

Title: PSI 2019/2020 - Classical Physics - Lecture 4

Speakers: David Kubiznak

Collection: PSI 2019/2020 - Classical Physics (Kubiznak)

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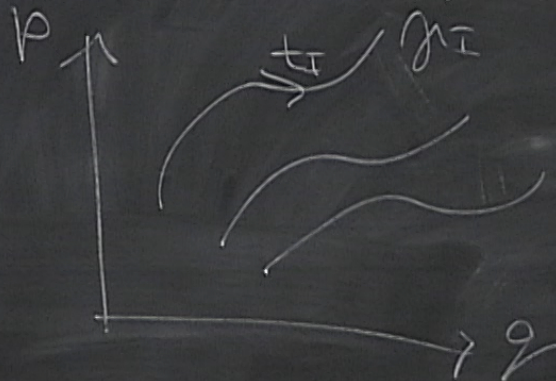
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f) HIDDEN SYMMETRIES

- GIVEN PHASE SPACE FUNCTION I , WE HAVE THE CORRESPONDING HAM. VECTOR FIELD

$$X_I = \{ \cdot, I \} = \frac{\partial I}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial I}{\partial q^i} \frac{\partial}{\partial p_i}$$

- GENERATES FLOW IN PHASE SPACE WITH TIME t_I .



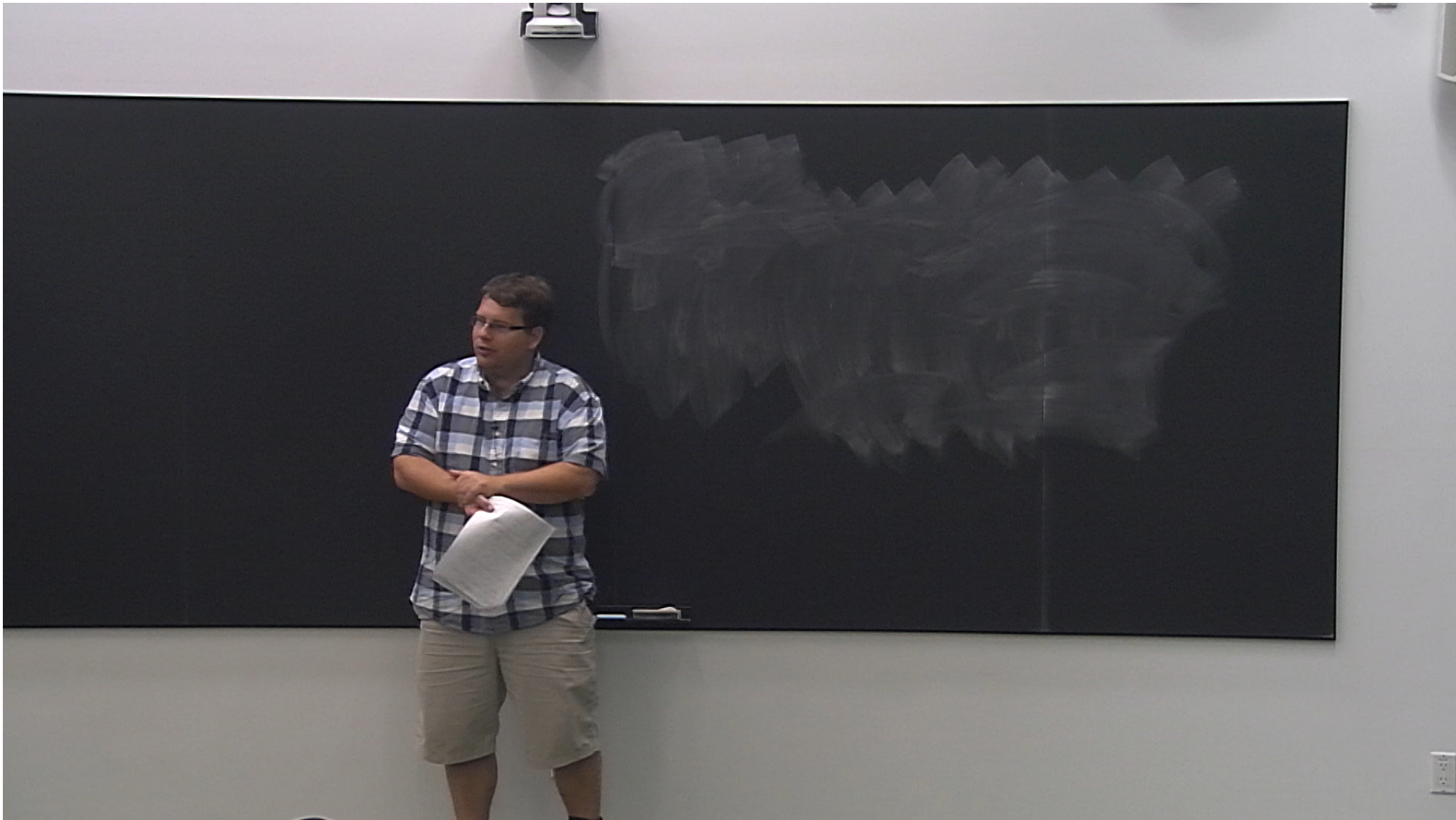
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- GENERATES FLOW IN PHASE SPACE WITH TIME t_I .





