Title: PSI 2019/2020 - Math for QFT - Lecture 1

Speakers: Dan Wohns

Collection: PSI 2019/2020 - Math for QFT (Wohns)

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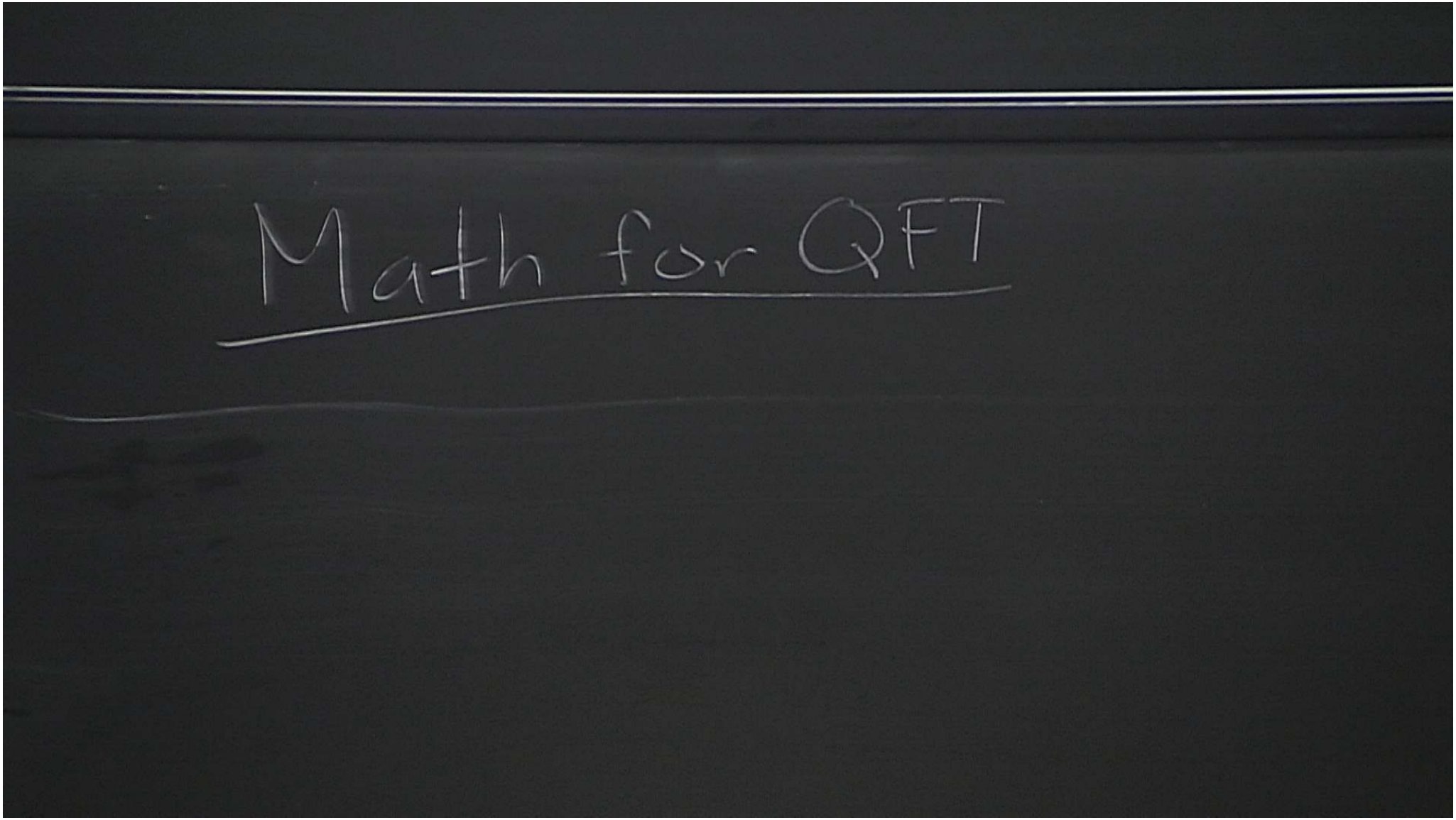
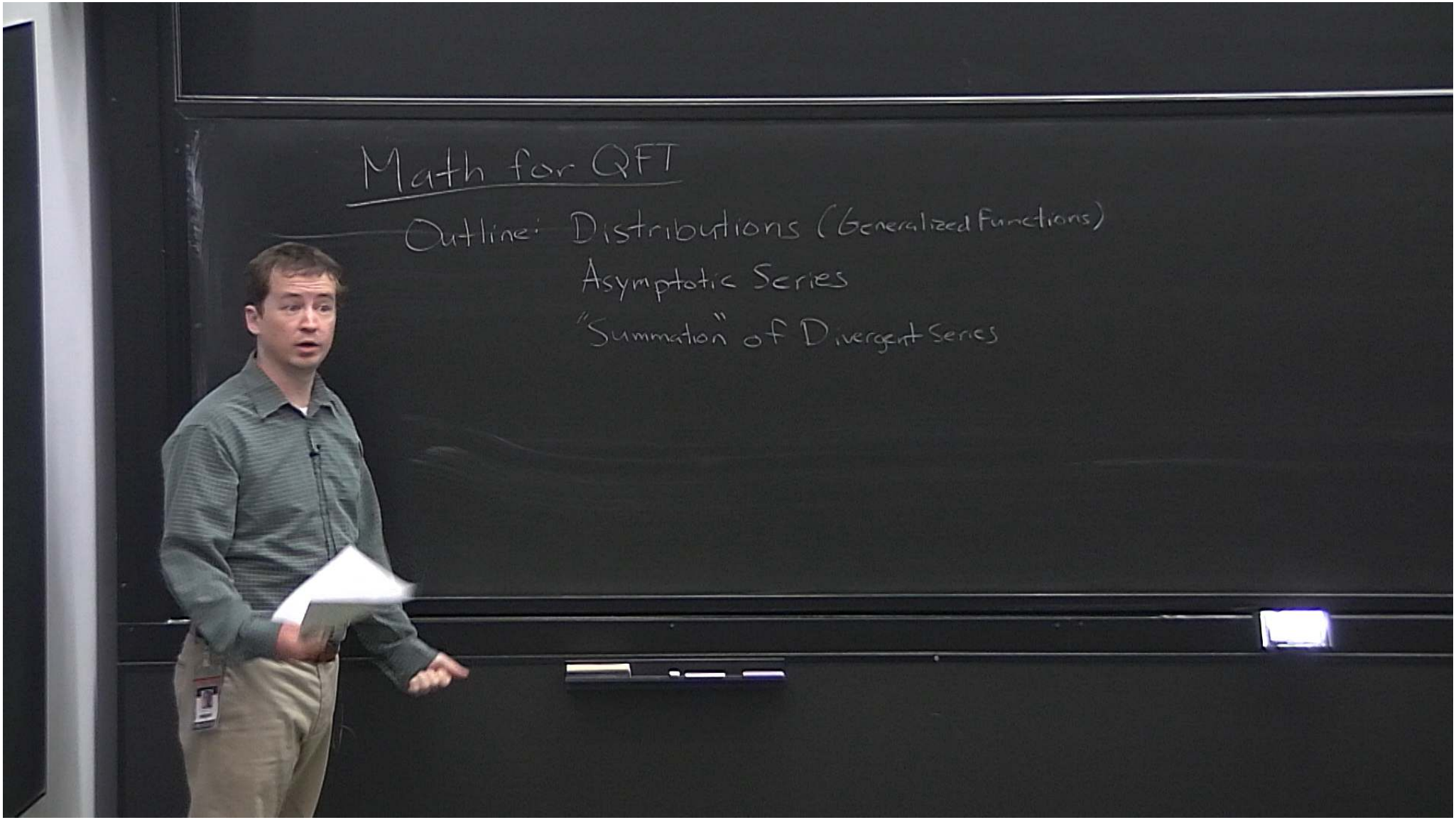

