

Title: PSI 2019/2020 - Math for QFT - Lecture 2

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Collection: PSI 2019/2020 - Math for QFT (Wohns)

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Application: Functional Derivatives  $\frac{\delta F}{\delta f(x)}$   
 $\leftarrow S[q(t), \dot{q}(t)]$

$$\delta S = \int dt \left[ \frac{\partial L}{\partial q(t)} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}(t)} \right] \delta q(t)$$

$$= \int dt \underbrace{\frac{\delta S}{\delta q(t)}}_{\text{distribution}} \underbrace{\delta q(t)}_{\text{test function } \delta q(t) = \epsilon \varphi(t)}$$

$$\delta S = \sum_i \frac{\partial S}{\partial q_i} \delta q_i$$

$$\frac{\delta F[f]}{\delta f(x)}(\varphi) = \lim_{\epsilon \rightarrow 0} \frac{F[f + \epsilon \varphi] - F[f]}{\epsilon} = \int_{-\infty}^{\infty} \frac{\delta F[f]}{\delta f(x)} \varphi(x) dx$$

Example:  $F[f] = f^2(x_0)$

$$\frac{\delta F[f]}{\delta f(x)}(\varphi) = \lim_{\epsilon \rightarrow 0} \frac{\cancel{f^2(x_0)} + 2\epsilon f(x_0)\varphi(x_0) + \cancel{\epsilon^2 \varphi(x_0)^2} - \cancel{f^2(x_0)}}{\epsilon}$$

$$= 2f(x_0)\varphi(x_0)$$

$$= \int_{-\infty}^{\infty} \boxed{2f(x_0)\delta(x-x_0)} \varphi(x) dx \rightarrow \frac{\delta F}{\delta f(x)} = 2f(x_0) \underbrace{\delta(x-x_0)}_{\delta x_0}$$

$\frac{\delta F[f]}{\delta f(x)}$  is linear

has a product rule  
 + chain rules

$$\frac{\delta(\lambda F[f] + M G[f])}{\delta f(x)} = \lambda \frac{\delta F[f]}{\delta f(x)} + M \frac{\delta G[f]}{\delta f(x)}$$

$$\frac{\delta(FG)}{\delta f(x)} = \frac{\delta F[f]}{\delta f(x)} G[f] + F[f] \frac{\delta G[f]}{\delta f(x)}$$

$$\frac{\delta F[g(f)]}{\delta f(x)} = \frac{\delta F[g(f)]}{\delta g(f(x))} \frac{dg(f(x))}{df(x)}$$

$$\frac{\delta g(F[f])}{\delta f(x)} = \frac{dg(F[f])}{dF[f]} \frac{\delta F[f]}{\delta f(x)}$$

# Application: Green's functions

$$\underset{\substack{\uparrow \\ \text{linear} \\ \text{differential} \\ \text{operator}}}{L} y = \underset{\substack{\nwarrow \nearrow \\ \text{functions}}}{f} \rightarrow y = \underbrace{L^{-1}}_{\text{Green's function}} f$$

Analogy:  
(Sometimes  
subtleties  
w/ bc conditions)

| finite dim                   | $\infty$ dim                      |
|------------------------------|-----------------------------------|
| vector $\vec{v}$             | function $f$                      |
| component $v_i$              | value of function at point $f(x)$ |
| matrix $M$                   | operator $\mathcal{O}$            |
| matrix element $M_{ij}$      | integral kernel $k(x,y)$          |
| identity matrix $\mathbb{1}$ | Dirac delta $\delta$              |

$$(M\vec{v})_i = \sum_j M_{ij} v_j$$

$$\mathcal{O}f(x) = \int k(x,y) f(y) dy$$

$$(MM^{-1}) = \mathbb{1} \rightarrow LL^{-1} = \delta$$

$$(MM^{-1})_{ij} = \delta_{ij} \quad L_x G(x, \xi) = \delta(x - \xi)$$

$$y(x) = \int G(x, \xi) f(\xi) d\xi$$

$$\begin{aligned} L_y(x) &= \int L_x G(x, \xi) f(\xi) d\xi \\ &= \int \delta(x - \xi) f(\xi) d\xi \\ &= f(x) \end{aligned}$$

$$\delta^1 = \delta$$

$$G(x, \xi) = \delta(x - \xi)$$

$$Y(\xi) d\xi$$

$$\xi) f(\xi) d\xi$$

$$f(\xi) d\xi$$

$$Ly = f$$

$$L = L_0 + \lambda L_1$$

$$y = y_0 + y_1 + \underbrace{\mathcal{O}(\lambda)}_{\mathcal{O}(\lambda)}$$

$$L_0 y_0 = f \rightarrow y_0 = L_0^{-1} f$$

$$Ly = f$$

$$\cancel{L_0 y_0} + \lambda L_1 y_0 + L_0 y_1 + \mathcal{O}(\lambda^2) = \cancel{f}$$

$$y_1 = -\lambda L_0^{-1} L_1 L_0^{-1} f$$

## Distributions

- product of two  
is not gener

- Next time: Asympt

## Distributions

- product of two distributions is not generally well-defined

- Next time: Asymptotics