Title: PSI 2019/2020 - Classical Physics - Lecture 3

Speakers: David Kubiznak

Collection: PSI 2019/2020 - Classical Physics (Kubiznak)

Date: August 19, 2019 - 9:00 AM

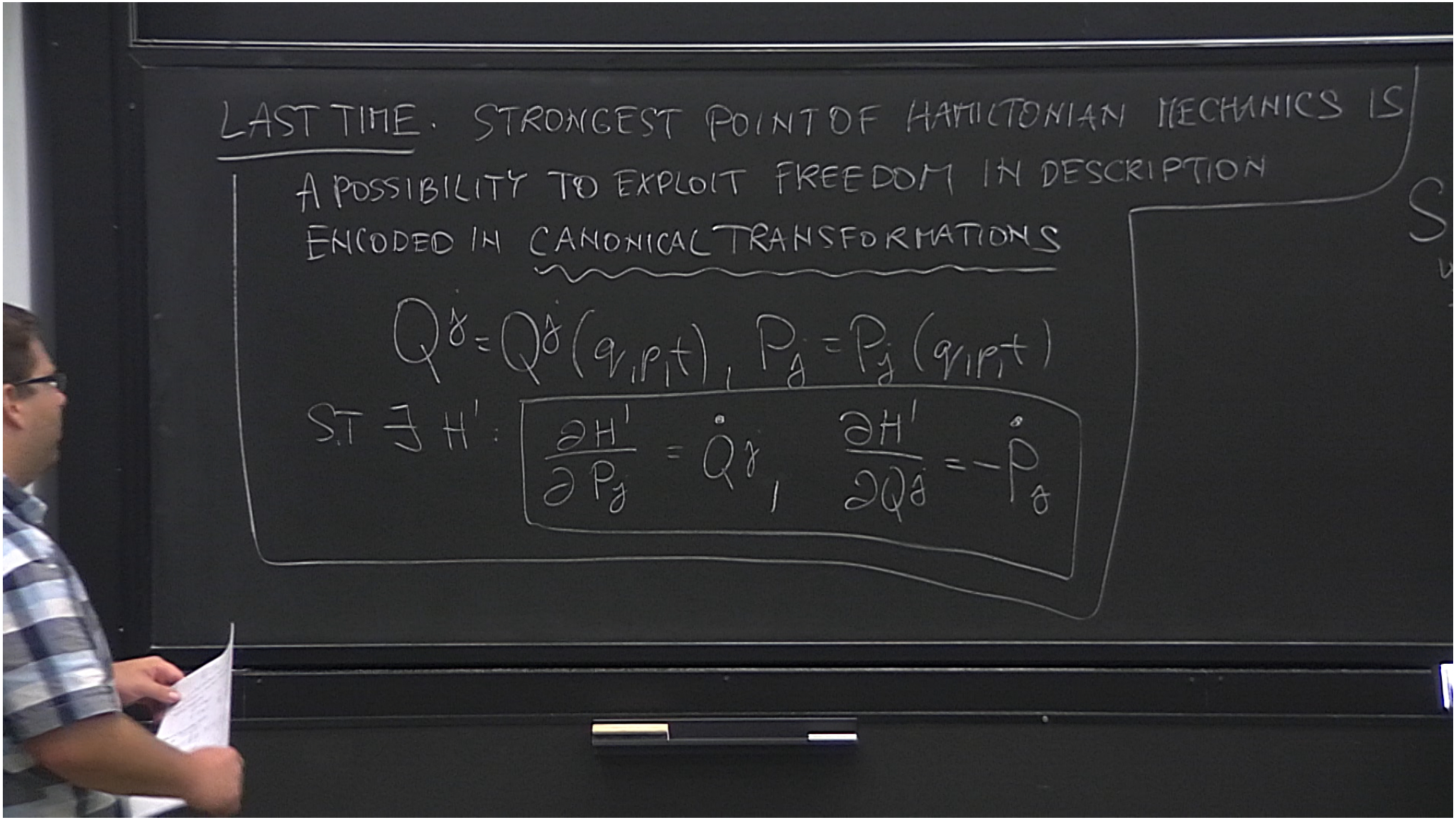
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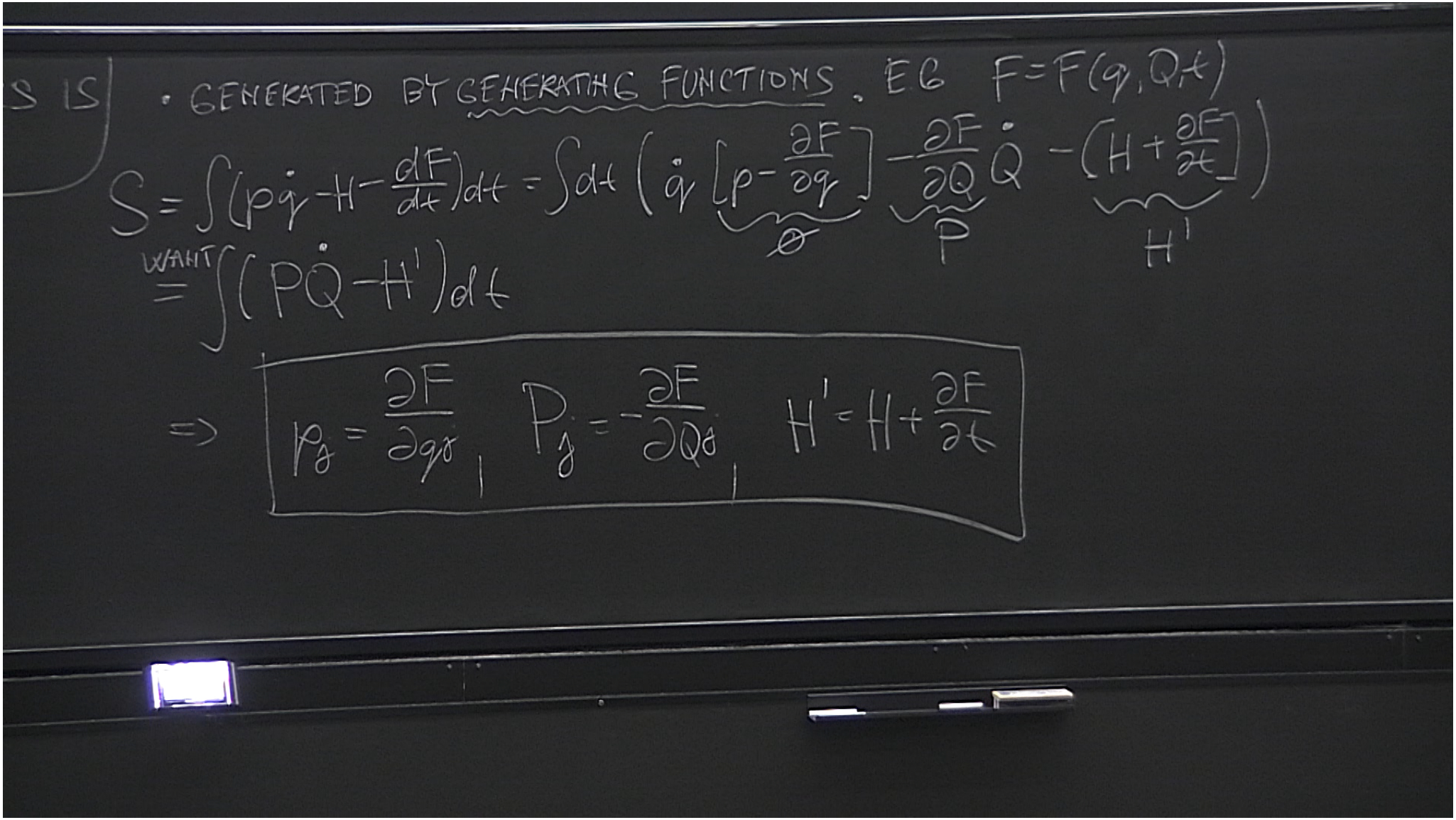
LAST TIME. STRONGEST POINT OF HAMILTONIAN MECHANICS IS · 6

A POSSIBILITY TO EXPLOIT FREEDOM IN DESCRIPTION ENCODED IN CANONICAL TRANSFORMATIONS

$$Q^j = Q^j(q_i, p_i, t), \quad P_j = P_j(q_i, p_i, t)$$

S.T $\exists H'$: $\frac{\partial H'}{\partial P_j} = \dot{Q}^j$, $\frac{\partial H'}{\partial Q^j} = -\dot{P}_j$


LAST TIME. STRONGEST POINT OF HAMILTONIAN MECHANICS IS
A POSSIBILITY TO EXPLOIT FREEDOM IN DESCRIPTION
ENCODED IN CANONICAL TRANSFORMATIONS
$Q^\alpha = Q^\alpha(q,p,t)$, $P_\alpha = P_\alpha(q,p,t)$
S.T $\exists H'$: $\frac{\partial H'}{\partial P_\alpha} = \dot{Q}^\alpha$, $\frac{\partial H'}{\partial Q^\alpha} = -\dot{P}_\alpha$

