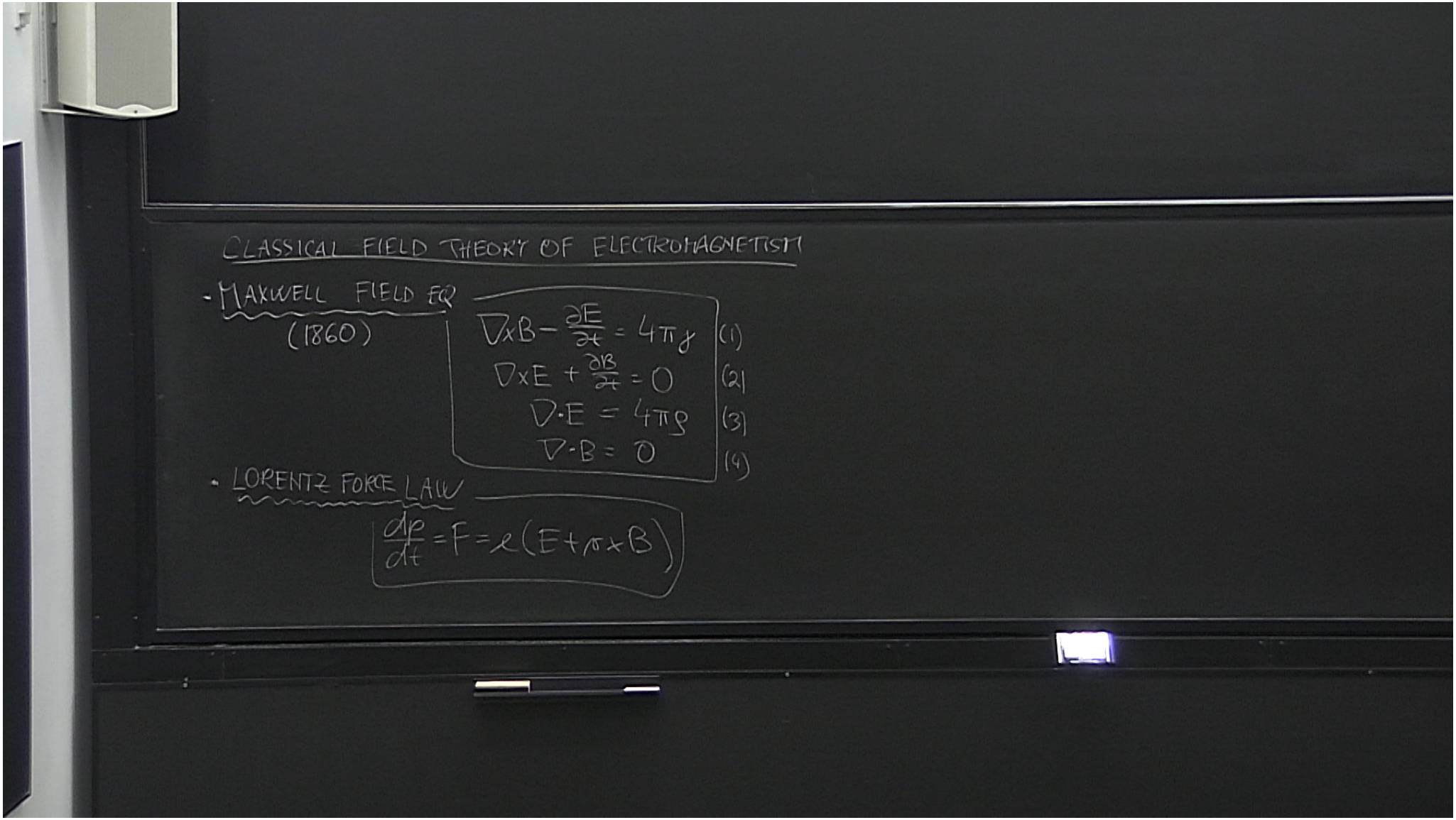
Title: PSI 2019/2020 - Classical Physics - Lecture 6

Speakers: David Kubiznak

Collection: PSI 2019/2020 - Classical Physics (Kubiznak)

Date: August 21, 2019 - 9:00 AM

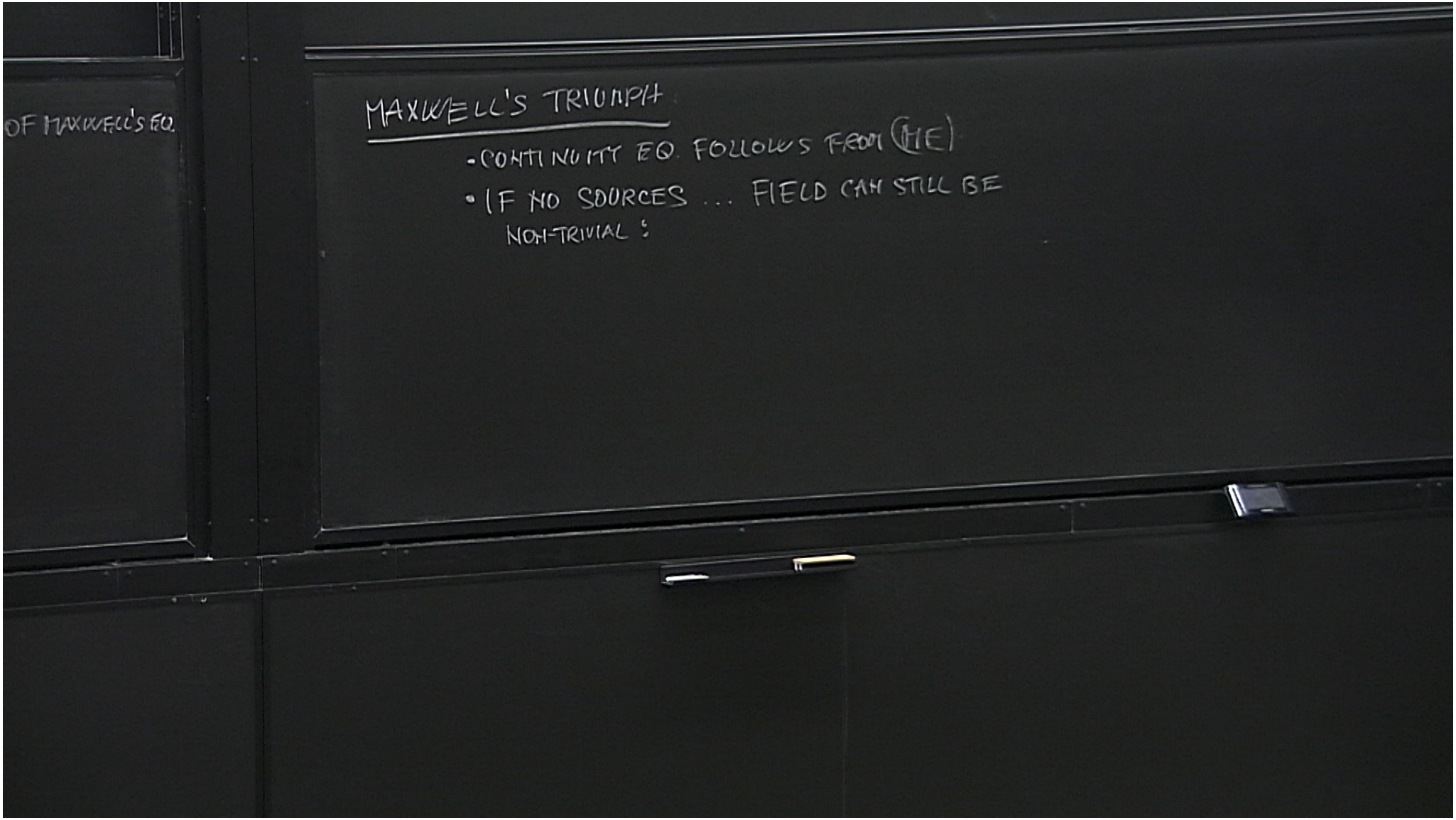
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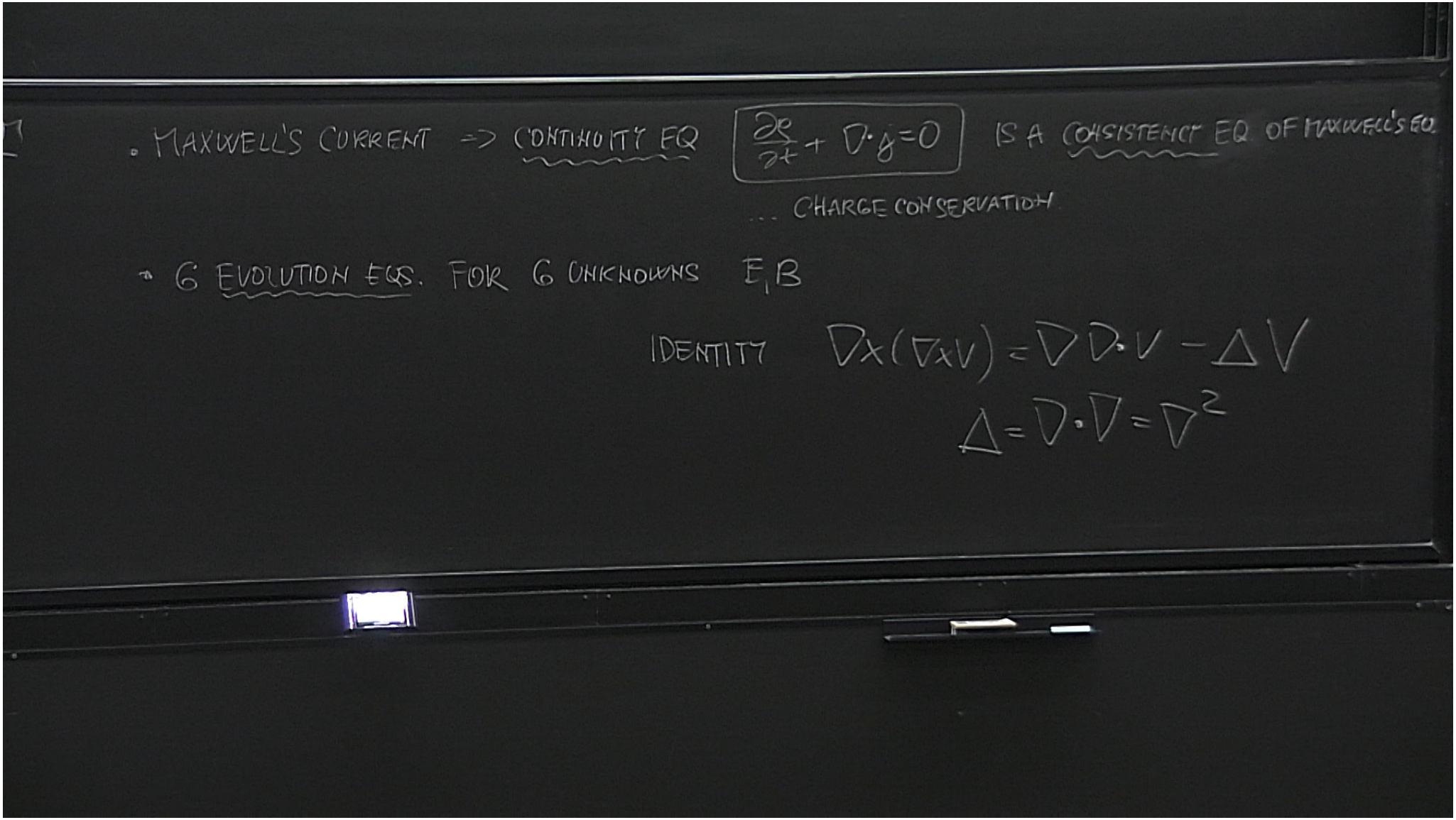

CLASSICAL FIELD THEORY OF ELECTROMAGNETISM
- MAXWELL FIELD EQ (1860)
$\nabla \times B - \frac{\partial E}{\partial t} = 4\pi j$ (1)
$\nabla \times E + \frac{\partial B}{\partial t} = 0$ (2)
$\nabla \cdot E = 4\pi \rho$ (3)
$\nabla \cdot B = 0$ (4)
- LORENTZ FORCE LAW
$\frac{dp}{dt} = F = e(E + v \times B)$

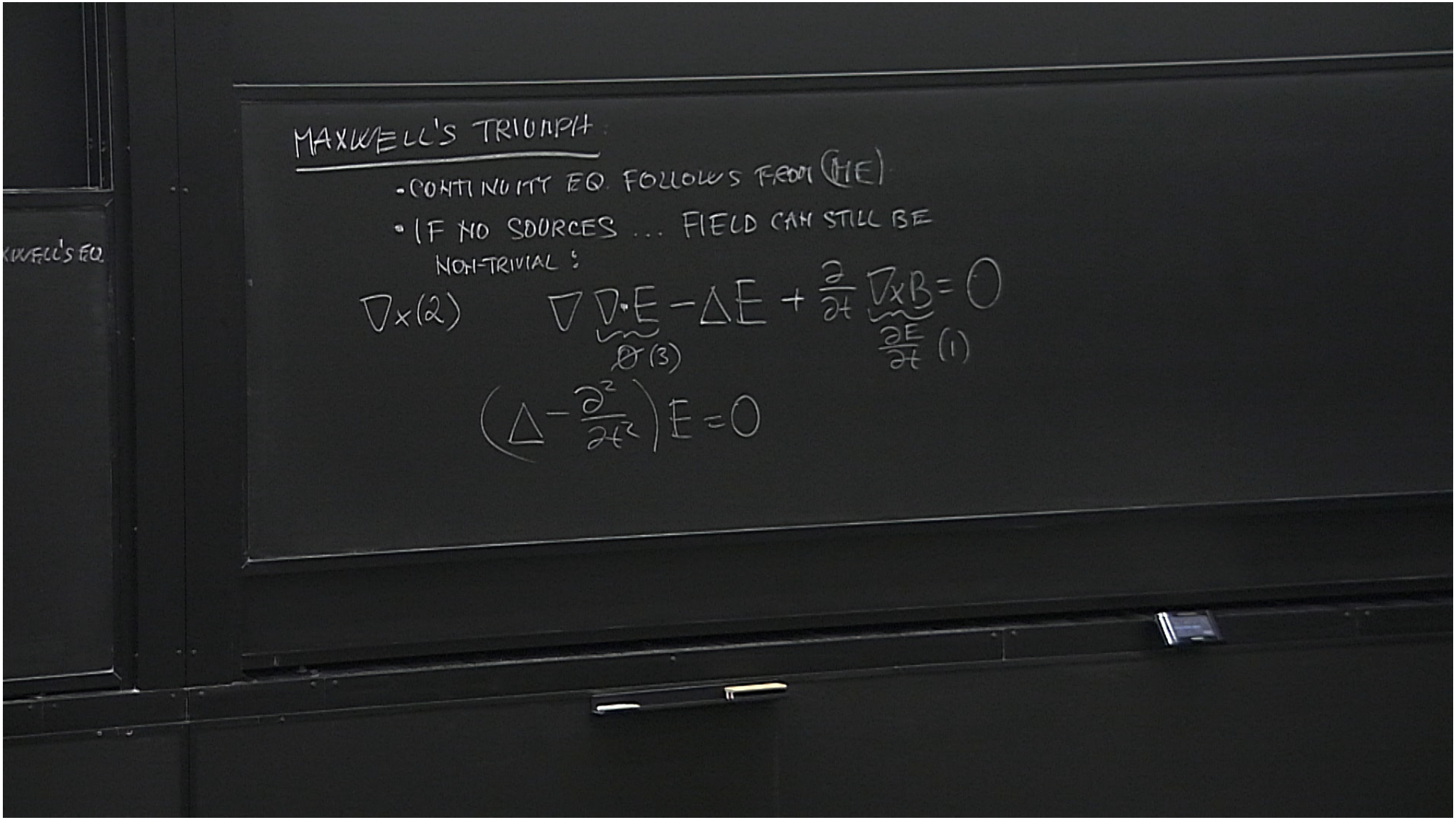
- MAXWELL'S CURRENT => CONTINUITY EQ $\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0}$ IS A CONSISTENCY EQ. OF MAXWELL'S EQ.

