Title: PSI 2019/2020 - Classical Physics - Lecture 2

Speakers: David Kubiznak

Collection: PSI 2019/2020 - Classical Physics (Kubiznak)

Date: August 16, 2019 - 9:00 AM

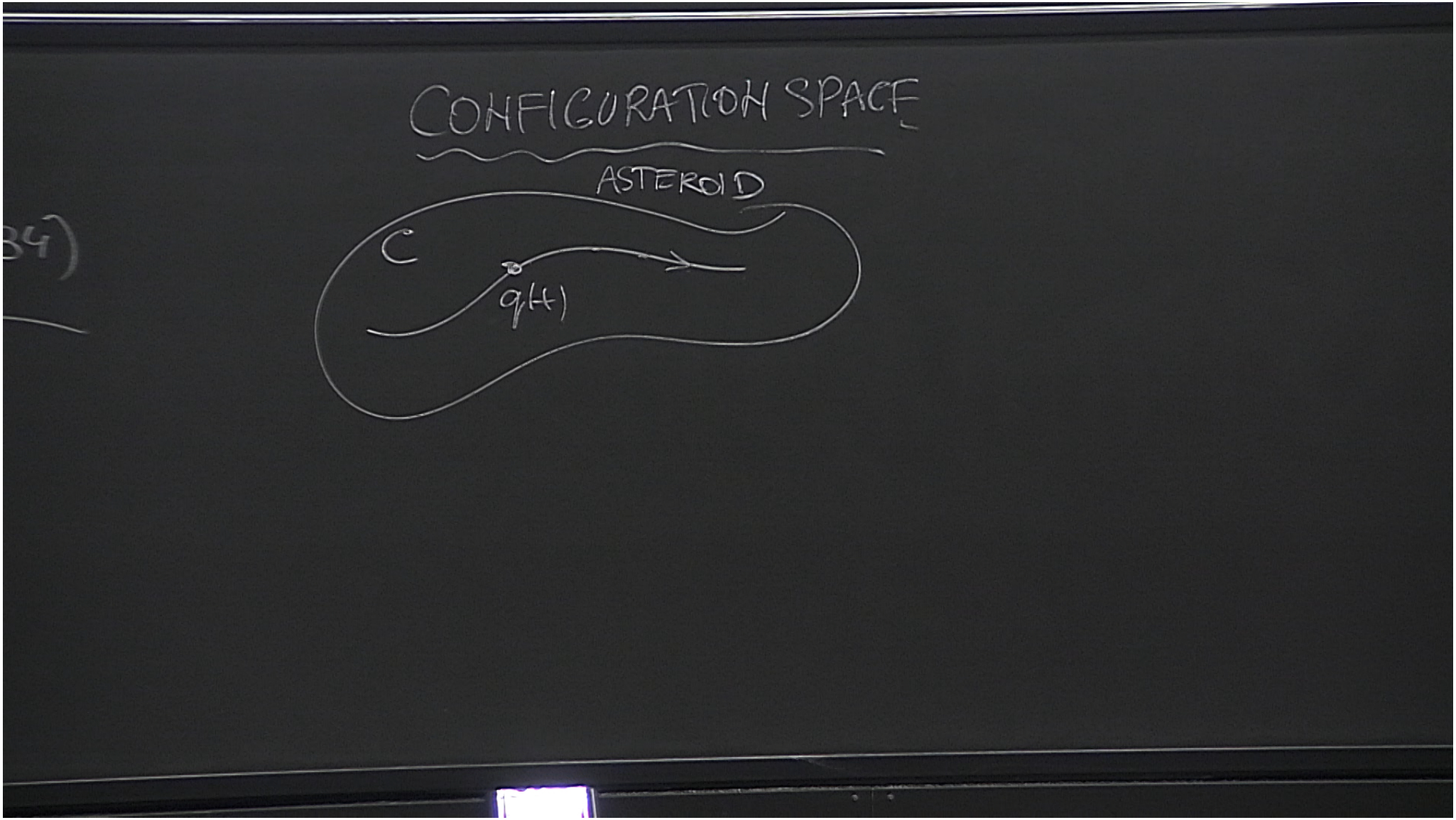
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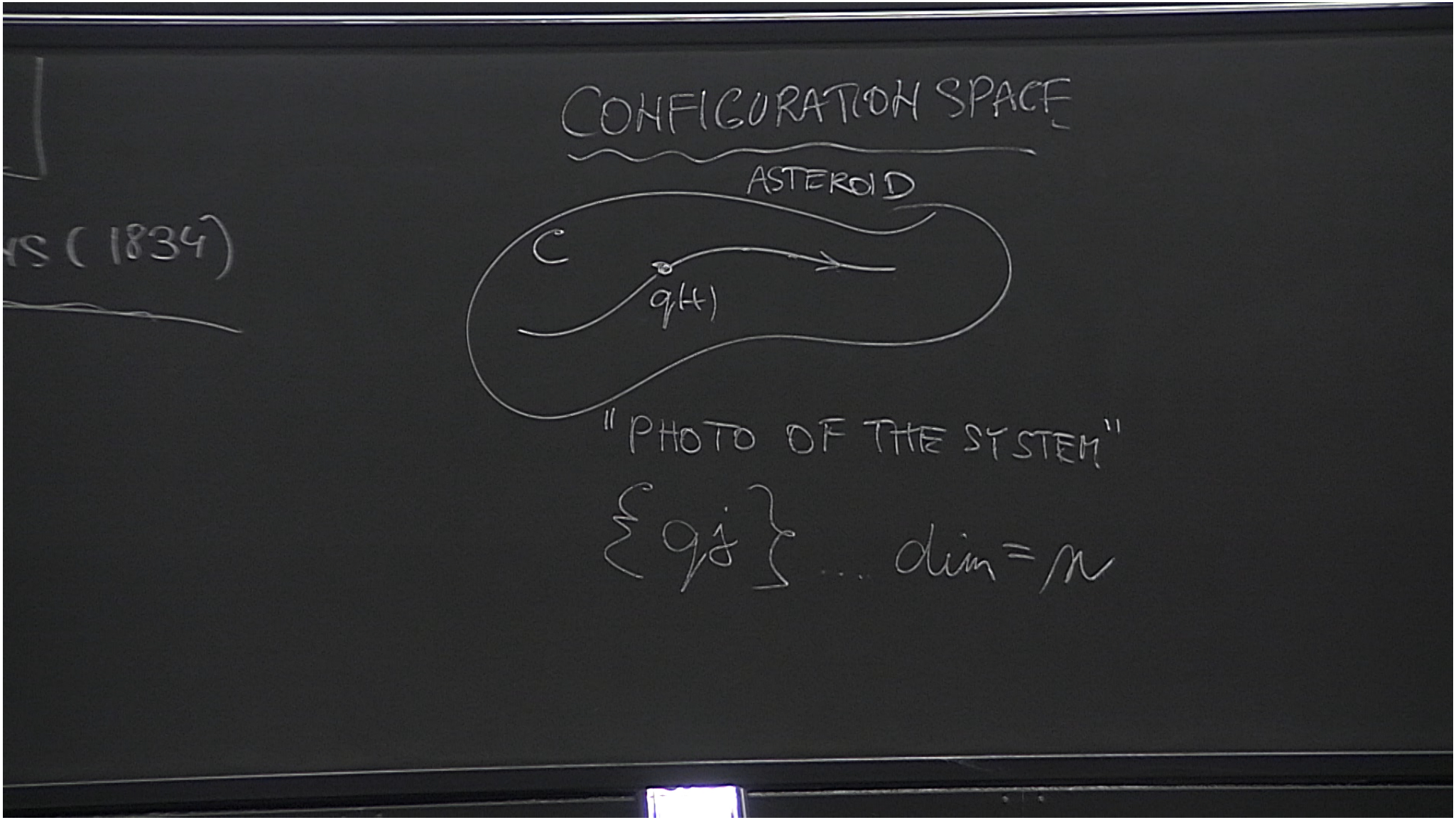
2) HAMILTONIAN MECHANICS