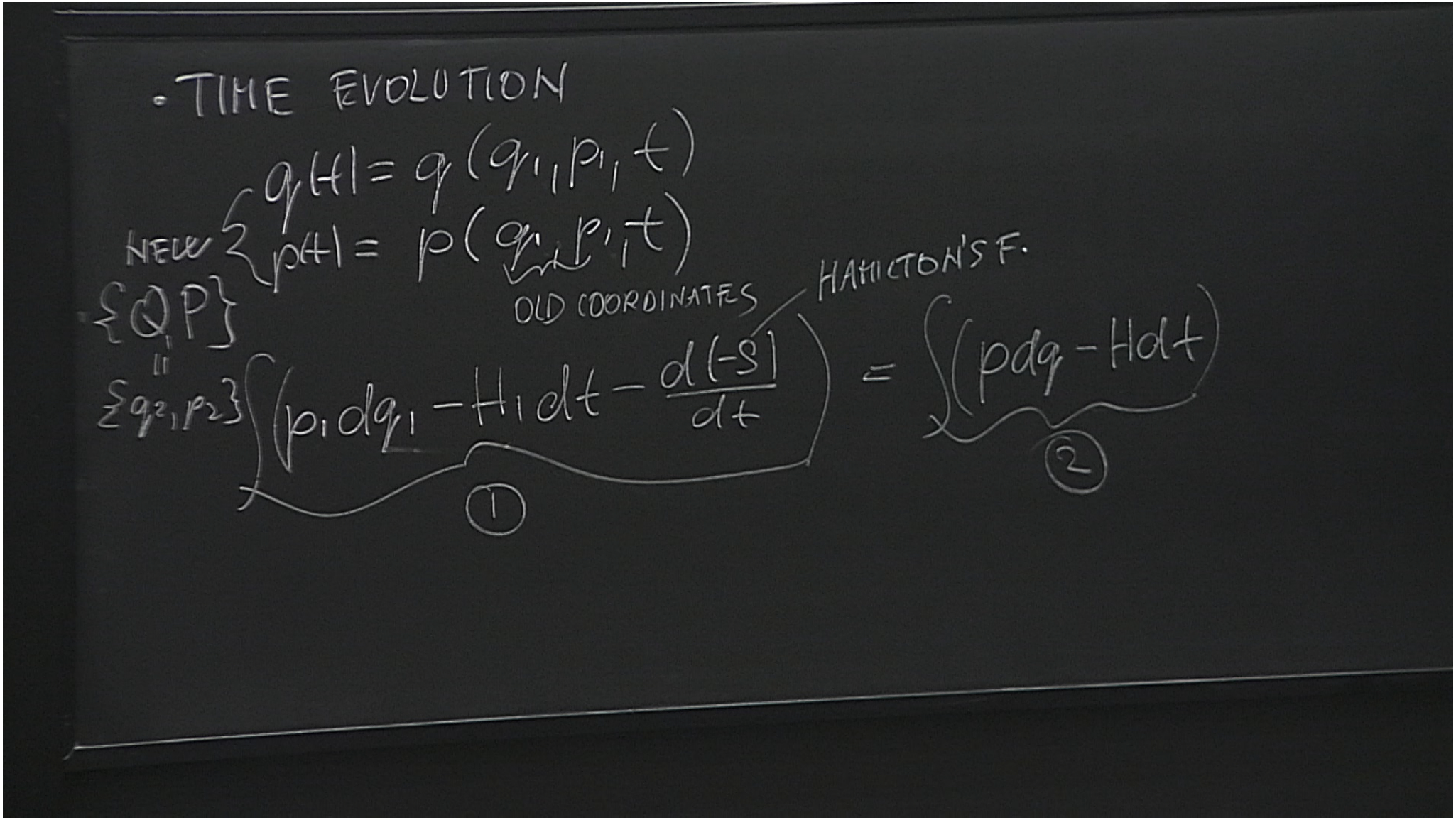

S IS
• GENERATED BY GENERATING FUNCTIONS. EG F=F(q,Q,t)
$S = \int (p\dot{q} - H - \frac{dF}{dt})dt = \int dt \left(\dot{q}\underbrace{[p - \frac{\partial F}{\partial q}]}_{0} - \underbrace{\frac{\partial F}{\partial Q}}_{P}\dot{Q} - \underbrace{(H + \frac{\partial F}{\partial t})}_{H'} \right)$
WANT $= \int (P\dot{Q} - H')dt$
$\Rightarrow$ $p_j = \frac{\partial F}{\partial q_j}$, $P_j = -\frac{\partial F}{\partial Q_j}$, $H' = H + \frac{\partial F}{\partial t}$

