

Title: PSI 2019/2020 - Classical Physics - Lecture 5

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Collection: PSI 2019/2020 - Classical Physics (Kubiznak)

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III) CLASSICAL FIELD THEORY

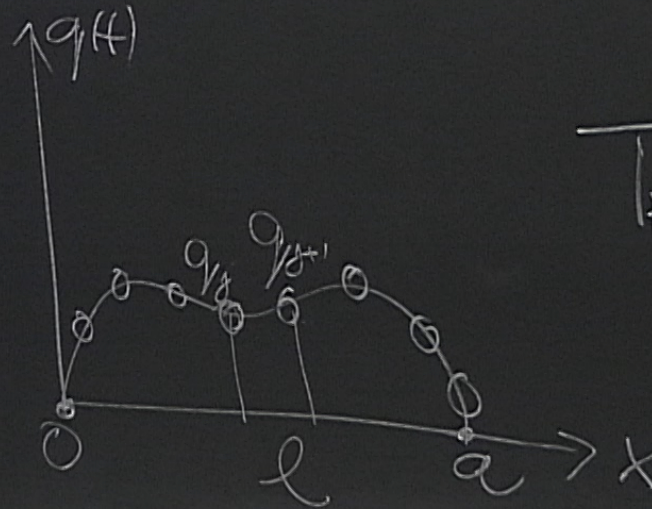
a) VIBRATIONS OF STRING: FROM DISCRETE
TO CONTINUUM

III) CLASSICAL FIELD THEORY

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a) VIBRATIONS OF STRING: FROM DISCRETE
TO CONTINUUM TRANVERSE VIBRATIONS.



DISCRETE

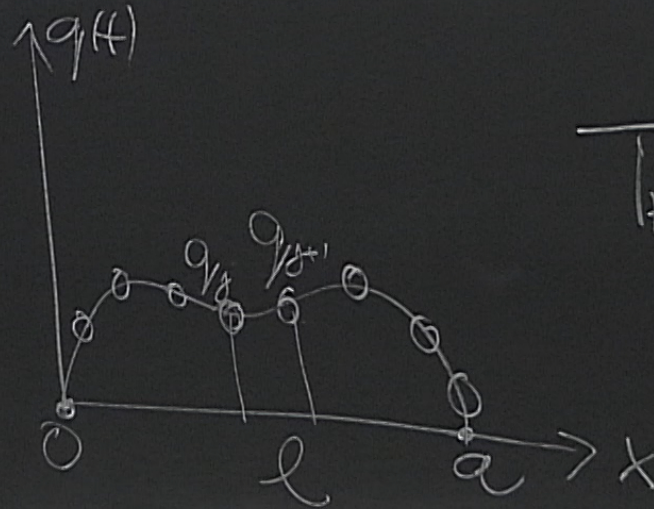
N IDENTICAL BEADS
 (EACH WITH MASS m)

LABELED BY $j = 1, \dots, N$

$$T_K = \frac{1}{2} \sum_{j=1}^N m \dot{q}_j^2$$

III) CLASSICAL FIELD THEORY

a) VIBRATIONS OF STRING: FROM DISCRETE
TO CONTINUUM TRANVERSE VIBRATIONS.



DISCRETE

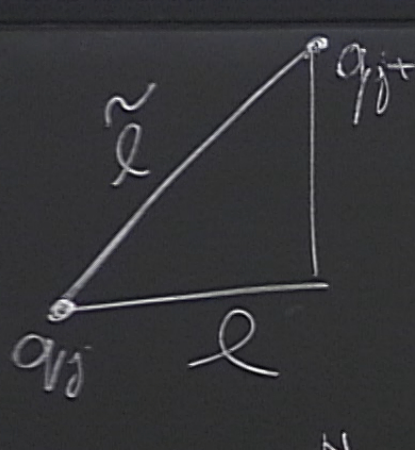
N IDENTICAL BEADS
 (EACH WITH MASS m)