a) HAMILTON'S CANONICAL EQUATIONS (1834)

- CANONICAL MOMENTUM

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$


CONFIGURATION SPACE
ASTEROID
C
q(t)


CONFIGURATION SPACE
ASTEROID
C
q(t)
"PHOTO OF THE SYSTEM"
{qi} ... dim = n

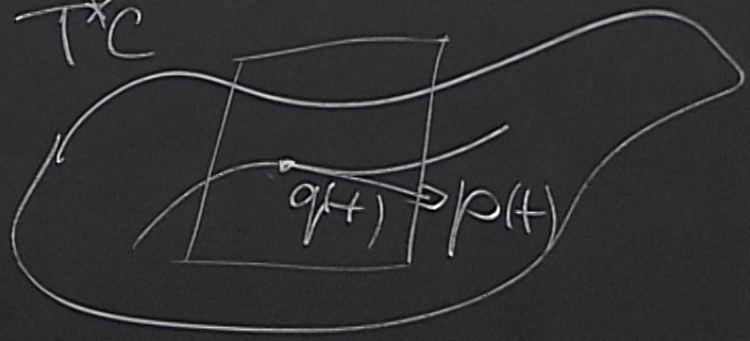
SPACE : C

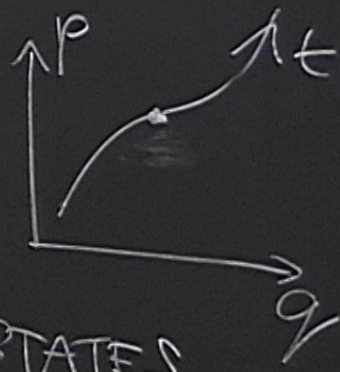
E SYSTEM"

dim = n

PHASE SPACE : T^*C

T^*C





DESCRIBES PHYSICAL STATES

$\{q^i, p_i\}$ dim = 2n

2) HAMILTONIAN MECHANICS

a) HAMILTON'S CANONICAL EQUATIONS (1834)

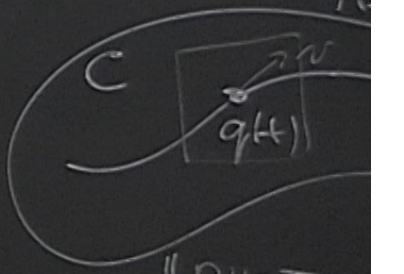
- CANONICAL MOMENTUM

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

IF INVERSION POSSIBLE

$$\dot{q}_i = \dot{q}_i(q, p)$$

CONFIGU



"PHOTO

$\{q_j\}$

- HAMILTONIAN = LEGENDRE TRANSF OF LAGRANGIAN

$$H(q^i, p_i, t) = p_j \dot{q}^j - L \qquad \dot{q}^i = \dot{q}^i(q^j, p_j, t)$$

TH: SYSTEM OF n (E-L) EQUATIONS IS EQUIVALENT TO $2n$ FIRST-ORDER HAMILTON'S EQUATIONS

$$\dot{p}_j = -\frac{\partial H}{\partial q^j}, \qquad \dot{q}^j = \frac{\partial H}{\partial p_j}$$