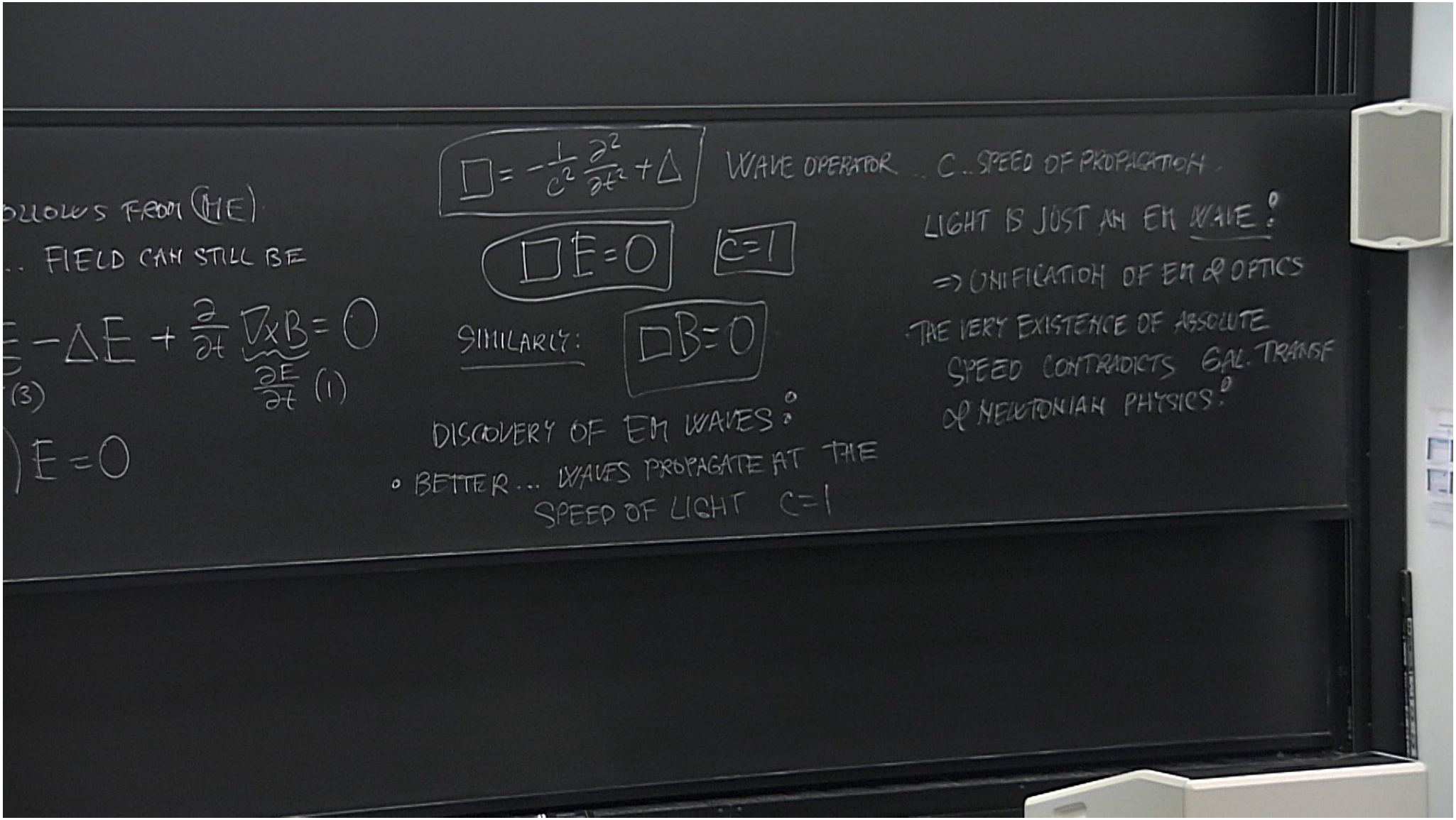
... CHARGE CONSERVATION

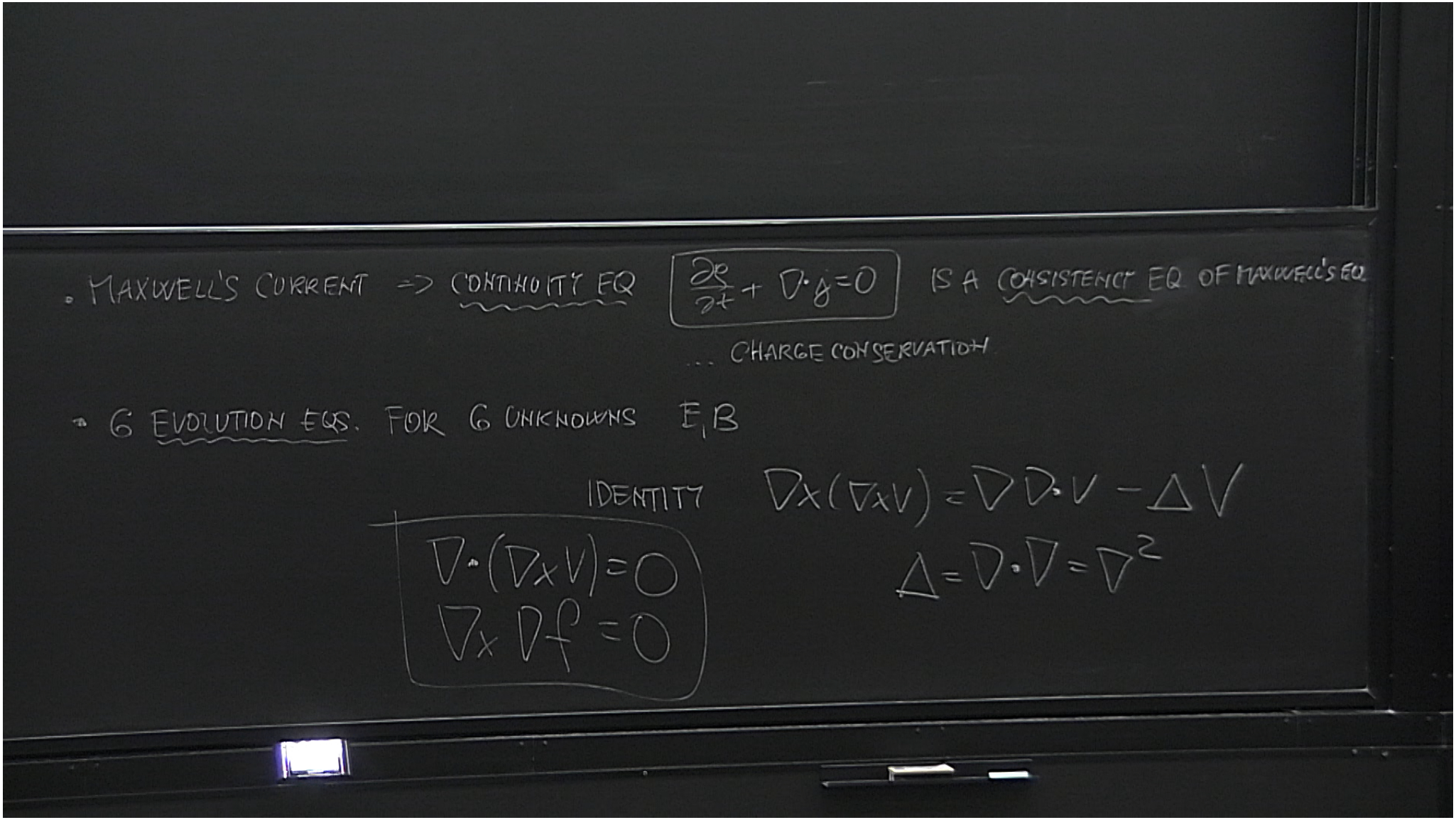
- 6 EVOLUTION EQS. FOR 6 UNKNOWNS $\vec{E}, \vec{B}$

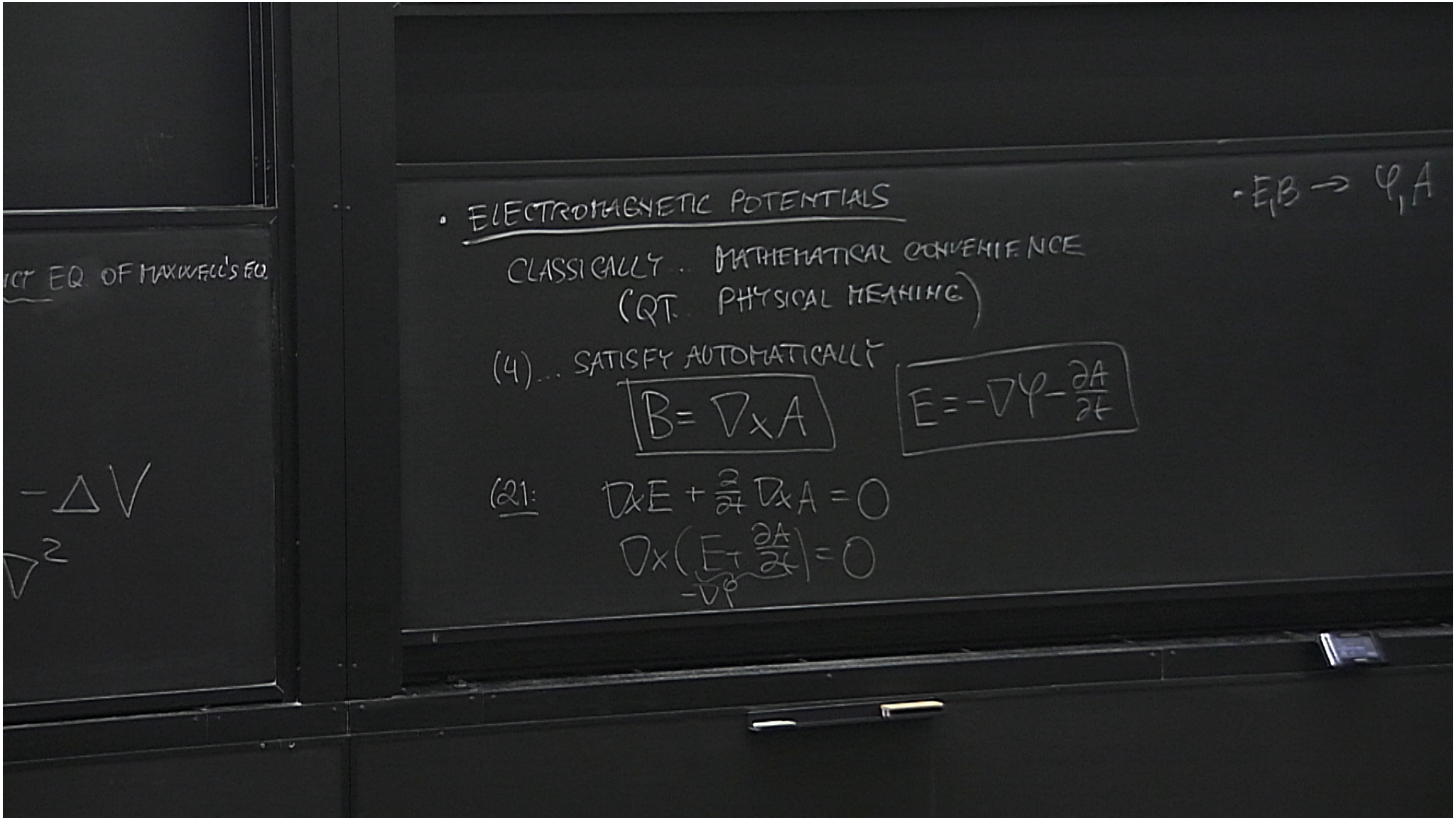

OF MAXWELL'S EQ.
MAXWELL'S TRIUMPH
• CONTINUITY EQ. FOLLOWS FROM (ME)
• IF NO SOURCES ... FIELD CAN STILL BE
NON-TRIVIAL !