Math for QFT


Math for QFT
Outline: Distributions (Generalized Functions)
Asymptotic Series
"Summation" of Divergent Series

# Distributions

Motivation: - point charge $\rho(\vec{x}) = \begin{cases} 0 & \vec{x} \neq 0 \\ \infty & \vec{x} = 0 \end{cases}$ $\int d^3x \rho(\vec{x}) = Q$

- $\frac{d^n}{dx^n}|x||_{x=0}$
- $\int_{-\infty}^{\infty} \frac{dx}{x}$
- $\frac{\delta F}{\delta f(x)}$

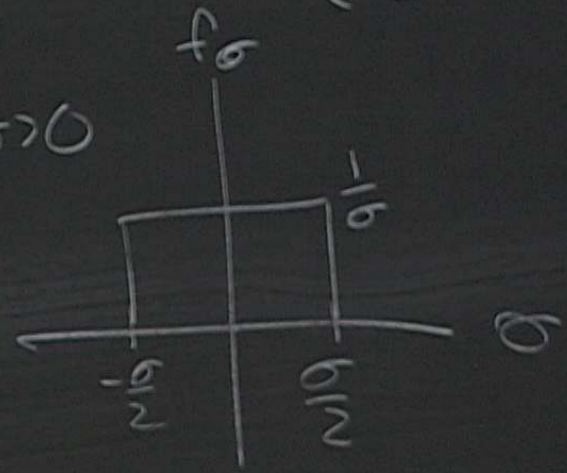
- $LG = \delta$

Most important example: $\delta$

$$\delta(x) \stackrel{?}{=} \lim_{\sigma \to 0} f_\sigma(x)$$

$$f_\sigma(x) = \begin{cases} \frac{1}{\sigma} & |x| \le \frac{\sigma}{2} \\ 0 & |x| > \frac{\sigma}{2} \end{cases}$$

$$\int_{-\infty}^{\infty} f_\sigma(x)\, dx = 1 \text{ for } \sigma > 0$$



# Definition: Function space

- vector space whose elements are functions with a norm that satisfies

  - positivity: $\|f\| \geq 0$
  - unique identity $\|f\| = 0 \Leftrightarrow f = 0$
  - triangle inequality $\|f+g\| \leq \|f\| + \|g\|$
  - homogeneity: $\|\lambda f\| = |\lambda| \|f\|$