- PROOF ACTION PRINCIPLE

$$0 = \delta S = \delta \int_1^2 L\,dt = \delta \int_1^2 (p\dot{q} - H)\,dt$$

$$= \int_1^2 \left(\delta p \dot{q} + p \delta \dot{q} - \frac{\partial H}{\partial q}\delta q - \frac{\partial H}{\partial p}\delta p \right) dt$$

BY PARTS $=$

$$\int_1^2 \left[\delta p \left(\dot{q} - \frac{\partial H}{\partial p} \right) + \delta q \left(-\frac{\partial H}{\partial q} - \dot{p} \right) \right] dt + \underbrace{\left[p \delta q \right]_1^2}_{0}$$

$$= \int_1^2 \left(\delta p \dot{q} + p \delta \dot{q} - \frac{\partial H}{\partial q} \delta q - \frac{\partial H}{\partial p} \delta p \right) dt$$

BY PARTS $$= \int_1^2 \left[\delta p \left(\dot{q} - \frac{\partial H}{\partial p} \right) + \delta q \left(-\frac{\partial H}{\partial q} - \dot{p} \right) \right] dt + \left[p \delta q \right]_1^2$$

$\dot{q} - \frac{\partial H}{\partial p} = 0$ $\quad -\frac{\partial H}{\partial q} - \dot{p} = 0$ $\quad p\delta q = 0$

$p_j = \frac{\partial g_8}{\partial q_j}$; $q_j = \frac{\partial g_8}{\partial p_j}$

REMARK: $\frac{dH}{dt} = \frac{\partial H}{\partial q_k}\dot{q} + \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$

$\frac{\partial H}{\partial p}$ $\quad -\frac{\partial H}{\partial q}$

IF $H \neq H(t)$ ... H .. INTEGRAL OF MOTION.

b) POISSON BRACKETS

DEF: LET f, g ARE PHASE SPACE FUNCTIONS => CAN. POISSON BRACKET

IS A NEW PHASE SPACE F.

$p_j = -\frac{\partial g_k}{\partial q^j}$; $q^j = \frac{\partial g_k}{\partial p_j}$

REMARK: $\frac{dH}{dt} = \frac{\partial H}{\partial q^k}\dot{q}^k + \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$

$\frac{\partial H}{\partial p}$ $\quad -\frac{\partial H}{\partial q}$

IF $H \neq H(t)$ ... H .. INTEGRAL OF MOTION.

b) POISSON BRACKETS

DEF: LET f, g ARE PHASE SPACE FUNCTIONS $\Rightarrow$ CAN. POISSON BRACKET IS A NEW PHASE SPACE F.

$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}$$

PROPS: a) FUNDAMENTAL BRACKETS

$$\{q^i, q^j\} = 0 = \{p_i, p_j\} \qquad \{q^i, p_j\} = \delta^i_j$$

BASIS FOR QUANTIZATION

i) ANTISYMMETRY $\{f, g\} = -\{g, f\}$, R

iii) LINEARITY $\{\alpha f + \beta g, h\} = \alpha\{f, h\} + \beta\{g, h\}$

iv) LEIBNITZ $\{fg, h\} = f\{g, h\} + \{f, h\}g$

POISSON BRACKET

$$\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i}$$

v) JACOBI $\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0$

REMARKS: $\frac{df}{dt} = \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p} + \frac{\partial f}{\partial t} = \{f,H\} + \frac{\partial f}{\partial t}$

($\dot{q} \to \frac{\partial H}{\partial p}$, $\dot{p} \to -\frac{\partial H}{\partial q}$)

- INTEGRAL OF MOTION $\boxed{\{I,H\} = 0}$

$\{\ \} = 0$

- LET I_1, I_2 BE INTEGRALS OF MOTION

$\{I_1, I_2\}$ IS ALSO INTEGRAL OF MOTION.

IN PRINCIPLE CAN GENERATE LOADS OF INTEGRALS OF MOTION

c) CANONICAL TRANSFORMATIONS (USE OF FREEDOM)