LABELED BY $j = 1, \dots, N$

$$T = \frac{1}{2} \sum_{j=1}^N m \dot{q}_j^2$$

$$V = \sum_{j=1}^{N-1} T (\Delta l)_j$$

↑
TENSION



$$(\Delta l)_j = (\tilde{l} - l)_j = \sqrt{l^2 + (q_{j+1} - q_j)^2} - l$$

$$= l \left(1 + \frac{1}{2} \frac{(q_{j+1} - q_j)^2}{l^2} \right) - l = \frac{l}{2} \left(\frac{q_{j+1} - q_j}{l} \right)^2$$

$$V = \sum_{j=1}^N \frac{l}{2} T \left(\frac{q_{j+1} - q_j}{l} \right)^2$$

$$L = T_K - V$$

DISCRETE

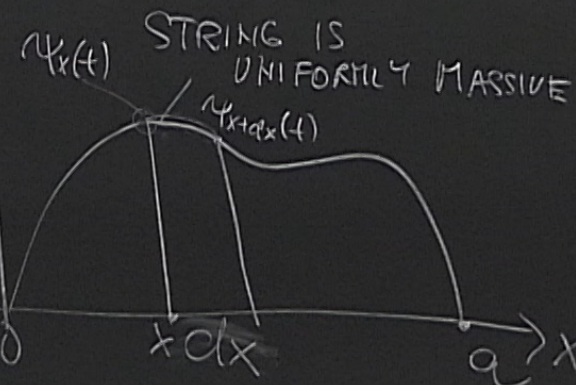
IDENTICAL BEADS
EACH WITH MASS
(m)

BY $j = 1 \rightarrow N$

Δx

CONTINUUM
 $\ell \rightarrow 0, N \rightarrow \infty$

$y(t)$



SO THAT

$$M = \sum_{j=1}^N m = \sum_j \mu \ell$$

$\rightarrow \int_0^a \mu dx$

$\uparrow$ UNIFORM MASS DENSITY

DISCRETE

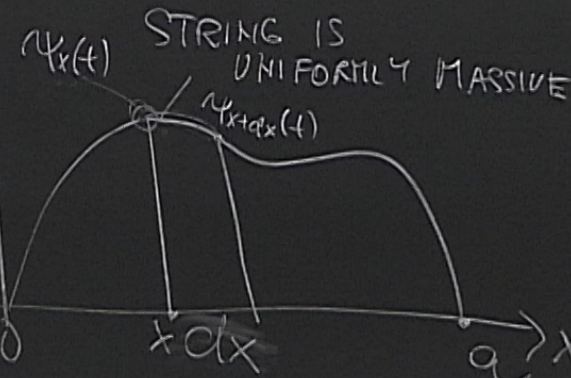
IDENTICAL BEADS
EACH WITH MASS
(m)

