Title: PSI 2019/2020 - Classical Physics - Lecture 7

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Collection: PSI 2019/2020 - Classical Physics (Kubiznak)

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MAXWELL'S THEORY

- $$\nabla \times B - \frac{\partial E}{\partial t} = 4\pi j \quad (1)$$
$$\nabla \times E + \frac{\partial B}{\partial t} = 0 \quad (2)$$
$$\nabla \cdot E = 4\pi\rho \quad (3)$$
$$\nabla \cdot B = 0 \quad (4)$$

- $$\frac{dp}{dt} = F = e(E + v \times B)$$

CONSEQUENCES.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0$$

$$\Box E = 0 = \Box B$$

$$\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$$

• POTENTIALS

$$B = \nabla \times A$$
$$E = -\nabla \varphi - \frac{\partial A}{\partial t}$$

SOLVES (2) & (4)

GAUGE FREEDOM

$$\varphi \to \varphi - \frac{\partial \Lambda}{\partial t}$$
$$A \to A + \nabla \Lambda$$

LORENZ GAUGE

$$\frac{\partial \varphi}{\partial t} + \nabla \cdot A = 0$$

MAXWELL'S THEORY

- $$\boxed{\begin{aligned} \nabla\times B - \frac{\partial E}{\partial t} &= 4\pi j \quad (1)\\ \nabla\times E + \frac{\partial B}{\partial t} &= 0 \quad (2)\\ \nabla\cdot E &= 4\pi\rho \quad (3)\\ \nabla\cdot B &= 0 \quad (4)\end{aligned}}$$
- $$\boxed{\frac{dp}{dt} = F = e\left(E + v\times B\right)}$$

CONSEQUENCES.

$$\boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot j = 0}$$

$$\boxed{\Box E = 0 = \Box B}$$

$$\boxed{\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2}$$

• POTENTIALS (φ, A)

$$\boxed{\begin{aligned} B &= \nabla\times A\\ E &= -\nabla\varphi - \frac{\partial A}{\partial t}\end{aligned}}$$

SOLVES (2) & (4)

GAUGE FREEDOM

$$\boxed{\begin{aligned} \varphi &\to \varphi - \frac{\partial\Lambda}{\partial t}\\ A &\to A + \nabla\Lambda\end{aligned}}$$

LORENZ GAUGE

$$\boxed{\frac{\partial\varphi}{\partial t} + \nabla\cdot A = 0}$$

(φ, A)

A

$\varphi - \frac{\partial A}{\partial t}$

(2) & (4)

GAUGE FREEDOM

$$\boxed{\begin{aligned} \varphi &\to \varphi - \frac{\partial \Lambda}{\partial t} \\ A &\to A + \nabla \Lambda \end{aligned}}$$

LORENZ GAUGE

$$\boxed{\frac{\partial \varphi}{\partial t} + \nabla \cdot A = 0}$$

IN THIS GAUGE

$$\boxed{\begin{aligned} \Box \varphi &= -4\pi\rho \\ \Box A &= -4\pi j \end{aligned}}$$

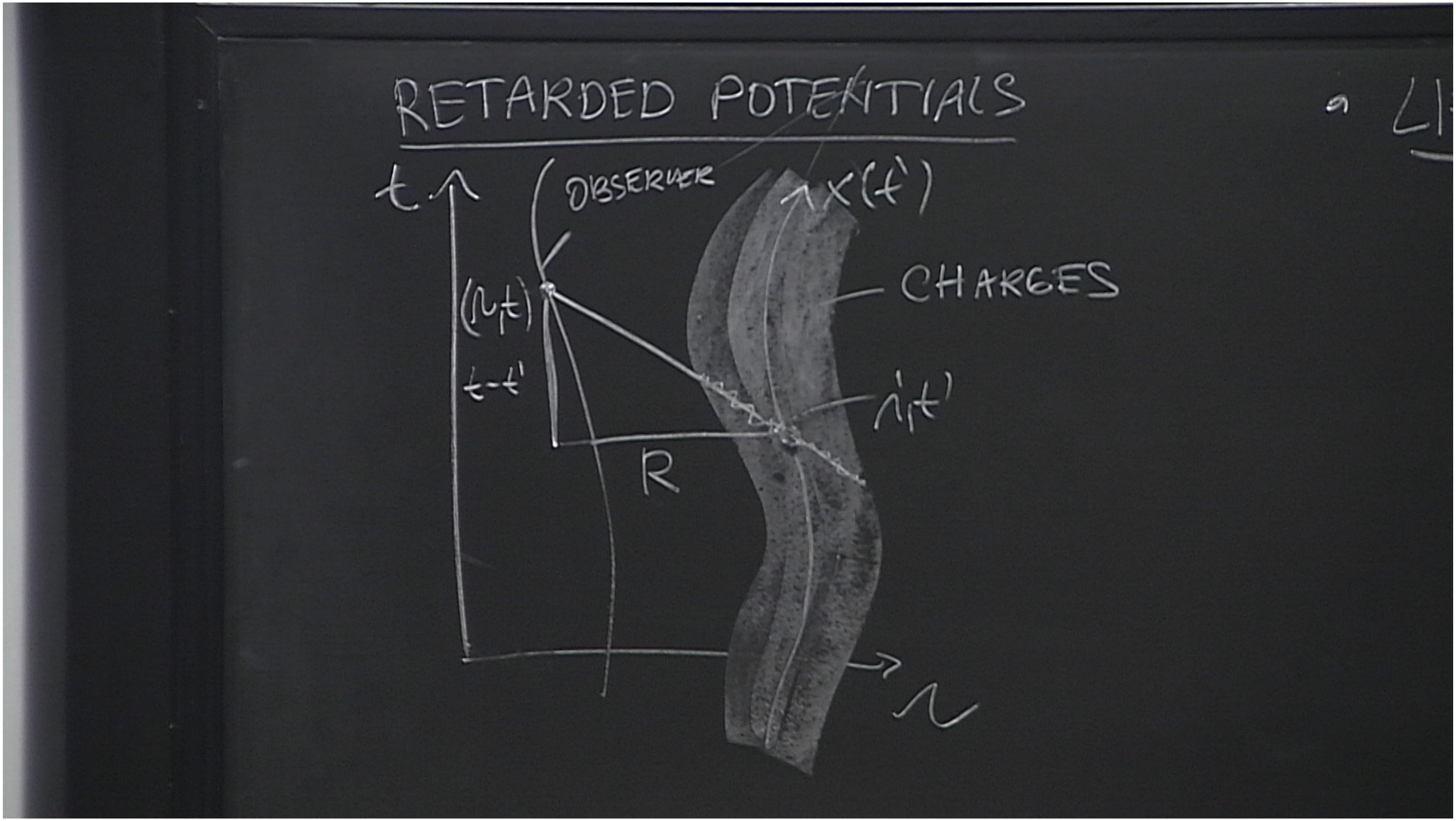
(1) & (3)

RATARDED POTENTIALS

$$\boxed{\varphi(r,t) = \int \frac{\delta(R-(t-t'))}{R}\, \rho(r',t')\, dr'\, dt'}$$

$$R = |r - r'|$$

$$\boxed{\begin{aligned} \varphi(r,t) &= \int dr' \, \frac{\rho(t-R, r)}{R} \\ A(r,t) &= \int dr' \, \frac{j(t-R, r)}{R} \end{aligned}}$$