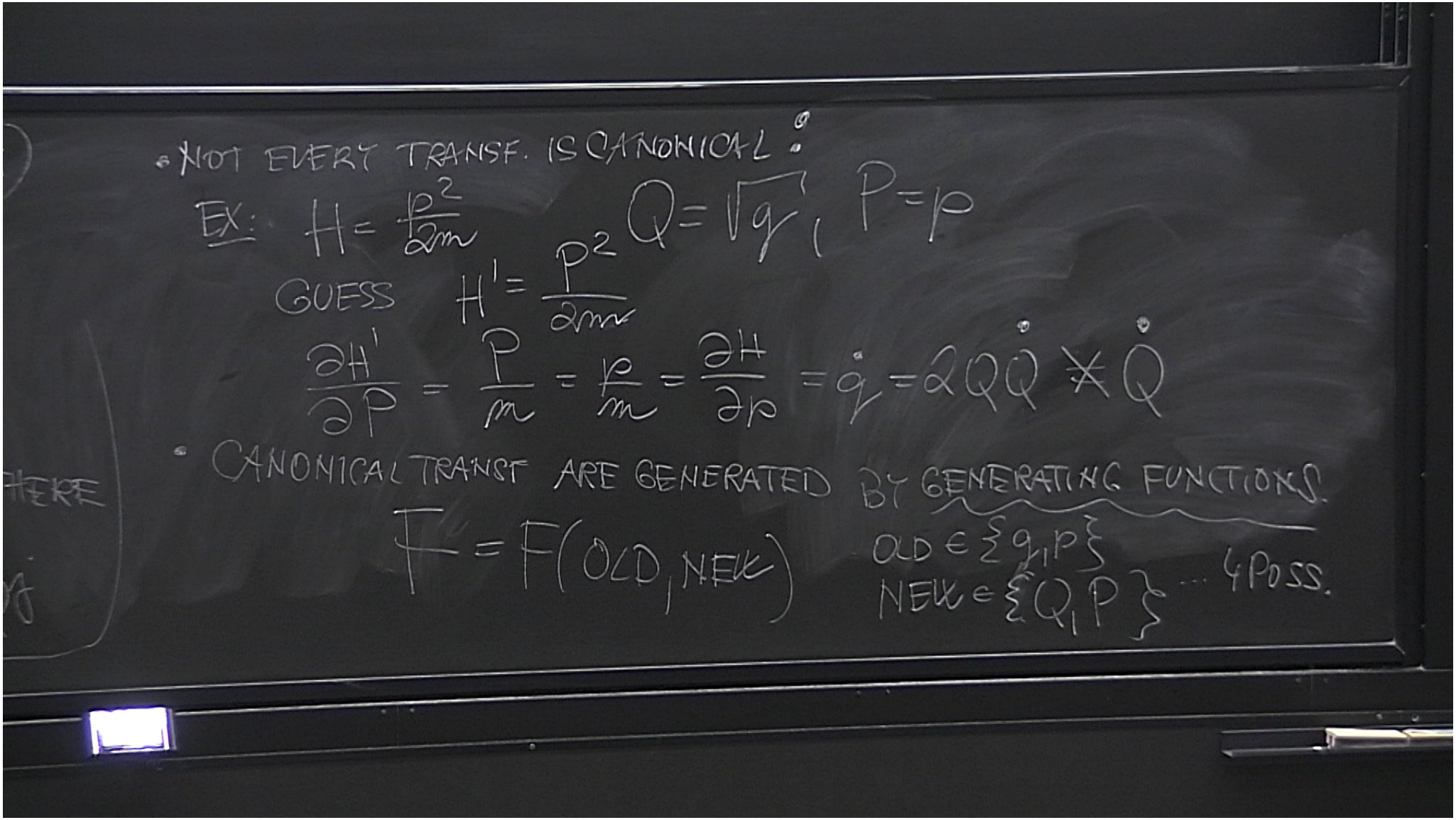
DEF: A CANONICAL TRANSF

$$Q^j = Q^j(q, p), \quad P_j = P_j(q, p)$$

(OLD)

PRESERVES THE FORM OF HAMILTON'S EQS. THERE EXISTS $H'(Q, P, t)$

$$\frac{\partial H'}{\partial Q^j} = -\dot{P}_j, \quad \frac{\partial H'}{\partial P_j} = \dot{Q}^j$$


• NOT EVERY TRANSF. IS CANONICAL!
EX: $H = \frac{p^2}{2m}$ $Q = \sqrt{q}$, $P = p$
GUESS $H' = \frac{P^2}{2m}$
$\frac{\partial H'}{\partial P} = \frac{P}{m} = \frac{p}{m} = \frac{\partial H}{\partial p} = \dot{q} = 2Q\dot{Q} \neq \dot{Q}$
• CANONICAL TRANST ARE GENERATED BY GENERATING FUNCTIONS.
$F = F(\text{OLD}, \text{NEW})$
OLD $\in \{q, p\}$
NEW $\in \{Q, P\}$ ... 4 Poss.

LET'S CHOOSE $F = F(q, Q, t)$ (WLOG)