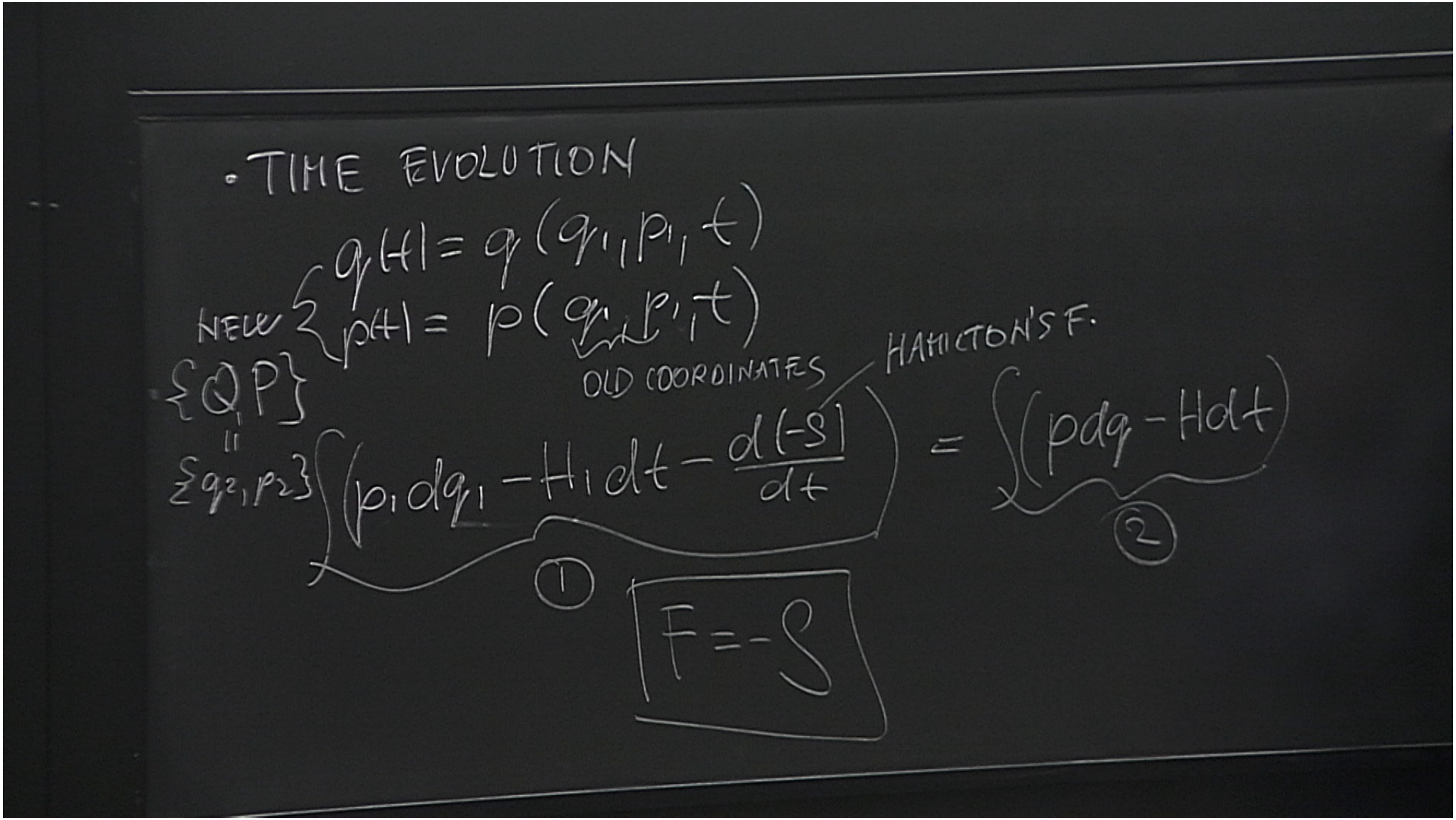
- TIME EVOLUTION

NEW $\{Q, P\}$

$$\begin{cases} q(t) = q(q_0, p_0, t) \\ p(t) = p(q_0, p_0, t) \end{cases}$$

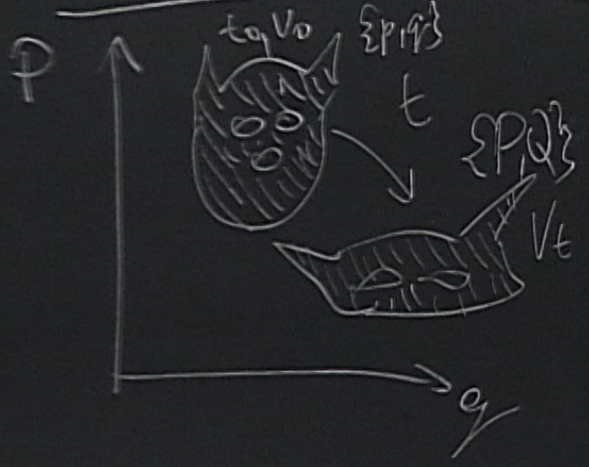
OLD COORDINATES


• TIME EVOLUTION
NEW
$q(t+1) = q(q_1, p_1, t)$
$p(t+1) = p(q_1, p_1, t)$
OLD COORDINATES
HAMILTON'S F.
$\{Q, P\}$
$\{q_2, p_2\}$
$p_1 dq_1 - H_1 dt - \frac{d(-S)}{dt}$
$= pdq - Hdt$
1
2


• TIME EVOLUTION
NEW
$Q_1 = q(q_1, p_1, t)$
$P_1 = p(q_1, p_1, t)$
OLD COORDINATES
HAMILTON'S F.
$\{Q, P\} = \{q_2, p_2\}$
$\int (p_1 dq_1 - H_1 dt - \frac{d(-S)}{dt}) = \int (p dq - H dt)$
1
2
$F = -S$

TIME EVOLUTION = CANONICAL TRANSFORMATION

- LIOUVILLE'S THEOREM: VOLUME OF THE PHASE SPACE REMAINS INVARIANT UNDER CANONICAL TRANSFORMATIONS (TIME EVOLUTION).

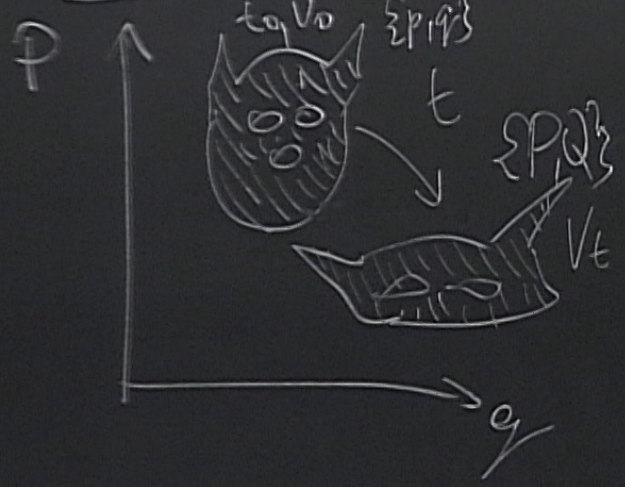


PROOF: $n=1$

$$V_t = \int dP\,dQ$$

TIME EVOLUTION = CANONICAL TRANSFORMATION

- **LIOUVILLE'S THEOREM**: VOLUME OF THE PHASE SPACE REMAINS INVARIANT UNDER CANONICAL TRANSFORMATIONS (TIME EVOLUTION).



PROOF: $N=1$

$$V_t = \int dP\,dQ = \int |JAC|\, dp\,dq = \int dp\,dq = V_0$$

$$JAC = \begin{vmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{vmatrix} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = \{Q, P\}_{q,p} = 1$$

d) HAMILTON-JACOBI THEORY

INGENIOUS IDEA: DO A SPECIAL CANONICAL TRANSFORMATION (GENERATED BY S) SO THAT $\boxed{H' = 0}$

$$\Rightarrow \boxed{Q\delta = \alpha\delta = \text{CONST}, \quad P_j = -\beta_j = \text{CONST}}$$

$F = -S$

q_j

JAC =

d) HAMILTON-JACOBI THEORY

INGENIOUS IDEA: DO A SPECIAL CANONICAL TRANSFORMATION (GENERATED BY S) SO THAT $\boxed{H' = 0}$

$$\Rightarrow \boxed{Q_\delta = \alpha_\delta = \text{CONST}, \quad P_j = \beta_j = \text{CONST}}$$