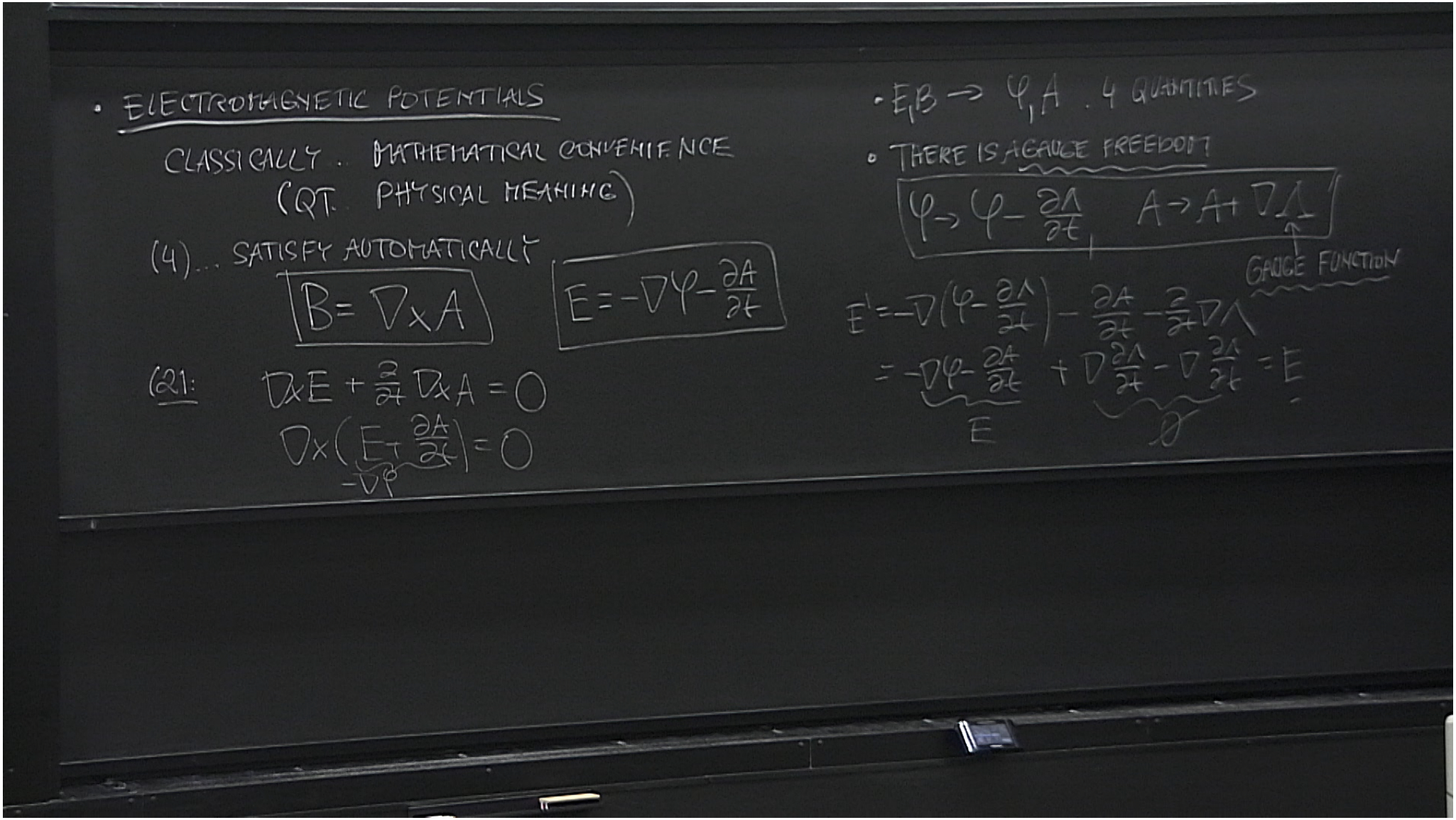

• MAXWELL'S CURRENT => CONTINUITY EQ $\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0$ IS A CONSISTENCY EQ OF MAXWELL'S EQ.
... CHARGE CONSERVATION
• 6 EVOLUTION EQS. FOR 6 UNKNOWNS E, B
IDENTITY $\nabla \times (\nabla \times V) = \nabla \nabla \cdot V - \Delta V$
$\Delta = \nabla \cdot \nabla = \nabla^2$

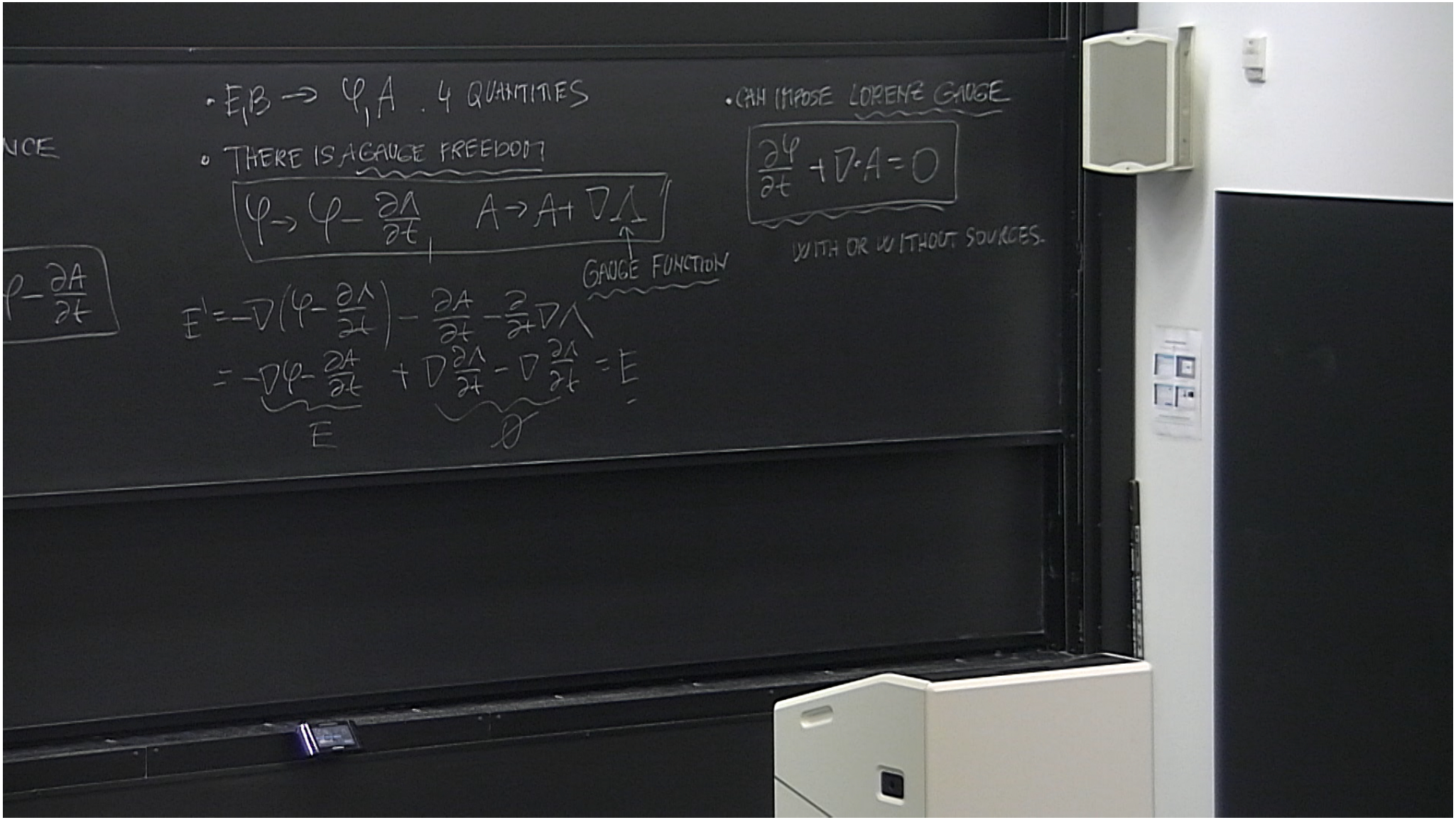

MAXWELL'S TRIUMPH
• CONTINUITY EQ. FOLLOWS FROM (ME)
• IF NO SOURCES ... FIELD CAN STILL BE NON-TRIVIAL !
$\nabla \times (2) \qquad \nabla \underbrace{\nabla \cdot E}_{0\,(3)} - \Delta E + \frac{\partial}{\partial t} \underbrace{\nabla \times B}_{\frac{\partial E}{\partial t}\,(1)} = 0$
$\left(\Delta - \frac{\partial^2}{\partial t^2}\right) E = 0$

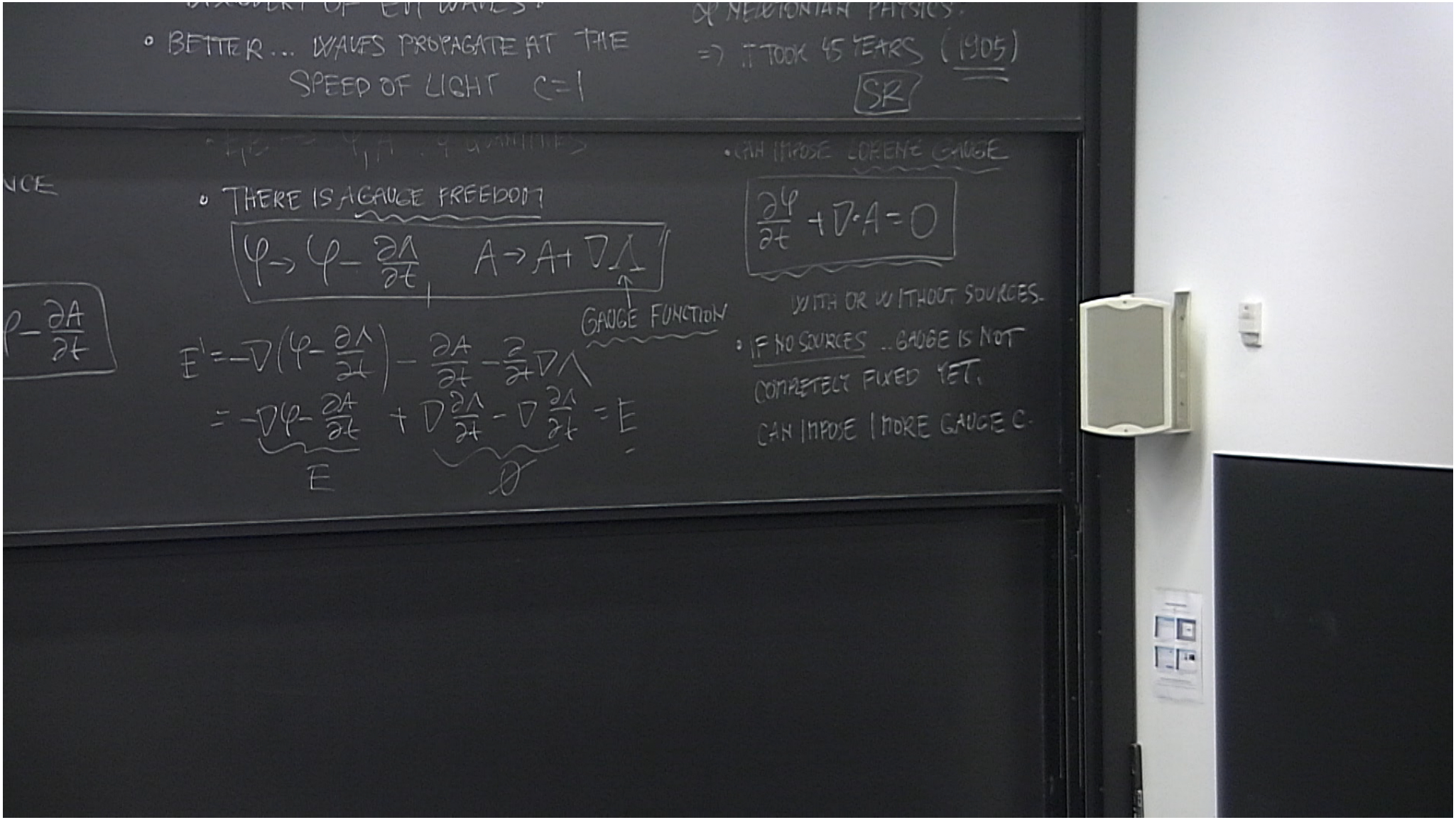

□ = −1/c² ∂²/∂t² + Δ
WAVE OPERATOR
C .. SPEED OF PROPAGATION
□E = 0
c = 1
SIMILARLY: □B = 0
LIGHT IS JUST AN EM WAVE!
⇒ UNIFICATION OF EM & OPTICS
THE VERY EXISTENCE OF ABSOLUTE SPEED CONTRADICTS GAL. TRANSF & NEWTONIAN PHYSICS!
DISCOVERY OF EM WAVES!
• BETTER ... WAVES PROPAGATE AT THE SPEED OF LIGHT c = 1
FIELD CAN STILL BE
−ΔE + ∂/∂t ∇×B = 0
∂E/∂t
(1)
(3)
E = 0