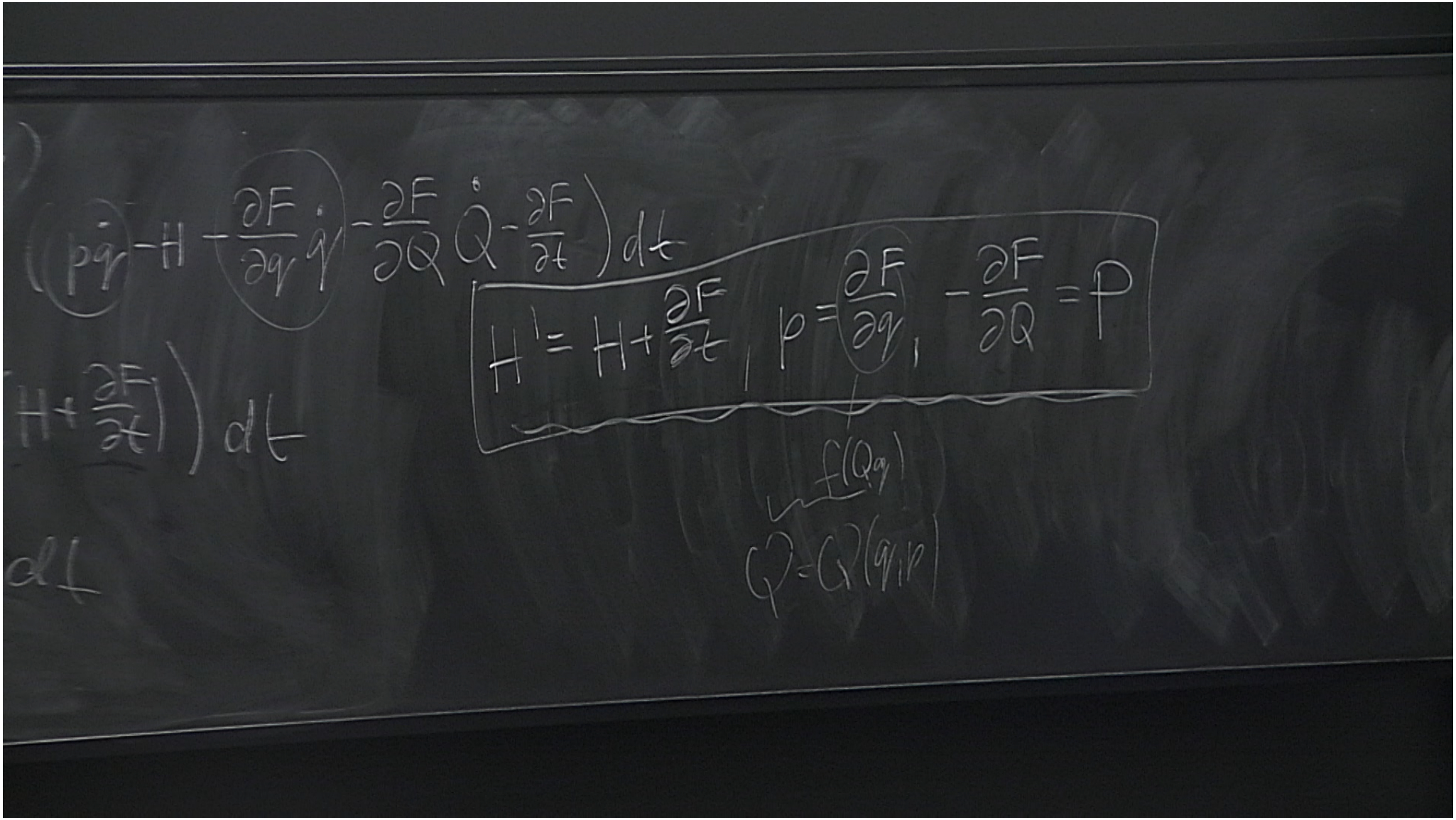
$$S = \int \left(p\dot{q} - H - \frac{dF}{dt} \right) dt = \int \left(p\dot{q} - H - \frac{\partial F}{\partial q}\dot{q} - \frac{\partial F}{\partial Q}\dot{Q} - \frac{\partial F}{\partial t} \right) dt$$