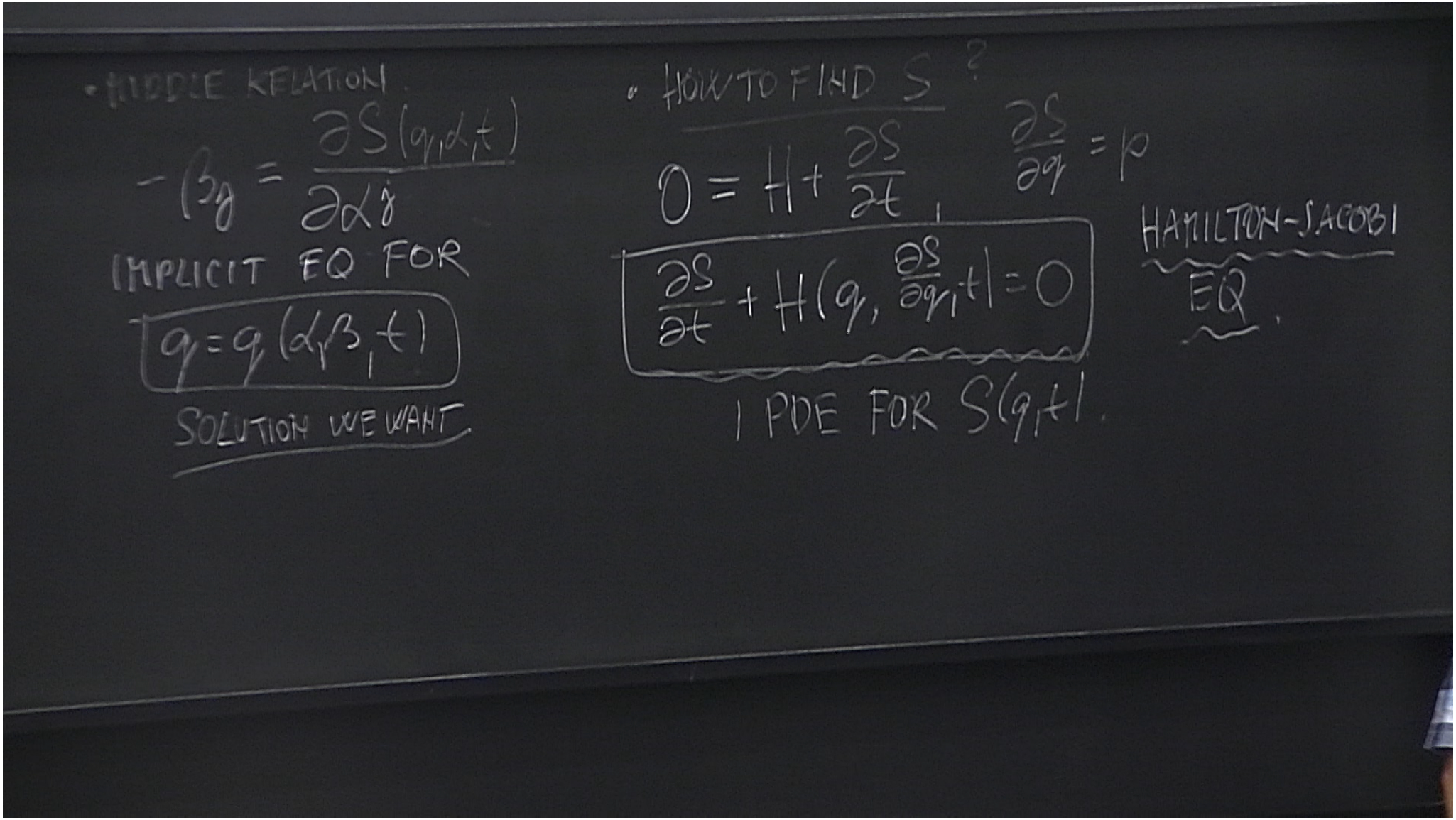
• MIDDLE RELATION

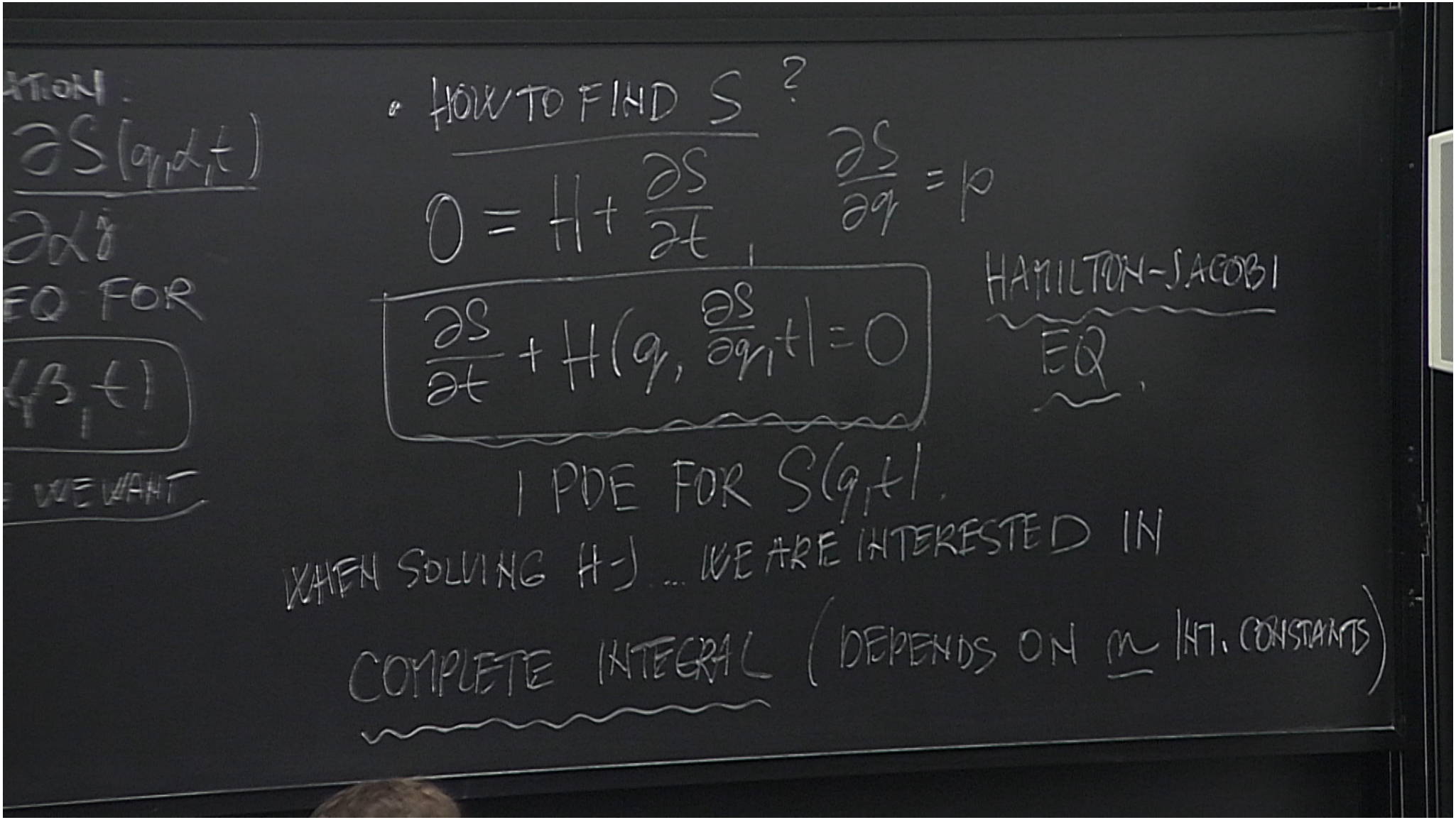
$$-\beta_\delta = \frac{\partial S(q, \alpha, t)}{\partial \alpha_\delta}$$

IMPLICIT EQ FOR

$$\boxed{q = q(\alpha, \beta, t)}$$

SOLUTION WE WANT


• MIDDLE RELATION.
$-\beta_\gamma = \frac{\partial S(q,\alpha,t)}{\partial \alpha_\gamma}$
IMPLICIT EQ FOR
$q = q(\alpha,\beta,t)$
SOLUTION WE WANT.
• HOW TO FIND S ?
$0 = H + \frac{\partial S}{\partial t}$, $\frac{\partial S}{\partial q} = p$
$\frac{\partial S}{\partial t} + H(q, \frac{\partial S}{\partial q}, t) = 0$
1 PDE FOR $S(q,t)$.
HAMILTON-JACOBI EQ.


• HOW TO FIND S ?
$0 = H + \frac{\partial S}{\partial t}$, $\frac{\partial S}{\partial q} = p$
$\frac{\partial S}{\partial t} + H(q, \frac{\partial S}{\partial q}, t) = 0$
HAMILTON-JACOBI EQ.
1 PDE FOR $S(q,t)$.
WHEN SOLVING H-J ... WE ARE INTERESTED IN
COMPLETE INTEGRAL (DEPENDS ON n INT. CONSTANTS)

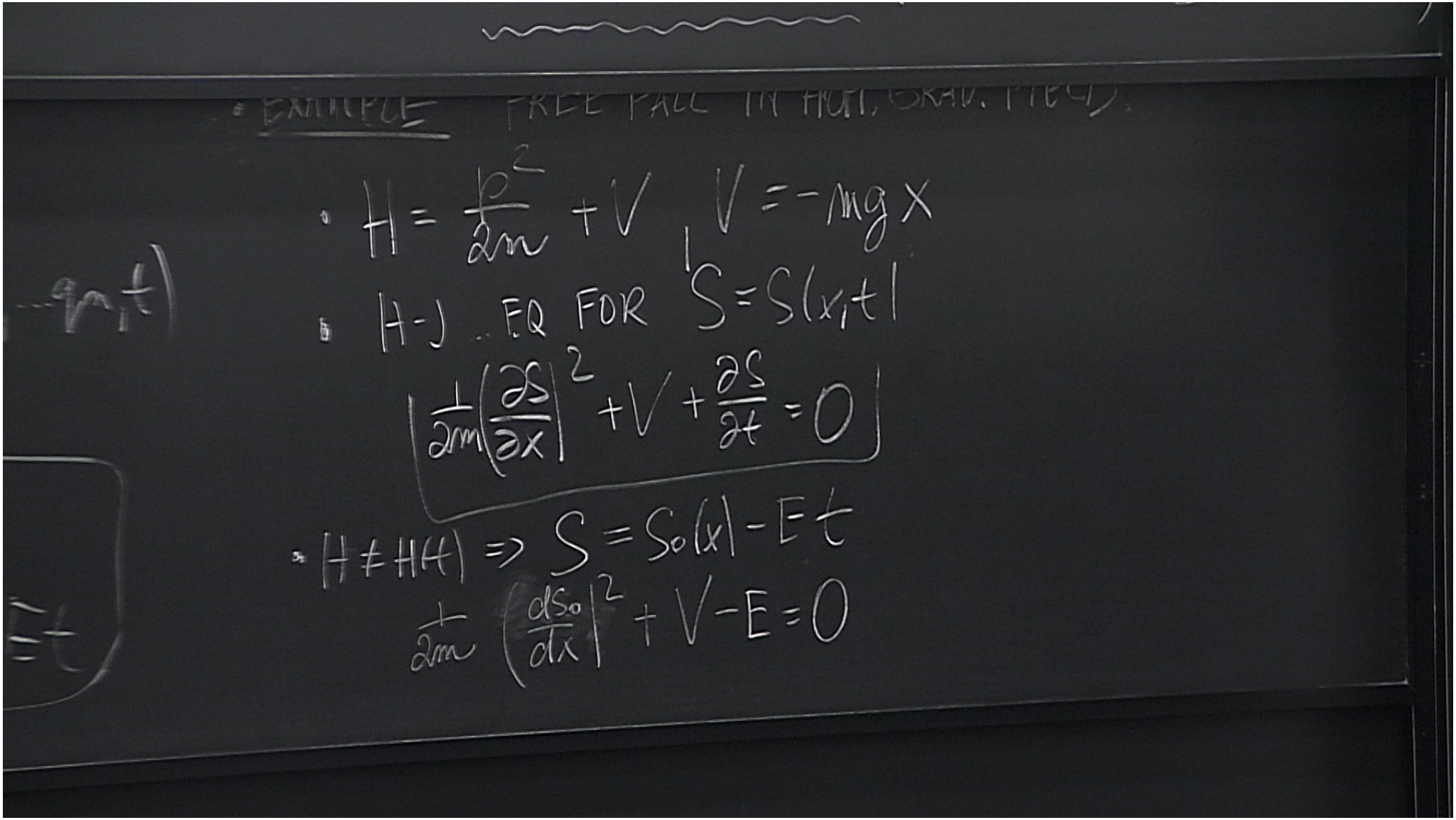
- TRICK IS TO USE SEPARATION OF VARIABLES