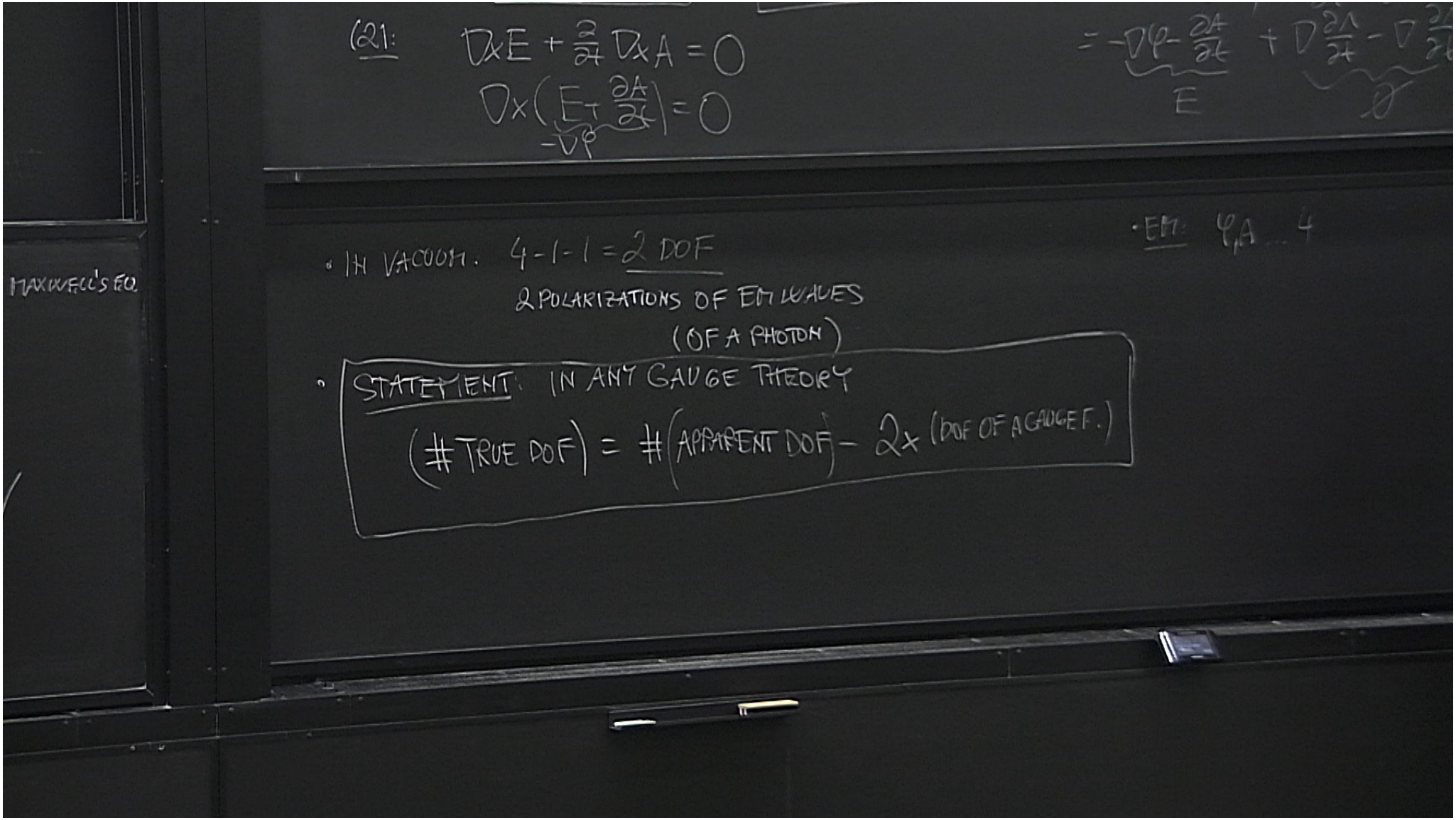

• MAXWELL'S CURRENT => CONTINUITY EQ $\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0$ IS A CONSISTENCY EQ OF MAXWELL'S EQ.
... CHARGE CONSERVATION
• 6 EVOLUTION EQS. FOR 6 UNKNOWNS E, B
IDENTITY $\nabla \times (\nabla \times V) = \nabla \nabla \cdot V - \Delta V$
$\nabla \cdot (\nabla \times V) = 0$
$\nabla \times \nabla f = 0$
$\Delta = \nabla \cdot \nabla = \nabla^2$

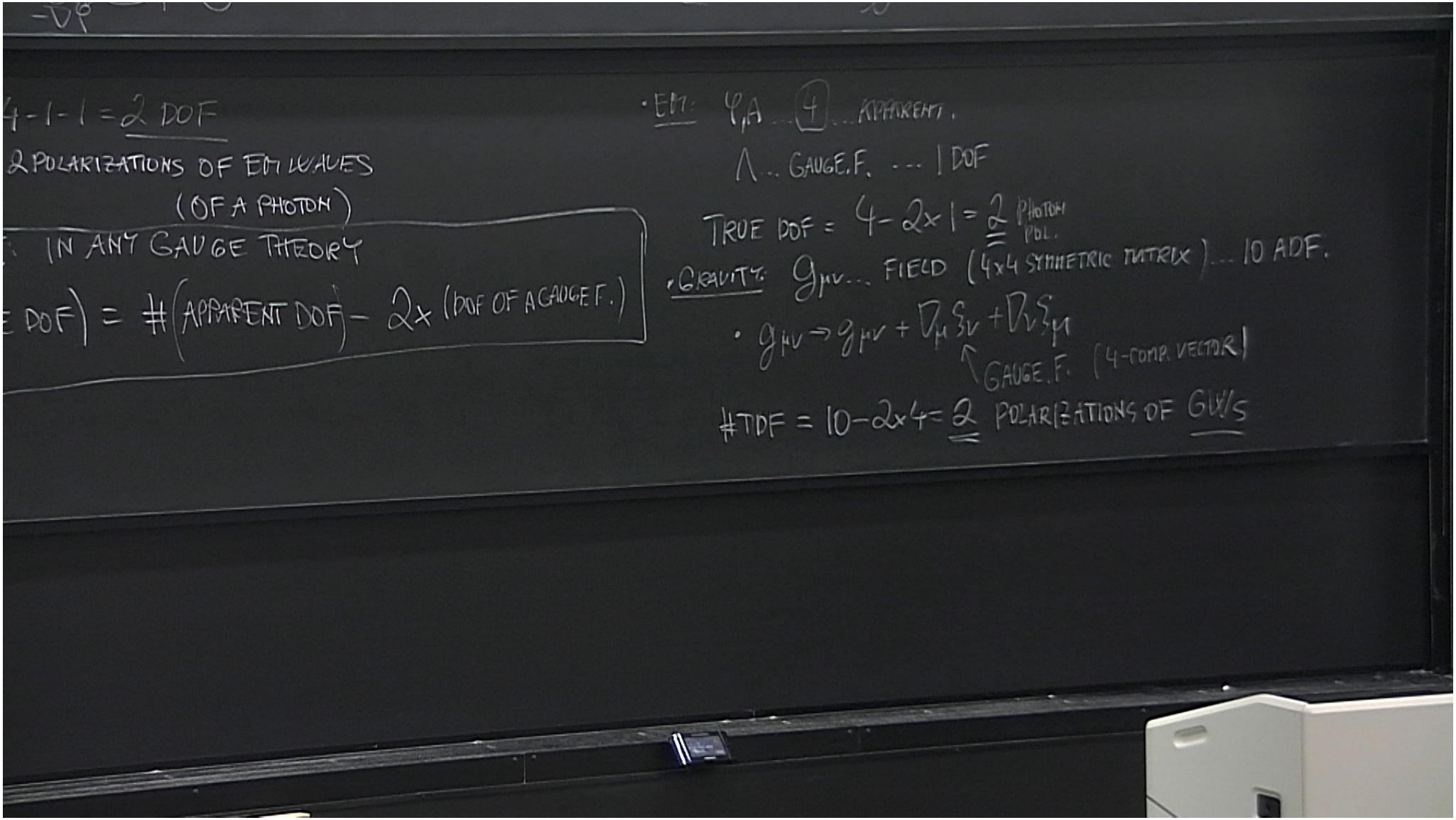

ICT EQ. OF MAXWELL'S EQ.
$-\Delta V$
∇^2
• ELECTROMAGNETIC POTENTIALS
• E,B → φ,A
CLASSICALLY .. MATHEMATICAL CONVENIENCE
(QT. PHYSICAL MEANING)
(4) ... SATISFY AUTOMATICALLY
$B = \nabla \times A$
$E = -\nabla\varphi - \frac{\partial A}{\partial t}$
(2): $\nabla \times E + \frac{\partial}{\partial t} \nabla \times A = 0$
$\nabla \times (E + \frac{\partial A}{\partial t}) = 0$
$-\nabla\varphi$

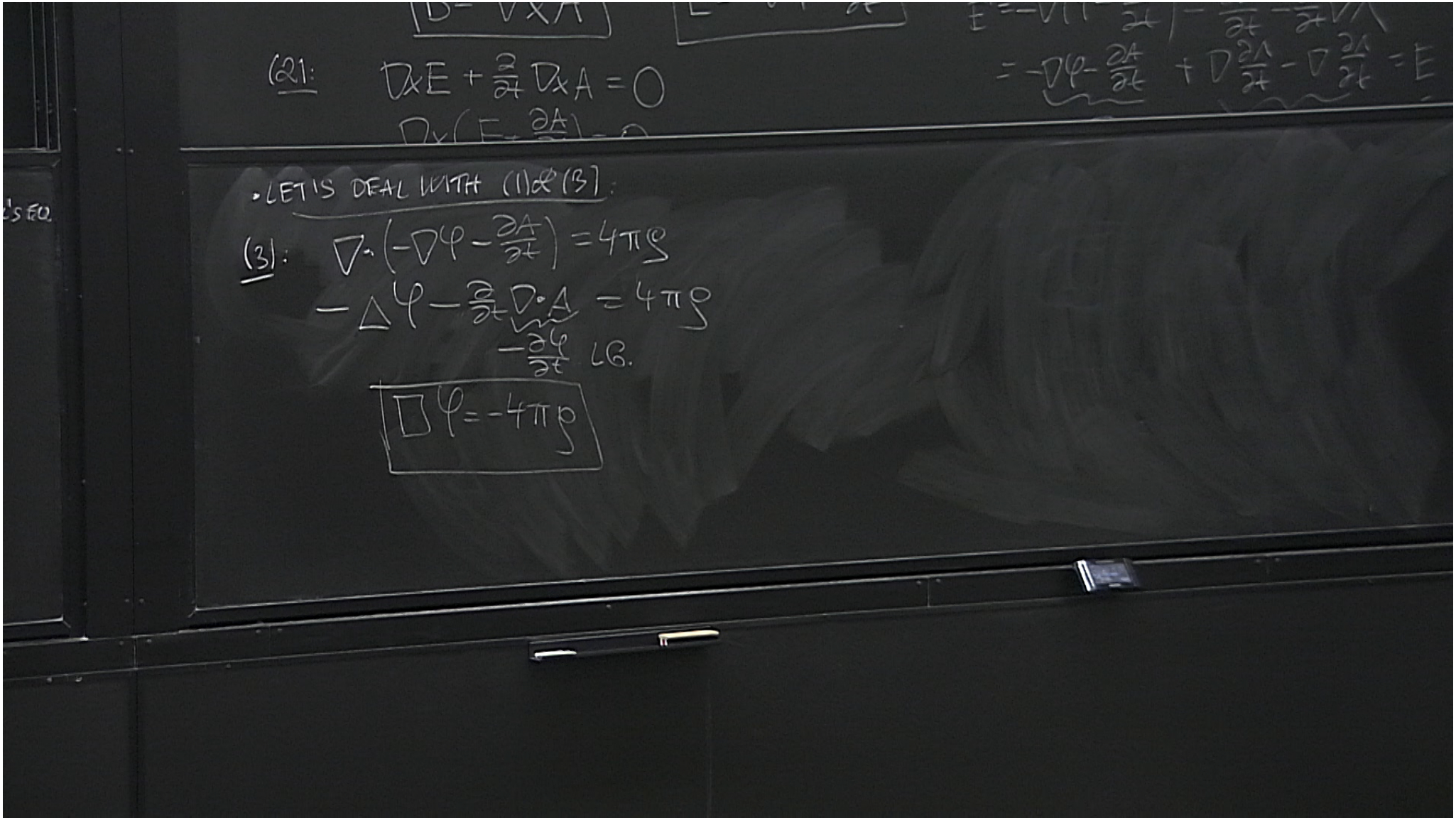

• ELECTROMAGNETIC POTENTIALS
CLASSICALLY .. MATHEMATICAL CONVENIENCE
(QT. PHYSICAL MEANING)
(4)... SATISFY AUTOMATICALLY
B = ∇×A
E = −∇φ − ∂A/∂t
(2): ∇×E + ∂/∂t ∇×A = 0
∇×(E + ∂A/∂t) = 0
−∇φ
• E,B → φ, A . 4 QUANTITIES
• THERE IS A GAUGE FREEDOM
φ → φ − ∂Λ/∂t , A → A + ∇Λ
GAUGE FUNCTION
E' = −∇(φ − ∂Λ/∂t) − ∂A/∂t − ∂/∂t ∇Λ
= −∇φ − ∂A/∂t + ∇ ∂Λ/∂t − ∇ ∂Λ/∂t = E
E
0

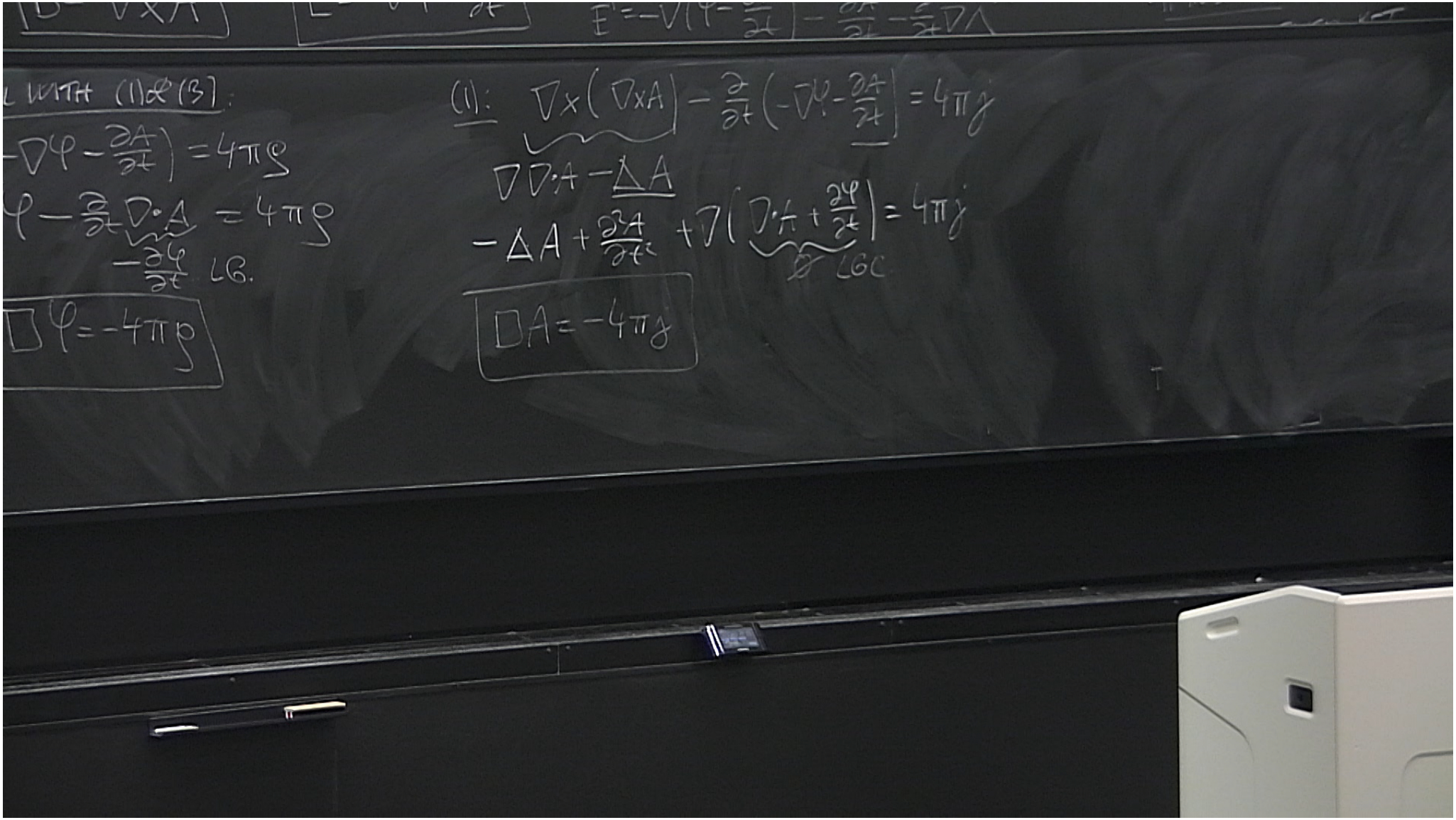

• E,B → φ,A . 4 QUANTITIES
• THERE IS A GAUGE FREEDOM
φ → φ − ∂Λ/∂t , A → A + ∇Λ
GAUGE FUNCTION
E' = −∇(φ − ∂Λ/∂t) − ∂A/∂t − ∂/∂t ∇Λ
= −∇φ − ∂A/∂t + ∇ ∂Λ/∂t − ∇ ∂Λ/∂t = E
E
0
• CAN IMPOSE LORENZ GAUGE
∂φ/∂t + ∇·A = 0
WITH OR WITHOUT SOURCES.