- LET THIS FLOW BE A SYMMETRY OF H :

$$\{H, I\} = \frac{dH}{dt} = 0$$

AT THE LEVEL OF PHASE SPACE

(MORE GENERAL THAN SYMMETRIES
OF CONFIGURATION SPACE)

- PHASE SPACE EQUIPPED WITH CANONICAL PROJECTION

$$\begin{aligned} \pi: (q, p) &\rightarrow q \\ \pi^*: X_I &\rightarrow \pi^*(X_I) = \left(\frac{\partial I}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial I}{\partial q^i} \frac{\partial}{\partial p_i} \right)_{X_I^i(q)} \end{aligned}$$

2 POSSIBILITIES:

$X_I^{(q)}$ $\left\{ \begin{array}{l} \text{FUNCTION OF } q \text{ ONLY} \dots \text{COMPONENTS OF VECTOR FIELD} \\ \text{ON CONF. SPACE} \\ \text{FUNCTION OF } p \text{ AS WELL} \dots \text{PROBLEM} \end{array} \right.$ ☹️

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EXPLICIT SYMMETRY ... ISOMETRY
(SEE THIS AS A FLOW IN CONF. SPACE)

HIDDEN SYMMETRY (GENUINE SYM. OF PHASE SPACE)
WHICH CANNOT BE SEEN IN CONF. SPACE

EXAMPLE OF HIDDEN SYMMETRY, $E = \frac{p^2}{2m}$ ENERGY.

$$X_E = \{ \cdot, E \} = \frac{\partial E}{\partial p} \frac{\partial}{\partial q} - \frac{\partial E}{\partial q} \frac{\partial}{\partial p} = \frac{p}{m} \frac{\partial}{\partial q} - \frac{\partial E}{\partial q} \frac{\partial}{\partial p}$$

$$\pi^*(X_E) = \frac{p}{m} \frac{\partial}{\partial q}$$

2 POSSIBILITIES

$X_I^{n(q)}$ $\left\{ \begin{array}{l} \text{FUNCTION OF } q \text{ ONLY} \dots \text{COMPONENTS OF VECTOR FIELD} \\ \text{ON CONF. SPACE} \end{array} \right.$

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(SEE THIS AS A FLOW IN CONF. SPACE)

$\left\{ \begin{array}{l} \text{FUNCTION OF } p \text{ AS WELL} \dots \text{PROBLEM} \end{array} \right.$

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$$\pi^*(X_E) = \left(\frac{p}{m} \frac{\partial}{\partial q} \right)_{f(p)} \dots \text{HIDDEN SYMMETRY}$$

2 POSSIBILITIES

$X_I^{1(q)}$ FUNCTION OF q ONLY

COMPONENTS OF VECTOR FIELD
ON CONF. SPACE

EXPLICIT SYMMETRY ... ISOMETRY
(SEE THIS AS A FLOW IN CONF. SPACE)

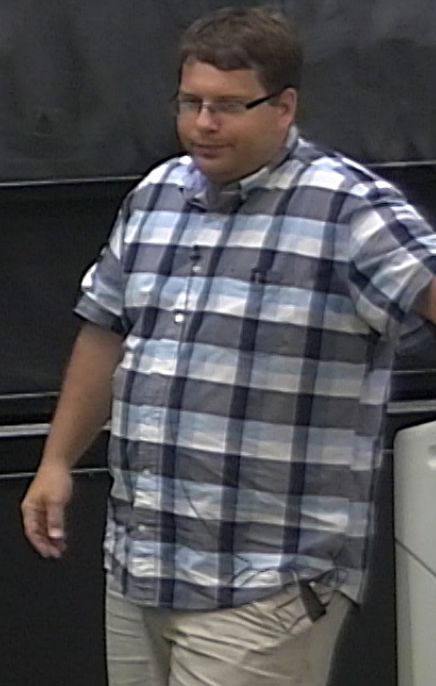
FUNCTION OF p AS WELL ... PROBLEM

HIDDEN SYMMETRY (GENUINE SYM. OF PHASE SPACE)
WHICH CANNOT BE SEEN IN CONF. SPACE

EXAMPLE OF HIDDEN SYMMETRY, $E = \frac{p^2}{2m}$ ENERGY.