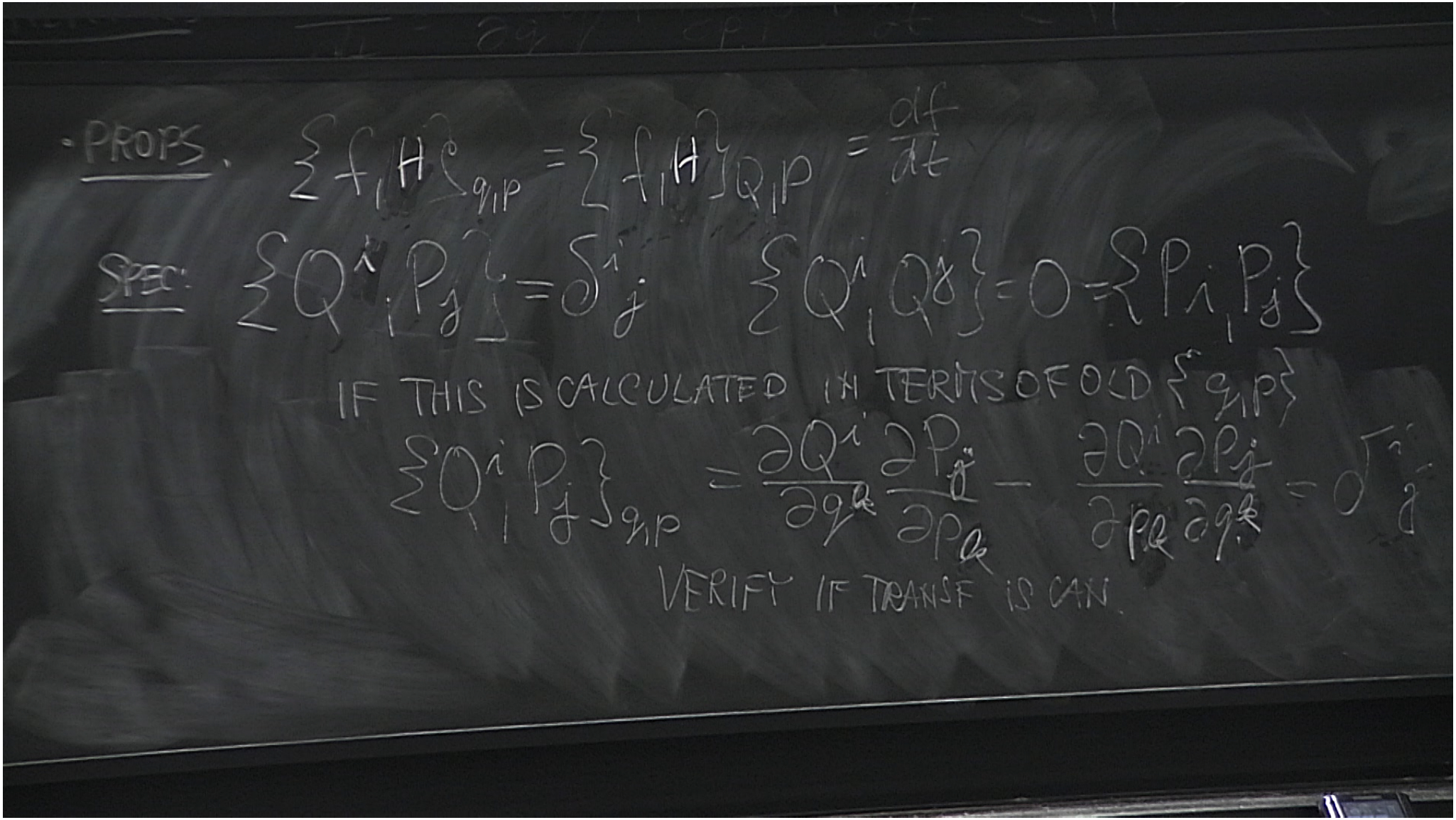
↑ FREEDOM

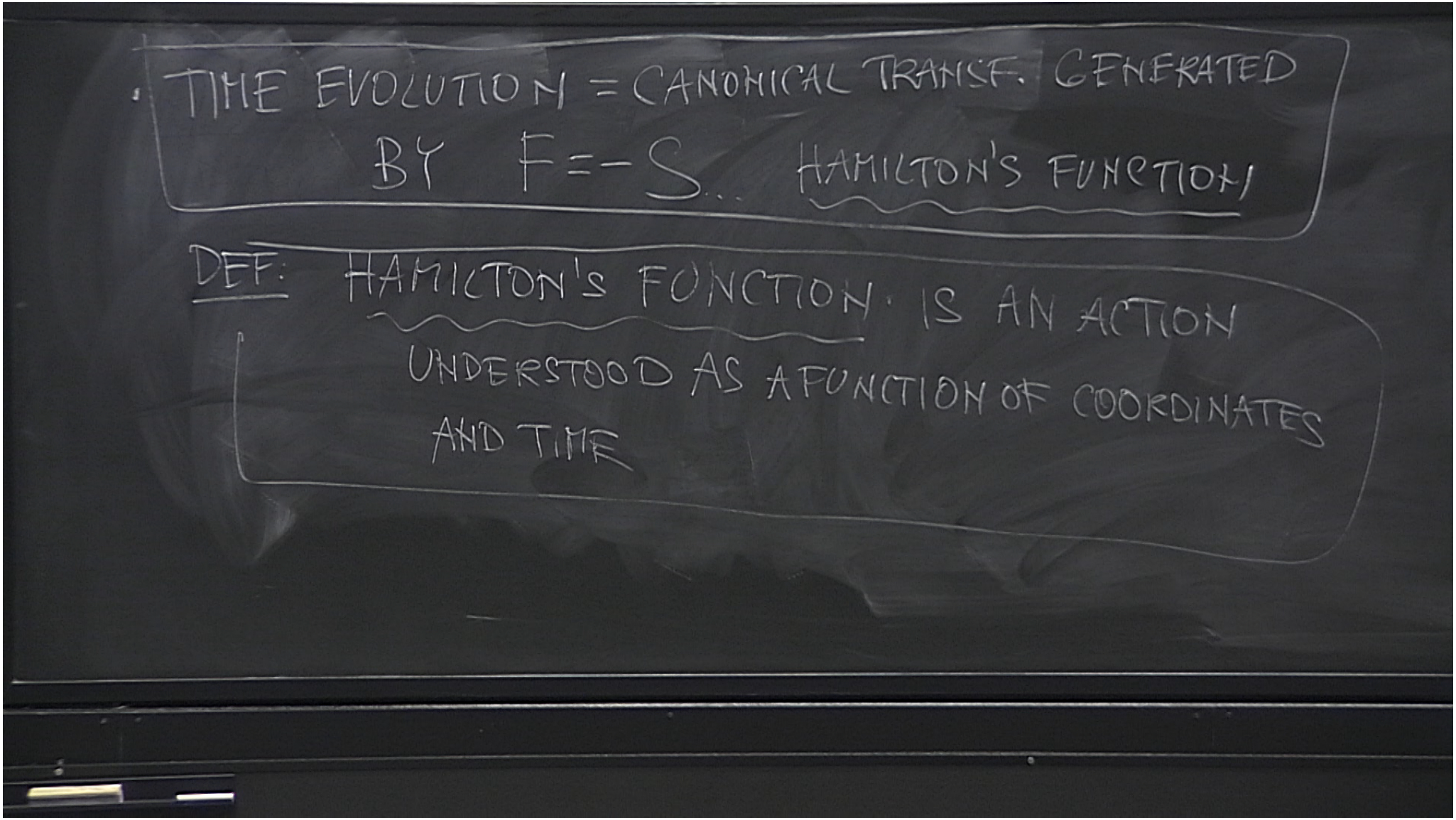
$$= \int \left(\dot{q}\left(p - \frac{\partial F}{\partial q}\right) - \frac{\partial F}{\partial Q}\dot{Q} - \left(H + \frac{\partial F}{\partial t}\right) \right) dt$$

INVARIANCE IN FORM = DEMAND

$$\int \left(P\dot{Q} - H'(P, Q, t) \right) dt$$


$\left(p\dot{q} - H - \frac{\partial F}{\partial q}\dot{q} - \frac{\partial F}{\partial Q}\dot{Q} - \frac{\partial F}{\partial t}\right)dt$
$H' = H + \frac{\partial F}{\partial t}$, $p = \frac{\partial F}{\partial q}$, $-\frac{\partial F}{\partial Q} = P$
$\left(H + \frac{\partial F}{\partial t}\right)dt$
dL
$f(Q,q)$
$Q = Q(q,p)$