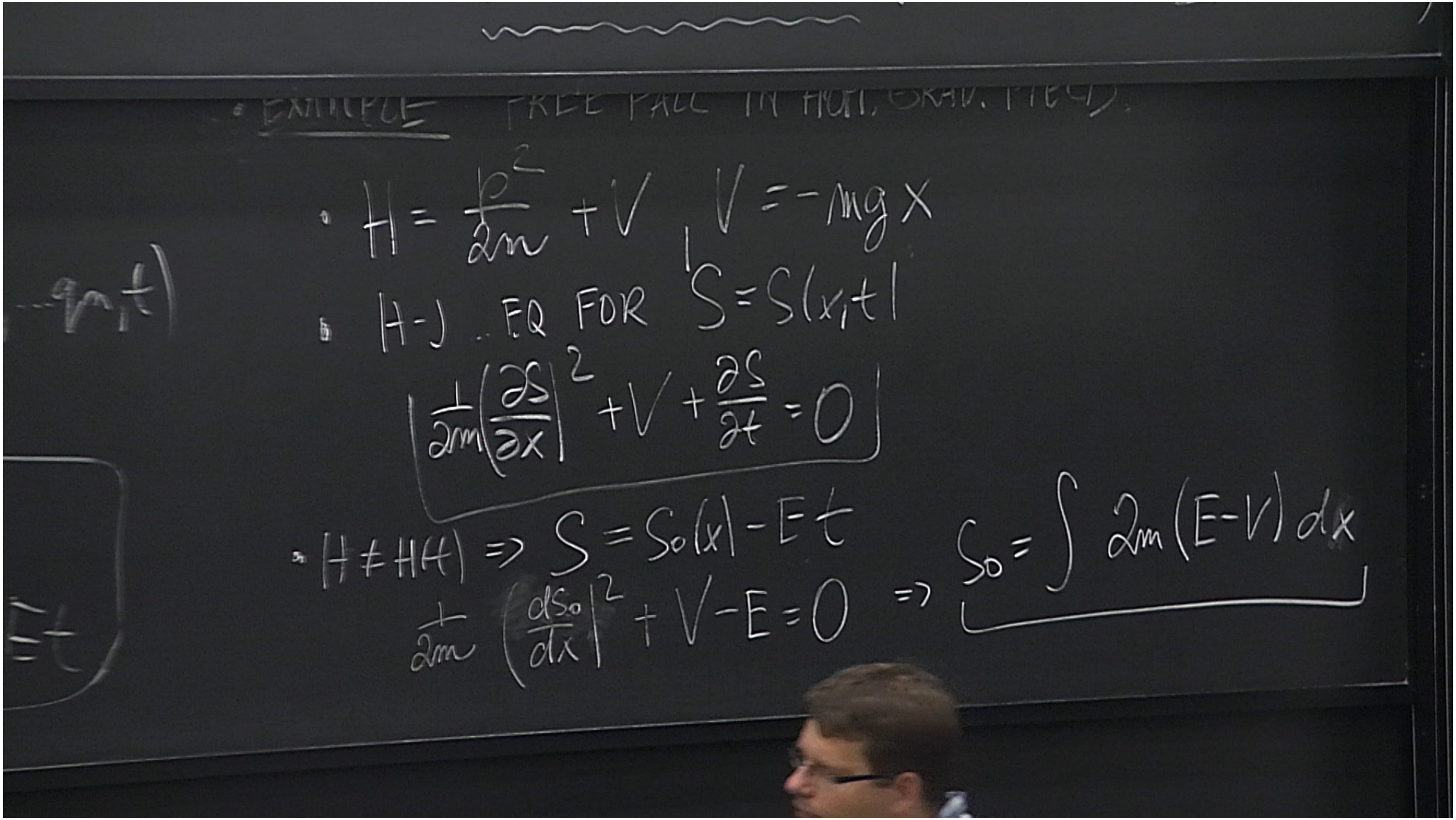
$H \neq H(t) \Rightarrow S(q,t) = S_0(q) - Et$ ← CONSTANT

$H \neq H(q_c) \Rightarrow S(q,t) = \alpha_c q_c + S_c(q_1, q_2, \ldots \hat{q_c}, \ldots q_n, t)$ ↑ CONSTANT

MORE GENERALLY WE CAN TRY ADDITIVE SEPARATION ANSATS

$$S(q_i,t) = S_1(q^1) + S_2(q^2) + \ldots + S_n(q^n) - Et$$


EXAMPLE FREE FALL IN HOR. GRAV. FIELD
$H = \frac{p^2}{2m} + V \quad V = -mgx$
H-J EQ FOR $S = S(x,t)$
$\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + V + \frac{\partial S}{\partial t} = 0$
$H \neq H(t) \Rightarrow S = S_0(x) - Et$
$\frac{1}{2m}\left(\frac{dS_0}{dx}\right)^2 + V - E = 0$


EXAMPLE FREE FALL IN HOM. GRAV. FIELD
$H = \frac{p^2}{2m} + V \quad V = -mgx$
H-J EQ FOR $S = S(x,t)$
$\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + V + \frac{\partial S}{\partial t} = 0$
$H \neq H(t) \Rightarrow S = S_0(x) - Et$
$\frac{1}{2m}\left(\frac{dS_0}{dx}\right)^2 + V - E = 0 \Rightarrow S_0 = \int 2m(E-V)\,dx$

$$S = S_0 - Et$$

$$\frac{\partial S}{\partial E} = \beta = -t + \int dx \frac{1}{\sqrt{2m(E-V)}} 2m$$

$$= \frac{1}{mg}\sqrt{2m(E-V)} - t = \beta$$

BY INVERSION

$$\Rightarrow$$

$$x = \frac{g}{2}(t+\beta)^2 - \frac{E}{mg}$$

PLICIT EQ FOR

$q = q(\alpha, \beta, t)$

SOLUTION WE WANT

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}, t\right) = 0$$

HAMILTON-JACOBI EQ.

1 PDE FOR $S(q,t)$.

WHEN SOLVING H-J ... WE ARE INTERESTED IN

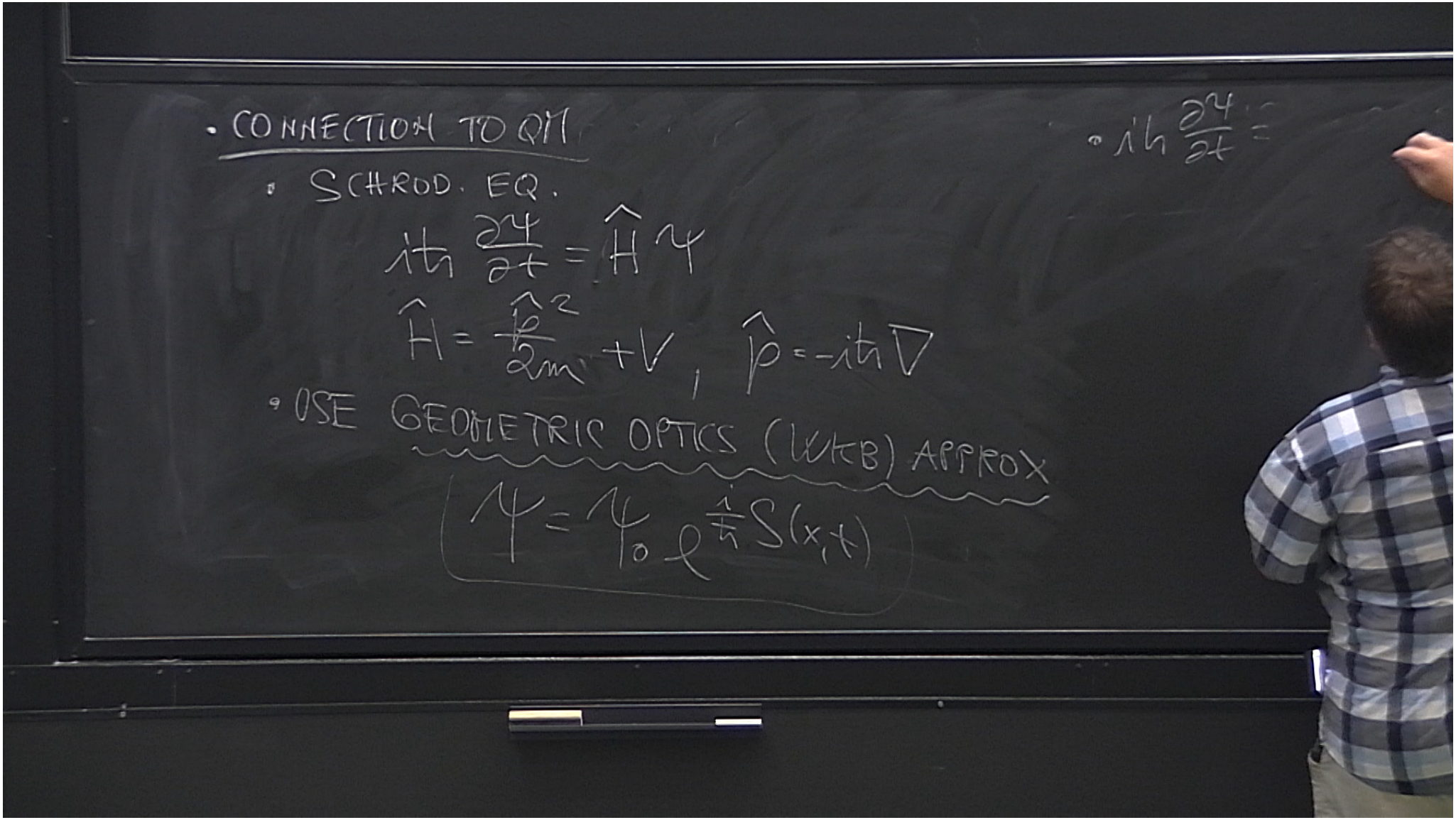
COMPLETE INTEGRAL (DEPENDS ON n INT. CONSTANTS)

