

Title: PSI 2019/2020 - Math for QFT - Lecture 1

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Collection: PSI 2019/2020 - Math for QFT (Wohns)

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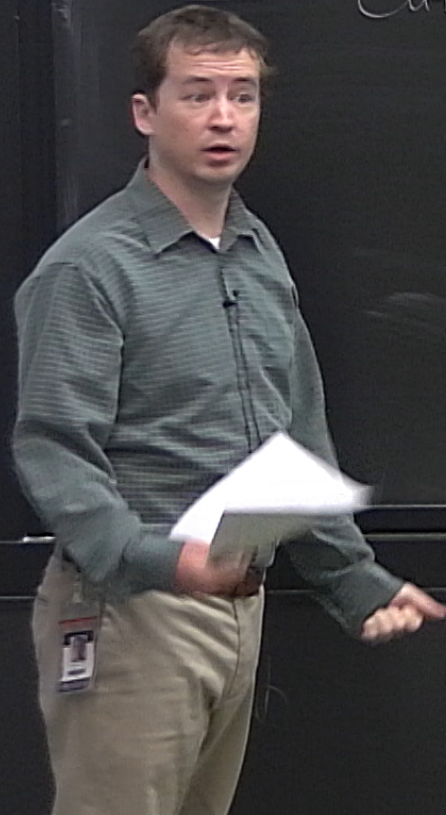
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Math for QFT

Math for QFT

Outline: Distributions (Generalized Functions)
Asymptotic Series
"Summation" of Divergent Series



Distributions

Motivation: - point charge $\rho(\vec{x}) = \begin{cases} 0 & \vec{x} \neq 0 \\ \infty & \vec{x} = 0 \end{cases} \quad \int d^3x \rho(\vec{x}) = Q$

- $\frac{d^n}{dx^n} |x| \Big|_{x=0}$

- $LG = \delta$

- $\int_{-\infty}^{\infty} \frac{dx}{x}$

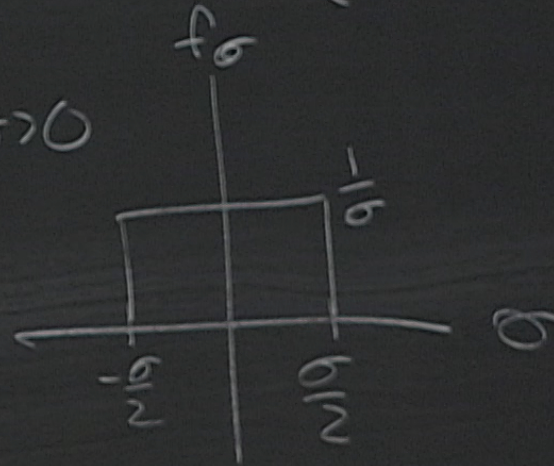
- $\frac{\delta F}{\delta f(x)}$

Most important example: δ

$$\delta(x) \stackrel{?}{=} \lim_{\sigma \rightarrow 0} f_{\sigma}(x)$$

$$\int_{-\infty}^{\infty} f_{\sigma}(x) dx = 1 \text{ for } \sigma > 0$$

$$f_{\sigma}(x) = \begin{cases} \frac{1}{\sigma} & |x| < \frac{\sigma}{2} \\ 0 & |x| > \frac{\sigma}{2} \end{cases}$$



Definition: Function space

- vector space whose elements are functions
with a norm that satisfies

- positivity: $\|f\| \geq 0$

- unique identity $\|f\| = 0 \Leftrightarrow f = 0$

- triangle inequality $\|f+g\| \leq \|f\| + \|g\|$

- homogeneity: $\|\lambda f\| = |\lambda| \|f\|$

important example : $L^2[a, b]$

$$\|f\|^2 = \int_a^b |f(x)|^2 dx$$

any f with $\|f\| < \infty$ is in $L^2[a, b]$

$$f_1(x) = 0 \rightarrow \|f_1\| = 0$$

$$f_2(x) = \begin{cases} 0 & x \neq \hat{x}_1 \\ 2 & x = \hat{x}_1 \end{cases} \quad \|f_2\| = 0$$

