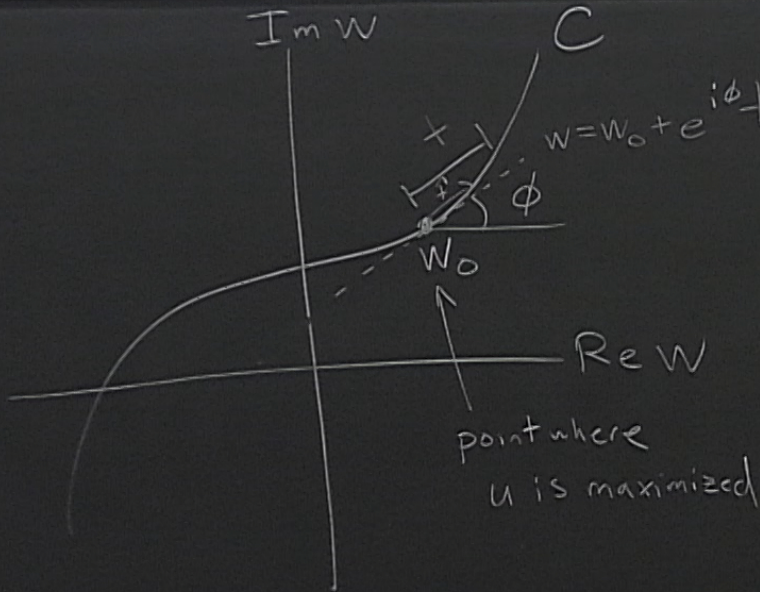
$$z = \frac{1}{\lambda}$$

expect dominant contributions from region near

where $u = \mathrm{Re}(f(w))$ is maximized

if $v = \mathrm{Im}(f(w))$ is constant $\rightarrow$ first deform contour into constant phase contour

otherwise $\rightarrow$ destructive interference



$w = w_0 + e^{i\phi} t$ along linearized C

C-R equation

at w_0: $\vec{\nabla} u = 0 \rightarrow \vec{\nabla} v = 0$

$$\boxed{f'(w_0) = 0}$$

w_0 is a saddle-point

along C: $\vec{\nabla} v \cdot \hat{t} = 0$

$\vec{\nabla} u \parallel \hat{t}$

steepest descent

As before $f(w) = f(w_0) + \frac{1}{2}(w-w_0)^2 f''(w_0) + \cdots$

$$= f(w_0) + \underbrace{\frac{1}{2}(w-w_0)^2 e^{i\delta_0}|f''(w_0)|}_{\text{want } = -\frac{1}{2}t^2|f''(w_0)|} + \cdots$$

$$\rightarrow e^{2i\phi} e^{i\delta_0} = -1$$

$$\boxed{2\phi + \delta_0 = \pm\pi}$$

global consider

ultimately unimportant

$$I(z) \approx e^{z f(w_0)} \int_{-t_1}^{t_1} g(w_0 + e^{i\phi} t)\, e^{-\frac{z}{2} t^2 |f''(w_0)|} \left.\frac{dw}{dt}\right|_0 dt$$

linear for $-t_1 < t < t_1$

change of variables

$$\approx e^{z f(w_0)} \int_{-\infty}^{\infty} \left[g(w_0) + \frac{g''(w_0)}{2!} e^{2i\phi} t^2 + \dots \right] e^{-\frac{z}{2} t^2 |f''(w_0)|}\, e^{i\phi}\, dt$$

$$\underset{z\to\infty}{\sim} \sqrt{\frac{2\pi}{z}}\, \frac{e^{z f(w_0)}}{\sqrt{-f''(w_0)}} \left[g(w_0) - \frac{1}{z} \frac{g''(w_0)}{1!\, 2 f''(w_0)} + \dots \right]$$

no absolute value

eventually higher orders in f become more import constant phase contours crucial for consistency of approximate