RETARDED POTENTIALS
t
OBSERVER
x'(t')
CHARGES
(r,t)
t-t'
r',t'
R
r

$$\varphi(r,t) = \int \frac{\delta(R-(t-t'))}{R}\, e\, \delta^{(3)}(r'-x(t'))\, dr\, dt'$$

$$= \int \frac{\delta(|r-x(t')|-t+t')}{|r-x(t')|}\, e\, dt'$$

$$\delta(f(t')) = \sum_{\substack{\text{ROOTS} \\ i}} \frac{\delta(t'-t_i)}{|f'(t_i)|}$$

$$f(t_i) = 0$$

$$(r,t) = \int \frac{\cdots}{R}\, e\, \delta(\cdots)$$

$$= \int \frac{\delta(|r - x(t')| - t + t')}{|r - x(t')|}\, e\, dt'$$

$$f(t_i) = 0$$

$$\boxed{\delta(f(t')) = \sum_{\text{ROOTS}} \frac{\delta(t' - t_i)}{|f'(t_i)|}}$$

IN CASE OF POINT PARTICLE

$$t_i =$$

$$\ldots = \int \frac{\ldots}{R}\, e\, \delta(\ldots)$$

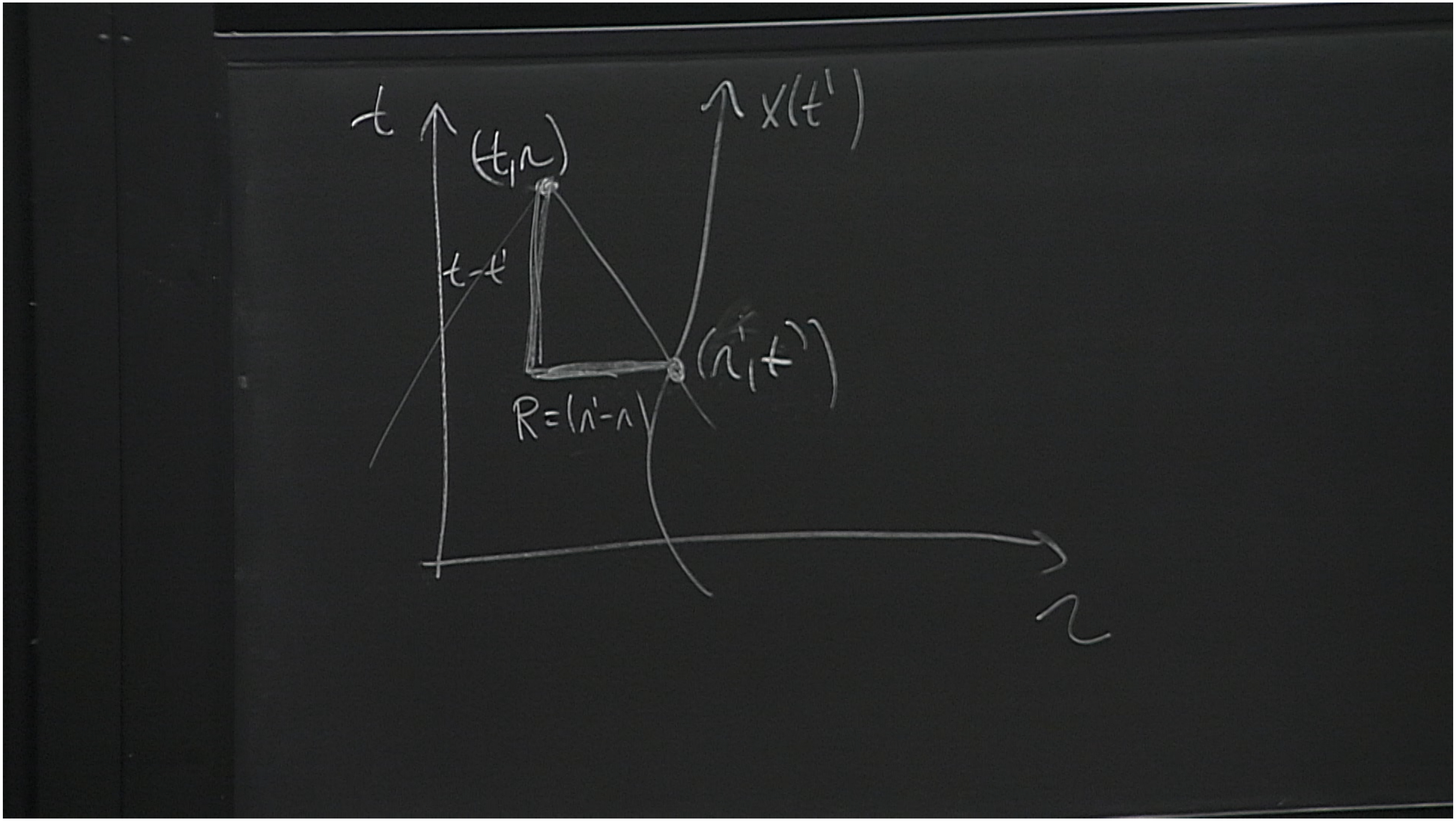
$$= \int \frac{\delta(|r - x(t')| - t + t')}{|r - x(t')|}\, e\, dt'$$

