

Title: PSI 2019/2020 - Math for QFT - Lecture 3

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Collection: PSI 2019/2020 - Math for QFT (Wohns)

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Asymptotics

Gamma

Asymptotic Series

Stirling's Approximation

Gamma

$\Gamma(\varepsilon)$

Gamma - generalization of $n!$

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt \quad \operatorname{Re}(z) > 0$$

Gamma - generalization of $n!$

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt \quad \operatorname{Re}(z) > 0$$

$$\frac{d}{dt}(t^z e^{-t}) = z t^{z-1} e^{-t} - t^z e^{-t}$$

$$\int_0^{\infty} dt \rightarrow t^z e^{-t} \Big|_0^{\infty} = z \Gamma(z) - \Gamma(z+1)$$

$$0 = z \Gamma(z) - \Gamma(z+1)$$

$$\boxed{z \Gamma(z) = \Gamma(z+1)} \leftarrow \text{impose for all } z$$

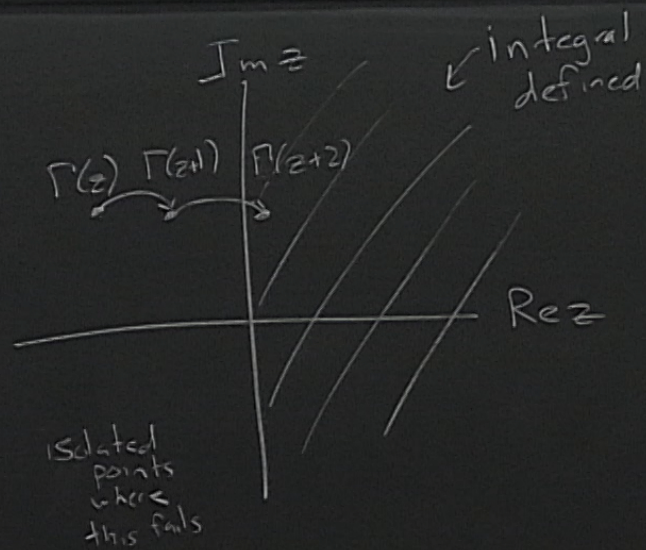
$$n(n-1)! = n!$$

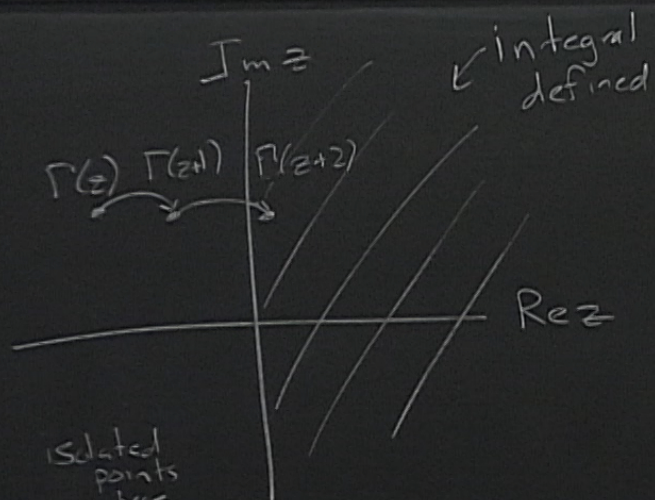
$$\operatorname{Re}(z) > 0$$

$$-t^z e^{-t}$$

$$(z+1)$$

$$\Gamma(z)$$





isolated points where this fails

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)} \quad \text{Re}(z)+n>0$$

↑ poles at $z = -n$

$$\Gamma(1) = 1$$

$$\Gamma(n+1) = n! \quad \text{for non-negative integer } n$$

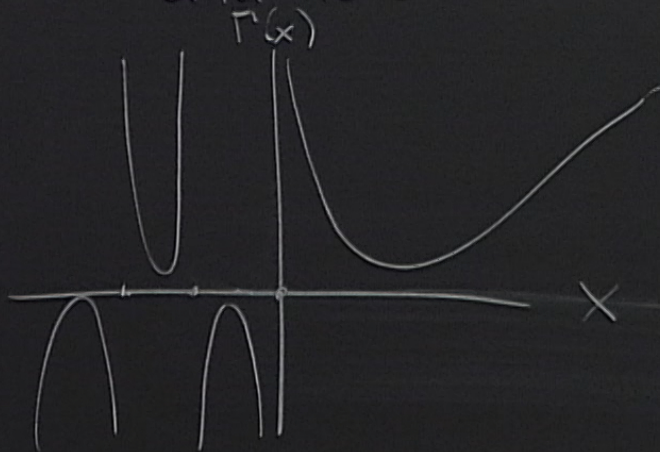
$$z = 1 - n + \epsilon$$

$$\Gamma(z) = \frac{\Gamma(1+\epsilon)}{(\epsilon-n+1)(\epsilon-n+2)\cdots(\epsilon-1)\epsilon}$$

$$\approx \frac{(-1)^{n+1}}{(n-1)!\epsilon} \quad \text{as } \epsilon \rightarrow 0$$