BY $j = 1 \rightarrow N$

Δx

CONTINUUM
 $\Delta x \rightarrow 0, N \rightarrow \infty$

$\psi(x, t)$



SO THAT

$$M = \sum_{j=1}^N m = \sum_j \mu \Delta x \rightarrow \int_0^a \mu dx$$

UNIFORM MASS DENSITY

$$\psi(x, t) \rightarrow \psi(x, t)$$

↑ LABEL WHERE ↑ LABEL WHEN

FIELD

PHYSICAL ENTITY THAT
DEPENDS ON SPACE & TIME

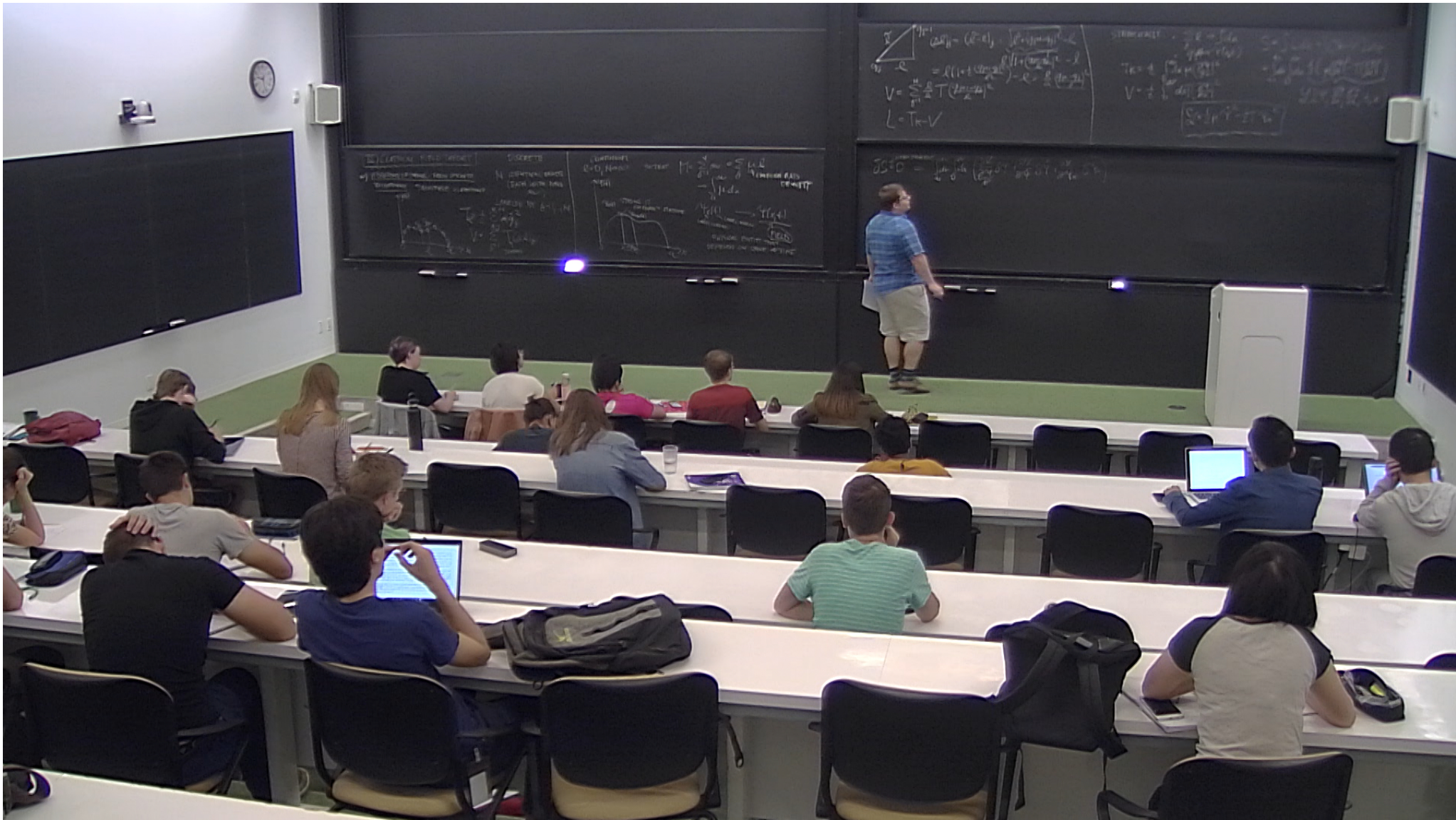
SYMBOLICALLY: $\sum_l \rightarrow \int dx$
 $q_j(t) \rightarrow \psi(x,t)$

$$T_K = \frac{1}{2} \int_0^a dx \mu \left(\frac{\partial \psi}{\partial t} \right)^2$$

$$V = \frac{1}{2} \int_0^a dx T \left(\frac{\partial \psi}{\partial x} \right)^2$$

$$S = \int L dt = \int (T_K - V) dt$$

$$= \int_{t_1}^{t_2} dt \int_0^a dx \underbrace{\frac{1}{2} \left(\mu \left(\frac{\partial \psi}{\partial t} \right)^2 - T \left(\frac{\partial \psi}{\partial x} \right)^2 \right)}_{\mathcal{L}(\psi, \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial t}, t, x)}$$



SYMBOLICALLY: $\sum \rightarrow \int dx$
 $q(x,t) \rightarrow \psi(x,t)$

$$T_K = \frac{1}{2} \int_0^a dx \mu \left(\frac{\partial \psi}{\partial t} \right)^2$$

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$$\mathcal{L} = \frac{1}{2} \mu \dot{\psi}^2 - \frac{1}{2} T \psi_x^2$$

$$L = T_k - V$$

$$\mathcal{L} = \frac{1}{2} \mu \dot{\psi}^2 - \frac{1}{2} T \psi_{xx}^2$$

$$\delta S \stackrel{\text{ACTION PRINCIPLE}}{=} 0 = \int_{t_1}^{t_2} dt \int_0^a dx \left(\underbrace{\frac{\partial \mathcal{L}}{\partial \psi}}_{p^t = \mu \dot{\psi}} \delta \psi + \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\psi}}}_{p^x = -T \psi_x} \delta \dot{\psi} + \frac{\partial \mathcal{L}}{\partial \psi_x} \delta \psi_x \right) = \underbrace{\int_{t_1}^{t_2} dt \int_0^a dx}_{\emptyset} \underbrace{(\dot{p}^t + p^x_x)}_{\text{ARBITRARY}} \delta \psi + \int_0^a dx [p^t \delta \psi]_{t_1}^{t_2} + \int_{t_1}^{t_2} dt [p^x \delta \psi]_0^a$$