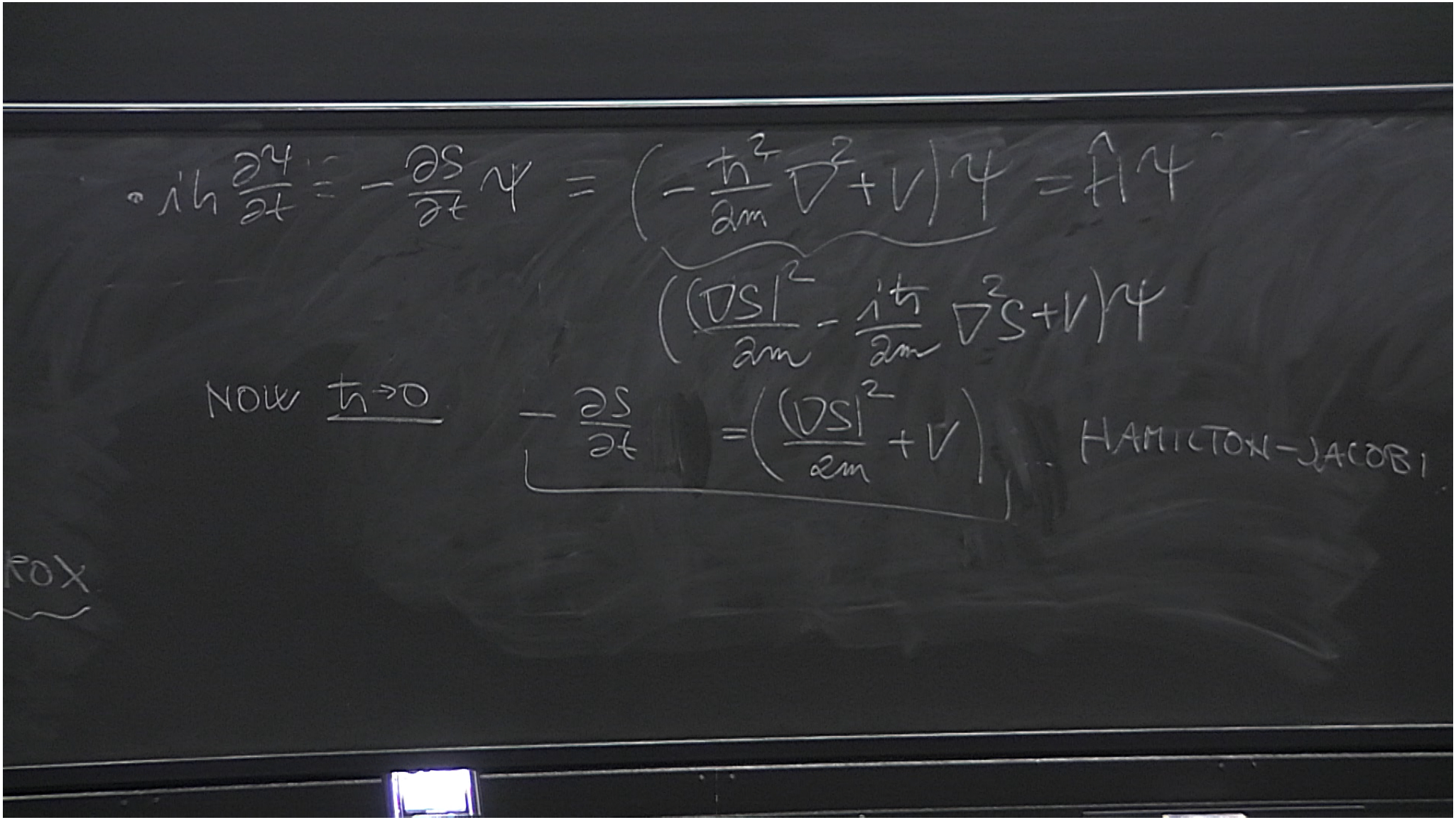
$\frac{\partial H}{\partial x}$ $\quad$ $\frac{\partial H}{\partial t} = 0$

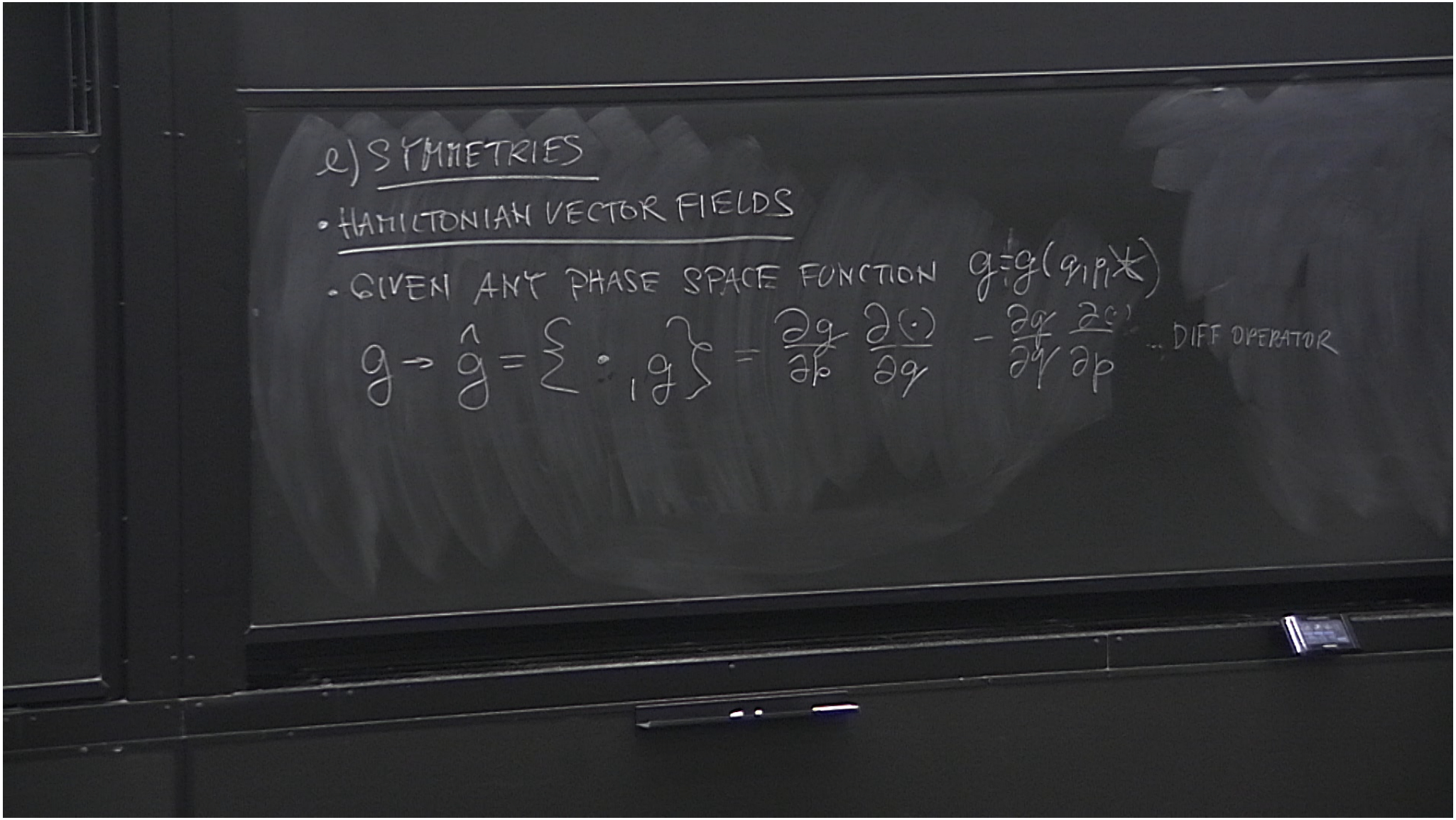
$\Rightarrow H \neq H(t) \Rightarrow S = S_0(x) - Et$

$$\frac{1}{2m}\left(\frac{dS_0}{dx}\right)^2 + V - E = 0 \Rightarrow S_0 = \int \sqrt{2m(E-V)}\, dx$$