important example: $L^2[a,b]$

$$\|f\|^2 = \int_a^b |f(x)|^2 dx$$

any $f$ with $\|f\| < \infty$ is in $L^2[a,b]$

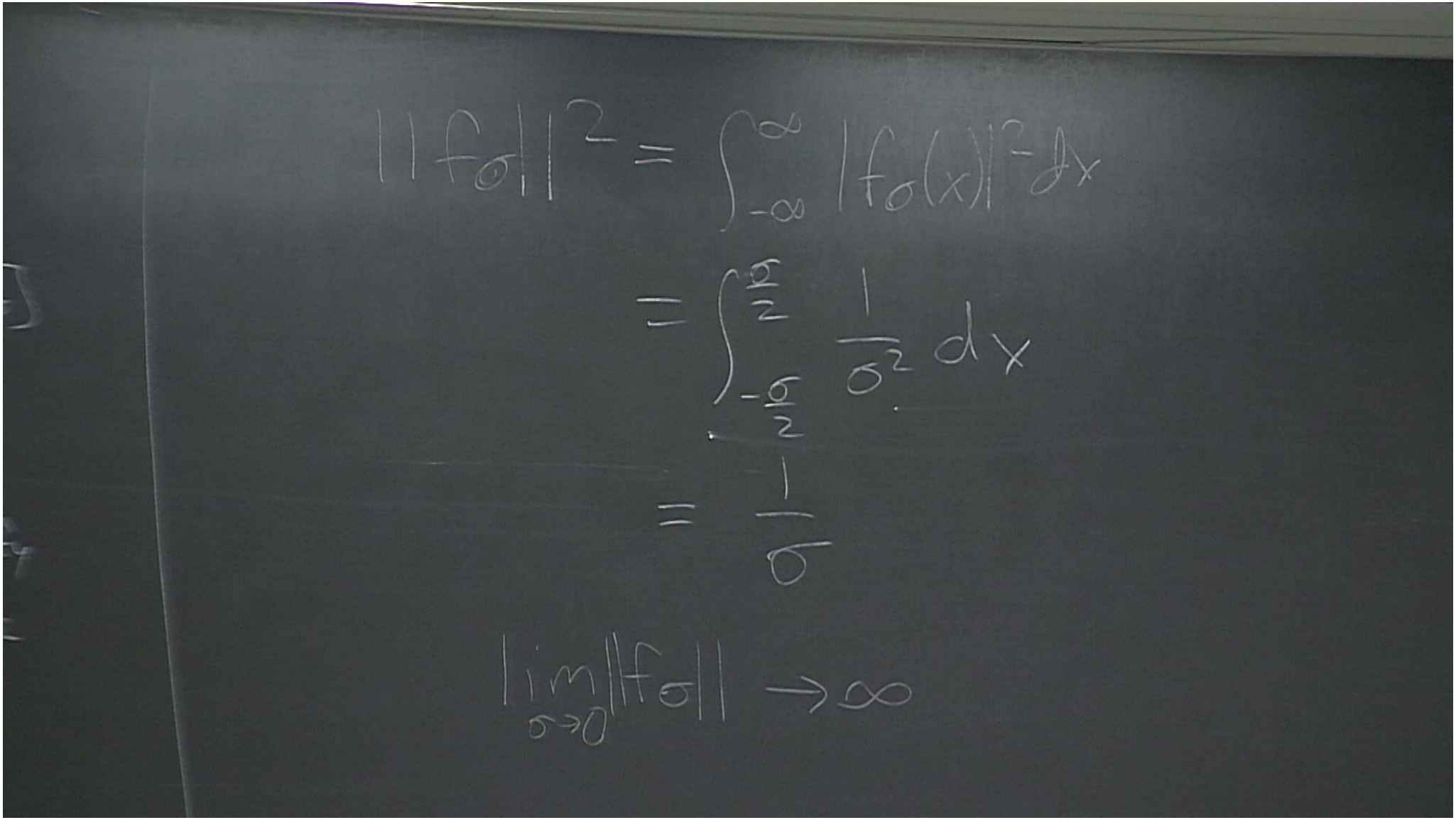
$f_1(x) = 0 \rightarrow \|f_1\| = 0$

must identify $f_1$ and $f_2$

$$f_2(x) = \begin{cases} 0 & x \neq \tilde{r}_1 \\ 27 & x = \tilde{r}_1 \end{cases} \quad \|f_2\| = 0$$

**Definition: Function space**

- vector space whose elements are equivalence classes of functions with a norm that satisfies
  - positivity: $\|f\| \geq 0$
  - unique identity $\|f\| = 0 \Leftrightarrow f = 0$
  - triangle inequality $\|f+g\| \leq \|f\| + \|g\|$
  - homogeneity: $\|\lambda f\| = |\lambda| \|f\|$


$\|f_\sigma\|^2 = \int_{-\infty}^{\infty} |f_\sigma(x)|^2 dx$
$= \int_{-\frac{\sigma}{2}}^{\frac{\sigma}{2}} \frac{1}{\sigma^2} dx$
$= \frac{1}{\sigma}$
$\lim_{\sigma \to 0} \|f_\sigma\| \to \infty$