$\Rightarrow$ which saddles are relevant

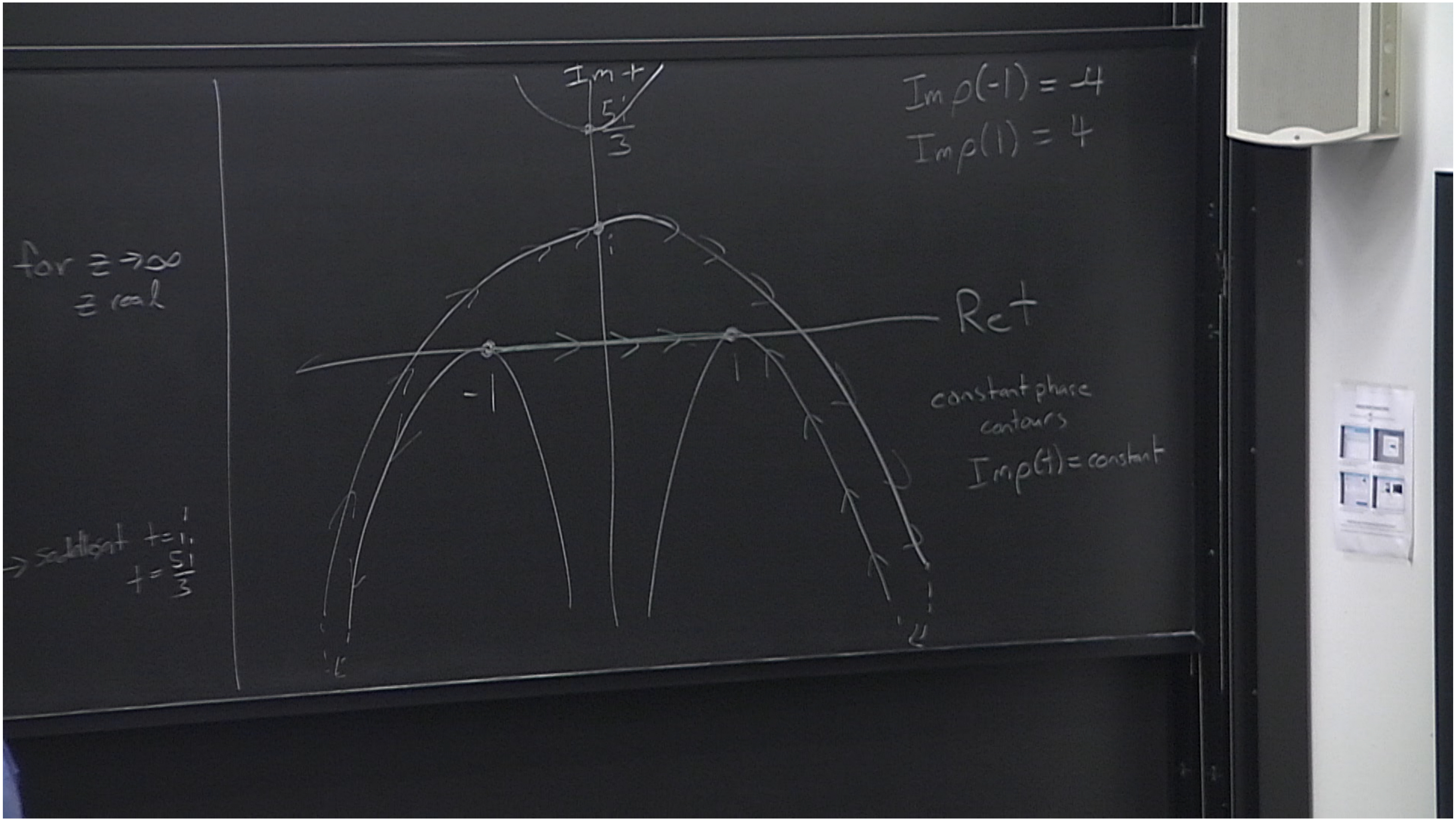
Example: $$I(z) = \int_0^1 dt\, e^{\underbrace{-4zt^2}_{\text{large when } t \lesssim \frac{1}{\sqrt{z}}}} \cos\big(\underbrace{5zt - zt^3}_{\text{when } t \sim \frac{1}{\sqrt{z}}\text{ destructive interference}}\big)$$ for $z \to \infty$, z real