$$\begin{aligned} \text{(E-L): } \dot{p}^t + p^x_x &= 0 \\ \mu \ddot{\psi} - T \psi_{xx} &= 0 \end{aligned}$$

$$\psi_{xx} - \frac{1}{a^2} \ddot{\psi} = 0$$

$$a = \sqrt{T/\mu}$$

$$L = T_k - V$$

$$\mathcal{L} = \frac{1}{2} \mu \dot{\psi}^2 - \frac{1}{2} T \psi_{,x}^2$$

$$\delta S \stackrel{\text{ACTION PRINCIPLE}}{=} 0 = \int_{t_1}^{t_2} dt \int_0^a dx \left(\underbrace{\frac{\partial \mathcal{L}}{\partial \psi}}_{p^t = \mu \dot{\psi}} \delta \psi + \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \delta \dot{\psi} + \underbrace{\frac{\partial \mathcal{L}}{\partial \psi_{,x}}}_{p^x = -T \psi_{,x}} \delta \psi_{,x} \right) = \int_{t_1}^{t_2} dt \int_0^a dx \underbrace{(\dot{p}^t + p^x_{,x})}_{\text{ARBITRARY}} \delta \psi + \int_0^a dx \underbrace{[p^t \delta \psi]_{t_1}^{t_2}}_{\text{?}} + \int_{t_1}^{t_2} dt \underbrace{[p^x \delta \psi]_0^a}_{\text{?}}$$

(E-L): $\dot{p}^t + p^x_{,x} = 0$
 $\mu \ddot{\psi} - T \psi_{,xx} = 0$

• 2ND TERM VANISHES IF
 $\delta \psi(x, t_1) = 0 = \delta \psi(x, t_2)$
 "INITIAL DATA"

$$\psi_{,xx} - \frac{1}{v^2} \ddot{\psi} = 0$$

$v = \sqrt{T/\mu}$ WAVES

$$L = T - V$$

$$\delta S \stackrel{\text{ACTION PRINCIPLE}}{=} 0 = \int_{t_1}^{t_2} \int_0^a dx \left(\underbrace{\frac{\partial \mathcal{L}}{\partial \psi}}_{p^t = m \dot{\psi}} \delta \psi + \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\psi}}}_{p^x = -T \psi_x} \delta \dot{\psi} + \frac{\partial \mathcal{L}}{\partial \psi_x} \delta \psi_x \right) = \int_{t_1}^{t_2} dt \int_0^a dx \left(\underbrace{(\dot{p}^t + p^x_x)}_{\text{ARBITRARY}} \delta \psi + \int_0^a dx \underbrace{[p^t \delta \psi]}_{\text{?}} \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt \underbrace{[p^x \delta \psi]}_0 \Big|_0^a \right)$$

(E-L): $\dot{p}^t + p^x_x = 0$
 $m \ddot{\psi} - T \psi_{xx} = 0$

$$\boxed{T \psi_{xx} - \frac{1}{a^2} \ddot{\psi} = 0}$$

$$a = \sqrt{\frac{T}{m}}$$

WAVES

2ND TERM VANISHES IF
 $\delta \psi(x, t_1) = 0 = \delta \psi(x, t_2)$
 "INITIAL DATA"

3RD TERM
 $[p^x \delta \psi]_0^a = 0$

$$(E-L): \quad \dot{p}^t + p^x_x = 0$$

$$M \ddot{\psi} - T \psi_{xx} = 0$$

$$\boxed{\psi_{xx} - \frac{1}{v^2} \ddot{\psi} = 0}$$

$$v = \sqrt{\frac{T}{\mu}} \quad \text{WAVES}$$

• 2ND TERM VANISHES IF
 $\delta\psi(x, t_1) = 0 = \delta\psi(x, t_2)$
"INITIAL DATA"