For $g_\sigma(x)$ such that

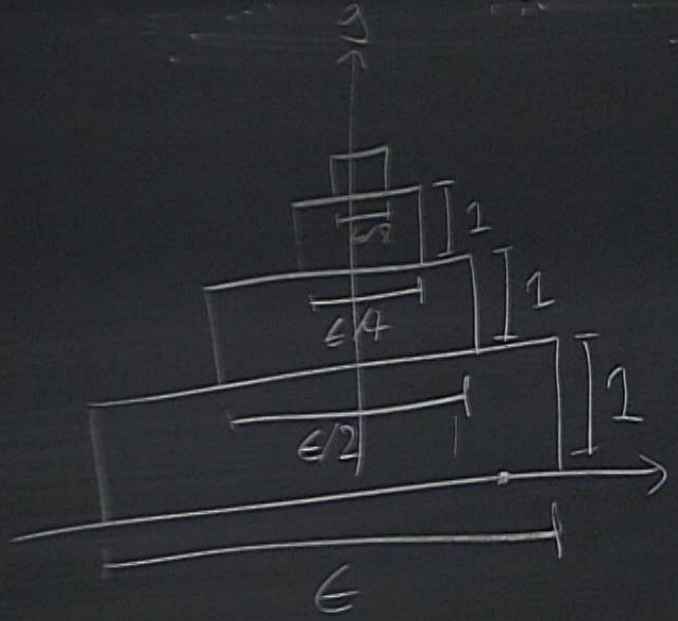
$\lim_{\sigma \to 0} g_\sigma(x) = 0$ for $x \neq 0$

$\int_{-\infty}^{\infty} g_\sigma(x)\,dx = 1$ for $\sigma > 0$.

For each $x$ and each $\epsilon$ we can find $\sigma_0$ such that $g_\sigma(x)$ is in the boxes for $\sigma < \sigma_0$

↑

for $f_\sigma$ $\sigma_0 = |x|$




$\lim_{\sigma \to 0} g_\sigma(x)$

$$\int_{-\infty}^{\infty} g(x)\,dx \leq A = \epsilon \cdot 1 \frac{\epsilon}{2} + \frac{\epsilon}{4} \cdots = 2\epsilon$$

$$\int_{-\infty}^{\infty} \lim_{\sigma \to 0} g_\sigma(x)\,dx = 0 \neq 1 = \lim_{\sigma \to 0} \int_{-\infty}^{\infty} g_\sigma(x)\,dx$$

$$\int_{-\infty}^{\infty} \delta(x)\, f(x)\, dx$$ - map from $f(x) \to \mathbb{R}$

Test functions: more restrictions → larger space of distributions

functions that distributions can act upon

For now: • $C^\infty$ (continuous w/ $\infty$ derivatives)

• compact support (vanish outside a finite interval)

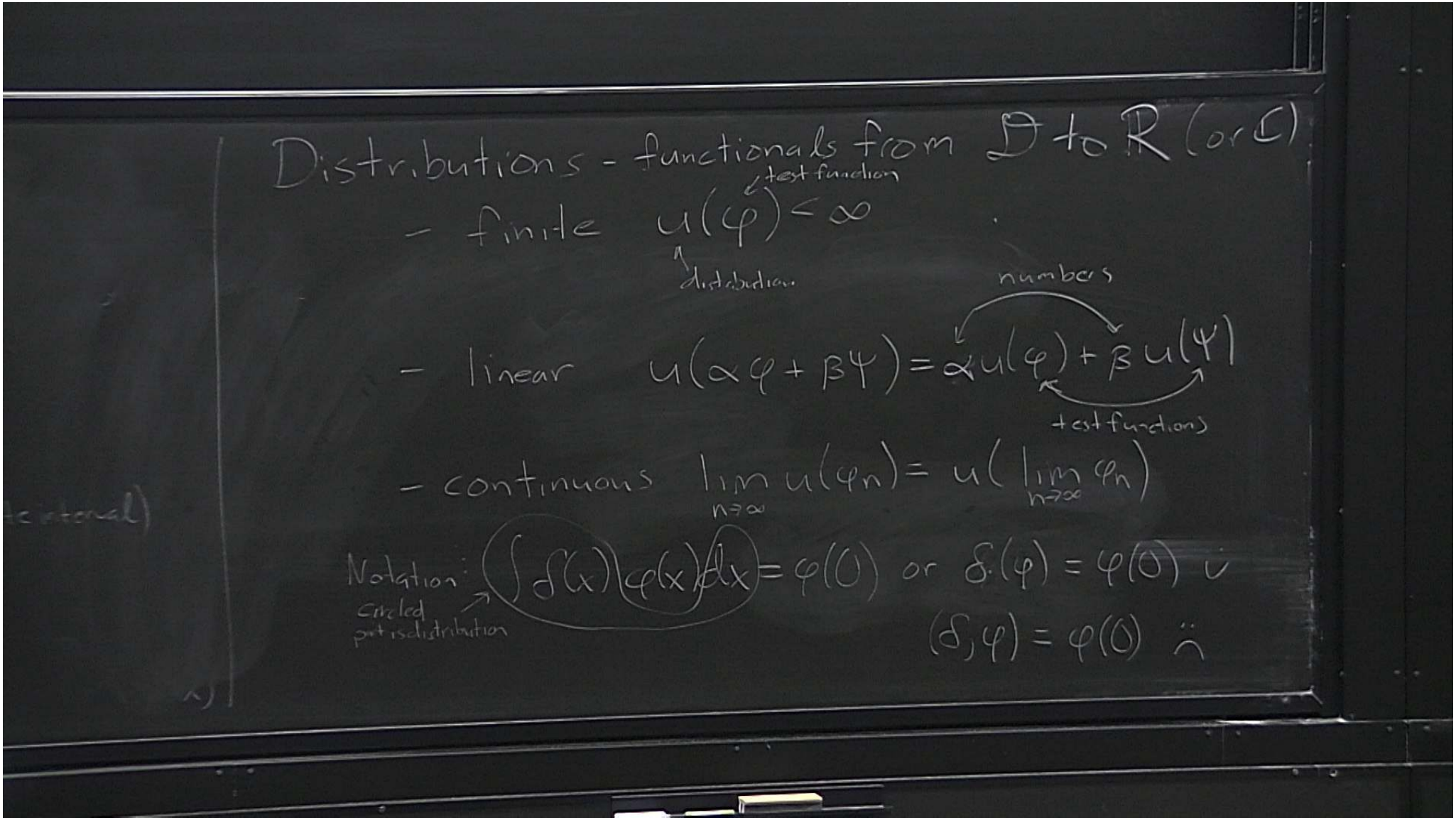
$$\begin{cases} e^{-\frac{1}{|x^2-a|}} & -\sqrt{a} \le x \le \sqrt{a} \\ 0 & \text{otherwise} \end{cases}$$