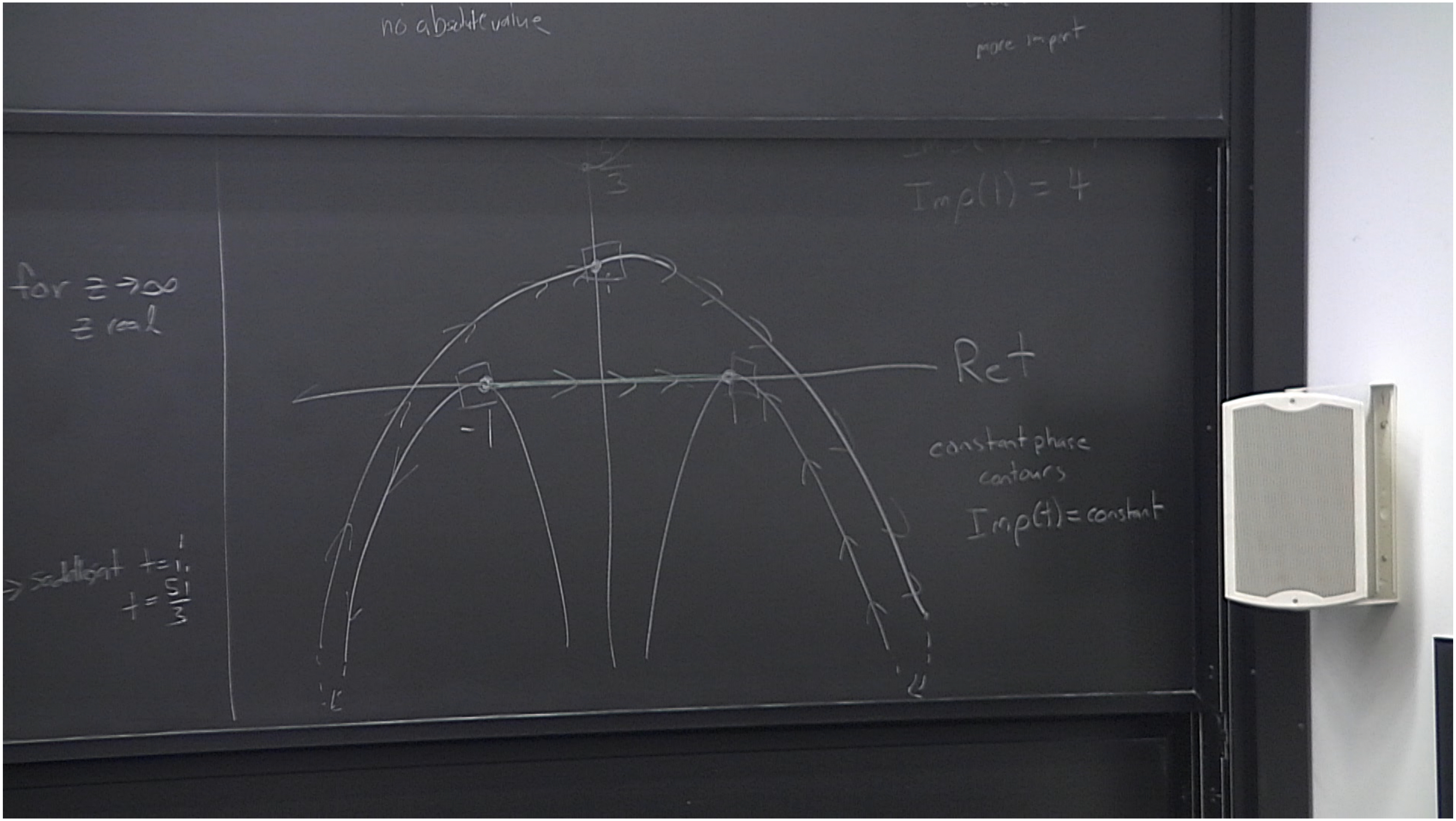
$$I(z) = \mathrm{Re} \int_0^1 dt\, e^{-4zt^2 + 5izt - izt^3}$$

$$= \frac{1}{2} \int_{-1}^1 dt \exp\Big[\qquad\Big]$$

$$= \frac{1}{2} e^{-2z} \int_{-1}^1 dt\, e^{z\rho(t)}$$

$$\rho(t) = -(t-i)^2 - i(t-i)^3 \rightarrow \text{saddlepoint } t = i, \quad t = \frac{5i}{3}$$


Im t
5i/3
Im ρ(-1) = -4
Im ρ(1) = 4
for z → ∞
z real
Re t
i
-1
1
constant phase
contours
Im ρ(t) = constant
→ saddlepoint t = i
t = 5i/3


no absolute value
more import
Im ρ(t) = 4
for z → ∞
z real
Re t
constant phase
contours
Im ρ(t) = constant
⇒ saddlepoint t = i,
t = 5i/3
3
-1

End points contribute

Theorem: $$I(x) = \int_a^b f(t)\, e^{ix\psi(x)}\, dt \underset{x\to\infty}{\sim} \frac{f(t)}{ix\psi'(t)}\, e^{ix\psi(t)}\Big|_{t=a}^{b}$$

if
- right hand side $\neq 0$
- $\frac{f}{\psi'}$ and ψ are cont. + diff.
- $\int_a^b |f(t)|\, dt < \infty$
- ψ is not constant in any finite interval

endpoint at $t=\pm 1$ contribute $\frac{-4\mp i}{68z}\, e^{(-4\mp 4i)z}$ (Theorem)

saddle-point $\frac{1}{2}\, e^{-2z}\sqrt{\frac{\pi}{z}}$ (saddle-point formula)

$$\lim_{z\to\infty} \frac{\left(\dfrac{e^{-(4\pm 4i)z}}{z}\right)}{\dfrac{e^{-2z}}{\sqrt{z}}} = 0 \quad \text{if } z \text{ is real}$$