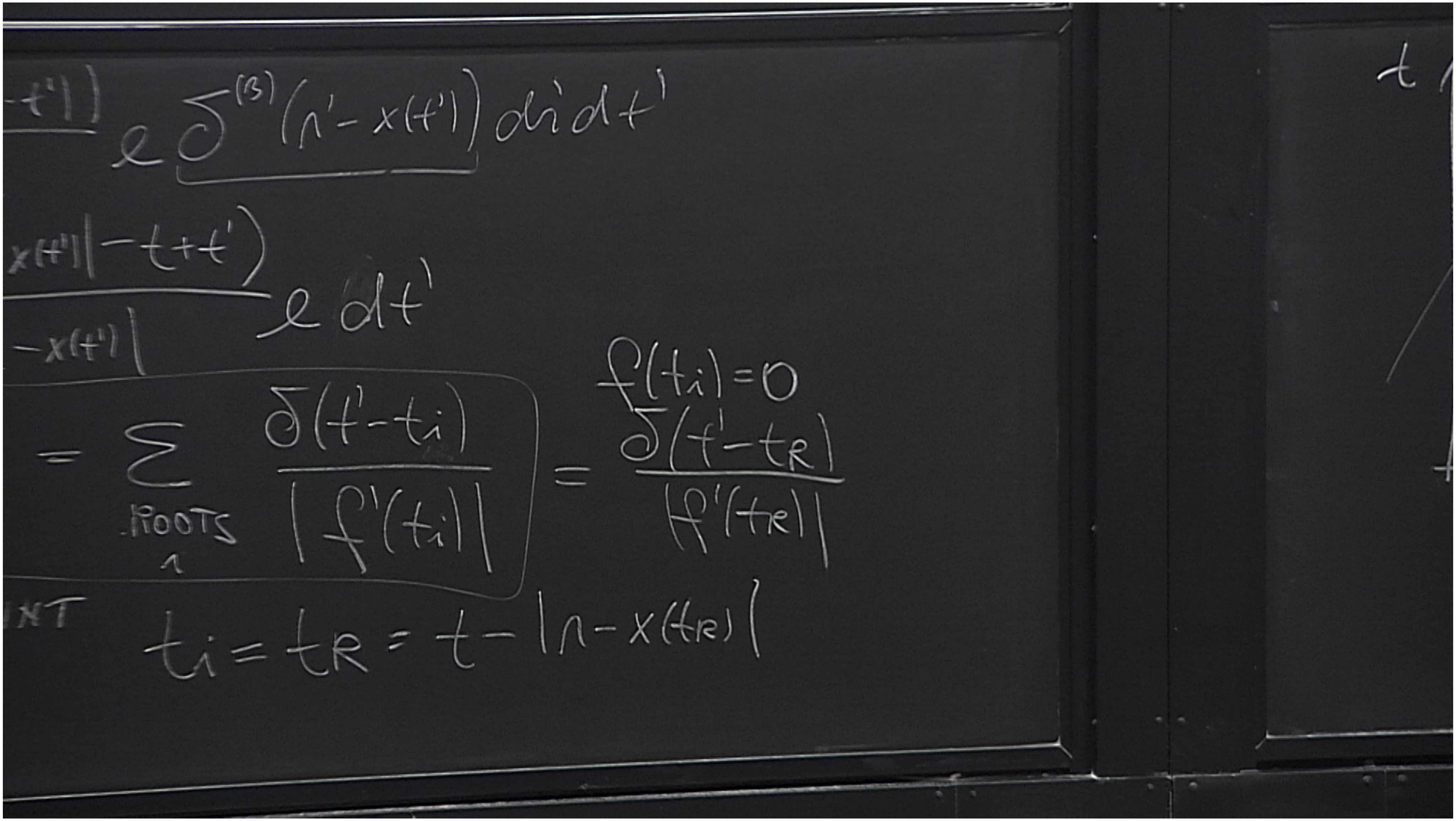
$$f(t_i) = 0$$

$$\boxed{\delta(f(t')) = \sum_{\text{ROOTS } i} \frac{\delta(t' - t_i)}{|f'(t_i)|}}$$

IN CASE OF POINT PARTICLE

$$t_i = t_R = t - |r - x(t_R)|$$


t
(t, r)
x(t')
t - t'
(r', t')
R = |r' - r|
r



-t')) ℓ δ^(3)(r' - x(t')) dr' dt'
x(t')| - t + t') / -x(t')| ℓ dt'
f(ti) = 0
= Σ ROOTS i δ(t' - ti) / |f'(ti)| = δ(t' - tR) / |f'(tR)|
INT
ti = tR = t - |r - x(tR)|

$$\varphi(r,t) = \int \frac{\delta(R-(t-t'))}{R}\, e\, \underline{\delta^{(3)}(r'-x(t'))}\, dr'\, dt'$$

$$= \int \frac{\delta(|r-x(t')|-t+t')}{|r-x(t')|}\, e\, dt'$$

$$f(t_i)=0$$

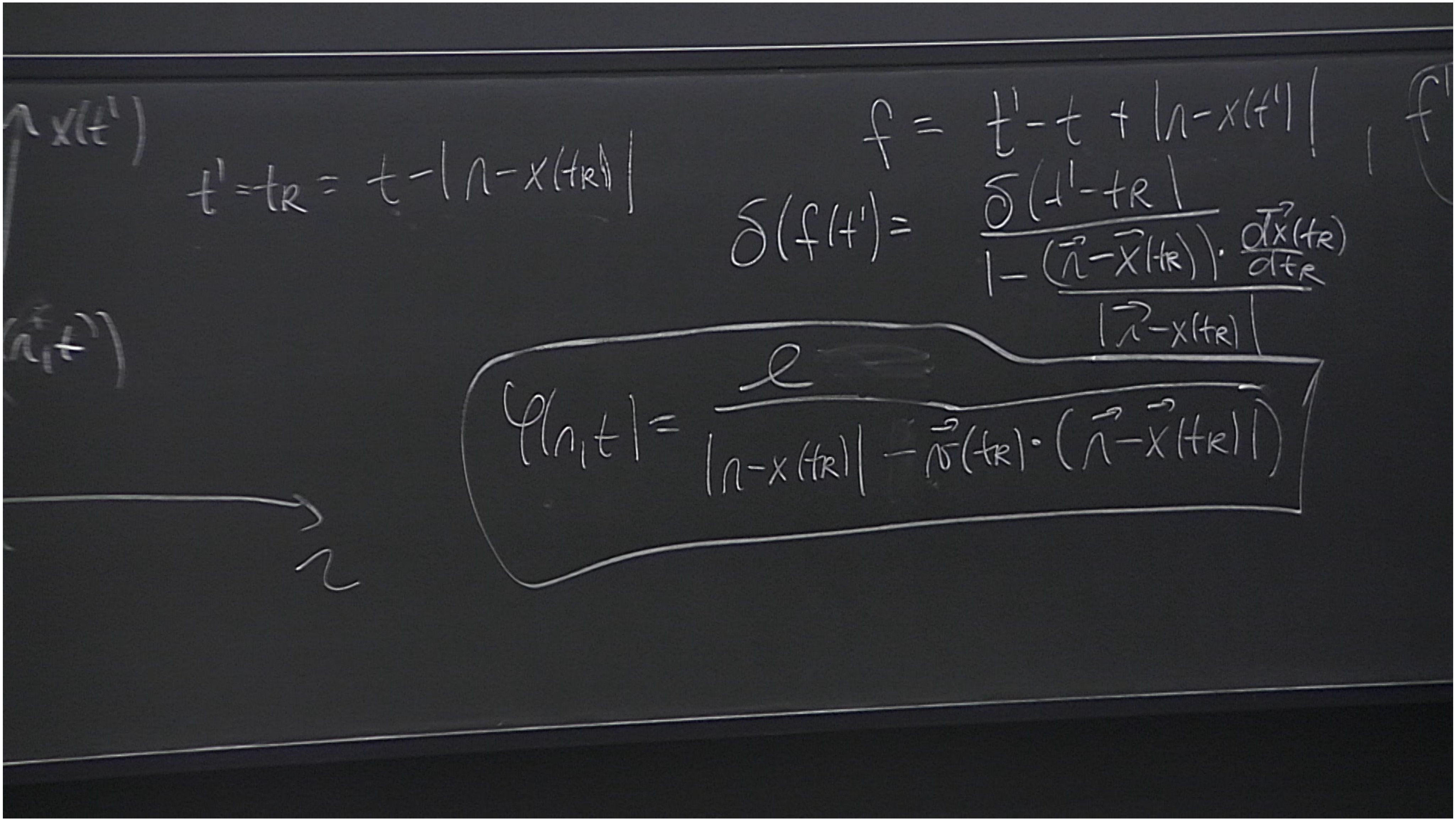
$$\boxed{\delta(f(t')) = \sum_{\text{ROOTS } i} \frac{\delta(t'-t_i)}{|f'(t_i)|}} = \frac{\delta(t'-t_R)}{|f'(t_R)|}$$