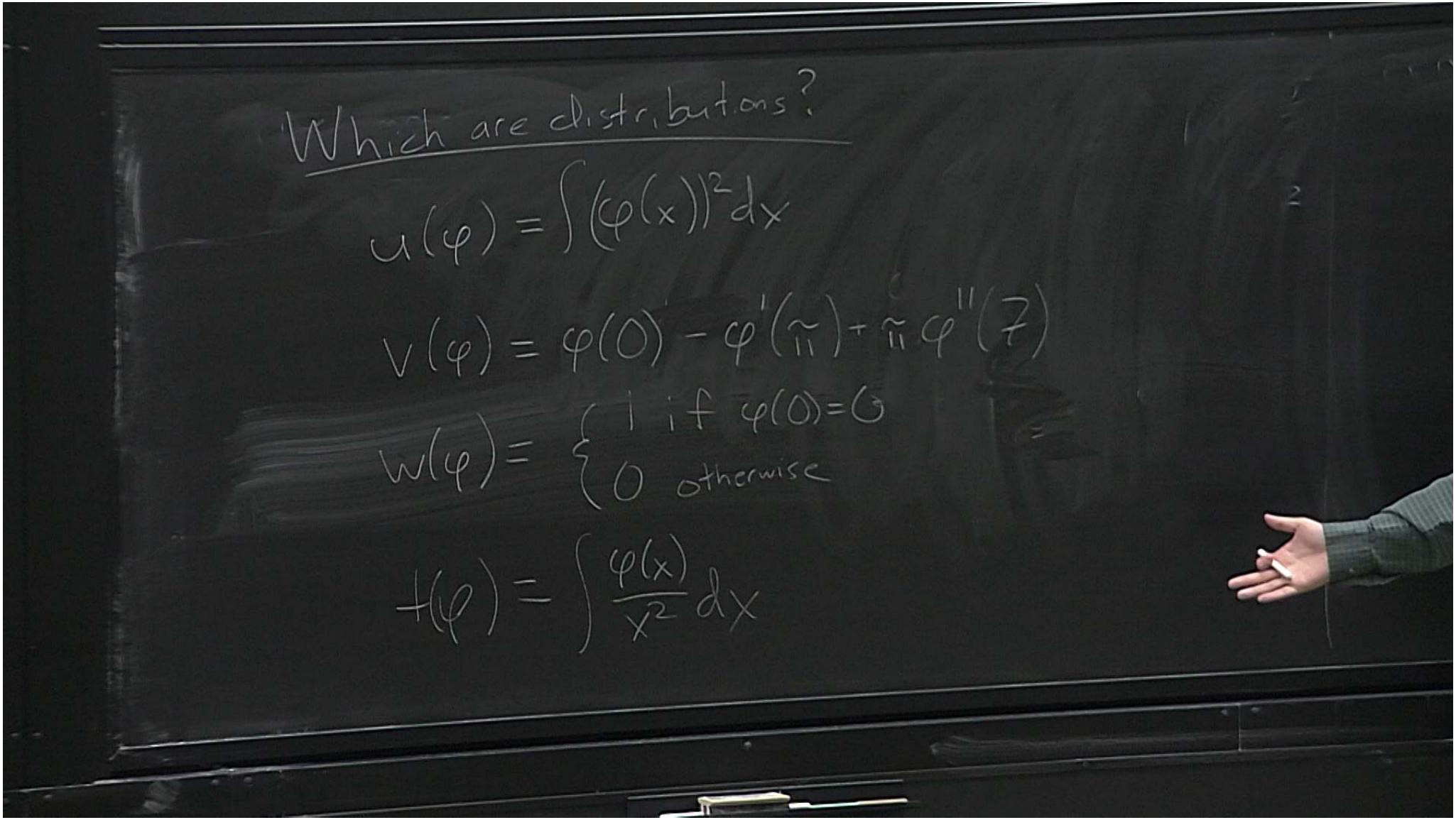
Distributions - functionals from $\mathcal{D}$ to $\mathbb{R}$ (or $\mathbb{C}$)