• BETTER... WAVES PROPAGATE AT THE SPEED OF LIGHT c=1
=) IT TOOK 45 YEARS (1905) SR
• THERE IS A GAUGE FREEDOM
$\varphi \to \varphi - \frac{\partial \Lambda}{\partial t}, \quad A \to A + \nabla\Lambda$
GAUGE FUNCTION
$E' = -\nabla(\varphi - \frac{\partial \Lambda}{\partial t}) - \frac{\partial A}{\partial t} - \frac{\partial}{\partial t}\nabla\Lambda$
$= \underbrace{-\nabla\varphi - \frac{\partial A}{\partial t}}_{E} + \underbrace{\nabla\frac{\partial \Lambda}{\partial t} - \nabla\frac{\partial \Lambda}{\partial t}}_{0} = E$
• CAN IMPOSE LORENZ GAUGE
$\frac{\partial \varphi}{\partial t} + \nabla \cdot A = 0$
WITH OR WITHOUT SOURCES.
• IF NO SOURCES ..GAUGE IS NOT COMPLETELY FIXED YET.
CAN IMPOSE MORE GAUGE C.


(21: $\nabla\times E + \frac{\partial}{\partial t}\nabla\times A = 0$
$\nabla\times\left(E + \frac{\partial A}{\partial t}\right) = 0$
$-\nabla\varphi$
$= -\nabla\varphi - \frac{\partial A}{\partial t} + \nabla\frac{\partial \Lambda}{\partial t} - \nabla\frac{\partial}{\partial t}$
E
χ
MAXWELL'S EQ.
• IN VACUUM. $4-1-1 = 2$ DOF
2 POLARIZATIONS OF EM WAVES
(OF A PHOTON)
• EM: φ, A ... 4
• STATEMENT: IN ANY GAUGE THEORY
(# TRUE DOF) = #(APPARENT DOF) − 2× (DOF OF A GAUGE F.)


4-1-1 = 2 DOF
2 POLARIZATIONS OF EM WAVES
(OF A PHOTON)
IN ANY GAUGE THEORY
DOF) = #(APPARENT DOF) - 2x (DOF OF A GAUGE F.)
· EM: φ,A ... 4 ... APPARENT.
Λ .. GAUGE F. ... 1 DOF
TRUE DOF = 4 - 2x1 = 2 PHOTON POL.
· GRAVITY: g_μν ... FIELD (4x4 SYMMETRIC MATRIX) ... 10 ADF.
· g_μν → g_μν + D_μ ξ_ν + D_ν ξ_μ
GAUGE F. (4-COMP. VECTOR)
#TDF = 10 - 2x4 = 2 POLARIZATIONS OF GW'S


B = ∇×A
(2): ∇×E + (1/c) ∂/∂t ∇×A = 0
∇×(E + (1/c) ∂A/∂t) = 0
E = -∇φ - (1/c) ∂A/∂t
= -∇φ - (1/c) ∂A/∂t + ∇ (1/c) ∂A/∂t - ∇ (1/c) ∂A/∂t = E
• LET IS DEAL WITH (1) & (3):
(3): ∇·(-∇φ - (1/c) ∂A/∂t) = 4πρ
-Δφ - (1/c) ∂/∂t ∇·A = 4πρ
-(1/c) ∂φ/∂t LG.
□φ = -4πρ


WITH (1) & (3):
$-\nabla\varphi - \frac{\partial A}{\partial t}) = 4\pi\rho$
$\varphi - \frac{\partial}{\partial t} \nabla\cdot A = 4\pi\rho$
$-\frac{\partial\varphi}{\partial t}$ LG.
$\Box\varphi = -4\pi\rho$
(1): $\nabla\times(\nabla\times A) - \frac{\partial}{\partial t}(-\nabla\varphi - \frac{\partial A}{\partial t}) = 4\pi j$
$\nabla\nabla\cdot A - \Delta A$
$-\Delta A + \frac{\partial^2 A}{\partial t^2} + \nabla(\nabla\cdot A + \frac{\partial\varphi}{\partial t}) = 4\pi j$
0 LGC
$\Box A = -4\pi j$

(4)... SATISFY AUTOMATICALLY

$$\boxed{B = \nabla \times A} \qquad \boxed{E = -\nabla\varphi - \frac{\partial A}{\partial t}}$$

$\varphi \to \varphi - \frac{\partial \Lambda}{\partial t}$, $A \to A + \nabla\Lambda$ — GAUGE FUNCTION

$$E' = -\nabla\left(\varphi - \frac{\partial \Lambda}{\partial t}\right) - \frac{\partial A}{\partial t} - \frac{\partial}{\partial t}\nabla\Lambda$$

• LET'S DEAL WITH (1) & (3):

(3): $\nabla \cdot \left(-\nabla\varphi - \frac{\partial A}{\partial t}\right) = 4\pi\rho$

$-\Delta\varphi - \frac{\partial}{\partial t}\nabla\cdot A = 4\pi\rho$

$-\frac{\partial \varphi}{\partial t}$ LG.

$$\boxed{\Box\varphi = -4\pi\rho}$$

$\varphi = \varphi_H + \varphi_S$ ← SPECIAL SOL. $\Box\varphi = -4\pi\rho$.

↑ WAVES $\Box\varphi = 0$

(1): $\nabla \times (\nabla \times A) - \frac{\partial}{\partial t}\left(-\nabla\varphi - \frac{\partial A}{\partial t}\right) = 4\pi j$

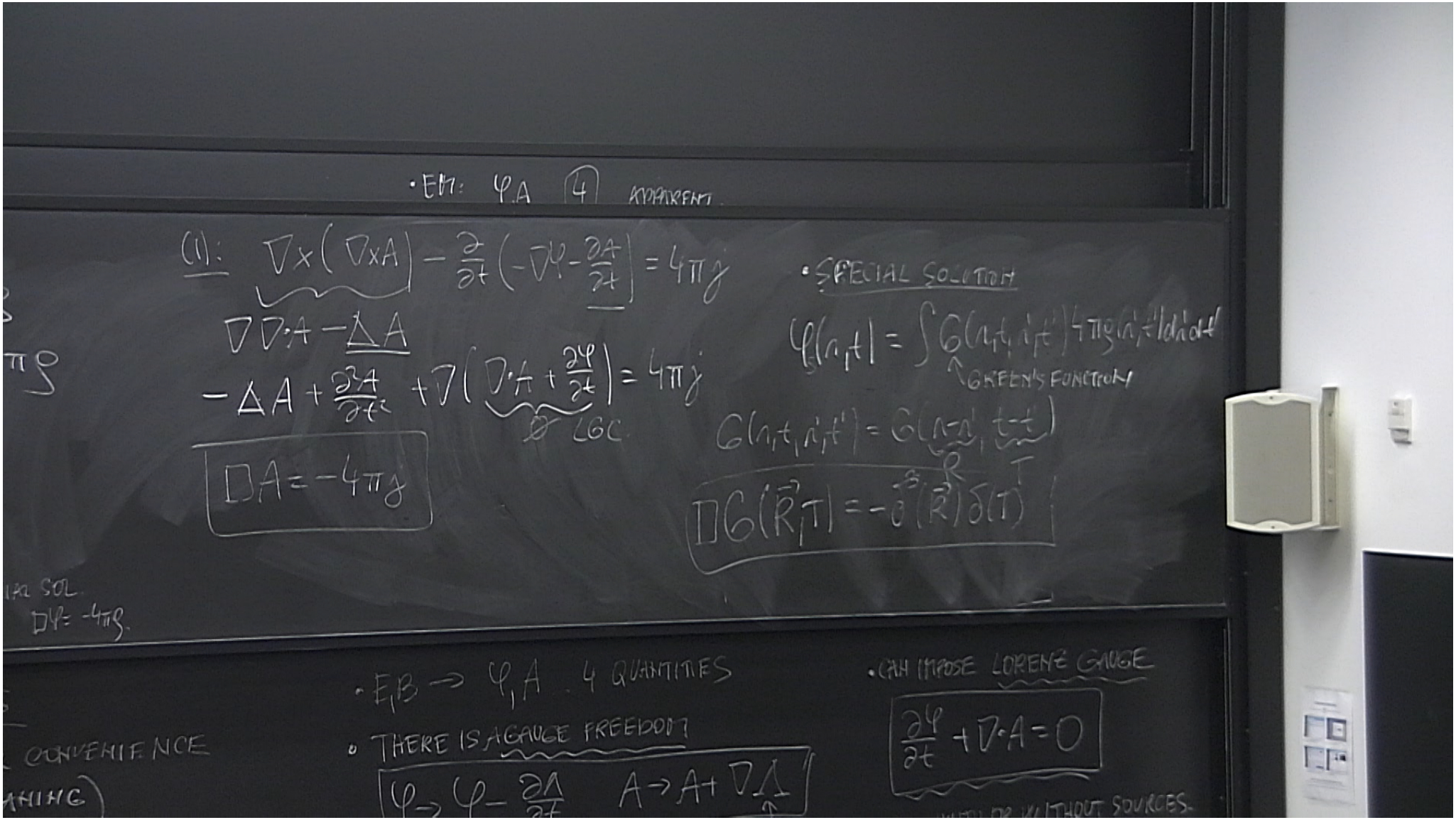
$\nabla\nabla\cdot A - \Delta A$

$-\Delta A + \frac{\partial^2 A}{\partial t^2} + \nabla\left(\nabla\cdot A + \frac{\partial\varphi}{\partial t}\right) = 4\pi j$

$\nabla\cdot A + \frac{\partial\varphi}{\partial t} = 0$ ← LGC

$$\boxed{\Box A = -4\pi j}$$

• SPEC


• EM: φ, A (4) APPARENT
(1): ∇×(∇×A) − ∂/∂t(−∇φ − ∂A/∂t) = 4πj
∇(∇·A) − ΔA
−ΔA + ∂²A/∂t² + ∇(∇·A + ∂φ/∂t) = 4πj
LGC
□A = −4πj
• SPECIAL SOLUTION
φ(r,t) = ∫ G(r,t; r',t') 4πρ(r',t') dr'dt'
GREEN'S FUNCTION
G(r,t,r',t') = G(r−r', t−t')
R T
□G(R,T) = −δ³(R)δ(T)
• E,B → φ, A 4 QUANTITIES
• THERE IS A GAUGE FREEDOM
φ → φ − ∂Λ/∂t A → A + ∇Λ
• CAN IMPOSE LORENZ GAUGE
∂φ/∂t + ∇·A = 0
WITH OR WITHOUT SOURCES.

$\varphi = \varphi_H + \varphi_S$ ← SPECIAL SOL. $\Box\varphi = -4\pi\rho$.