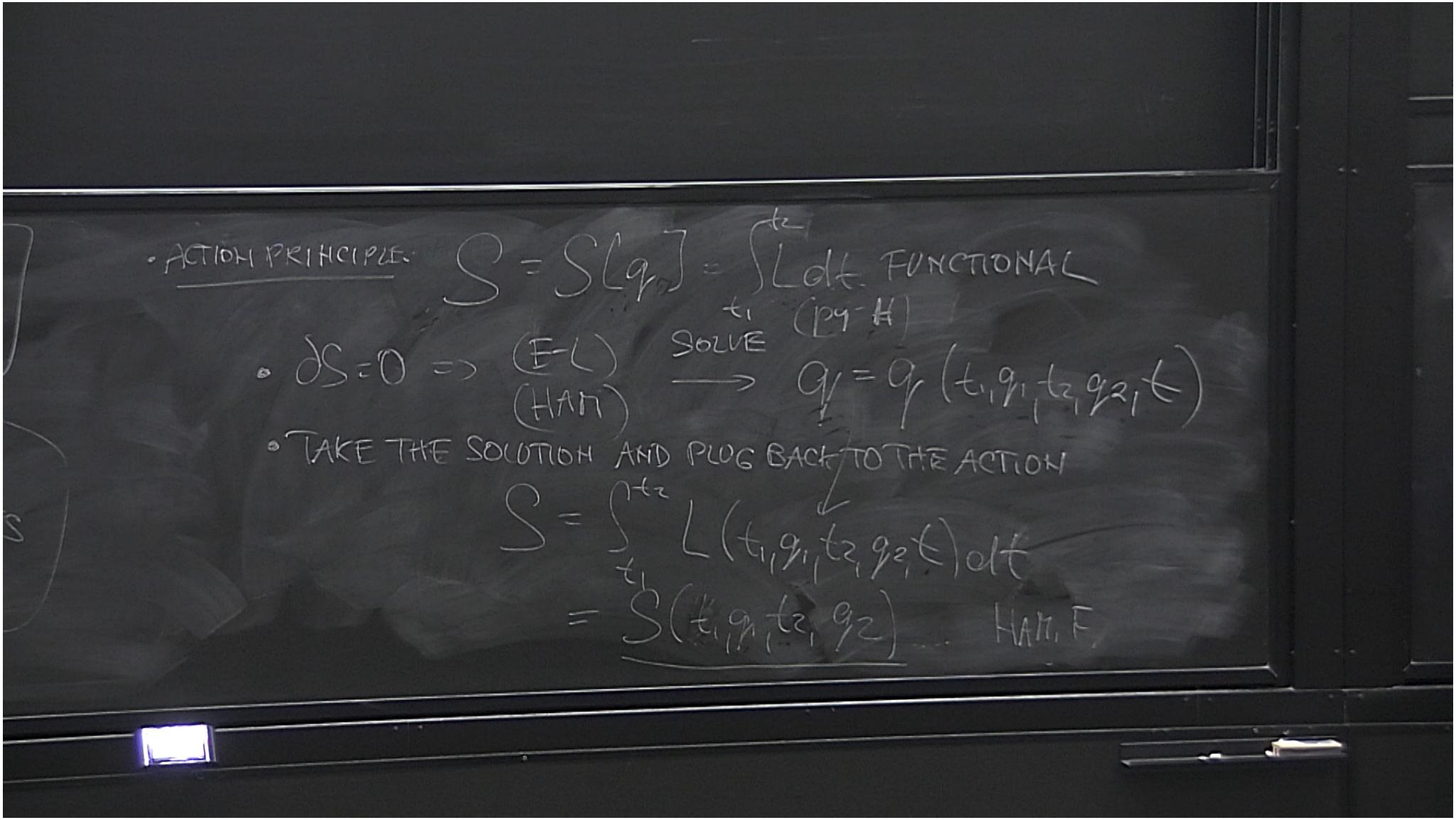

• PROPS.
$\{f, H\}_{q,p} = \{f, H\}_{Q,P} = \frac{df}{dt}$
SPEC:
$\{Q^i, P_j\} = \delta^i_j \quad \{Q^i, Q^j\} = 0 = \{P_i, P_j\}$
IF THIS IS CALCULATED IN TERMS OF OLD $\{q,p\}$
$\{Q^i, P_j\}_{q,p} = \frac{\partial Q^i}{\partial q^k}\frac{\partial P_k}{\partial p_j} - \frac{\partial Q^i}{\partial p_k}\frac{\partial P_k}{\partial q^j} = \delta^i_j$
VERIFY IF TRANSF IS CAN.


TIME EVOLUTION = CANONICAL TRANSF. GENERATED
BY F=-S... HAMILTON'S FUNCTION
DEF: HAMILTON'S FONCTION · IS AN ACTION
UNDERSTOOD AS A FUNCTION OF COORDINATES
AND TIME

- ACTION PRINCIPLE: $S = S[q] = \int_{t_1}^{t_2} L\,dt$ FUNCTIONAL
- $\delta S = 0 \Rightarrow$ (E-L) / (HAM) $\xrightarrow{\text{SOLVE}}$ $q_i = q_i(t_1, q_1, t_2, q_2, t)$
- TAKE THE SOLUTION AND PLUG BACK TO THE ACTION

$$S = \int_{t_1}^{t_2} L(t_1, q_1, t_2, q_2, t)\,dt$$

$$= S(t_1, q_1, t_2, q_2) \quad \text{... HAM. F.}$$


• ACTION PRINCIPLE: $S = S[q_i] = \int_{t_1}^{t_2} L\,dt$ FUNCTIONAL
(PY-H)
• $\delta S = 0 \Rightarrow$ (E-L) / (HAM) SOLVE $\longrightarrow q_i = q_i(t_1, q_1, t_2, q_2, t)$
• TAKE THE SOLUTION AND PLUG BACK TO THE ACTION
$S = \int_{t_1}^{t_2} L(t_1, q_1, t_2, q_2, t)\,dt$
$= S(t_1, q_1, t_2, q_2)$ ... HAM. F.