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$$\Rightarrow \pi^*(X_E) = \left(\frac{p}{m} \frac{\partial}{\partial q} \right)_{f(p)} \dots \text{HIDDEN SYMMETRY}$$



PTS OF VECTOR FIELD
CONF. SPACE

EXPLICIT SYMMETRY ... ISOMETRY
(I SEE THIS AS A FLOW IN CONF. SPACE)

PROBLEM ☹

HIDDEN SYMMETRY (GENUINE SYM. OF PHASE SPACE)
WHICH CANNOT BE SEEN IN CONF. SPACE

ENERGY.

$$E = \frac{p^2}{2m} = \frac{\partial E}{\partial p} \frac{\partial p}{\partial q} = \frac{\partial E}{\partial q}$$

SYMMETRY

MOMENTUM CONSERVED $I = p$

$$X_p = \left\{ \cdot, p \right\} = \frac{\partial p}{\partial p} \frac{\partial}{\partial q} - \frac{\partial p}{\partial q} \frac{\partial}{\partial p} = \frac{\partial}{\partial q}$$
$$\Pi^* X_p = 1 \frac{\partial}{\partial q}$$

EXAMPLE KEPLER PROBLEM

- $\vec{F} = f(r) \hat{n} \dots$ CENTRAL FORCE

1 HIDDEN: $\vec{E}$

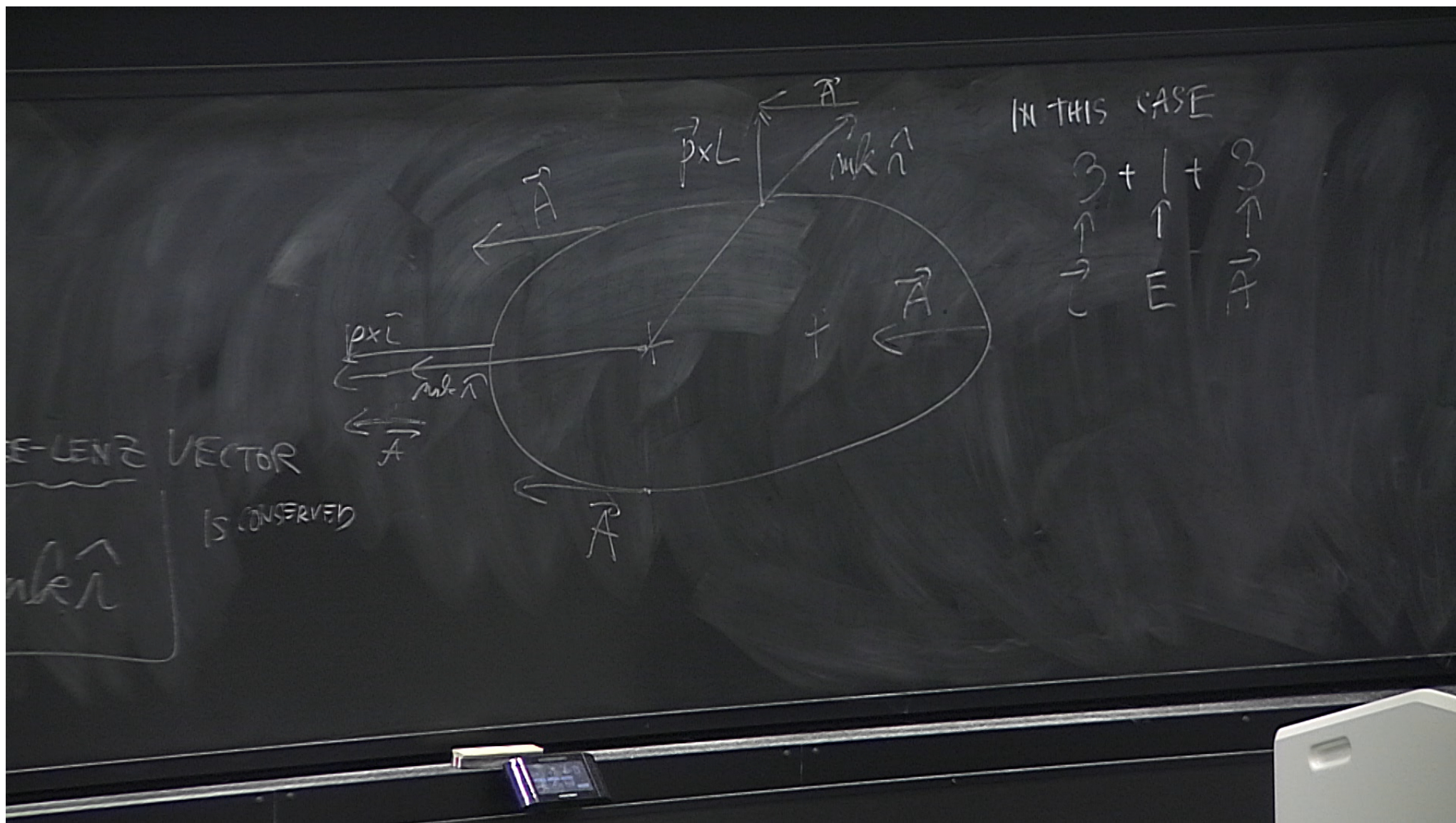
3 ISOMETRIES $\{L_x, L_y, L_z\} = \vec{L}$