WAVES $\Box\varphi = 0$

PROOF: $\Box\varphi = \int (\Box_{x,t} G)\, 4\pi\rho(x',t')\, dx'dt' = -4\pi\rho(x,t)$

$\Box_{x,t} G = -\delta(R)\delta(T)$

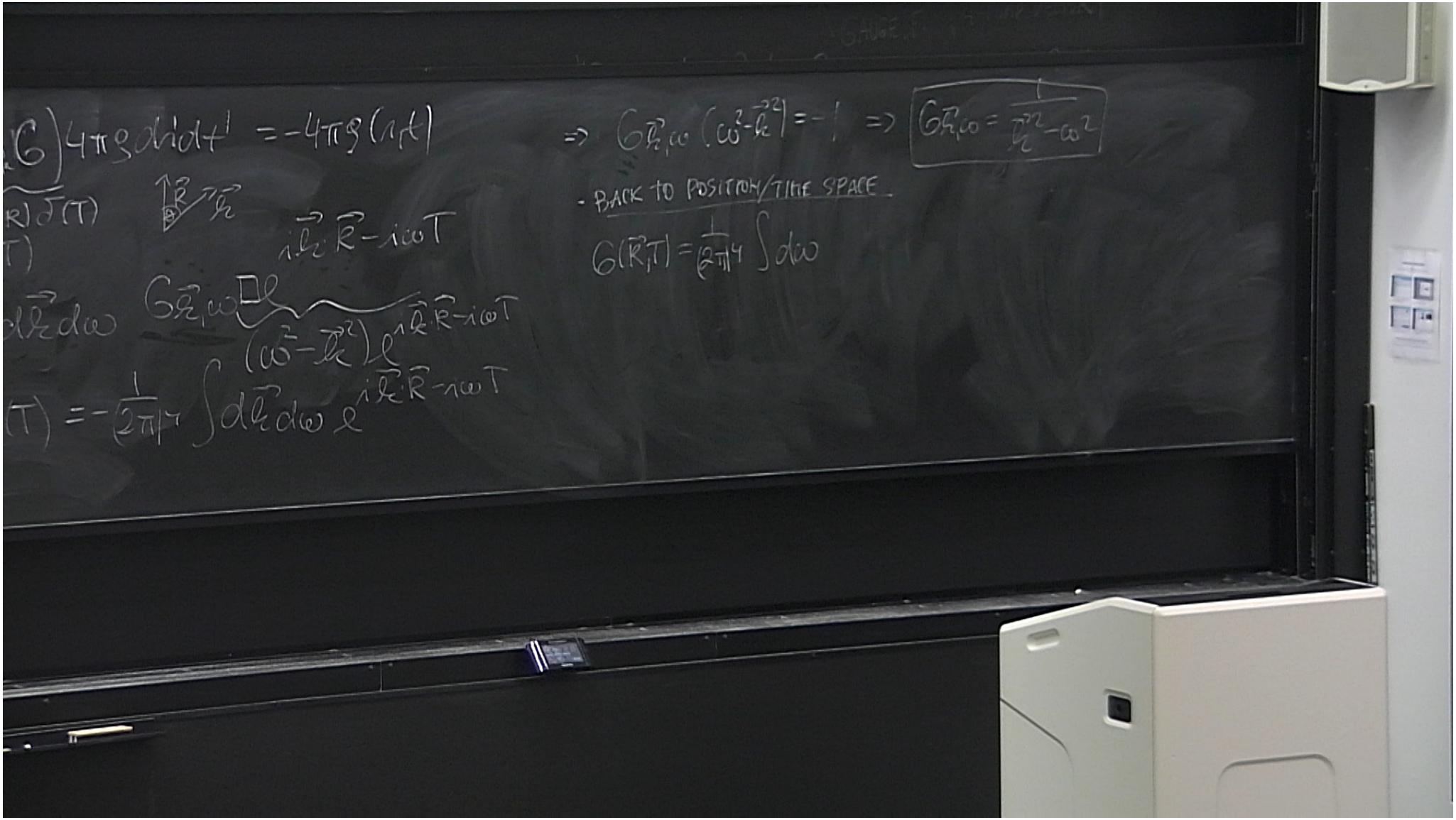
OF MAXWELL'S EQ.

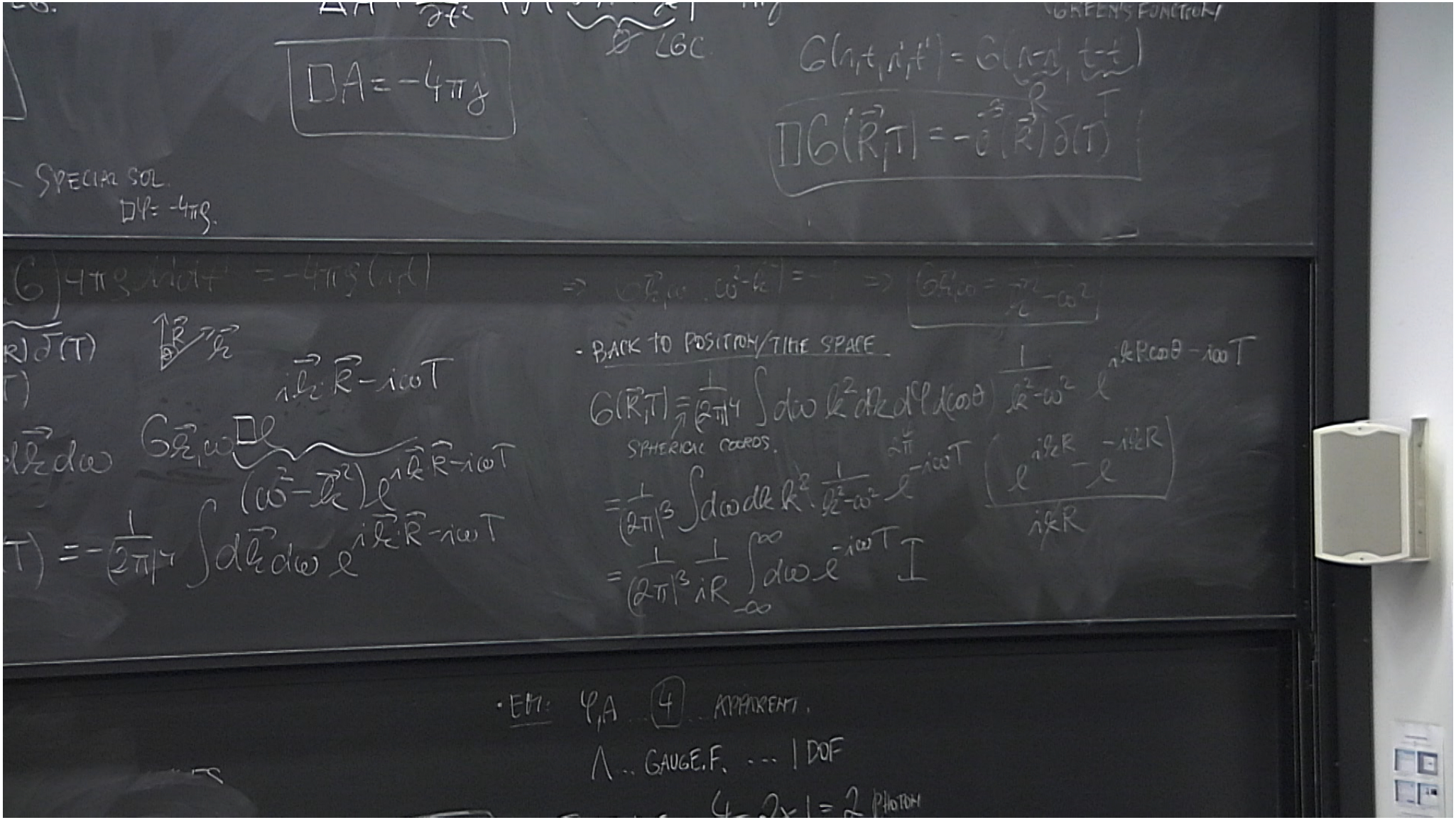
LET'S FIND G $\quad G(\vec{R},T)$

$$\Box G(\vec{R},T) = \frac{1}{(2\pi)^4}\int d\vec{k}\,d\omega\; G_{\vec{k},\omega}\,\underbrace{\Box e^{i\vec{k}\vec{R}-i\omega T}}_{(\omega^2-\vec{k}^2)\,e^{i\vec{k}\vec{R}-i\omega T}}$$

$$= -\delta(\vec{R})\delta(T) = -\frac{1}{(2\pi)^4}\int d\vec{k}\,d\omega\; e^{i\vec{k}\vec{R}-i\omega T}$$

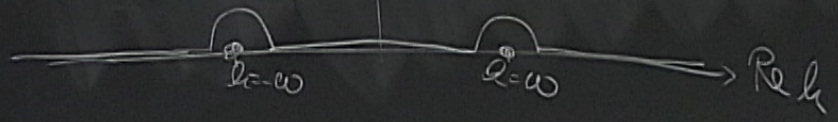
V


$\Box G) 4\pi\delta(t-t') = -4\pi\delta(1,t)$
$\Rightarrow G_{\vec{k},\omega}(\omega^2-\vec{k}^2) = -1 \Rightarrow G_{\vec{k},\omega} = \frac{1}{\vec{k}^2-\omega^2}$
- BACK TO POSITION/TIME SPACE -
$G(\vec{R},T) = \frac{1}{(2\pi)^4}\int d\omega$
$i\vec{k}\cdot\vec{R}-i\omega T$
$(\omega^2-\vec{k}^2)e^{i\vec{k}\cdot\vec{R}-i\omega T}$
$= -\frac{1}{(2\pi)^4}\int d\vec{k}\,d\omega\, e^{i\vec{k}\cdot\vec{R}-i\omega T}$

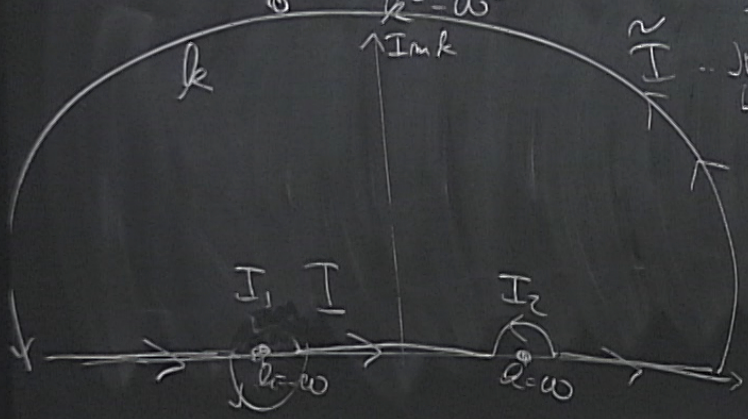

LGC
GREEN'S FUNCTION
SPECIAL SOL.
BACK TO POSITION/TIME SPACE
SPHERICAL COORDS.
GAUGE.F. ... 1 DOF
PHOTON

$$I = \int_0^{\infty} \frac{e^{ikR} - e^{-ikR}}{k^2 - \omega^2} k\,dk = \int_{-\infty}^{\infty} \frac{k e^{ikR}}{k^2 - \omega^2}\,dk$$

k



$$I = \int_0^{\infty} \frac{e^{ikR} - e^{-ikR}}{k^2 - \omega^2} k\,dk = \int_{-\infty}^{\infty} \frac{k e^{ikR}}{k^2 - \omega^2} dk \qquad R>0$$



JORDAN'S LEMMA

$$I + \tilde{I} - I_1 - I_2 = 2\pi i \sum_{RES} = 0$$

$\tilde{I} \to 0$

$$I = I_1 + I_2 = i\pi \left(\mathrm{Res}_{-\omega} f + \mathrm{Res}_{+\omega} f \right)$$

$$= \frac{i\pi}{2} \left(e^{i\omega R} + e^{-i\omega R} \right)$$

$$\lim_{k \to \omega} f(k=\omega) = \frac{k e^{ikR}}{(k-\omega)(k+\omega)} (k-\omega) = \frac{e^{i\omega R}}{2}$$