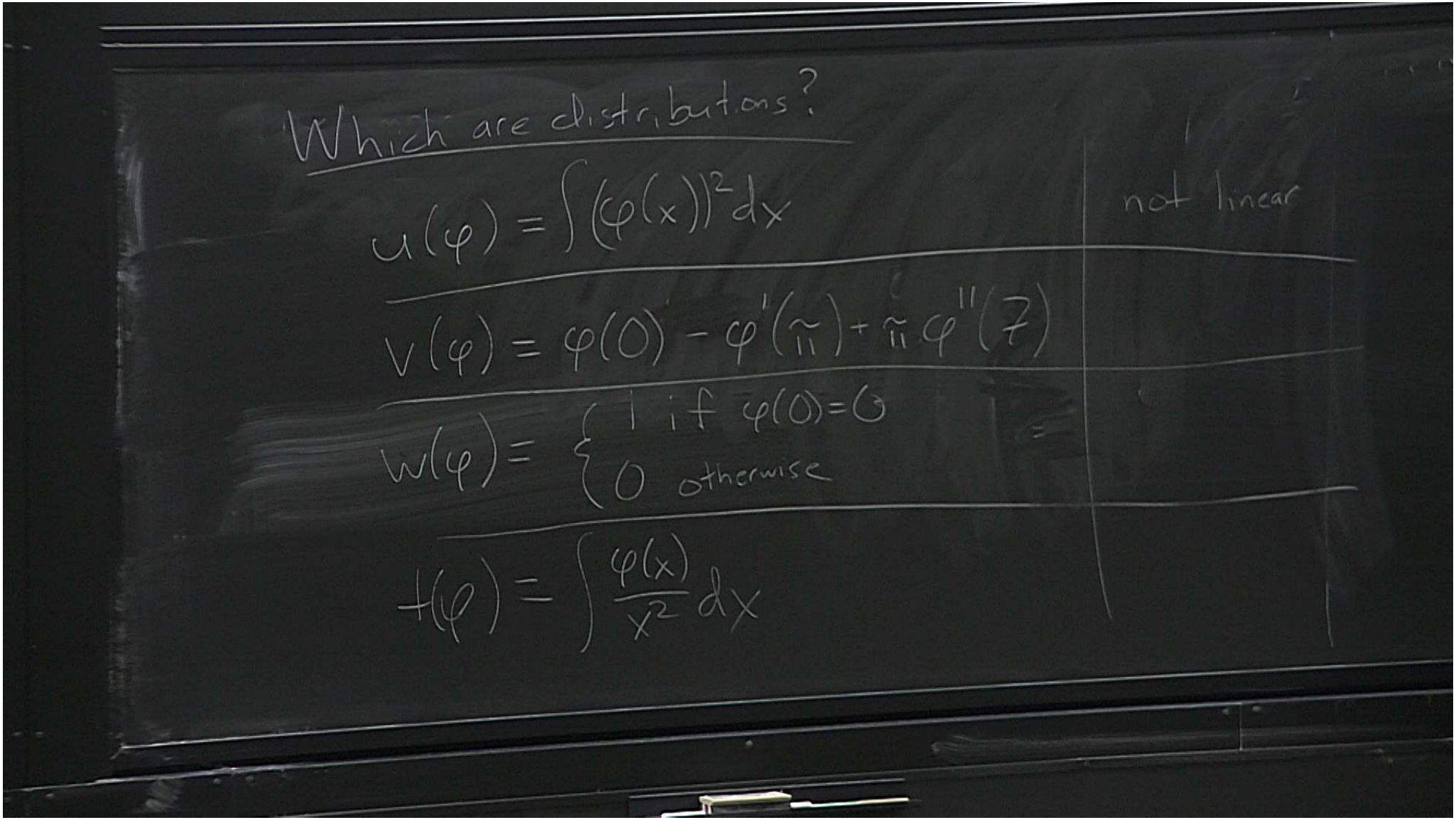
- finite $u(\varphi) < \infty$ ($\varphi$: test function; $u$: distribution)

- linear $u(\alpha\varphi + \beta\psi) = \alpha u(\varphi) + \beta u(\psi)$ (numbers; test functions)

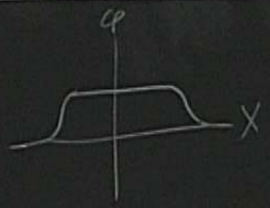
- continuous $\lim_{n\to\infty} u(\varphi_n) = u(\lim_{n\to\infty} \varphi_n)$