$= t$


• CONNECTION TO QM
• SCHROD. EQ.
$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$
$\hat{H} = \frac{\hat{p}^2}{2m} + V$, $\hat{p} = -i\hbar\nabla$
• USE GEOMETRIC OPTICS (WKB) APPROX
$\psi = \psi_0 e^{\frac{i}{\hbar}S(x,t)}$
• $i\hbar \frac{\partial \psi}{\partial t} =$


$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\partial S}{\partial t}\psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = \hat{H}\psi$
$\left(\frac{|\nabla S|^2}{2m} - \frac{i\hbar}{2m}\nabla^2 S + V\right)\psi$
NOW $\hbar \to 0$
$-\frac{\partial S}{\partial t} = \left(\frac{|\nabla S|^2}{2m} + V\right)$
HAMILTON-JACOBI


e) SYMMETRIES
• HAMILTONIAN VECTOR FIELDS
• GIVEN ANY PHASE SPACE FUNCTION $g = g(q,p,t)$
$g \to \hat{g} = \{\cdot, g\} = \frac{\partial g}{\partial p}\frac{\partial(\cdot)}{\partial q} - \frac{\partial g}{\partial q}\frac{\partial(\cdot)}{\partial p}$ ... DIFF OPERATOR

e) SYMMETRIES

- HAMILTONIAN VECTOR FIELDS

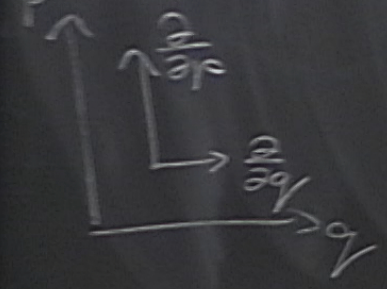
- GIVEN ANY PHASE SPACE FUNCTION $g = g(q, p, t)$

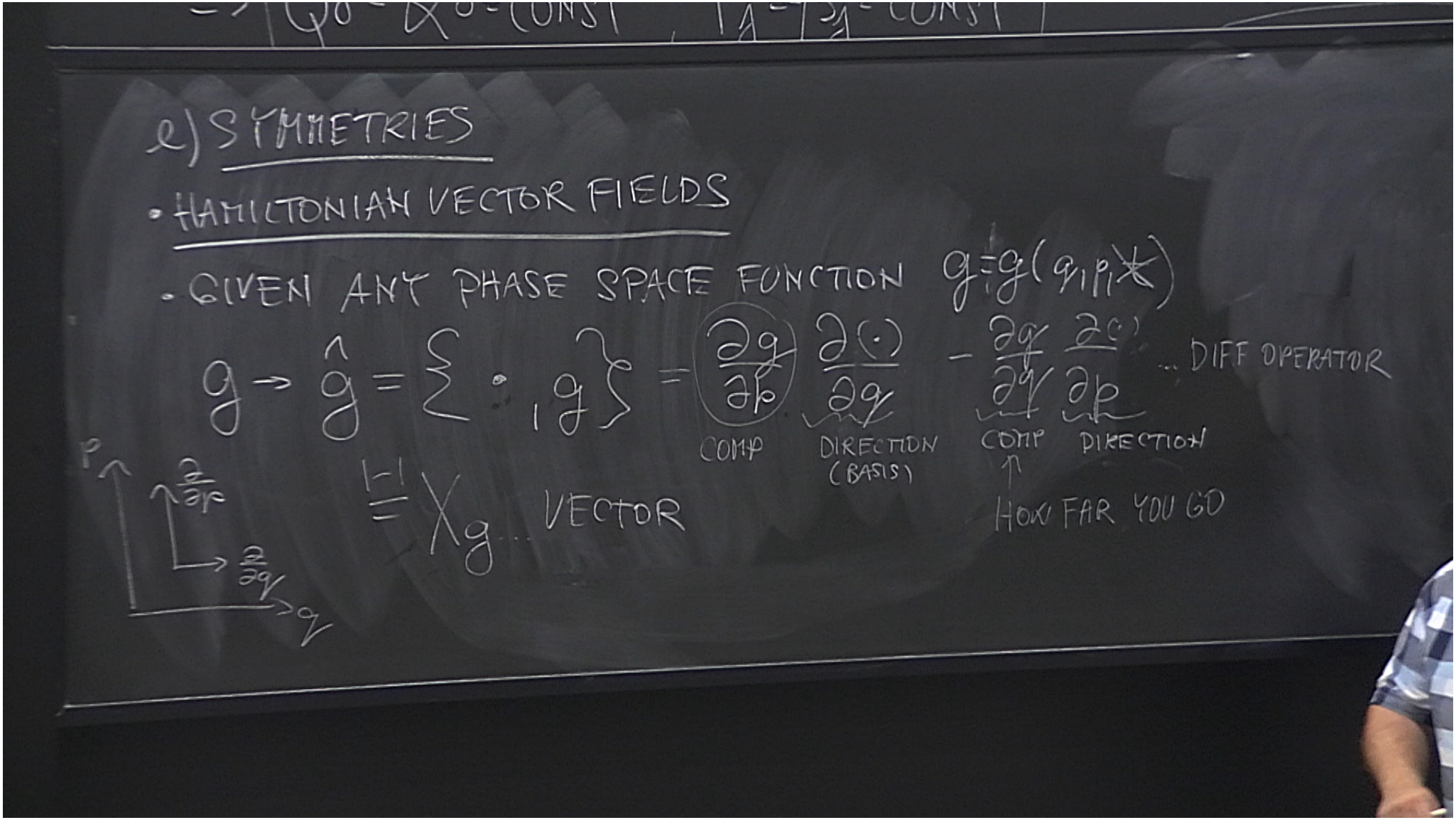
$$g \to \hat{g} = \{\cdot, g\} = \frac{\partial g}{\partial p}\frac{\partial (\cdot)}{\partial q} - \frac{\partial g}{\partial q}\frac{\partial (\cdot)}{\partial p} \quad \ldots \text{DIFF OPERATOR}$$

COMP — DIRECTION (BASIS) — COMP — DIRECTION

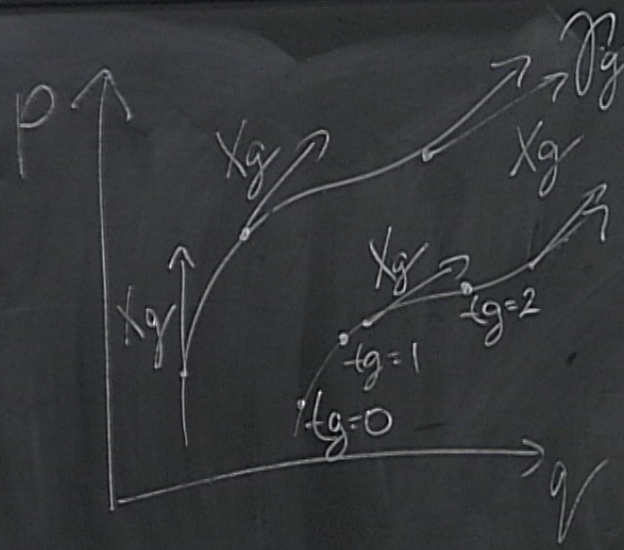
↑ HOW FAR YOU GO

$$= X_g \ldots \text{VECTOR}$$




e) SYMMETRIES
• HAMILTONIAN VECTOR FIELDS
• GIVEN ANY PHASE SPACE FUNCTION g=g(q,p)
g → ĝ = {·, g} = ∂g/∂p ∂(·)/∂q − ∂g/∂q ∂(·)/∂p ... DIFF OPERATOR
COMP
DIRECTION (BASIS)
COMP
DIRECTION
HOW FAR YOU GO
p
∂/∂p
∂/∂q
q
≡ X_g ... VECTOR

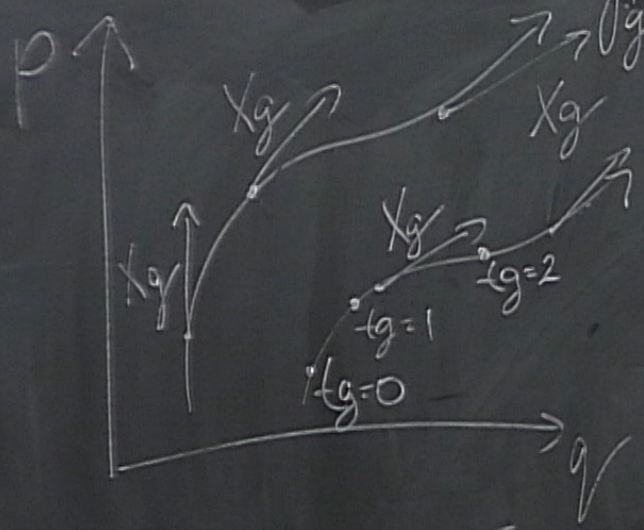
WHEN SOLVING H-J ... WE ARE INTERESTED IN



CURVES PARAMETRIZED BY 'TIME' t_g.