IN CASE OF POINT PARTICLE

$$t_i = t_R = t - |r - x(t_R)|$$

$$f = t' - t + |r - x(t')| \,, \quad f' = 1 - \frac{(r - x(t'))\cdot \frac{dx}{dt'}}{|r - x(t')|}$$

$$\delta(f(t')) = \frac{\delta(t' - t_R)}{}$$


$x(t')$
$t' = t_R = t - |r - x(t_R)|$
$f = t' - t + |r - x(t')|$
$\delta(f(t')) = \dfrac{\delta(t' - t_R)}{1 - \dfrac{(\vec{r} - \vec{x}(t_R)) \cdot \frac{d\vec{x}(t_R)}{dt_R}}{|\vec{r} - x(t_R)|}}$
$\boxed{\varphi(r,t) = \dfrac{e}{|r - x(t_R)| - \vec{v}(t_R) \cdot (\vec{r} - \vec{x}(t_R))}}$

$t - |r - x(t_R)|$

$$\delta(f(t')) = \frac{\delta(t' - t_R)}{1 - \dfrac{(\vec{r} - \vec{x}(t_R)) \cdot \dfrac{d\vec{x}(t_R)}{dt_R}}{|\vec{r} - x(t_R)|}}$$

$$\boxed{\varphi(r,t) = \frac{e}{|r - x(t_R)| - \vec{v}(t_R) \cdot (\vec{r} - \vec{x}(t_R))}} \qquad (RH1)$$

$$f = t' - t + |r - x(t')|$$

$$f' = 1 - \frac{(r - x(t')) \cdot \frac{dx}{dt'}}{|r - x(t')|}$$

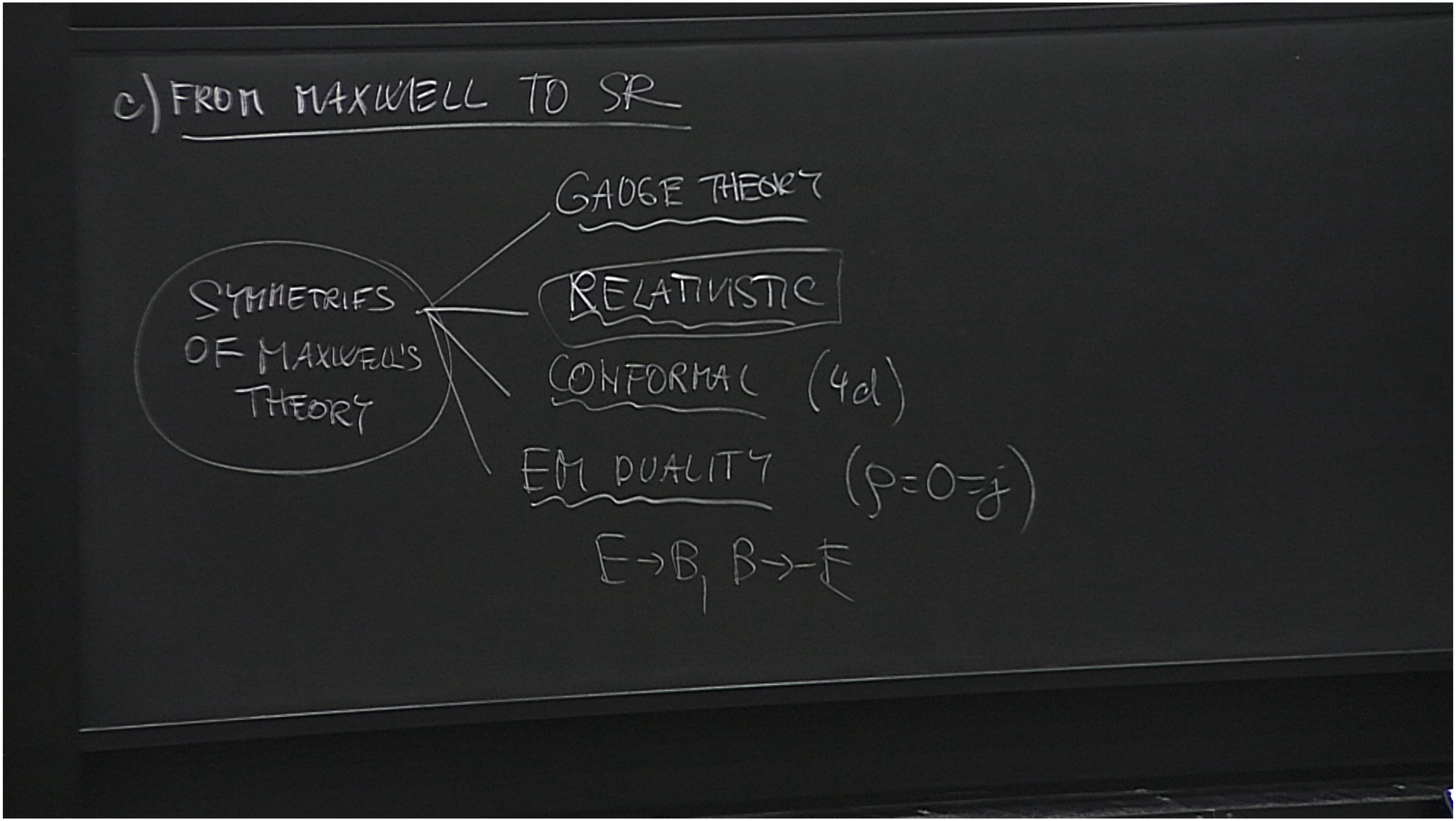
$$|x| = \sqrt{\vec{x} \cdot \vec{x}}$$

$$|x|' = \frac{1}{\sqrt{x \cdot x}}\, x \cdot \frac{dx}{dt}$$

$$) = \frac{\delta(t' - t_R)}{1 - \frac{(\vec{r} - \vec{x}(t_R)) \cdot \frac{d\vec{x}(t_R)}{dt_R}}{|\vec{r} - x(t_R)|}}$$

$$- \vec{v}(t_R) \cdot (\vec{r} - \vec{x}(t_R)|)$$

(RH1)


c) FROM MAXWELL TO SR
SYMMETRIES OF MAXWELL'S THEORY
GAUGE THEORY
RELATIVISTIC
CONFORMAL (4d)
EM DUALITY (ρ=0=j)
E→B, B→-E