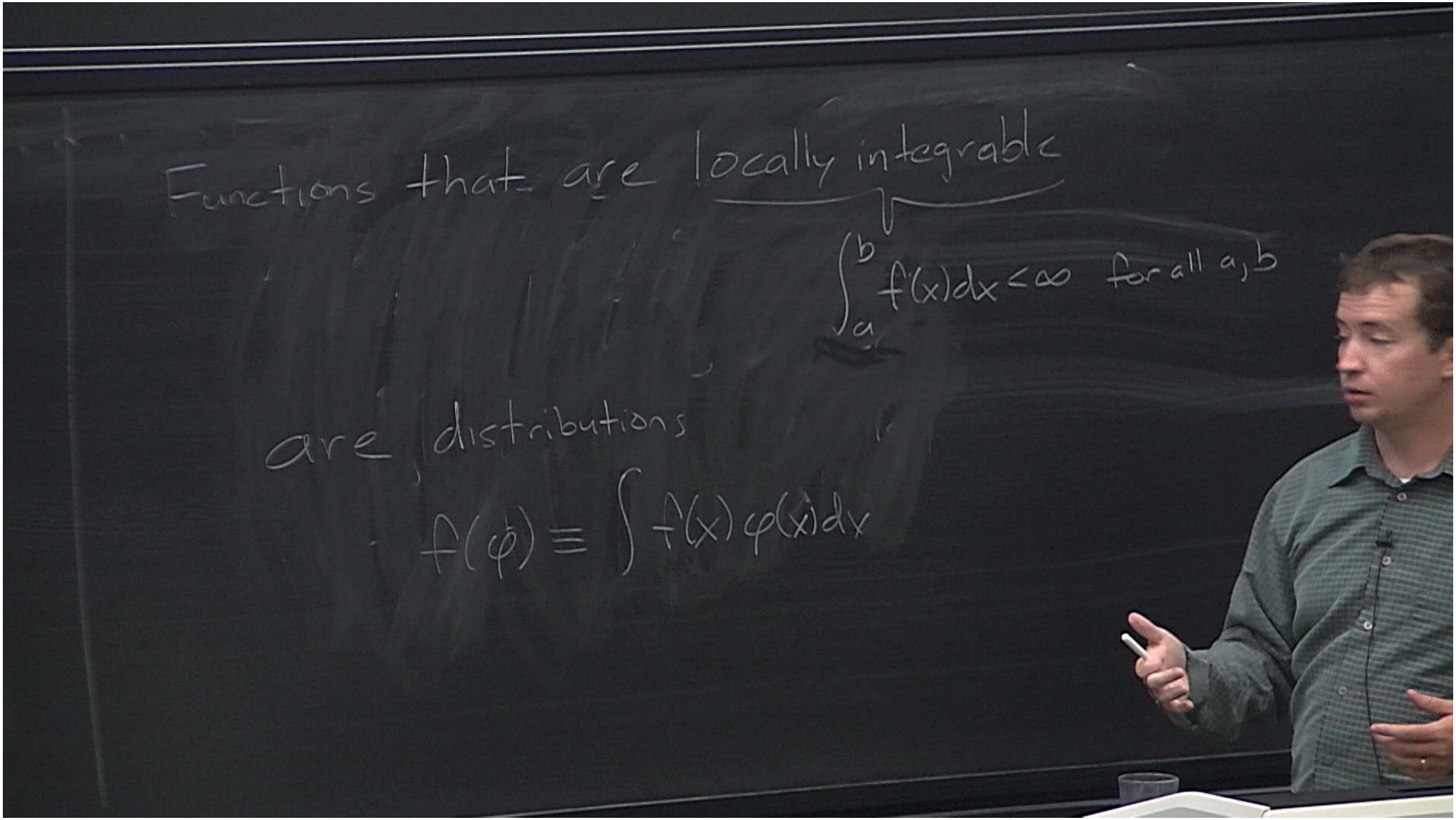

Distributions - functionals from $\mathcal{D}$ to $\mathbb{R}$ (or $\mathbb{C}$)
test function
- finite $u(\varphi) < \infty$
distribution
numbers
- linear $u(\alpha\varphi + \beta\psi) = \alpha u(\varphi) + \beta u(\psi)$
test functions
- continuous $\lim_{n\to\infty} u(\varphi_n) = u(\lim_{n\to\infty} \varphi_n)$
Notation: $\int \delta(x)\varphi(x)dx = \varphi(0)$ or $\delta(\varphi) = \varphi(0)$ or $(\delta, \varphi) = \varphi(0)$ ∴
circled part is distribution


Which are distributions?
$u(\varphi) = \int (\varphi(x))^2 dx$
$v(\varphi) = \varphi(0) - \varphi'(\pi) + \pi\varphi''(7)$
$w(\varphi) = \begin{cases} 1 \text{ if } \varphi(0)=0 \\ 0 \text{ otherwise} \end{cases}$
$t(\varphi) = \int \frac{\varphi(x)}{x^2} dx$


Which are distributions?
$u(\varphi) = \int (\varphi(x))^2 dx$
not linear
$v(\varphi) = \varphi(0) - \varphi'(\tilde{\pi}) + \tilde{\pi}\,\varphi''(7)$
$w(\varphi) = \begin{cases} 1 \text{ if } \varphi(0)=0 \\ 0 \text{ otherwise} \end{cases}$
$t(\varphi) = \int \frac{\varphi(x)}{x^2} dx$

## Which are distributions?

| | |
|---|---|
| $u(\varphi) = \int (\varphi(x))^2 dx$ | not linear |
| $v(\varphi) = \varphi(0) - \varphi'(\tilde{\pi}) + \tilde{\pi}\,\varphi''(7)$ | ✓ |
| $w(\varphi) = \begin{cases} 1 \text{ if } \varphi(0)=0 \\ 0 \text{ otherwise} \end{cases}$ | not linear, continuous |
| $t(\varphi) = \int \frac{\varphi(x)}{x^2} dx$   | not finite |