OF ELECTROMAGNETISM

$$\frac{\partial E}{\partial t} = 4\pi\rho \quad (1)$$

$$+\frac{\partial B}{\partial t} = 0 \quad (2)$$

$$\cdot E = 4\pi\rho \quad (3)$$

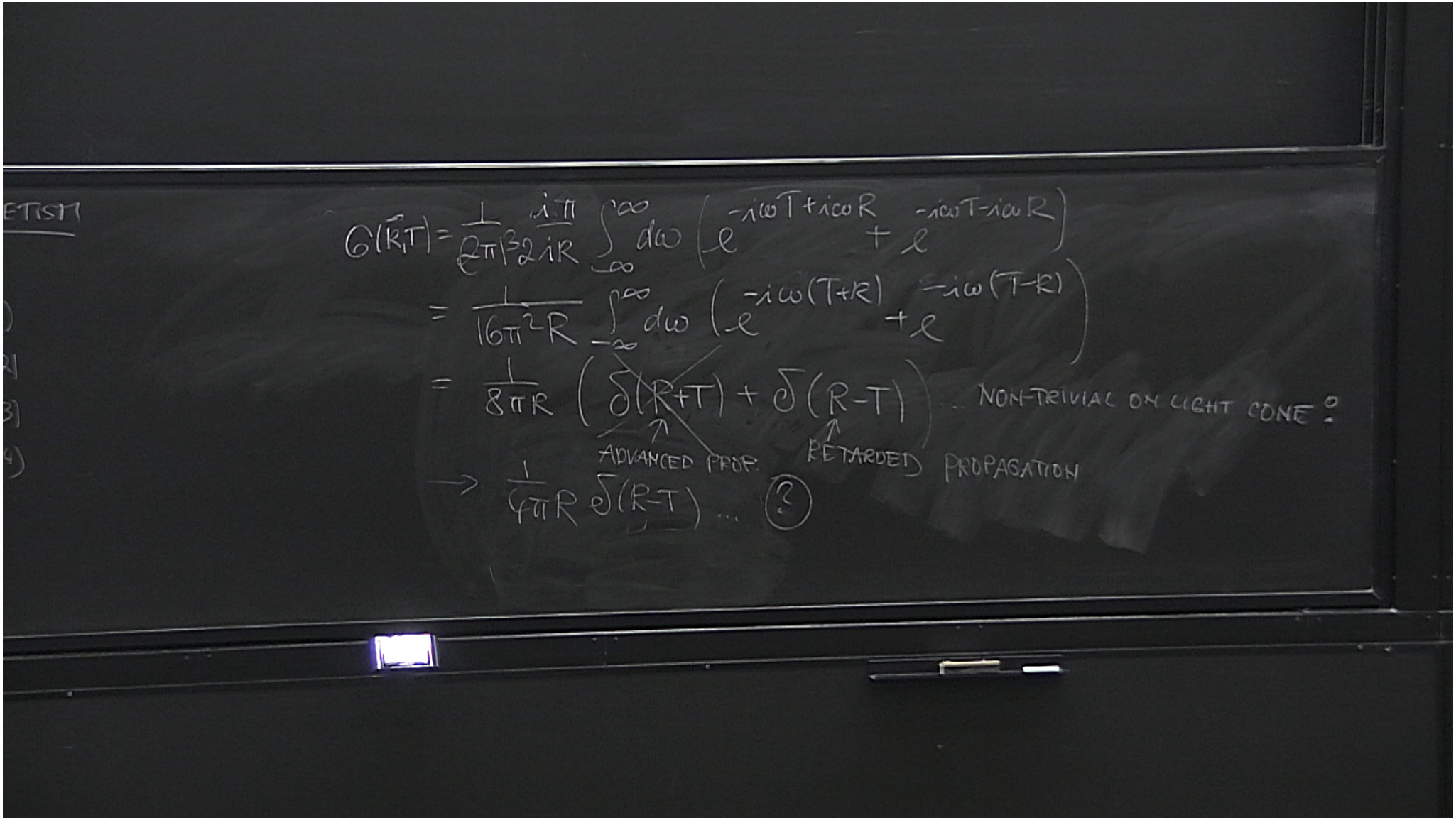
$$\nabla \cdot B = 0 \quad (4)$$

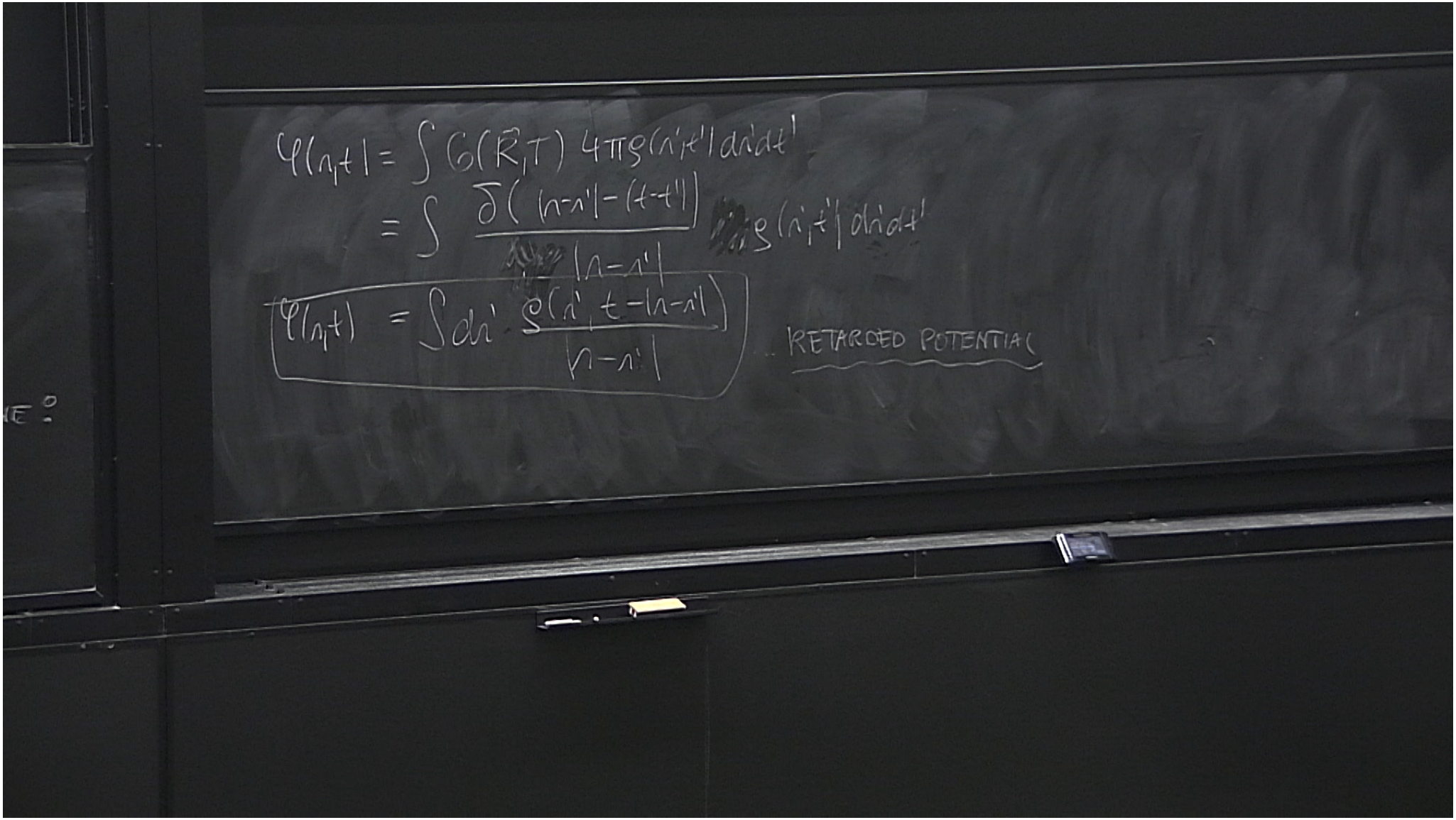
$$E + v \times B)$$

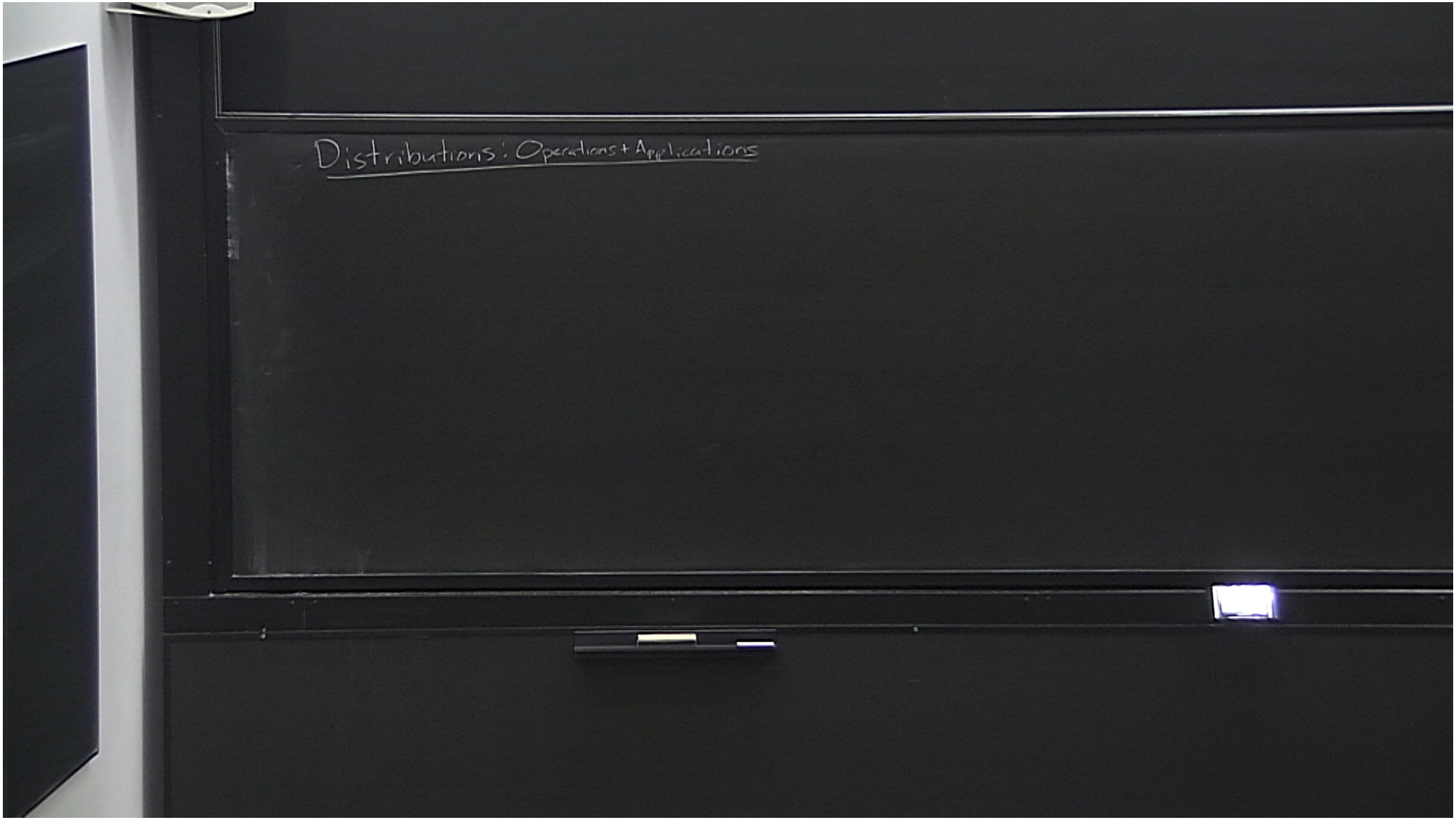
$$G(\vec{R},T) = \frac{1}{(2\pi)^3} \frac{i\pi}{2iR} \int_{-\infty}^{\infty} d\omega \left(e^{-i\omega T + i\omega R} + e^{-i\omega T - i\omega R} \right)$$

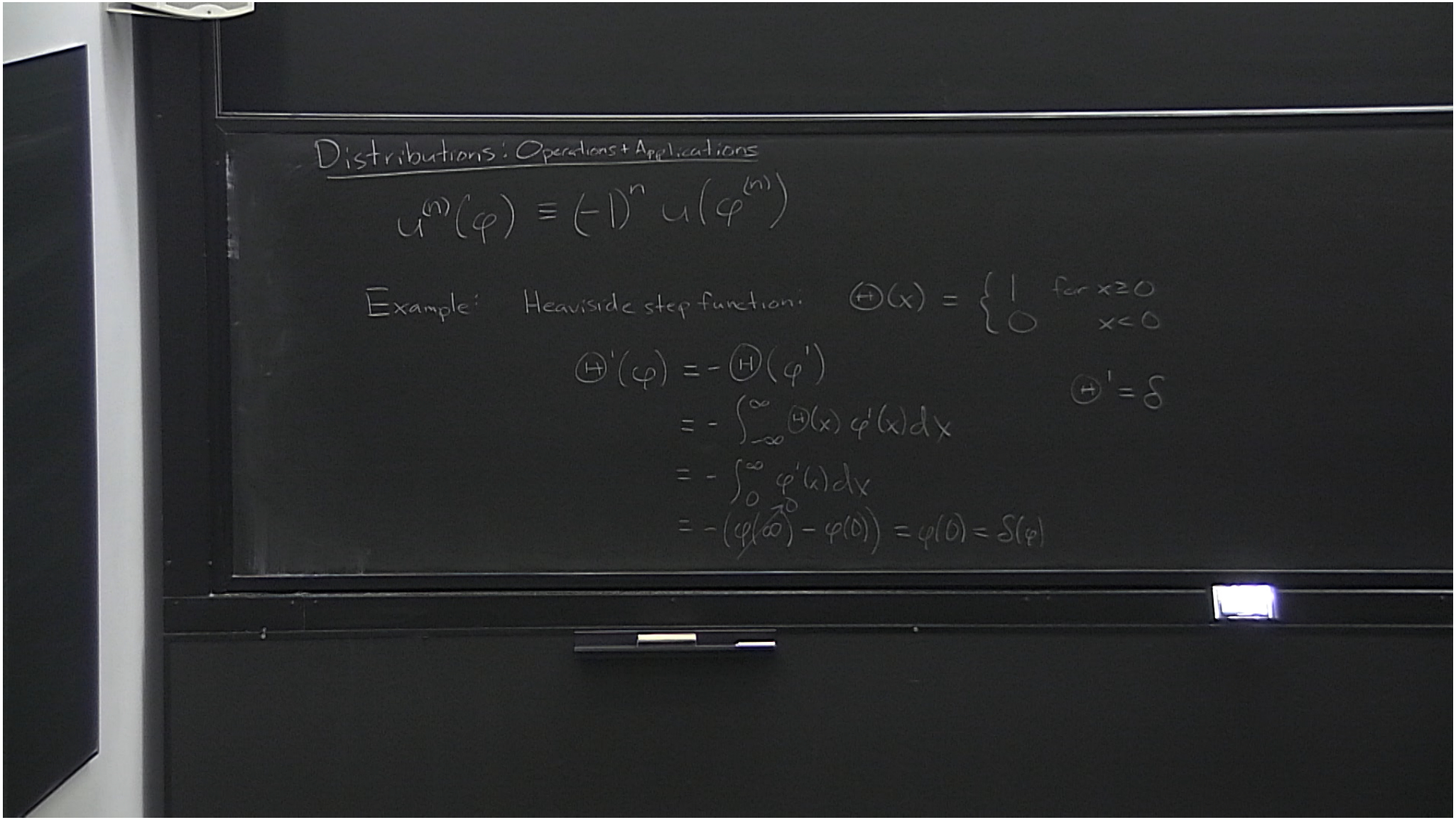
$$= \frac{1}{16\pi^2 R} \int_{-\infty}^{\infty} d\omega \left(e^{-i\omega(T+R)} + e^{-i\omega(T-R)} \right)$$

$$= \frac{1}{8\pi R} \left(\delta(R+T) + \delta(R-T) \right)$$


ETIST
$G(R,T) = \frac{1}{(2\pi)^3} \frac{i\pi}{2iR} \int_{-\infty}^{\infty} d\omega \left(e^{-i\omega T + i\omega R} + e^{-i\omega T - i\omega R} \right)$
$= \frac{1}{16\pi^2 R} \int_{-\infty}^{\infty} d\omega \left(e^{-i\omega(T+R)} + e^{-i\omega(T-R)} \right)$
$= \frac{1}{8\pi R} \left(\delta(R+T) + \delta(R-T) \right)$
NON-TRIVIAL ON LIGHT CONE !
ADVANCED PROP.
RETARDED PROPAGATION
$\rightarrow \frac{1}{4\pi R} \delta(R-T)$ ... (?)