F OPERATOR

HAMILTON-JACOBI



CURVES PARAMETRIZED BY 'TIME' t_g.

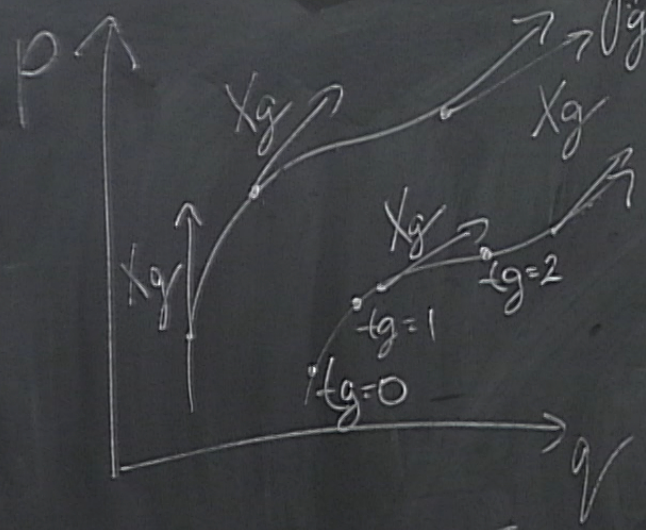
- t_g... TIME FLOW GENERATED BY g.
- X_g... HAMILTONIAN VECTOR FLOW OF g.
- γ_g... INTEGRAL CURVES OF X_g.

HAMILTONIAN IS NOT SPECIAL $g=H$

$\frac{\partial S}{\partial t}$ FOR

$0 = H + \frac{\partial S}{\partial t}$

HAMILTON-JACOBI



CURVES PARAMETRIZED BY 'TIME' t_g.

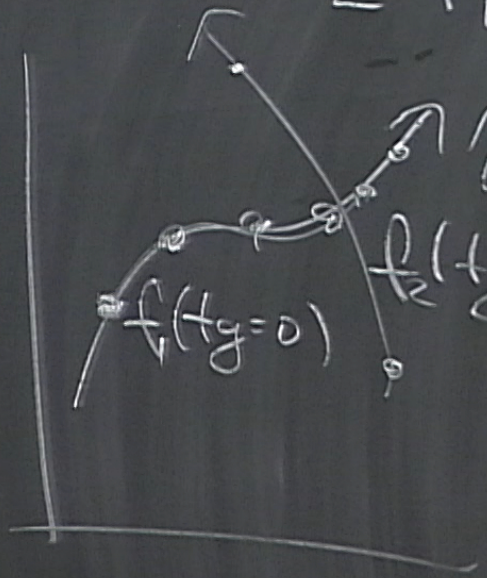
- t_g... TIME FLOW GENERATED BY g.
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- γ_g... INTEGRAL CURVES OF X_g.

FF OPERATOR

HAMILTONIAN IS NOT SPECIAL $g=H$

EXCEPT $t_H = t$

SPEC: $\{f, g\} = X_g(f) = \frac{df}{dt_g}$



TIME EVOLUTION OF f ALONG γ_g.

- SYMMETRY ~~FLOW OF g~~ IS A OF PHASE SPACE FUNCTION f

IF $\{f, g\} = 0 = \frac{df}{dt_g} = 0$

"f DOES NOT CHANGE ALONG FLOW OF g"

$f_1 = f_2$

NOETHER (V4 - GEOMETRIC HAMILTONIAN VERSION)

LET THERE BE ASYMMETRY OF HAMILTONIAN H GENERATED BY I (PHASE SPACE FUNCTION). THEN I IS AN INTEGRAL OF MOTION.

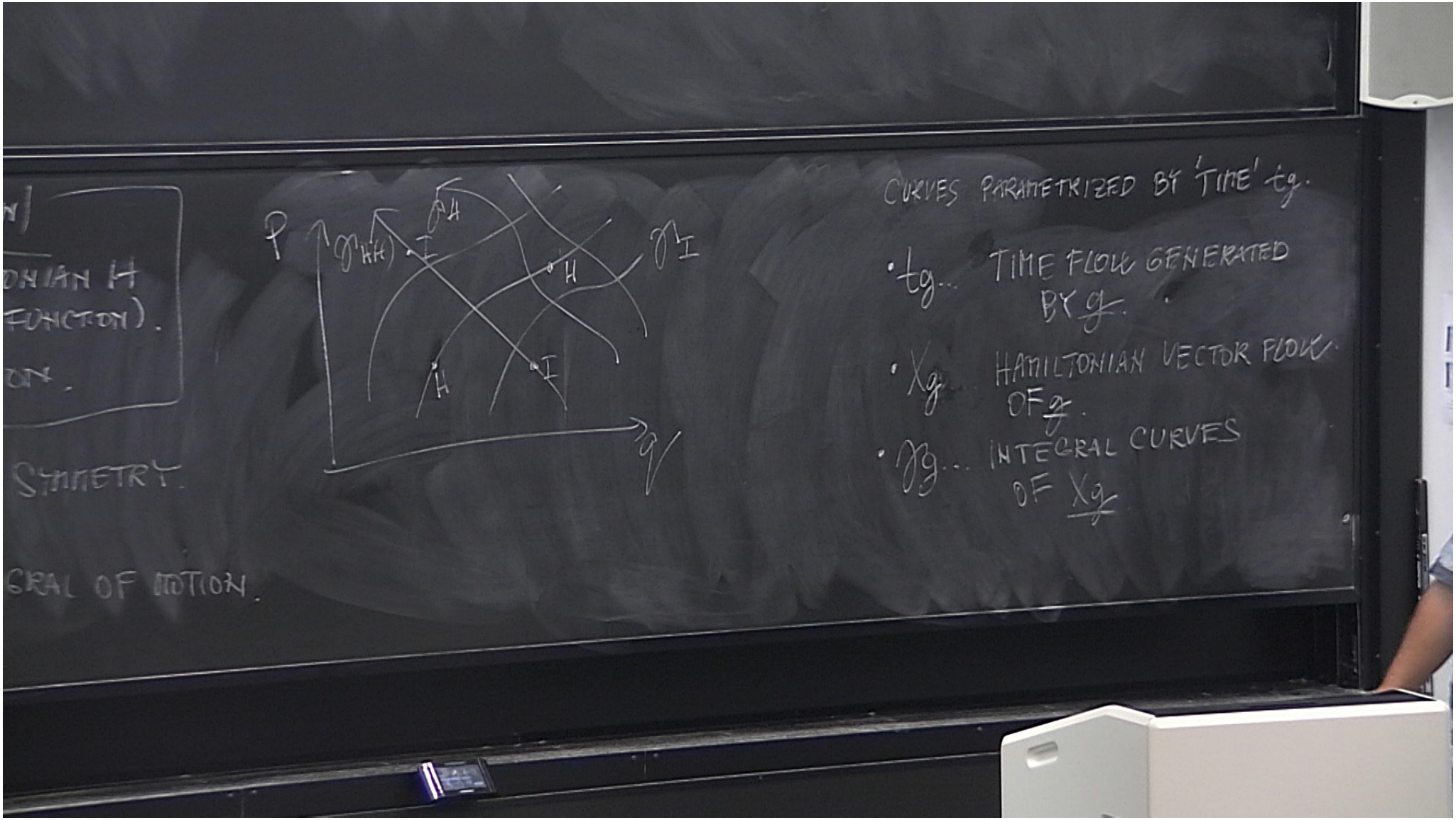
PROOF: $\{H, I\} = 0 = \frac{dH}{dt_I}$ ... SYMMETRY.

NOETHER (V4 - GEOMETRIC HAM VERSION)

LET THERE BE ASYMMETRY OF HAMILTONIAN H GENERATED BY I (PHASE SPACE FUNCTION). THEN I IS AN INTEGRAL OF MOTION.

PROOF: $\{H, I\} = 0 = \frac{dH}{dt_I}$ ... SYMMETRY.

ANTISYM. $\Rightarrow \{I, H\} = 0 = \frac{dI}{dt}$ .. INTEGRAL OF MOTION.


ONIAN H
FUNCTION).
ON.
SYMMETRY.
GRAL OF MOTION.
P
H
I
H
I
q
CURVES PARAMETRIZED BY 'TIME' t_g.
• t_g... TIME FLOW GENERATED BY g.
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