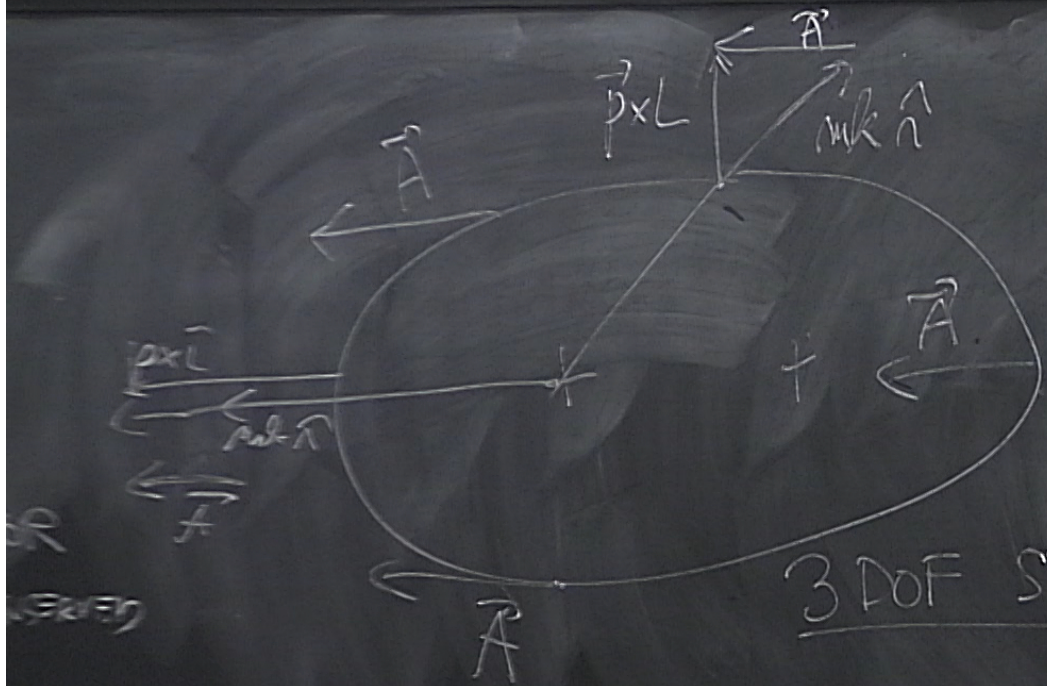
- $f = -\frac{k}{r^2}$

A NEW LAPLACE-RUNGE-LENZ VECTOR

$$\vec{A} = \vec{p} \times \vec{L} - mk\hat{n}$$

IS CONSERVED





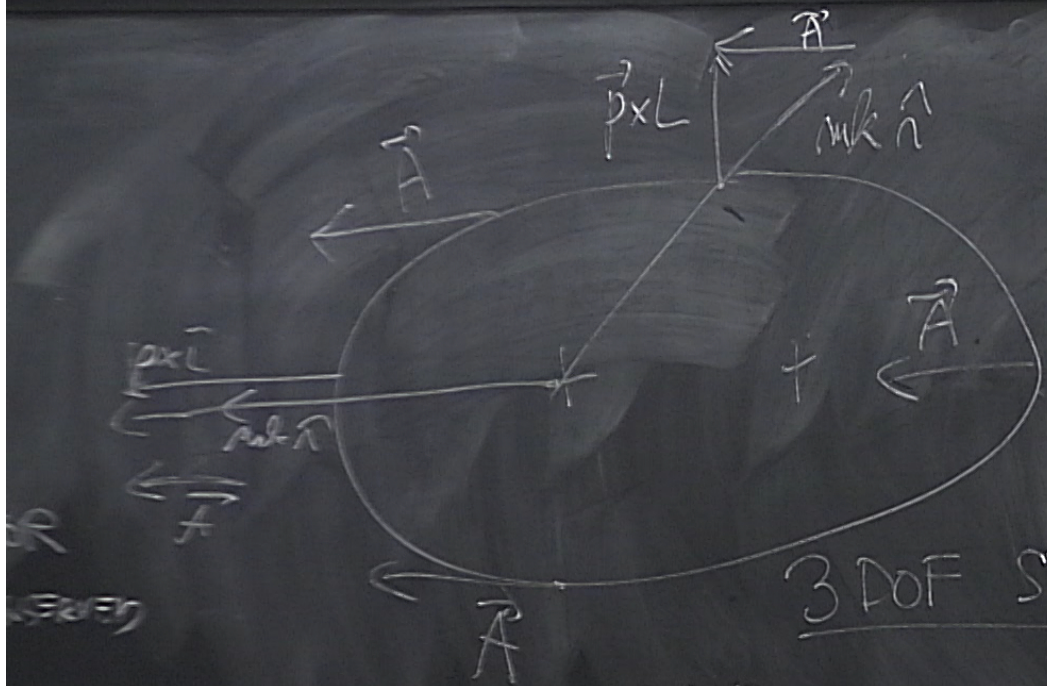
IN THIS CASE

$$3 + 1 + 3 - 2 = 5 \text{ INDEP}$$

$\uparrow$ $\uparrow$ $\uparrow$ $\uparrow$
 $\vec{L}$ E $\vec{A}$ CONSTRAINTS

$$\vec{A} \cdot \vec{L} = 0$$

$$\vec{A}^2 = m^2 k^2 + 2mEL^2$$



IN THIS CASE

$$\begin{array}{ccccccc}
 3 & + & 1 & + & 3 & - & 2 & = & 5 \text{ INDEP} \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 \vec{L} & & E & & \vec{A} & & \text{CONSTRAINTS} & &
 \end{array}$$

$$\vec{A} \cdot \vec{L} = 0$$

$$\vec{A}^2 = m^2 k^2 + 2mE\vec{L}^2$$

3 DOF SYSTEM