• EMERGENCE OF SPACETIME

HAVING PREDICTED THE UNIVERSAL SPEED

$c = (1)$ → CAN UNIFY x & t → SPACETIME

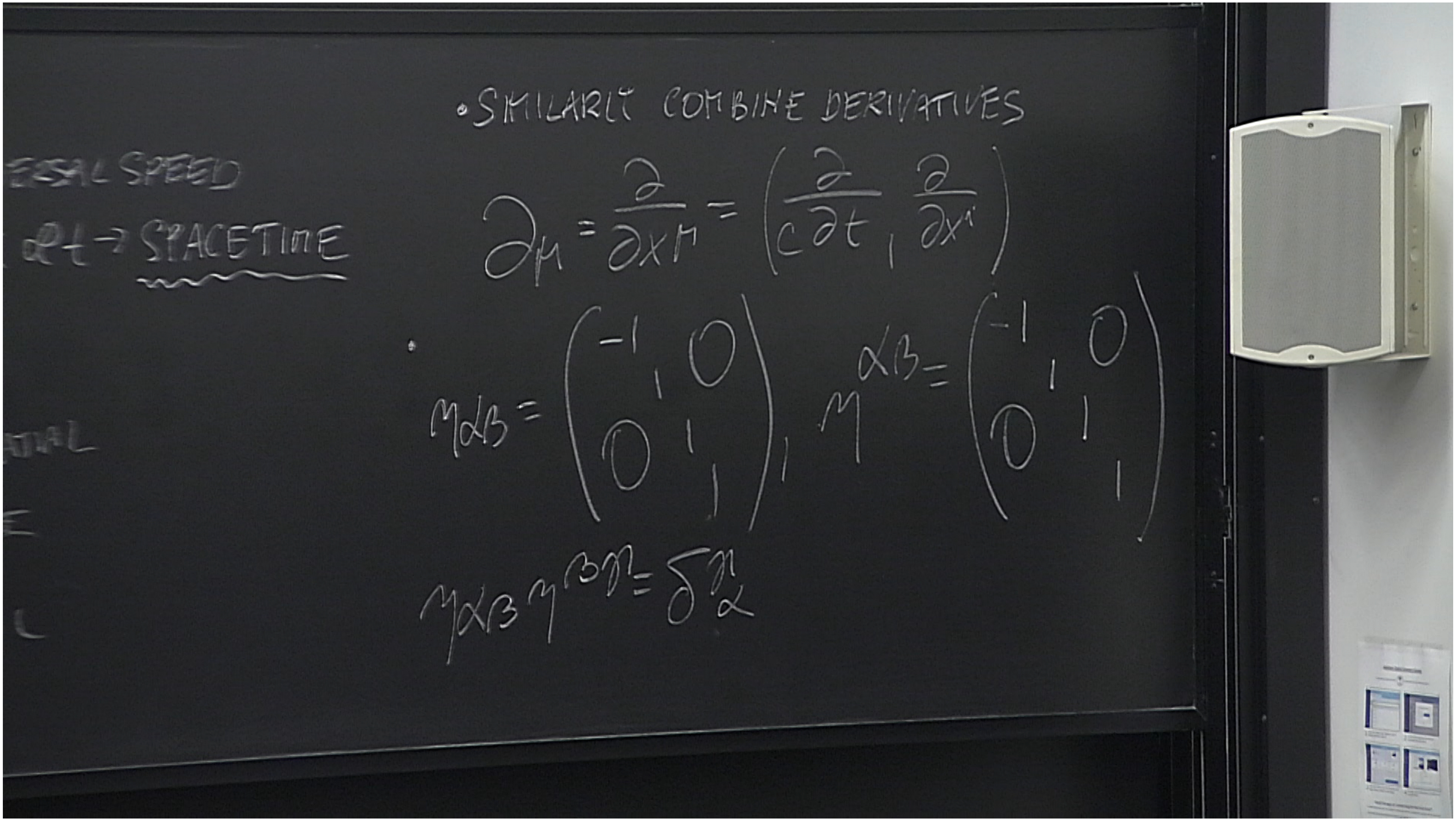
$$x^{\mu} = (ct, x^{i})$$

SPACETIME

SPATIAL

$\mu = 0, 1, 2, 3 \ldots$ SPACETIME

$i = 1, 2, 3 \ldots$ SPATIAL



• SIMILARLY COMBINE DERIVATIVES
$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{c\,\partial t}, \frac{\partial}{\partial x^i}\right)$
• $\eta_{\alpha\beta} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}, \quad \eta^{\alpha\beta} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$
$\eta_{\alpha\beta}\eta^{\beta\gamma} = \delta^\gamma_\alpha$
→ SPACETIME

(t, x^i)

SPATIAL

... SPACETIME

... SPATIAL

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}, \quad \eta^{\alpha\beta} =$$

$$\eta_{\alpha\beta}\,\eta^{\beta\gamma} = \delta^{\gamma}_{\alpha}$$

$$ds^2 = -\eta_{\alpha\beta}\, dx^\alpha dx^\beta$$

- CAN USE TO LOWER & RISE INDICES

V^μ ... VECTORS, CO-VECTORS ω_μ

$$\omega^\mu = \eta^{\mu\nu}\omega_\nu$$

$$V_\mu = \eta_{\mu\nu} V^\nu$$

$$V^\mu = (V^0, V^i)$$

$$V_\mu = (-V^0, V^i)$$

$$\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta = \eta_{\alpha\beta}\partial^\alpha\partial^\beta = -\partial_t^2 + \sum_i \partial_{x^i}^2$$

- ELECTROMAGNETISM IN RELATIVISTIC FORM

- $A^\mu = (\varphi, \vec{A})$

- LORENZ GAUGE $\boxed{\partial_\mu A^\mu = 0} = \frac{\partial \varphi}{\partial t} + \partial_i A^i$

- $A_\mu = (-\varphi, \vec{A})$ $\boxed{A_\mu \to A_\mu + \partial_\mu \Lambda}$

$-\varphi \to -\varphi + \partial_t \Lambda$

$\sum_i \partial^2_{x^i}$

- INTRODUCING $J^\mu = (\rho, \vec{j})$ $\boxed{\Box A^\mu = -4\pi J^\mu}$

$$\boxed{A^\mu = \int d^3 r' \, \frac{J^\mu(t - |r - r'|, r')}{|r - r'|}}$$

- LORENTZ TRANSFORMATIONS

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu \, x^\nu$$

OLD

NEW

CONSTANT 4×4

$$x'^\mu \to x^\mu = \tilde{\Lambda}^\mu{}_\nu \, x^\nu$$

n-n'|,n')

IS

$^\mu{}_\nu X^\nu$

ANT 4x4

$^\mu{}_\nu x^\nu$

Λ & $\tilde{\Lambda}$ ARE INVERSE

$$\Lambda^\mu{}_\nu \tilde{\Lambda}^\nu{}_\alpha = \delta^\mu{}_\alpha$$

$$\Lambda^\mu{}_\nu \tilde{\Lambda}^\alpha{}_\mu = \delta^\alpha{}_\nu$$

$$\partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu}$$

$$|\Lambda - \Lambda'|$$

- LORENTZ TRANSFORMATIONS

$$\underbrace{x^{\mu}}_{\text{OLD}} \rightarrow \underbrace{x'^{\mu}}_{\text{NEW}} = \underbrace{\Lambda^{\mu}{}_{\nu}}_{\text{CONSTANT } 4\times 4} x^{\nu}$$

$$x'^{\mu} \rightarrow x^{\mu} = \tilde{\Lambda}^{\mu}{}_{\nu} x'^{\nu}$$

$$\Lambda^{\mu}{}_{\nu}\tilde{\Lambda}^{\nu}{}_{\alpha} = \delta^{\mu}{}_{\alpha}$$

$$\Lambda^{\mu}{}_{\nu}\tilde{\Lambda}^{\alpha}{}_{\mu} = \delta^{\alpha}{}_{\nu}$$

$$=$$

$$\partial'_{\mu} = \frac{\partial}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\nu}}$$

$$\tilde{\Lambda}^{\nu}{}_{\mu}$$

OR

$\delta^{\alpha}{}_{\nu}$

$\frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\nu}}$

$\tilde{\Lambda}^{\nu}{}_{\mu}$

- ANY OBJECT WITH INDEX DOWN

$$\boxed{\begin{aligned} \omega'_{\mu} &= \tilde{\Lambda}^{\nu}{}_{\mu}\, \omega_{\nu} \\ V'^{\mu} &= \Lambda^{\mu}{}_{\nu}\, V^{\nu} \end{aligned}}$$

... CO-VECTOR

... VECTOR

• ANY OBJECT WITH INDEX DOWN

$$\boxed{\begin{aligned} \omega'_\mu &= \tilde{\Lambda}^\nu{}_\mu\, \omega_\nu \quad \text{... CO-VECTOR} \\ V'^\mu &= \Lambda^\mu{}_\nu\, V^\nu \quad \text{... VECTOR} \end{aligned}}$$

$\frac{\partial}{\partial x^\nu}$

• IS A^μ A VECTOR?