must identify
 f_1 and f_2

Definition: Function space

equivalence classes of

- vector space whose elements are $\sim$ functions
with a norm that satisfies

• positivity: $\|f\| \geq 0$

• unique identity $\|f\| = 0 \Leftrightarrow f = 0$

• triangle inequality $\|f+g\| \leq \|f\| + \|g\|$

• homogeneity: $\|\lambda f\| = |\lambda| \|f\|$

$$\|f_\sigma\|^2 = \int_{-\infty}^{\infty} |f_\sigma(x)|^2 dx$$

$$= \int_{-\frac{\sigma}{2}}^{\frac{\sigma}{2}} \frac{1}{\sigma^2} dx$$

$$= \frac{1}{\sigma}$$

$$\lim_{\sigma \rightarrow 0} \|f_\sigma\| \rightarrow \infty$$

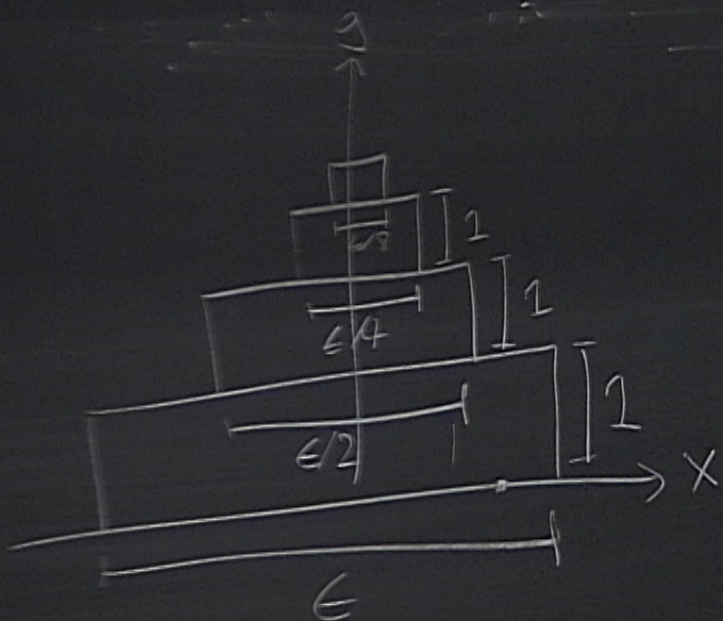
For $g_\sigma(x)$ such that

$$\lim_{\sigma \rightarrow 0} g_\sigma(x) = 0 \text{ for } x \neq 0$$

$$\int_{-\infty}^{\infty} g_\sigma(x) dx = 1 \text{ for } \sigma > 0.$$

For each x and each ϵ we can find σ_0 such
that $g_\sigma(x)$ is in the boxes for $\sigma < \sigma_0$

↑
for σ $\sigma_0 = |x|$



$$\int_{-\infty}^{\infty} g(x) dx \leq A = \epsilon + \frac{\epsilon}{2} + \frac{\epsilon}{4} + \dots = 2\epsilon$$

$\lim_{\sigma \rightarrow 0} g_{\sigma}(x)$

$$\int_{-\infty}^{\infty} \lim_{\sigma \rightarrow 0} g_{\sigma}(x) dx = 0 \neq 1 = \lim_{\sigma \rightarrow 0} \int_{-\infty}^{\infty} g_{\sigma}(x) dx$$

$$\int_{-\infty}^{\infty} \delta(x) f(x) dx - \text{map from } f(x) \rightarrow \mathbb{R}$$

Test functions:

functions that
distributions
can act upon

more restrictions $\rightarrow$ larger space of distributions

For now: $\bullet C^\infty$ (continuous w/ ∞ derivatives)

$\bullet$ Compact support (vanish outside a finite interval)

$$\begin{cases} e^{-\frac{1}{|x-a|}} & \text{if } a-\epsilon \leq x \leq a+\epsilon \\ 0 & \text{otherwise} \end{cases}$$

Distributions - functionals from $\mathcal{D}$ to $\mathbb{R}$ (or $\mathbb{C}$)

- finite $u(\varphi) < \infty$
 $\uparrow$ test function
 $\uparrow$ distribution

- linear $u(\alpha\varphi + \beta\psi) = \alpha u(\varphi) + \beta u(\psi)$
 $\nwarrow$ numbers $\nearrow$
 $\nwarrow$ test functions $\nearrow$

- continuous $\lim_{n \rightarrow \infty} u(\varphi_n) = u(\lim_{n \rightarrow \infty} \varphi_n)$

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Notation: $\int \delta(x) \varphi(x) dx = \varphi(0)$ or $\delta \cdot \varphi = \varphi(0) \cdot 1$
circled part is distribution
 $(\delta, \varphi) = \varphi(0) \quad \ddots$

Which are distributions?

$$u(\varphi) = \int (\varphi(x))^2 dx$$

$$v(\varphi) = \varphi(0) - \varphi'(\tilde{\eta}) + \tilde{\eta} \varphi''(7)$$

$$w(\varphi) = \begin{cases} 1 & \text{if } \varphi(0) = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$t(\varphi) = \int \frac{\varphi(x)}{x^2} dx$$



Which are distributions?