$$2n-1 = 5 \text{ MAXIMALLY SUPERINTEGRABLE}$$

FOR COMPLETE INTEGRALS,
n INTEGRALS (n-DOF)



9) CONSTRAINTS

- LET NOT ALL q, p BE INDEPENDENT, THAT IS AT ANY t WE HAVE A CONSTRAINT

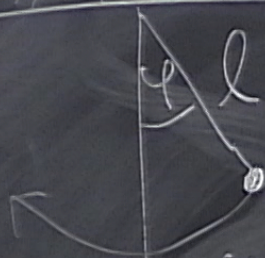
$$\boxed{\phi(q^i, p_i) = 0}$$

- SINCE THIS IS TRUE FOR ALL t IT IS A CONSERVED QUANTITY.

$\Rightarrow$ EXTRA SYMMETRY

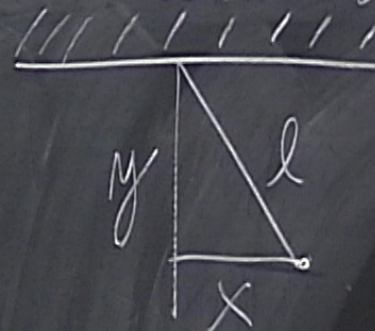
EXAMPLE:

CLEVER DESCR. - OCCAM



- $n=1$ (φ)
- PHASE SPACE (φ, p_φ)
- NO CONSTRAINTS

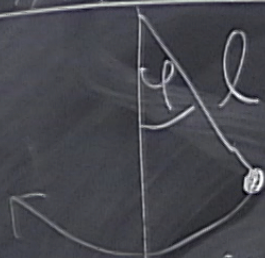
CONSTRAINED SYSTEM



- $n=2$

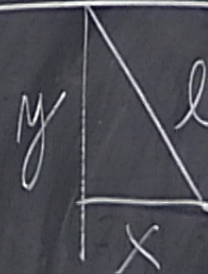
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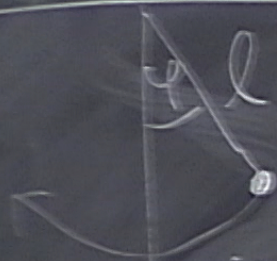
CONSTRAINED SYSTEM