• 3RD TERM $\left[p^x \delta\psi \right]_0^a = 0$ BOUNDARY DATA
 DIRICHLET $\boxed{\delta\psi/a = 0 = \delta\psi/0} \Rightarrow \dot{\psi}/a = 0 = \dot{\psi}/0$
 NEUMANN $p^x/a = 0 = p^x/0$

$$(E-L): \quad \dot{p}^t + p^x_x = 0$$

$$M \ddot{\psi} - T \psi_{xx} = 0$$

$$\left[\psi_{xx} - \frac{1}{\alpha^2} \ddot{\psi} = 0 \right]$$

$$\alpha = \sqrt{T/\mu}$$

WAVES

• 2ND TERM VANISHES IF
 $\delta\psi(x, t_1) = 0 = \delta\psi(x, t_2)$
"INITIAL DATA"

• 3RD TERM
 $\left[p^x \delta\psi \right]_0^a = 0$

BOUNDARY DATA

DIRICHLET

$$\delta\psi/a = 0 = \delta\psi/0$$

$$\Rightarrow \dot{\psi}/a = 0 = \dot{\psi}/0$$

NEUMANN

$$p^x/a = 0 = p^x/0$$

$$\Rightarrow \psi_x/a = 0 = \psi_x/0$$

$$(E-L): \quad \dot{p}^t + p^x_x = 0$$

$$M \ddot{\psi} - T \psi_{xx} = 0$$

$$\boxed{\psi_{xx} - \frac{1}{v^2} \ddot{\psi} = 0}$$

$$v = \sqrt{T/\mu}$$

WAVES

$$p^t = M \dot{\psi}$$

$$p^x = -T \psi_x$$

• 2ND TERM VANISHES IF

$$\delta \psi(x, t_1) = 0 = \delta \psi(x, t_2)$$

"INITIAL DATA"

• 3RD TERM

$$\left[p^x \delta \psi \right]_0^a = 0$$

ARBITRARY

BOUNDARY DATA

DIRICHLET

$$\delta \psi / a = 0 = \delta \psi / 0$$

$$\Rightarrow \dot{\psi} / a = 0 = \dot{\psi} / 0$$

NEUMANN

$$p^x / a = 0 = p^x / 0$$

$$\Rightarrow \psi_{,x} / a = 0 = \psi_{,x} / 0$$

$$\lambda = \sqrt{T/\mu}$$

WAVES

NEUMANN

REMARK: MOMENTUM CARRIED BY STRING

$$p = (p^t, p^x) \dots \frac{\partial \mathcal{L}}{\partial t} + \frac{\partial \mathcal{L}}{\partial x} = 0$$

$$P = \int_0^a \mu \dot{\psi}^2 dx = \int_0^a p^t dx \dots \text{CHARGE} \nearrow \text{CURRENT}$$

$$\dot{P} = \int_0^a \mu \ddot{\psi}^2 dx = \int_0^a \mu \dot{\psi}^2 \frac{d}{dx} \psi_x dx = \mu \dot{\psi}^2 \left[\psi_x \right]_0^a$$

NEUMANN: $\dot{P} = 0$

$$\sigma = \sqrt{T/\mu}$$

WAVES

NEUMANN

REMARK: MOMENTUM CARRIED BY STRING

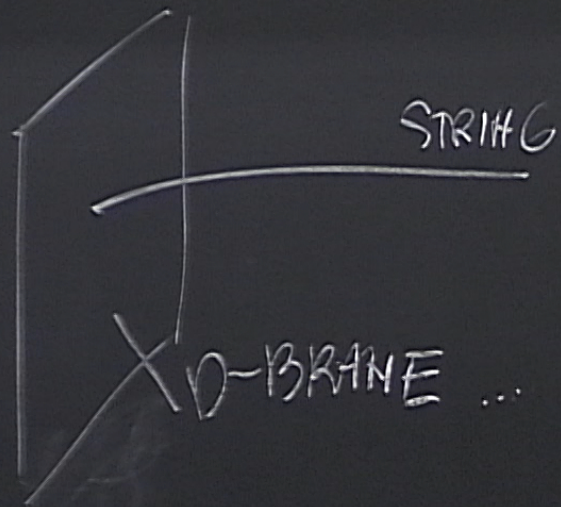
$$p = (p^t, p^x) \dots \frac{\partial \mathcal{L}}{\partial t} + \frac{\partial \mathcal{L}}{\partial x} = 0$$

$$P = \int_0^a \mu \dot{\psi} dx = \int_0^a p^t dx \dots \text{CHARGE} \nearrow \text{CURRENT}$$

$$\dot{P} = \int_0^a \mu \ddot{\psi} dx = \int_0^a \mu \sigma^2 \frac{d}{dx} \psi_x dx = \mu \sigma^2 \left[\psi_x \right]_0^a$$

NEUMANN: $\boxed{\dot{P} = 0}$

DIRICHLET: $\boxed{\dot{P} \neq 0}$ FIXED ENDPOINTS CAN PROVIDE P.