Residue at $z = -n$ is $\frac{(-1)^n}{n!}$

$\Gamma(z)$ has no other poles
and no zeros



Asymptotics

Motivation: - expansion in large or small parameter
- saddle-point approximation

Toy model: $Z(\lambda) = \int_{-\infty}^{\infty} e^{-x^2 - \lambda x^4} dx$

$$= \int_{-\infty}^{\infty} e^{-x^2} \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n x^{4n}}{n!} dx$$

Call $u = x^2$

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$$= \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{n!} \int_0^{\infty} e^{-u} u^{2n-1/2} \frac{du}{2} \cdot 2$$

$$Z(\lambda) \stackrel{?}{=} \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \Gamma(2n + \frac{1}{2})$$

$$\Gamma(2n + \frac{1}{2}) \approx (2n)! > (n!)^2$$

→ coefficients grow faster than $n!$

→ series diverges

$$Z(\lambda) \not\sim \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \Gamma(2n + \frac{1}{2})$$

$$\Gamma(2n + \frac{1}{2}) \approx (2n)! > (n!)^2$$

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Definitions

1. $f(x) \ll_{x \rightarrow x_0} g(x) \iff \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 0$

"much less than"

$$x \ll_{x \rightarrow 0} \frac{1}{x}$$

$$x^2 \ll_{x \rightarrow 0} -1$$

2. $f(x) \sim_{x \rightarrow x_0} g(x) \iff f(x) - g(x) \ll_{x \rightarrow x_0} g(x) \iff \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 1$

"asymptotic to"

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"asymptotic to"

$$\cos x \sim_{x \rightarrow \pi} -1$$

$$x^2 \not\sim_{x \rightarrow 0} 0$$

$$x^2 \not\sim_{x \rightarrow 0} x$$

$$3. \quad y(x) \underset{x \rightarrow x_0}{\sim} \sum_{n=0}^{\infty} a_n (x-x_0)^n \iff y(x) - \sum_{n=0}^N a_n (x-x_0)^n \underset{x \rightarrow x_0}{\ll} (x-x_0)^{N+1} \text{ for all } N$$

↑
"has an asymptotic series"

$$3. \quad y(x) \underset{x \rightarrow x_0}{\sim} \sum_{n=0}^{\infty} a_n (x-x_0)^n \Leftrightarrow y(x) - \sum_{n=0}^N a_n (x-x_0)^n \underset{x \rightarrow x_0}{\ll} (x-x_0) \quad \text{for all } N$$

$\uparrow$ "has an asymptotic series"

Comments: • convergent series are better approx. w/ more terms

- asymptotic series are better approx. as $x \rightarrow x_0$ w/ fixed number of terms
- taking too many terms usually makes approximation worse

• If $F(\lambda) \underset{\lambda \rightarrow 0}{\sim} \sum_{n=0}^{\infty} a_n \lambda^n$ then $F(\lambda) + e^{-1/\lambda^2} \underset{\lambda \rightarrow 0}{\sim} \sum_{n=0}^{\infty} a_n \lambda^n$

$$e^{-1/\lambda^2} \underset{\lambda \rightarrow 0}{\ll} \lambda^N$$

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$\uparrow$ "has an asymptotic series" for all N

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$$e^{-1/\lambda^2} \underset{\lambda \rightarrow 0}{\ll} \lambda^N$$

$\uparrow$
non-perturbative term

$$X^2 \not\sim_{X \rightarrow 0} 0$$

$$X^2 \not\sim_{X \rightarrow 0} X$$

$$F(\lambda) \underset{\lambda \rightarrow 0}{\sim} \sum_{n=0}^{\infty} a_n \lambda^n$$

unique
 ↙
 ↘
 non-unique
 (at least w/o extra
 info about F)

non-perturbative physics - tunneling or vacuum decay

$$f(z) \underset{z \rightarrow z_0}{\sim} g(z) \quad \text{generally only valid}$$

in wedge $a < \arg(z - z_0) < b$

$$e^{-1/\lambda^2} \underset{\lambda \rightarrow 0}{\rightarrow} 0 \quad \lambda \text{ real}$$

$$e^{-1/\lambda^2} \underset{\lambda \rightarrow 0}{\rightarrow} \infty \quad \lambda \text{ imaginary}$$