- $n=2$
- PHASE SPACE (x, y, p_x, p_y)
- CONSTRAINT

$$\Phi = x^2 + y^2 - l^2 = 0$$

CLEVER DESCR. - OCCAM



$n=1$ (φ)

PHASE SPACE

NO CONSTRAINTS

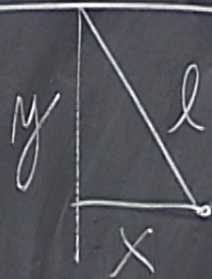
(φ, p_φ)

ELIMINATE

$$x = l \cos \varphi$$

$$y = l \sin \varphi$$

CONSTRAINED SYSTEM



$n=2$

PHASE SPACE (x, y, p_x, p_y)

CONSTRAINT

$$\Phi = x^2 + y^2 - l^2 = 0$$

2 REASONS

• EXTRA SYMMETRY

• CAN CALCULATE CONSTRAINT
FORCE

• TO DEAL WITH CONSTRAINTS -- USE METHOD OF LAGRANGE MULTIPLIERS

$$\underset{\text{TOTAL}}{\overset{\uparrow}{H}} = \underset{\text{ORIGINAL}}{H} + \sum_{j=1}^m \underset{\substack{\uparrow \text{LAGRANGE} \\ \text{MULT.}}}{\lambda_j(H)} \phi_j$$

CE
POSS. I: ENLARGE PHASE SPACE ... $(N+M)$ DOF.
 λ_J TREATED AS NEW COORDINATES.

AT LAGRANGIAN LEVEL $L_T = L - \lambda_J \phi_J$

$$\delta S_T = \int (\delta L - \lambda_J \delta \phi_J - \phi_J \delta \lambda_J) dt = \int \left[\underbrace{\left(\frac{\partial L_T}{\partial \dot{q}^i} - \frac{d}{dt} \left(\frac{\partial L_T}{\partial \dot{q}^i} \right) \right)}_0 \delta q^i - \underbrace{\phi_J \delta \lambda_J}_0 \right] dt$$

(E-L) $\frac{\partial L_T}{\partial \dot{q}^i} - \frac{d}{dt} \left(\frac{\partial L_T}{\partial \dot{q}^i} \right) = 0, \quad \phi_J = 0$

RANCE

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(E-L) $\left(\frac{\partial L_T}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L_T}{\partial \dot{q}_i} \right) = 0, \phi_J = 0 \right) \rightarrow \text{SOLVE FOR } \left[\frac{\partial}{\partial q_i} \right]$

$\downarrow_n$ $\downarrow_m$

$$\text{IF } \phi(q_i, \dot{q}_i) = 0$$

$$\frac{\partial L}{\partial q_i} - \lambda \frac{\partial \phi}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0$$

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = \lambda \frac{\partial \phi}{\partial q_i} = F_i$$

GENERALIZED FORCE
OF CONSTRAINT

IT IS A CONSERVED QUANTITY.

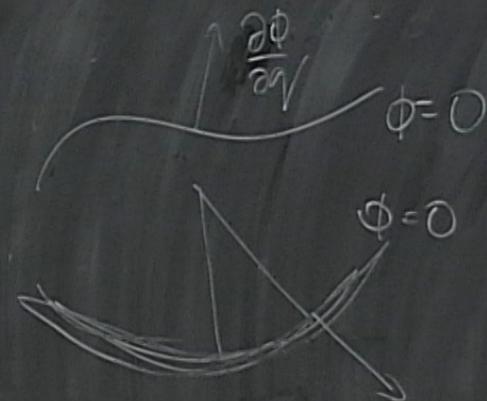
→ Extra

IF $\phi(q_i, \dot{q}_i) = 0$

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GENERALIZED FORCE
OF CONSTRAINT