$\partial_\mu A^\mu$ = SCALAR

$$\partial'_\mu A'^\mu = \tilde{\Lambda}^\nu{}_\mu\, \partial_\nu\, \Lambda^\mu{}_\alpha A^\alpha = \partial_\alpha A^\alpha$$

DEMANDING THAT $\Box$ PRESERVES ITS FORM UNDER LORENTZ TRANSFORMATION

INVARIANCE IN FORM

$$\Box = \eta_{\alpha\beta}\partial^{\alpha}\partial^{\beta} = \eta'_{\mu\nu}\partial'^{\mu}\partial'^{\nu} = \eta_{\mu\nu}\partial'^{\mu}\partial'^{\nu}$$

DUMMY

$$= -\partial_t^2 + \nabla^2 \rightarrow -\partial_{t'} + \nabla'^2$$

PRESERVES ITS FORM UNDER

...ATION

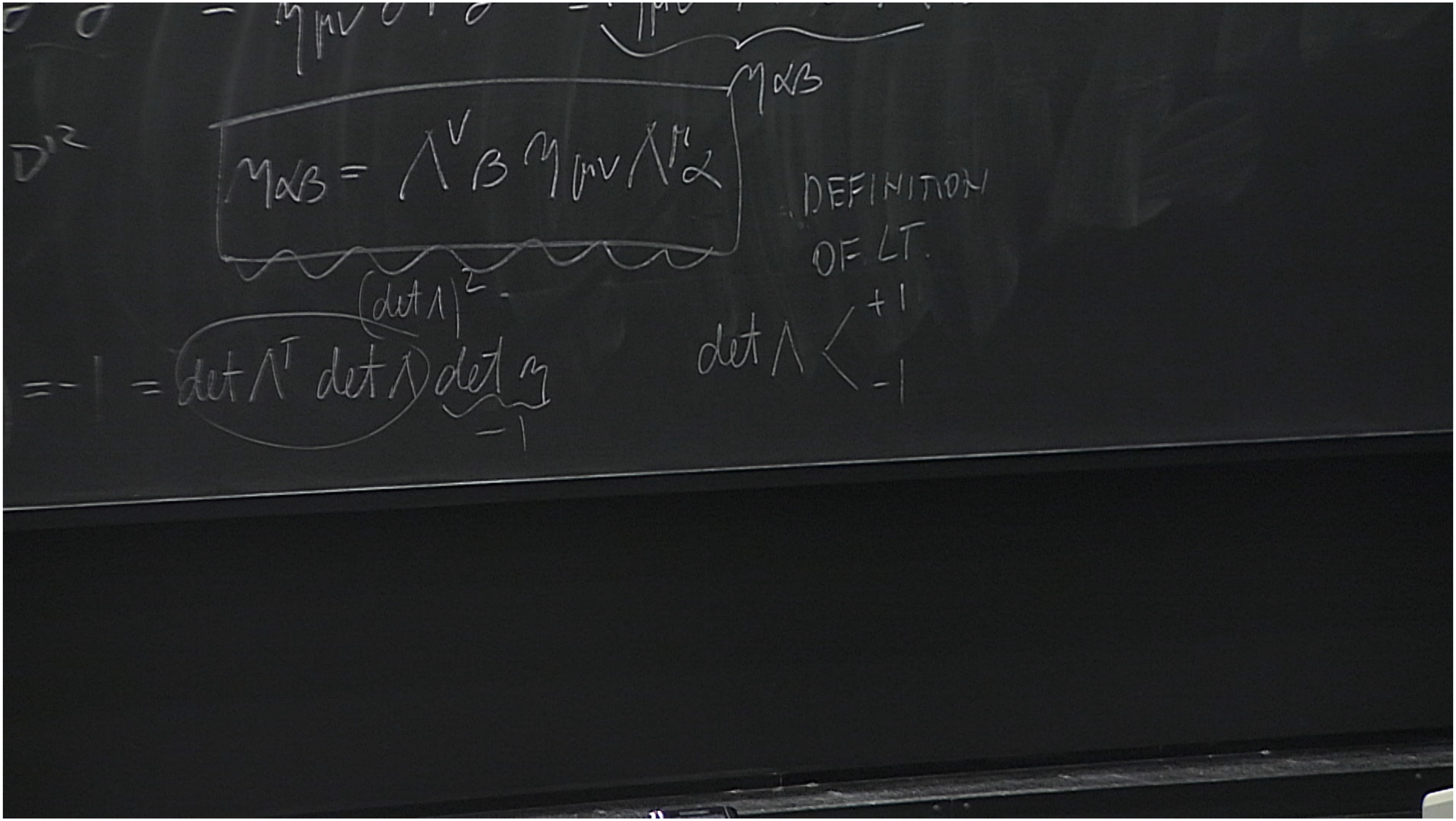
INVARIANCE IN FORM

$$= \eta'^{\mu\nu}\partial'_\mu\partial'_\nu \overset{\checkmark}{=} \eta_{\mu\nu}\partial'^\mu\partial'^\nu = \underbrace{\eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta}_{\eta_{\alpha\beta}}\partial^\alpha\partial^\beta$$

$$\rightarrow -\partial_{t'}^2 + \nabla'^2$$

$$\boxed{\eta_{\alpha\beta} = \Lambda^\nu{}_\beta\,\eta_{\mu\nu}\,\Lambda^\mu{}_\alpha}$$

DEFINITION OF LT.


ηαβ = Λ^ν_β η_μν Λ^μ_α
DEFINITION OF LT.
(det Λ)^2
= -1 = det Λ^T det Λ det η
-1
det Λ +1
-1

$$\eta_{\alpha\beta}$$

$$\boxed{\eta_{\alpha\beta} = \Lambda^{\nu}{}_{\beta}\, \eta_{\mu\nu}\, \Lambda^{\mu}{}_{\alpha}}$$

DEFINITION OF LT.

$$(\det\Lambda)^2 =$$

$$\det\Lambda^T \det\Lambda \underbrace{\det\eta}_{-1}$$

$$\det\Lambda \begin{cases} +1 \ \dots \ \text{PROPER} \\ -1 \end{cases}$$

MAXWELL EQS:

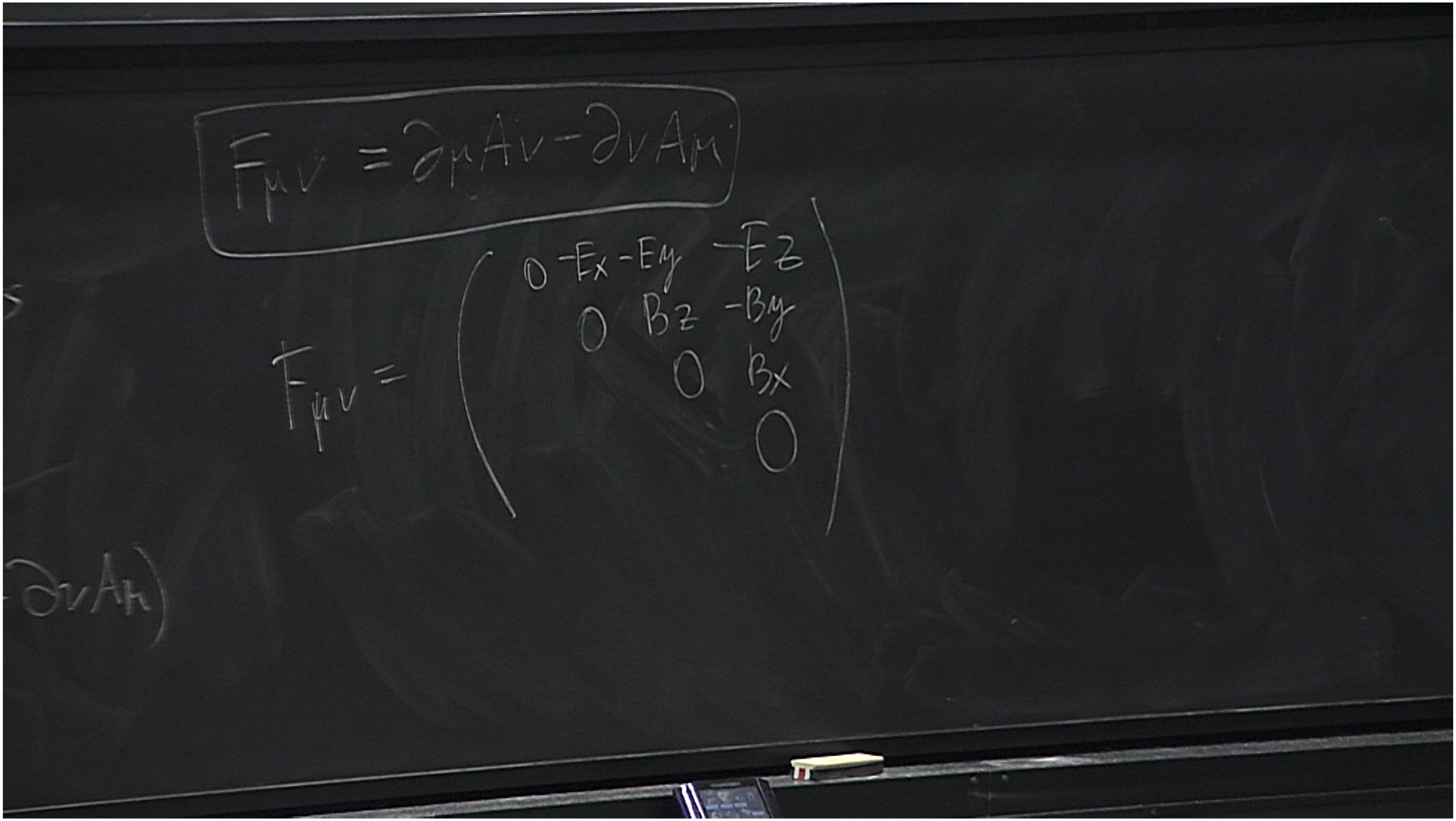
$\boxed{E,B}$ ... 6

$$\partial_\mu A_\nu \quad \text{.... 16 COMPTS}$$

$$\partial_{(\mu}A_{\nu)} \quad + \quad \partial_{[\mu}A_{\nu]}$$

$$\frac{1}{2}\left(\partial_\mu A_\nu + \partial_\nu A_\mu\right) \qquad \frac{1}{2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)$$

$$10 \qquad\qquad 6$$



$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
$F_{\mu\nu} =$
0 −Ex −Ey −Ez
0 Bz −By
0 Bx
0
∂νAμ)

E_z
$-B_y$
B_x
0

• CHECK THE GAUGE TRANSF.

$$A_\mu \to A_\mu + \partial_\mu \Lambda$$

$$F_{\mu\nu} \to \underline{\partial_\mu (A_\nu} + \partial_\nu \Lambda) - \underline{\partial_\nu (A_\mu} + \partial_\mu \Lambda) = \underline{F_{\mu\nu} + 0}.$$

$$\partial_\mu \partial_\nu \Lambda - \partial_\nu \partial_\mu \Lambda = 0$$

<u>(2)|2 |4|:</u>

$$F_{[\mu\nu,\lambda]_{\mathrm{CYCL}}} = 0 = F_{[\mu\nu,\lambda]}$$

$$\| $$

$$F_{\mu\nu,\lambda} + F_{\lambda\mu,\nu} + F_{\nu\lambda,\mu} = 0$$

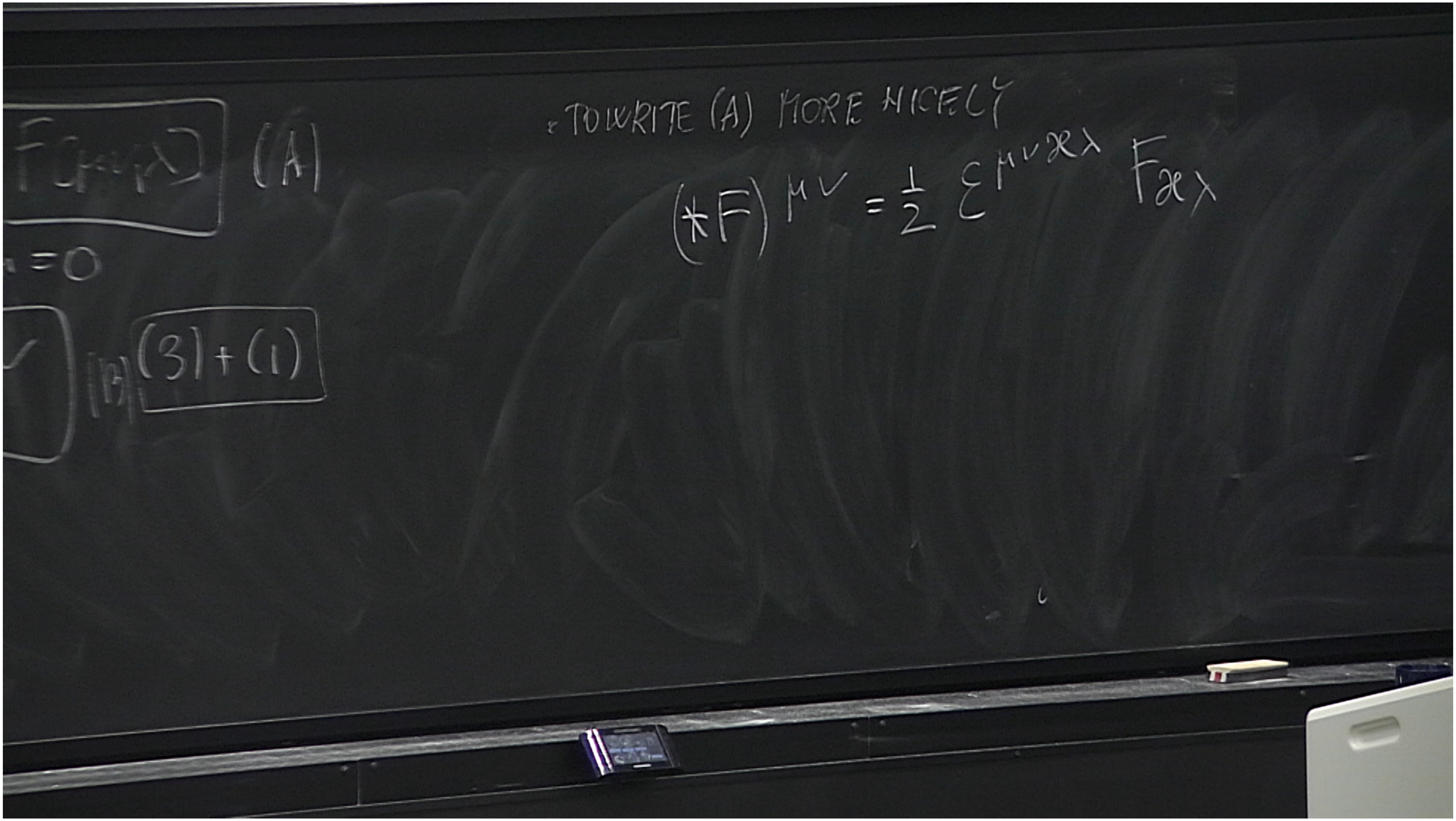
(2)|(2)(4):