decay

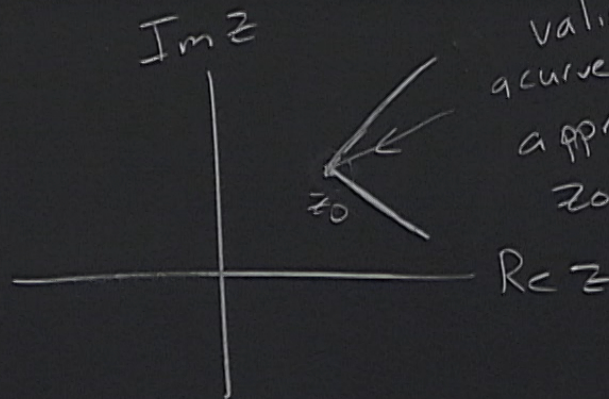
non-perturbative

(z) generally only valid

in wedge $a < \arg(z - z_0) < b$

λ real

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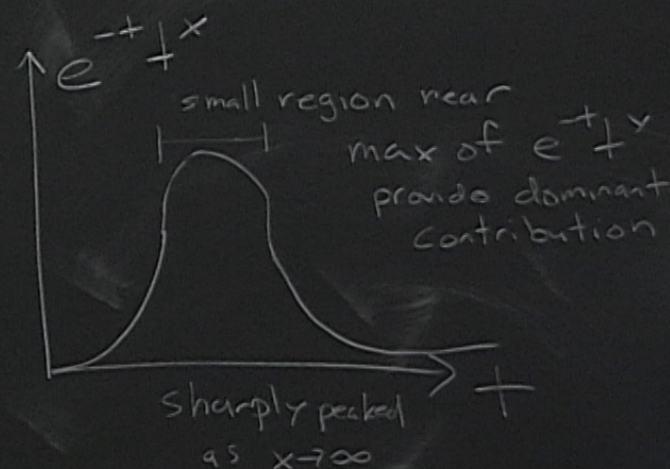
$f(z) \sim_{z \rightarrow z_0} g(z)$
 valid for
 a curve that
 approaches
 z_0 from inside
 this wedge

Example: Stirling's Approximation

$\Gamma(x+1) = x!$ if x is integer for large real x

$$\Gamma(x+1) = \int_0^{\infty} e^{-t} t^x dt$$

$$\begin{aligned} t &= wx \\ &= x^{x+1} \int_0^{\infty} e^{-x(w - \ln w)} dw \\ &\quad \underbrace{e^{-x f(w)}}_{\leftarrow \text{dominated near minimum of } f} \end{aligned}$$



$$f'(w) = 1 - \frac{1}{w} = 0 \text{ at } w=1$$

$$f(w) = 1 + \frac{1}{2}(w-1)^2 + \dots$$

$$\Gamma(x+1) \approx x^{x+1} e^{-x} \int_0^{\infty} \underbrace{e^{-\frac{x}{2}(w-1)^2 + \dots}}_{\text{Gaussian}} dw$$

$$\approx x^{x+1} e^{-x} \sqrt{\frac{2\pi}{x}}$$

more terms

$$e^{-\frac{x}{2}(w-1)^2 + c_3(w-1)^3 + c_4(w-1)^4 + \dots}$$

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more terms

$$e^{-\frac{x}{2}(w-1)^2 + c_3(w-1)^3 + c_4(w-1)^4 + \dots} = e^{-\frac{x}{2}(w-1)^2} \left(1 + a_1(w-1) + a_2(w-1)^2 + \dots \right)$$

Taylor expand

$$x^2 \sim x$$

$$\frac{\Gamma(x+1)}{\sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x}} \sim_{x \rightarrow \infty} 1 + \frac{1}{12x} + \dots$$

$$x^2 \sim x$$

$$\frac{\Gamma(x+1)}{\sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x}} \sim_{x \rightarrow \infty} 1 + \frac{1}{12x} + \dots$$

Steps: - Identify region where integrand is peaked

- Approximate integral with Gaussian integral
- Evaluated

$$f'(w) = 1 - \frac{1}{w} = 0 \text{ at } w=1$$

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$$\Gamma(x+1) \approx x^{x+1} e^{-x} \int_0^{\infty} \underbrace{e^{-\frac{x}{2}(w-1)^2 + \dots}}_{\text{Gaussian (after extending l. limits)}} dw$$

$$\approx x^{x+1} e^{-x} \sqrt{\frac{2\pi}{x}}$$

more terms

$$e^{-\frac{x}{2}(w-1)^2 + c_3(w-1)^3 + c_4(w-1)^4 + \dots} = e^{-\frac{x}{2}(w-1)^2} \underbrace{\left(1 + a_1(w-1) + a_2(w-1)^2 + \dots \right)}_{\text{Taylor expand}}$$

non-perturba

Next time: Apply this method to

$$\oint_C g(w) e^{-zf(w)} dw$$

ε

ser