$\boxed{\text{saddle-point dominates}}$

$$\boxed{I(z) \underset{z\to\infty}{\sim} \frac{1}{2}\, e^{-2z}\sqrt{\frac{\pi}{z}}}$$

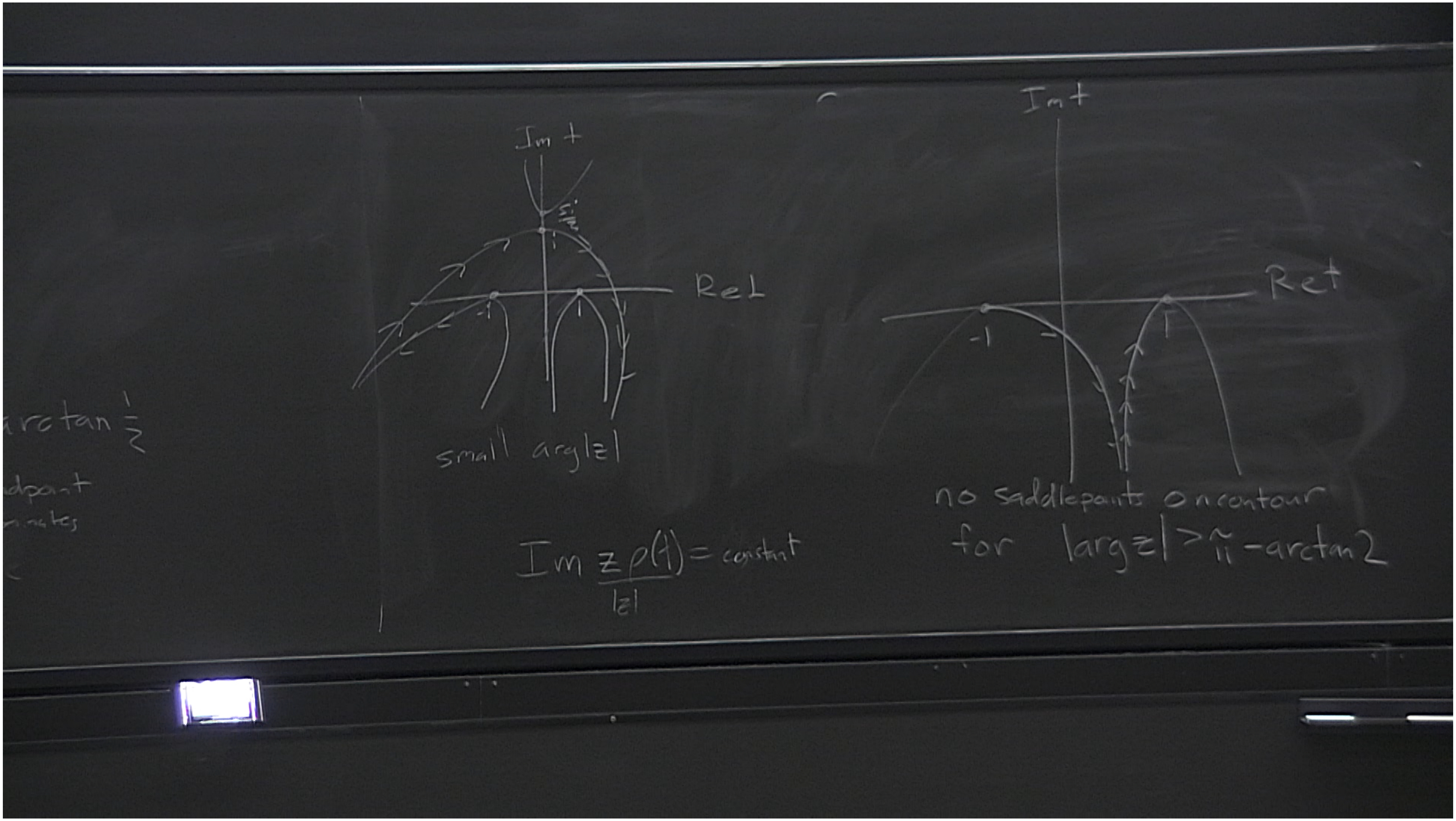
Stokes' Phenomenon

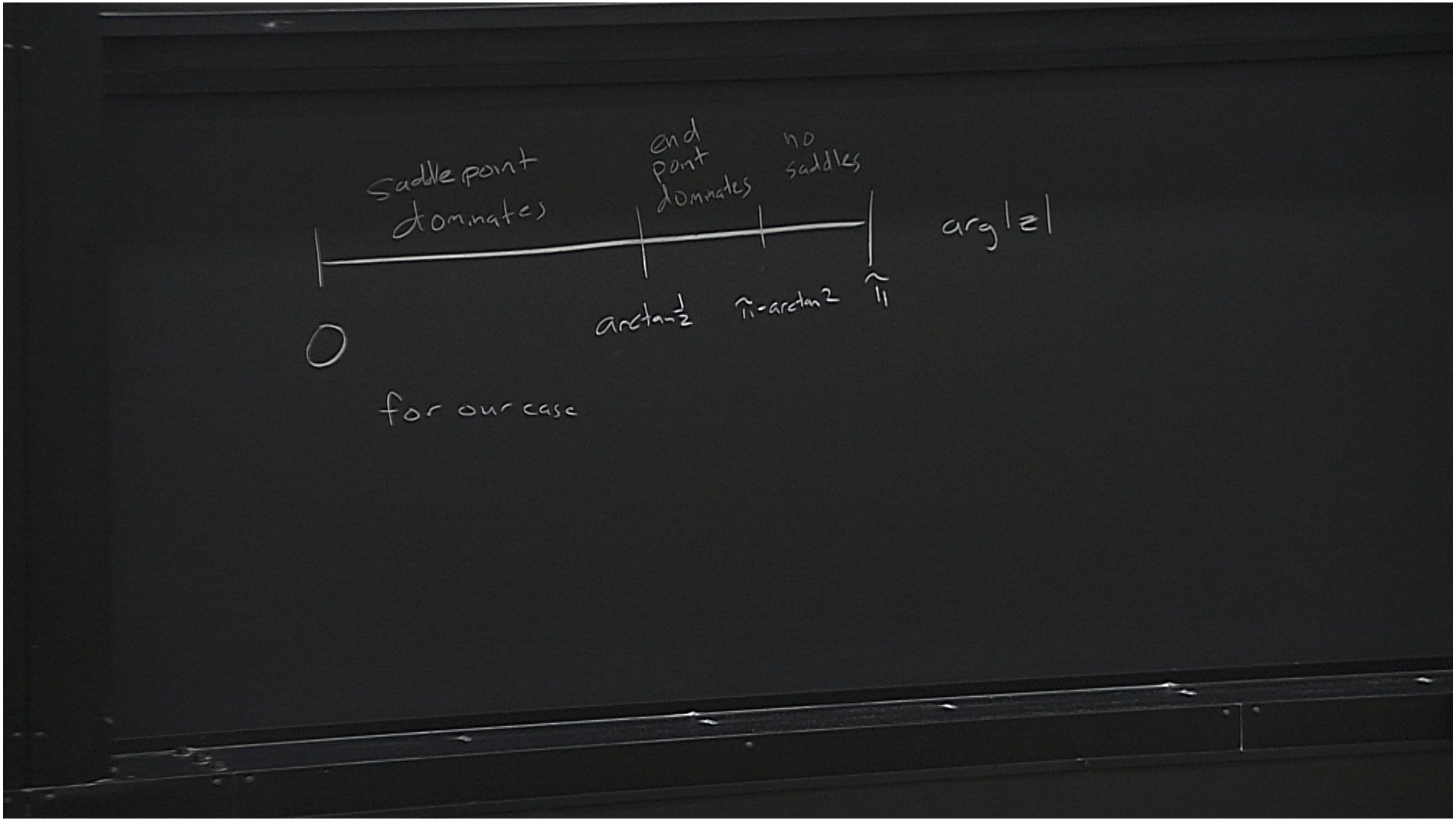
- abrupt change in asymptotics
- caused by an exchange in dominance in terms

  or terms becoming relevant/irrelevant

$$\lim_{z \to \infty} \frac{e^{(-4 \pm 4i)z}}{\sqrt{z}\, e^{-2z}} = 0 \quad \text{if} \quad |\arg z| < \arctan \tfrac{1}{2}$$

otherwise endpoint dominates


Im t
Re t
small arg|z|
no saddlepoints on contour
for |arg z| > π - arctan 2


saddle point dominates
end point dominates
no saddles
arg|z|
0
arctan 1/2
π - arctan 2
π
for our case