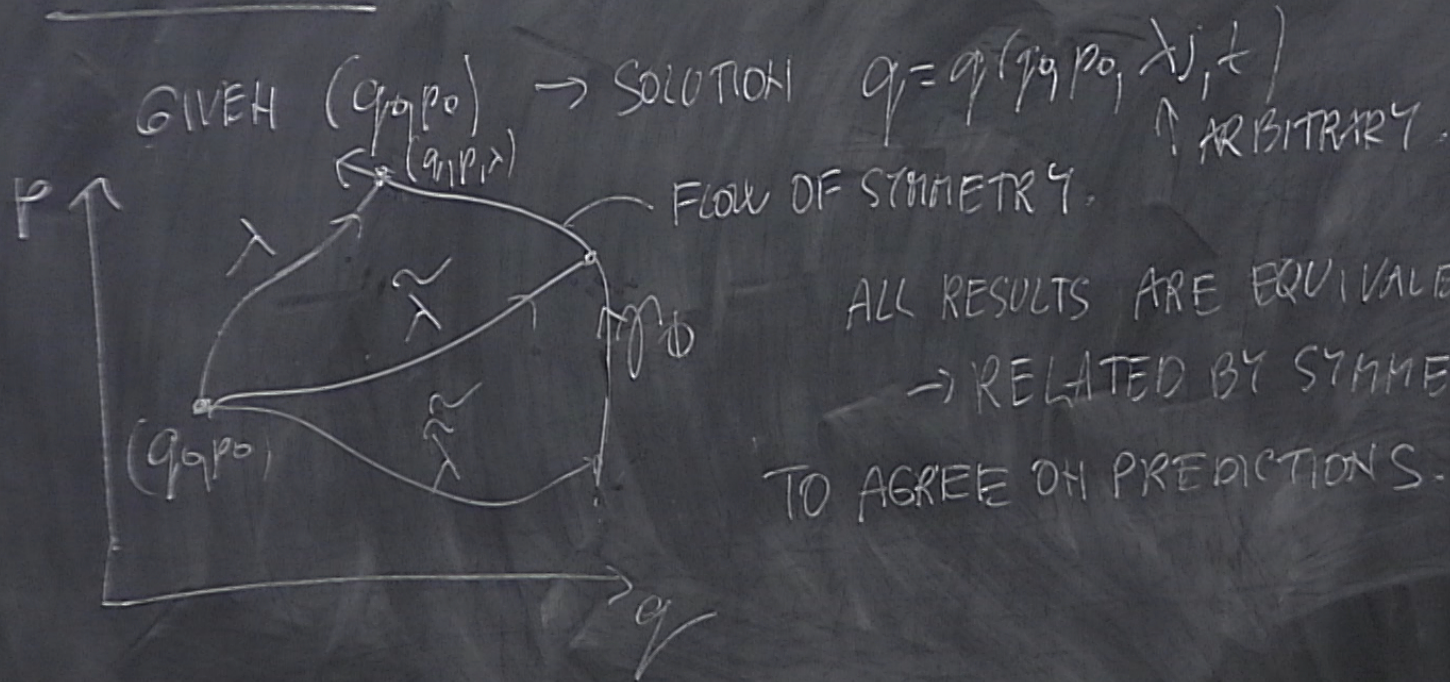
EXPLICIT SYMMETRY
(SEE THIS)