TAKE DIFF. OF HAMILTON'S FUNCTION

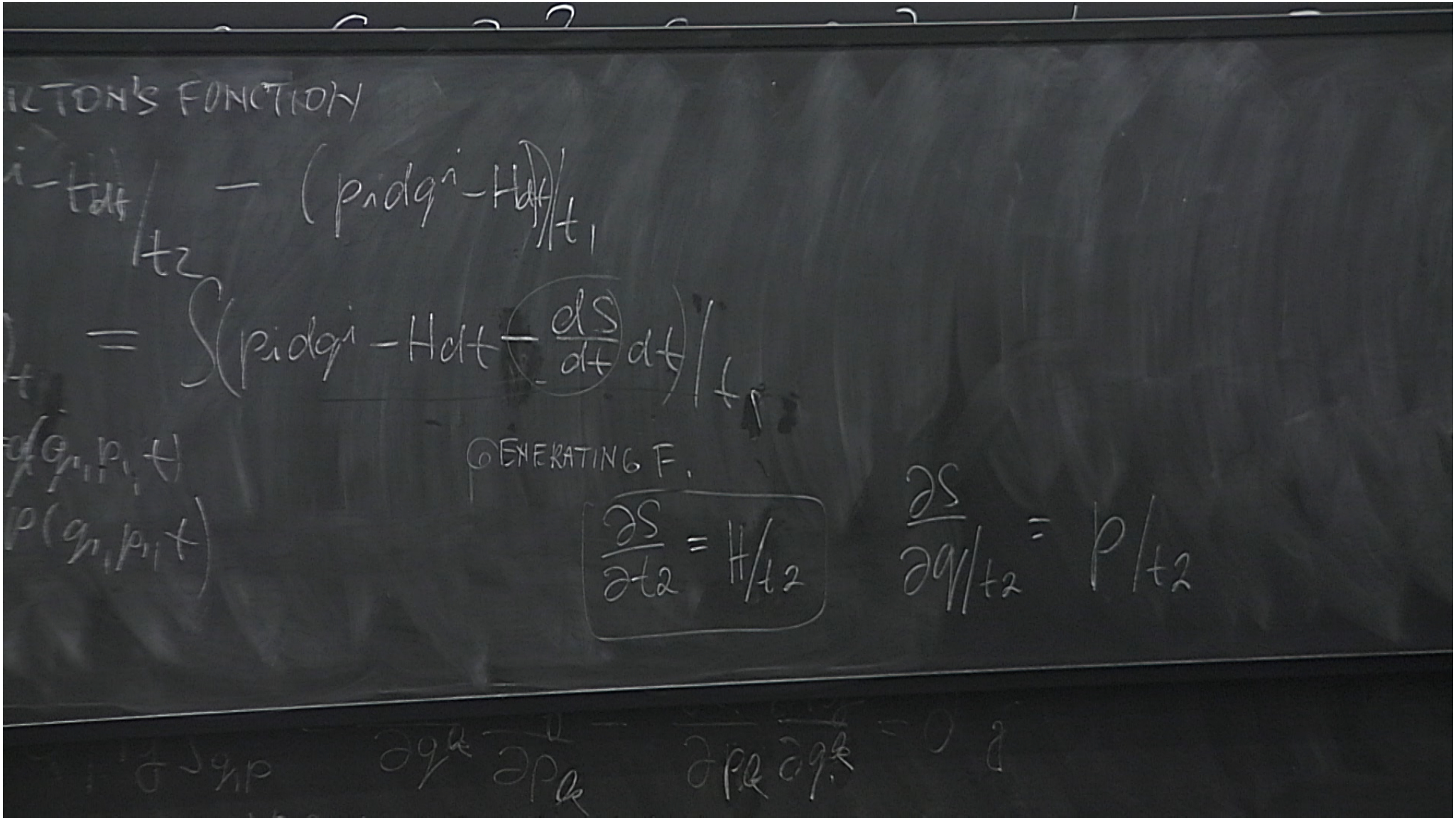
$$dS = (p_i dq^i - H dt)\big|_{t_2} - (p_i dq^i - H dt)\big|_{t_1}$$

$$\int (p_i dq^i - H dt)_t = \int \left(p_i dq^i - H dt - \frac{dS}{dt} dt\right)\Big|_{t_1}$$

$$q = q(q_1, p_1, t)$$
$$p = p(q_1, p_1, t)$$

GENERATING F.

$$\frac{\partial S}{\partial t_2} = H\big|_{t_2}$$


ILTON'S FUNCTION
$\int (p_i dq^i - H dt)\Big|_{t_1}^{t_2} = S\left(p_i dq^i - H dt - \frac{dS}{dt} dt\right)\Big|_{t_1}$
GENERATING F.
$\frac{\partial S}{\partial t_2} = H|_{t_2}$
$\frac{\partial S}{\partial q}\Big|_{t_2} = p|_{t_2}$