Functions that are locally integrable
$\int_a^b f(x)dx < \infty$ for all a, b
are distributions
$f(\phi) \equiv \int f(x)\,\varphi(x)dx$

# Distributional Derivatives

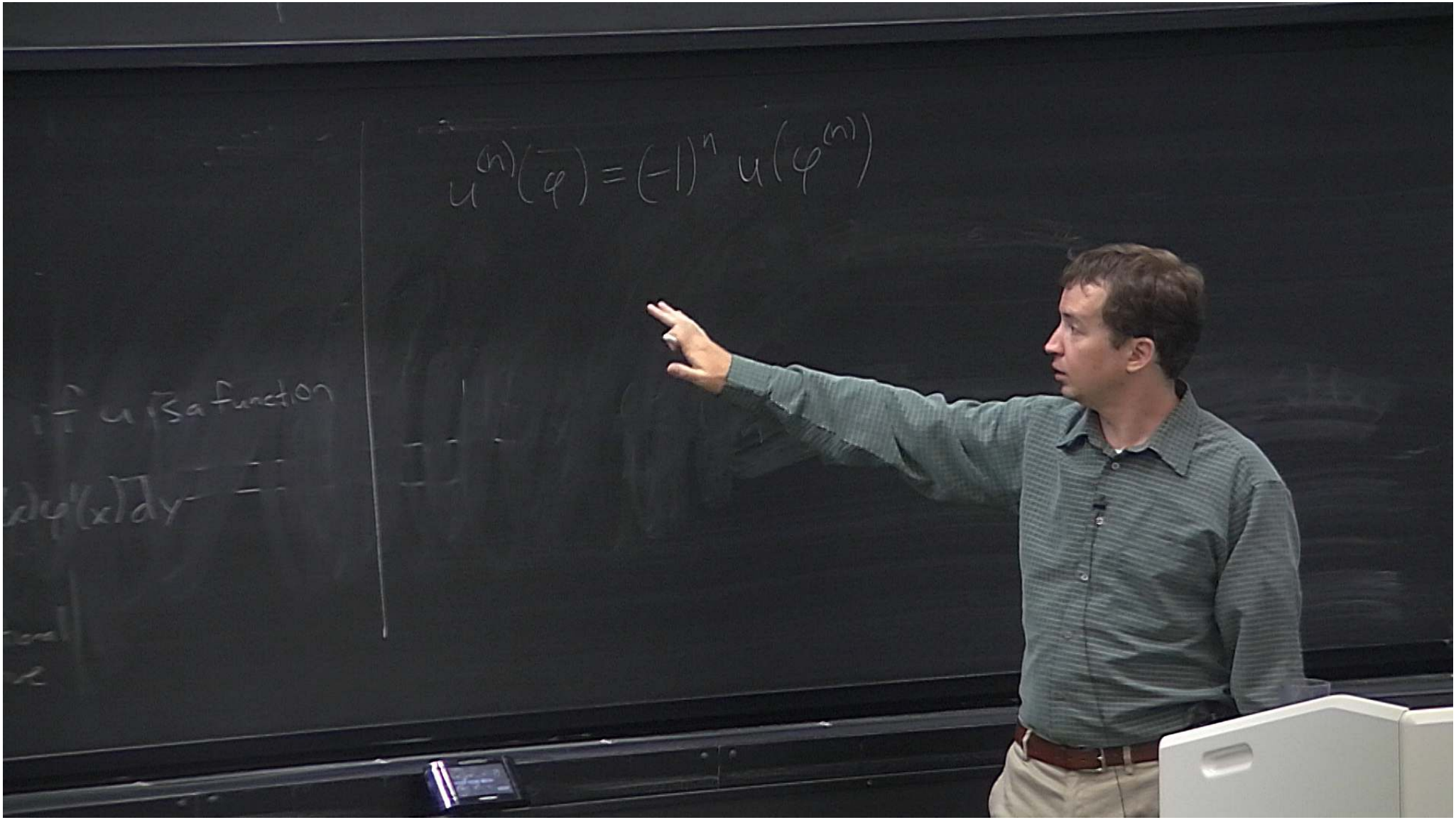
$$u'(x) \overset{?}{=} \lim_{\epsilon \to 0} \frac{u(x+\epsilon) - u(x)}{\epsilon}$$

$$u'(\varphi) = \int_{-\infty}^{\infty} u'(x)\varphi(x)\,dx \quad \text{if } u \text{ is a function}$$

$$= u(x)\varphi(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(x)\varphi'(x)\,dx$$

$$\boxed{u'(\varphi) \equiv -u(\varphi')}$$

weak or distributional derivative


$u^{(n)}(\varphi) = (-1)^n u(\varphi^{(n)})$
if u is a function

# Distributional Derivatives

$$u'(x) \overset{?}{=} \lim_{\epsilon \to 0} \frac{u(x+\epsilon) - u(x)}{\epsilon}$$

$$u'(\varphi) \equiv \int_{-\infty}^{\infty} u'(x)\varphi(x)\,dx \quad \text{if } u \text{ is a function}$$

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weak or distributional derivative