$\varphi(r,t) = \int G(\vec{R},T)\, 4\pi\rho(r',t')\, dr'dt'$
$= \int \frac{\delta(|r-r'| - (t-t'))}{|r-r'|} \rho(r',t')\, dr'dt'$
$\varphi(r,t) = \int dr' \frac{\rho(r', t-|r-r'|)}{|r-r'|}$
RETARDED POTENTIAL


Distributions: Operations + Applications


Distributions: Operations + Applications
$u^{(n)}(\varphi) \equiv (-1)^n u(\varphi^{(n)})$
Example: Heaviside step function: $\Theta(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & x < 0 \end{cases}$
$\Theta'(\varphi) = -\Theta(\varphi')$
$= -\int_{-\infty}^{\infty} \Theta(x)\, \varphi'(x)\, dx$
$= -\int_{0}^{\infty} \varphi'(x)\, dx$
$= -(\varphi(\infty) - \varphi(0)) = \varphi(0) = \delta(\varphi)$
$\Theta' = \delta$

$\Theta(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & x < 0 \end{cases}$

$\Theta' = \delta$

$\int \varphi'(x)\,dx$

$\int \ldots dx$

$\ldots \varphi(0)) = \varphi(0) = \delta(\varphi)$

Example: $(\ln|x|)'$

$\int_0^a \frac{1}{x}\,dx \to \infty$ $\quad \frac{1}{x}$ not locally integrable

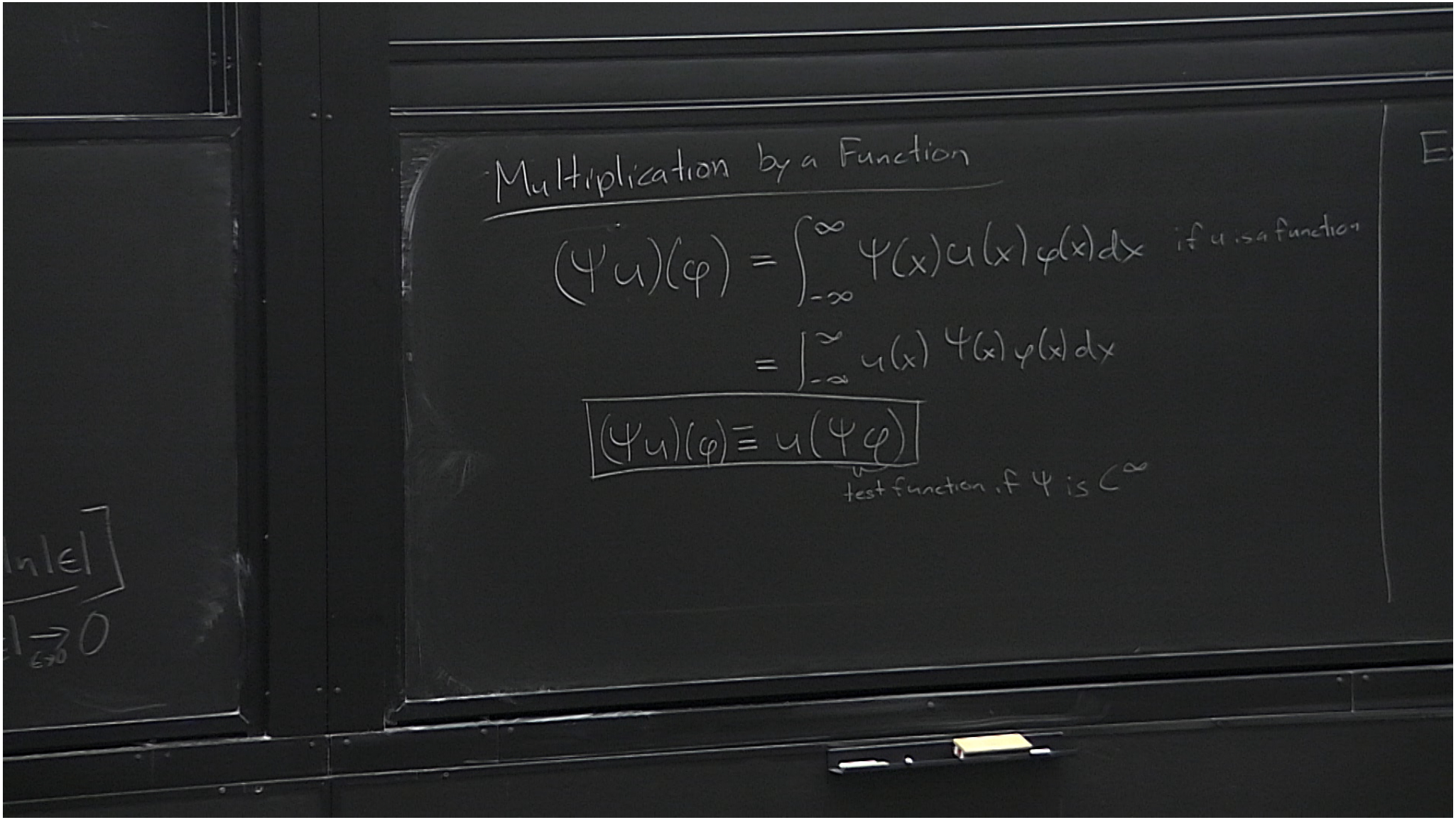
$$(\ln|x|)(\varphi) \equiv \lim_{\epsilon \to 0} \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right) \varphi(x) \ln|x|\,dx$$

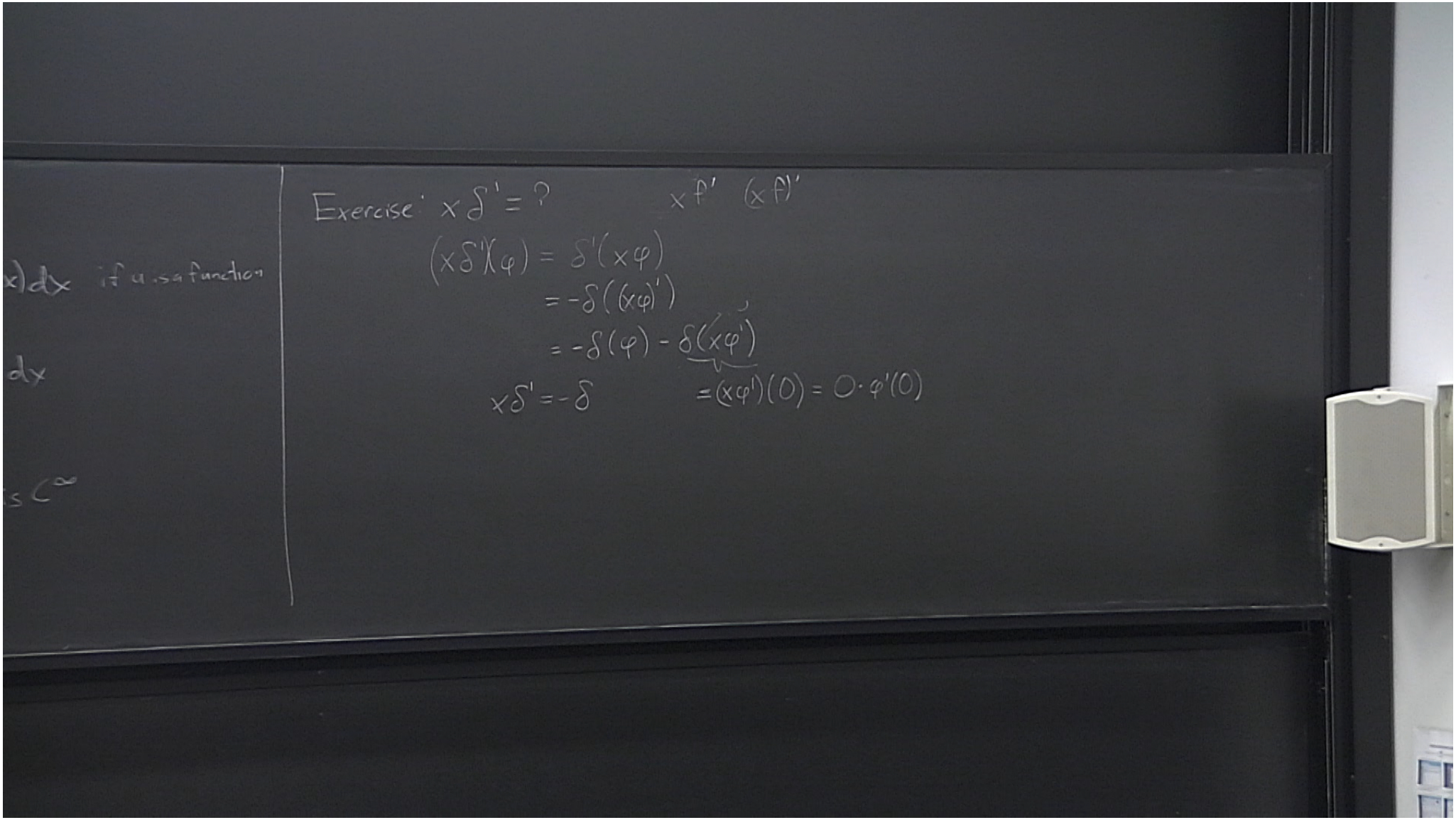
↑ cut-out part vanishes

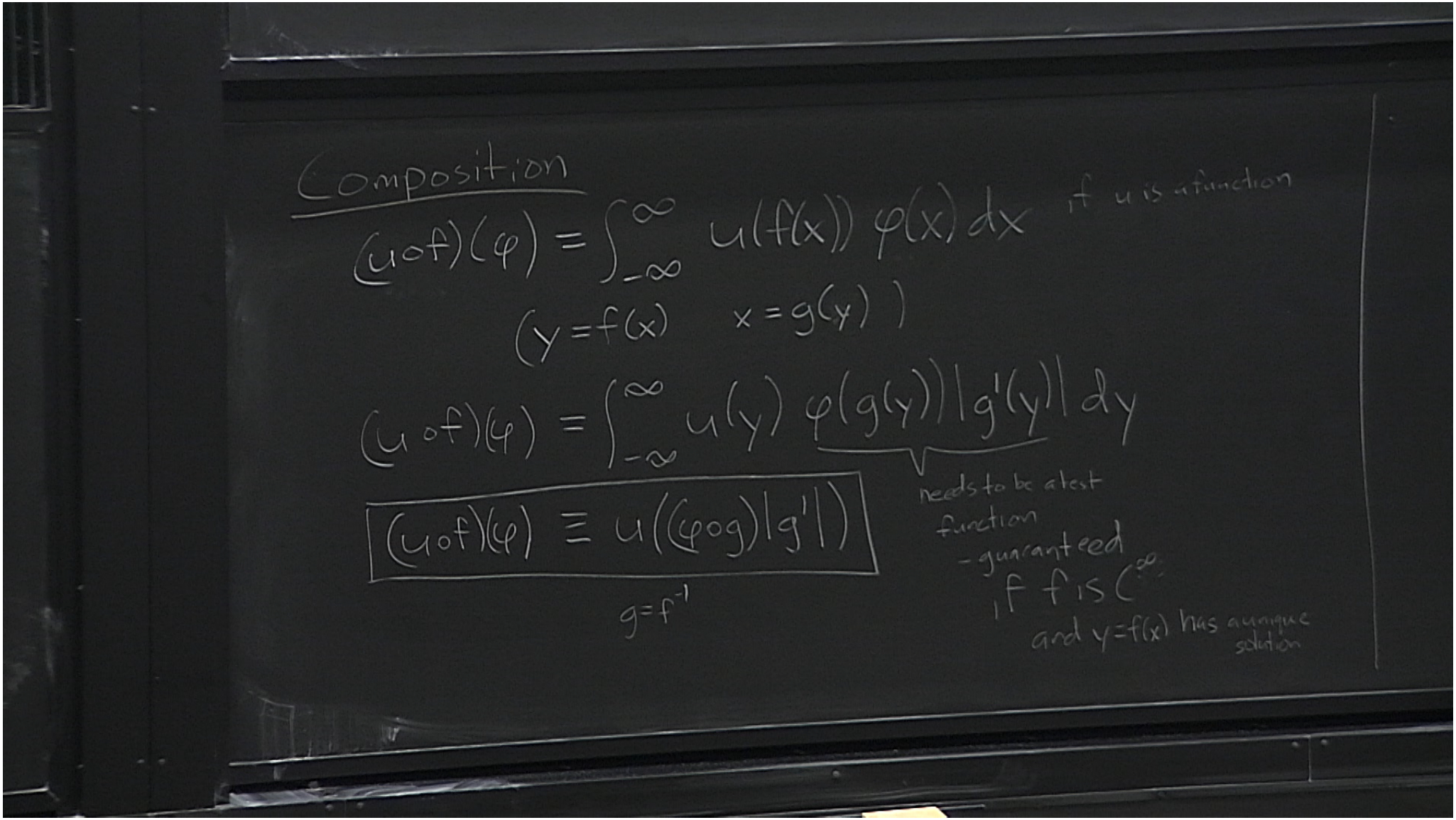
$$(\ln|x|)'(\varphi) = -\lim_{\epsilon \to 0} \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right) \varphi'(x) \ln|x|\,dx$$

$$= \lim_{\epsilon \to 0} \left[\left(\int_{-\infty}^{-\epsilon} - \int_{\epsilon}^{\infty} \right) \frac{1}{x} \varphi(x)\,dx + \underbrace{\varphi(\epsilon)\ln|\epsilon| - \varphi(-\epsilon)\ln|\epsilon|}_{\approx 2\varphi'(0)\epsilon\ln|\epsilon| \xrightarrow{\epsilon \to 0} 0} \right]$$

$$\equiv (PV\tfrac{1}{x})(\varphi)$$


Multiplication by a Function
$(\psi u)(\varphi) = \int_{-\infty}^{\infty} \psi(x) u(x) \varphi(x) dx$ if u is a function
$= \int_{-\infty}^{\infty} u(x) \psi(x) \varphi(x) dx$
$(\psi u)(\varphi) \equiv u(\psi\varphi)$
test function if ψ is C^∞


x)dx if u.s a function
dx
is C∞
Exercise: xδ' = ?
xf' (xf)'
(xδ')(φ) = δ'(xφ)
= -δ((xφ)')
= -δ(φ) - δ(xφ')
xδ' = -δ
= (xφ')(0) = 0·φ'(0)


Composition
$(u \circ f)(\varphi) = \int_{-\infty}^{\infty} u(f(x))\,\varphi(x)\,dx$ if u is a function
$(y = f(x) \quad x = g(y))$
$(u \circ f)(\varphi) = \int_{-\infty}^{\infty} u(y)\,\varphi(g(y))\,|g'(y)|\,dy$
needs to be a test function
- guaranteed if f is C^∞ and $y = f(x)$ has a unique solution
$\boxed{(u \circ f)(\varphi) \equiv u((\varphi \circ g)|g'|)}$
$g = f^{-1}$

$$(\delta \circ f)(\varphi) = \sum_i \frac{\delta_{x_i}}{|f'(x_i)|}(\varphi)$$

$f(x_i) = 0$

$\delta_{x_i}(\varphi) = \varphi(x_i)$

Works if f has only simple zeros

$f(x) = x^2 - 1 \rightarrow (\delta \circ f)(\varphi) = \left(\frac{\delta_{-1}}{2} + \frac{\delta_1}{2}\right)(\varphi)$

$f(x) = x^2 \rightarrow (\delta \circ f)$ undefined

Composition

$$(u \circ f)(\varphi) = \int_{-\infty}^{\infty} u(f(x))\,\varphi(x)\,dx$$ if u is a function

$(y = f(x) \quad x = g(y))$

$$(u \circ f)(\varphi) = \int_{-\infty}^{\infty} u(y)\,\varphi(g(y))\,|g'(y)|\,dy$$

$$\boxed{(u \circ f)(\varphi) \equiv u((\varphi \circ g)|g'|)}$$

$g = f^{-1}$

needs to be a test function - guaranteed if f is C^∞ and $y = f(x)$ has a unique solution