HIGHER DIMENSIONAL OBJECTS.



b) MAXWELL'S THEORY:

• GENERAL STRUCTURE OF CFTs.

IDEA. REPLACE 'ACTION AT A DISTANCE'
WITH LOCAL INTERACTIONS VIA THE
CONCEPT OF FIELD (PHYSICAL ENTITY AT EACH
POINT OF SPACE & TIME)

2 TYPES OF EQUATIONS

FIELD EQUATIONS (DISTR OF MATTER $\rightarrow$ DETERMINES FIELDS)

EQUATIONS OF MOTION FOR MATTER

(HOW MATTER MOVES IN GIVEN FIELDS)

NEED TO SOLVE SIMULTANEOUSLY... PAIR!

SOMETIMES USE TEST FIELD APPROX.

2 TYPES OF EQUATIONS

FIELD EQUATIONS (DISTR OF MATTER $\rightarrow$ DETERMINES FIELDS)

EQUATIONS OF MOTION FOR MATTER

(HOW MATTER MOVES IN GIVEN FIELDS)

NEED TO SOLVE SIMULTANEOUSLY ... HAIR?

SOMETIMES USE TEST MATTER APPROX:



MERCURY
 $\uparrow$
TEST MATTER

ELECTROMAGNETISM.

FIELD EQS = MAXWELL EQS (1860)

AMPERE & MAXWELL:

FARADAY:

GAUSS:

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 4\pi \mathbf{j}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{E} = 4\pi \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

• EQS OF

• EQS OF MOTION . LORENZ FORCE LAW

$$\frac{dp}{dt} = F = q(E + v \times B)$$

$\nabla \times V$... CORL ✓

$\nabla \cdot V$... DIV V

∇f ... GRAD f

$$\begin{aligned}\nabla \cdot \nabla \times V &= 0 \\ \nabla \times \nabla f &= 0\end{aligned}$$

• EQS OF MOTION . LORENZ FORCE LAW ✓

$$\frac{dp}{dt} = F = q(E + v \times B)$$

- (1) $\nabla \times V \dots$ CURL ✓
- (2) $\nabla \cdot V \dots$ DIV ✓
- (3) $\nabla f \dots$ GRAD f
- (4)

$$\begin{aligned} \nabla \cdot \nabla \times V &= 0 \\ \nabla \times \nabla f &= 0 \end{aligned}$$

• REMARKS

① CHARGE CONSERVATION

MAXWELL ADDED $-\frac{\partial E}{\partial t}$.. SO THAT WE HAVE
CONSISTENT EQUATIONS (CONSERVED CURRENT)

$\nabla \cdot (1)$:

• EQS OF MOTION • LORENZ FORCE LAW ✓

$$\frac{dp}{dt} = F = q(E + v \times B)$$

- (1) $\nabla \times V \dots$ CURL ✓
- (2) $\nabla \cdot V \dots$ DIV ✓
- (3)
- (4) $\nabla f \dots$ GRAD f

$$\begin{aligned} \nabla \cdot \nabla \times V &= 0 \\ \nabla \times \nabla f &= 0 \end{aligned}$$

• REMARKS

① CHARGE CONSERVATION

MAXWELL ADDED $-\frac{\partial E}{\partial t}$.. SO THAT WE HAVE
CONSISTENT EQUATIONS (CONSERVED CURRENT)

$$\nabla \cdot (1): 0 - \frac{\partial}{\partial t} \underbrace{\nabla \cdot E}_{4\pi \rho} = 4\pi \nabla \cdot j \quad (3)$$

• EQS OF MOTION . LORENZ FORCE LAW

$$\frac{dp}{dt} = F = q(E + v \times B)$$

$\nabla \times V$... CURL V

$\nabla \cdot V$... DIV V

∇f ... GRAD f

$$\begin{aligned} \nabla \cdot \nabla \times V &= 0 \\ \nabla \times \nabla f &= 0 \end{aligned}$$

• REMARKS

① CHARGE CONSERVATION

MAXWELL ADDED $-\frac{\partial E}{\partial t}$.. SO THAT WE HAVE CONSISTENT EQUATIONS (CONSERVED CURRENT)

$$\nabla \cdot (1): \quad 0 - \frac{\partial}{\partial t} \underbrace{\nabla \cdot E}_{4\pi \rho} = 4\pi \nabla \cdot j \quad (3)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0$$

CONTINUITY EQ.