EN SYMMETRY

$$-\frac{\partial E}{\partial q} \frac{\partial L}{\partial R}$$

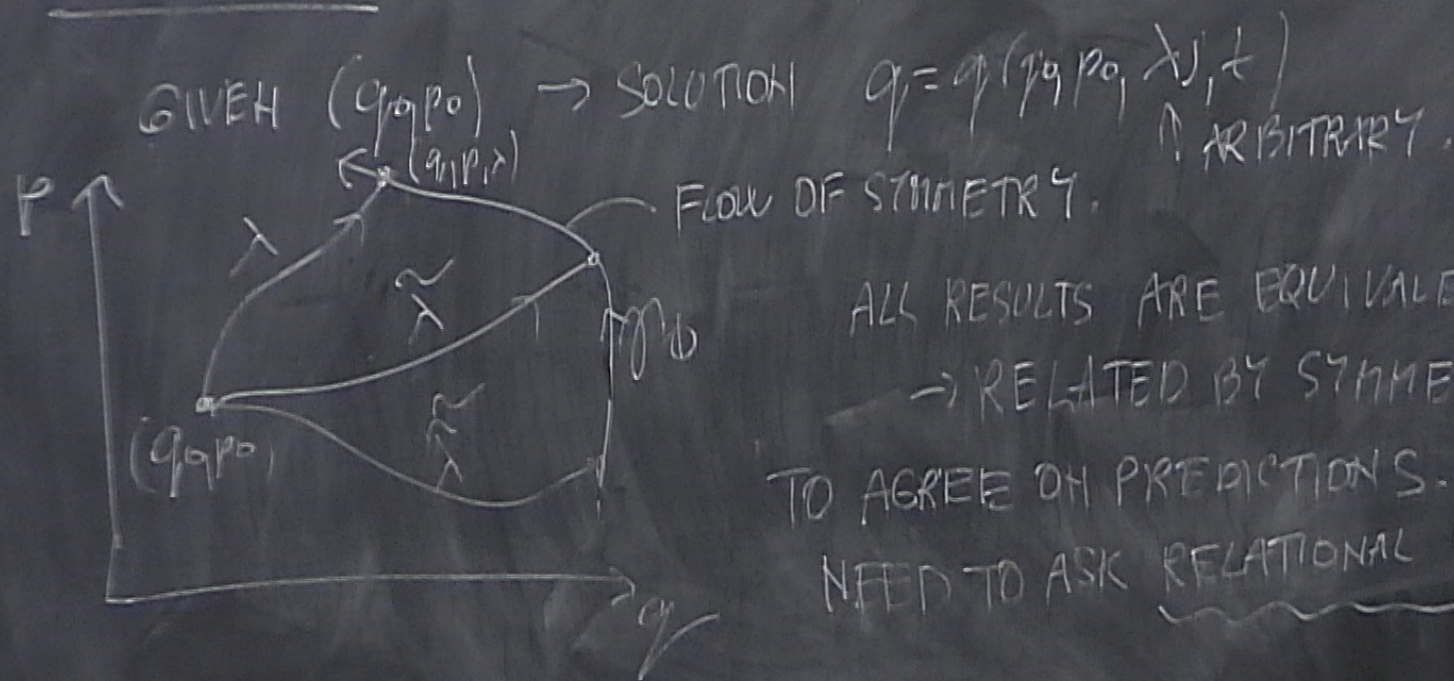
TO CONSTRAINTS

• POSSIBILITY 2: LIVE WITH λ_j AS ARBITRARY FUNCTIONS?



ALL RESULTS ARE EQUIVALENT
 $\rightarrow$ RELATED BY SYMMETRY.
 TO AGREE ON PREDICTIONS.

• POSSIBILITY 2: LIVE WITH λ_j AS ARBITRARY FUNCTIONS!



ALL RESULTS ARE EQUIVALENT
 $\rightarrow$ RELATED BY SYMMETRY.
 TO AGREE ON PREDICTIONS.
 NEED TO ASK RELATIONAL QUESTIONS

LAGRANGE

POSS. I: ENLARGE PHASE SPACE ... $(N+M)$ DOF.
 λ_j TREATED AS NEW COORDINATES.

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$$(E-L) \quad \left(\frac{\partial L_T}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L_T}{\partial \dot{q}_i} \right) = 0, \quad \phi_j = 0 \right) \rightarrow \text{SOLVE FOR } \left[\frac{\partial}{\partial q_i}, \lambda_j \right]$$

$\downarrow n$ $\downarrow m$

• ONE SOURCE OF CONSTRAINTS

$L \rightarrow H$ LEGENDRE.

$$p = \frac{\partial L}{\partial \dot{q}}$$

$$H = p\dot{q} - L \quad / \quad \dot{q} = \dot{q}(q, p)$$

INVERT $\dot{q} = \dot{q}(q, p)$

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$L \rightarrow H$ LEGENDRE

$$p = \frac{\partial L}{\partial \dot{q}}$$

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INVERT $\dot{q} = \dot{q}(q, p)$

MAY NOT BE POSSIBLE

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$$

1) INTEGRABLE SYSTEMS

DEF: DYN. SYSTEM WITH n DOF IS
COMPLETELY INTEGRABLE IF IT POSSESSES

n GLOBAL, INDEPENDENT INTEGRALS
OF MOTION $F_i(q, p) = f_i$, $\{H, F_i\} = 0$,
THAT ARE IN INVOLUTION

$$\{F_i, F_j\} = 0 \quad \forall i, j$$

f) INTEGRABLE SYSTEMS

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THAT ARE IN INVOLUTION

$$\{F_i, F_j\} = 0 \quad \forall i, j$$

LIOUVILLE'S THEOREM. SOLUTION FOR A COMP. INT.
SYSTEM CAN BE OBTAINED BY

QUADRATURE (FINITE # OF INTEGRAT.)
& ALGEBRAIC OPS.

* PROOF IS CONSTRUCTIVE

| INTEGRAL + ALG. OPS

$$(q, p) \rightarrow (F, \psi)$$

↑ ↑
INT. OF MOT. ANGLE VAR.