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not linear

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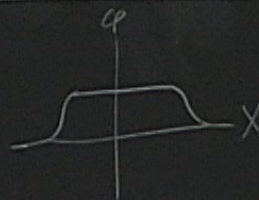
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✓

$$w(\varphi) = \begin{cases} 1 & \text{if } \varphi(0) = 0 \\ 0 & \text{otherwise} \end{cases}$$

not linear, continuous

$$t(\varphi) = \int \frac{\varphi(x)}{x^2} dx$$



not finite

Functions that are locally integrable

$$\int_a^b f(x) dx < \infty \text{ for all } a, b$$

are distributions

$$f(\phi) \equiv \int f(x) \phi(x) dx$$

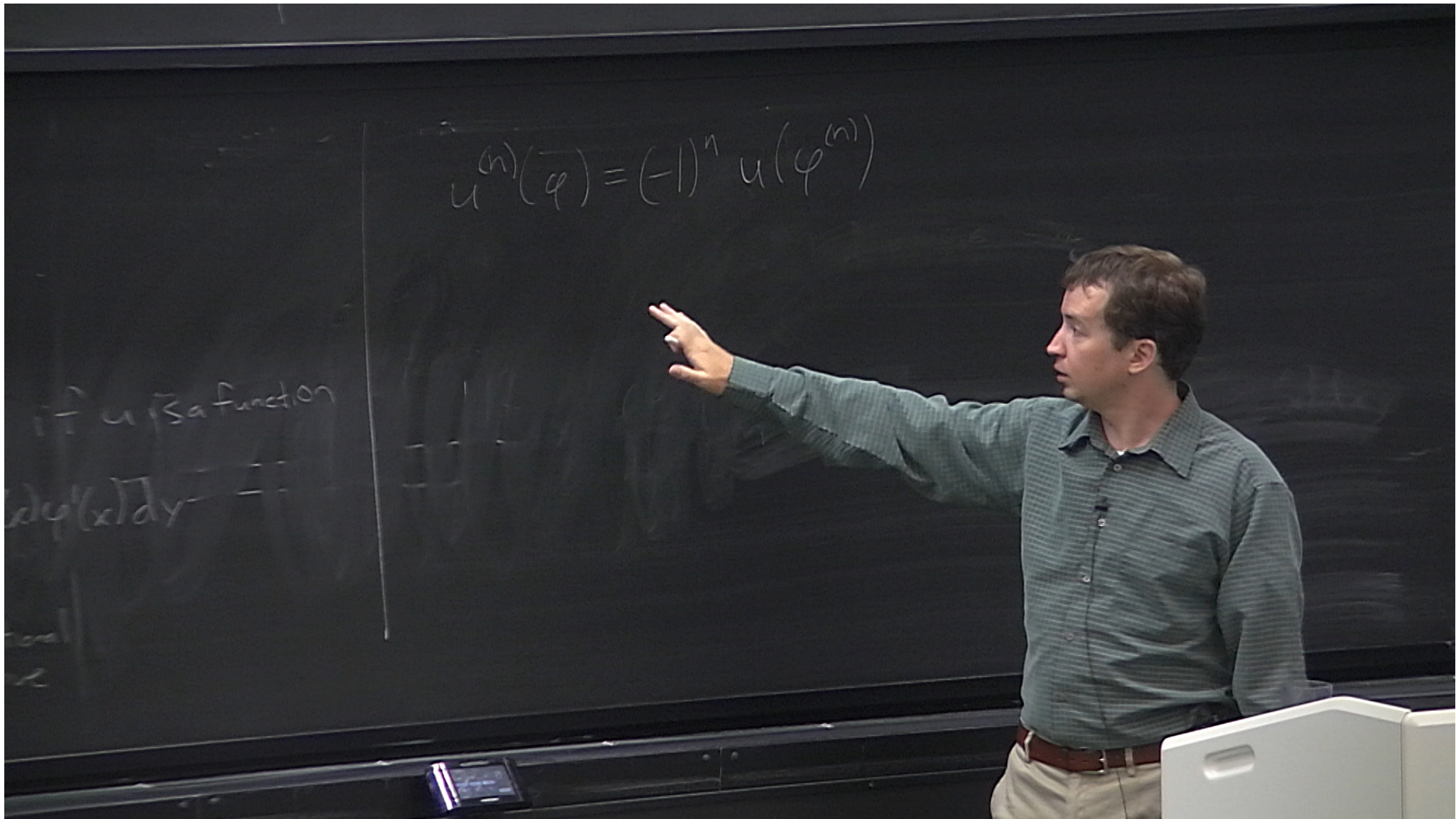
Distributional Derivatives

$$\cancel{u'(x) \stackrel{?}{=} \lim_{\epsilon \rightarrow 0} \frac{u(x+\epsilon) - u(x)}{\epsilon}}$$

$$\begin{aligned} u'(\varphi) &= \int_{-\infty}^{\infty} u'(x) \varphi(x) dx, \text{ if } u \text{ is a function} \\ &= \cancel{u(x) \varphi(x)} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(x) \varphi'(x) dx \end{aligned}$$

$$\boxed{u'(\varphi) \equiv -u(\varphi')}$$

weak or distributional
derivative



Distributional Derivatives

$$\cancel{u'(x) = \lim_{\epsilon \rightarrow 0} \frac{u(x+\epsilon) - u(x)}{\epsilon}}$$

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weak or distributional
derivative