$$\boxed{F_{[\mu\nu,\lambda]_{cycl}} = 0 = F_{[\mu\nu,\lambda]}}$$

$$F_{\mu\nu,\lambda} + F_{\lambda\mu,\nu} + F_{\nu\lambda,\mu} = 0$$

$$\boxed{\partial_\mu F^{\mu\nu} = -4\pi J^\nu}$$

$\boxed{(3) + (1)}$


(A)
TO WRITE (A) MORE NICELY
$(*F)^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\kappa\lambda}F_{\kappa\lambda}$
$=0$
$(3)+(1)$

(A)

TO WRITE (A) MORE NICELY

$$(*F)^{\mu\nu} = \frac{1}{2}\underbrace{\varepsilon^{\mu\nu\kappa\lambda}}_{\text{LEVI-CIVITA}} F_{\kappa\lambda}$$

$$\varepsilon^{0123} = +1$$

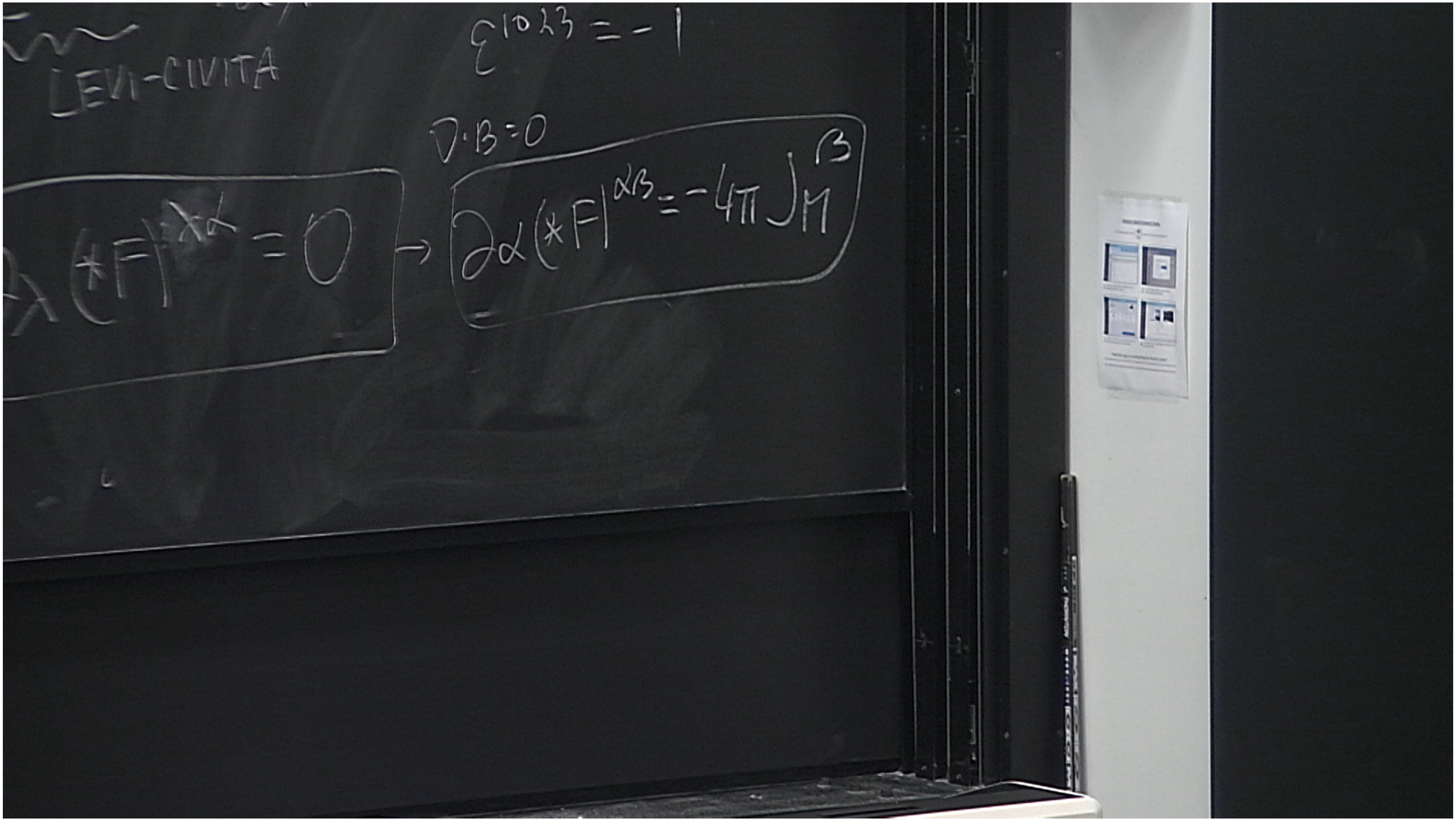
$$\varepsilon^{1023} = -1$$

(1)

$$\frac{1}{2}\varepsilon^{\alpha\mu\nu\lambda}\underbrace{F_{[\mu\nu,\lambda]}}_{\partial_\lambda F_{\mu\nu}} = 0$$

$$\boxed{\partial_\lambda (*F)^{\lambda\alpha} = 0}$$

$$\partial_\lambda\underbrace{\left(\frac{1}{2}\varepsilon^{\alpha\mu\nu\lambda}F_{\mu\nu}\right)}_{*F^{\alpha\lambda}}$$


LEVI-CIVITA
$\varepsilon^{1023} = -1$
$\nabla \cdot B = 0$
$\partial_\lambda (*F)^{\lambda\alpha} = 0 \rightarrow \partial_\alpha (*F)^{\alpha\beta} = -4\pi J_M^\beta$