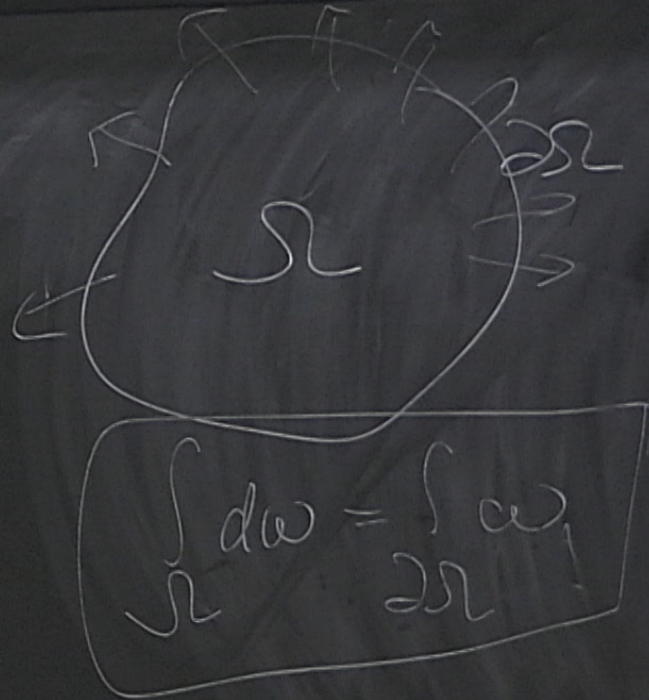
✓ HIGHER DIMENSIONAL OBJECTS

LET'S DEFINE CHARGE:

$$Q = \int_{\Omega} \rho \, dV$$

$$-\dot{Q} = - \int_{\Omega} \frac{\partial \rho}{\partial t} \, dV = \int_{\Omega} \nabla \cdot \mathbf{j} \, dV$$

$$= \int_{\partial\Omega} \mathbf{j} \cdot d\mathbf{S} = \text{CURRENT FLUX THROUGH } \partial\Omega$$



• HOW MANY EQS?

• 8 EQS

• UNKNOWNS: E, B 6

• EQS (3) & (4) CONSTRAINT
EQS

"INITIAL CONDITIONS"

$\nabla \cdot (2)$: $\boxed{\vec{\nabla} \cdot \vec{B} = 0}$

$\Leftrightarrow (4)$

↑ AT BIG BANG WE HAD
NO MAGNETIC MONOPOLES,

$\nabla \cdot (1)$: $\vec{\nabla} \cdot \vec{E} = 4\pi \rho$

• HOW MANY EQS?

• 8 EQS

• UNKNOWNS: E, B 6

• EQS (3) & (4) CONSTRAINT EQS ... "INITIAL CONDITIONS"

$\nabla \cdot (2)$: $\boxed{\nabla \cdot \mathbf{B} = 0}$ $\Leftrightarrow$ (4) $\uparrow$ AT BIG BANG WE HAD

$\nabla \cdot (1)$: $\nabla \cdot \mathbf{E} = 4\pi \rho$ NO MAGNETIC MONOPOLES,
 $\frac{\partial}{\partial t}(\nabla \cdot \mathbf{E} - 4\pi \rho) = 0$

• HOW MANY EQS?

• 8 EQS

• UNKNOWNS: E, B 6

• EQS (3) & (4) CONSTRAINT ... "INITIAL CONDITIONS"
EQS

$\nabla \cdot (2)$: $\boxed{\nabla \cdot \frac{\partial}{\partial t} \mathbf{D} \cdot \mathbf{B} = 0}$

$\Leftrightarrow (4)$

↑ AT BIG BANG WE HAD

$\nabla \cdot (1)$: $\nabla \cdot \frac{\partial}{\partial t} \mathbf{D} \cdot \mathbf{E} = 4\pi \nabla \cdot \mathbf{j} = -4\pi \rho$

NO MAGNETIC MONOPOLES,

$\frac{\partial}{\partial t} (\nabla \cdot \mathbf{E} - 4\pi \rho) = 0 \Leftrightarrow (3)$

