Title: SDP / Quantum Lecture Series

Speakers: Jamie Sikora
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Lecture 4: Entanglement & Entropies

Separability Problem

Two quantum states $\rho \in Pos(X)$, $\sigma \in Pos(Y)$

$\rho \otimes \sigma \in Pos(X \otimes Y)$

Lecture 4: Entanglement & Entropies

Separability Problem

Two quantum states $\rho \in Pos(X)$, $\sigma \in Pos(Y)$

$\rho \otimes \sigma \in Pos(X \otimes Y)$ Product state

$SEP(X:Y) = conv\{ \rho \otimes \sigma : \rho \in D(X), \sigma \in D(Y) \}$ (quantum states)

$= \{ \sum_k p_k \rho_k \otimes \sigma_k : \rho_k \in D(X), \sigma_k \in D(Y) \}$

If $\rho \in D(X \otimes Y) \setminus SEP(X:Y)$, then ρ is said to be entangled. Asking whether ρ is separable or entangled is the separability problem

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$\in Pos(X \otimes Y)$ Product state

$= conv\{\rho \otimes \sigma : \rho \in D(X), \sigma \in D(Y)\}$
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$= \{\sum_k p_k \rho_k \otimes \sigma_k : \rho_k \in D(X), \sigma_k \in D(Y)\}$

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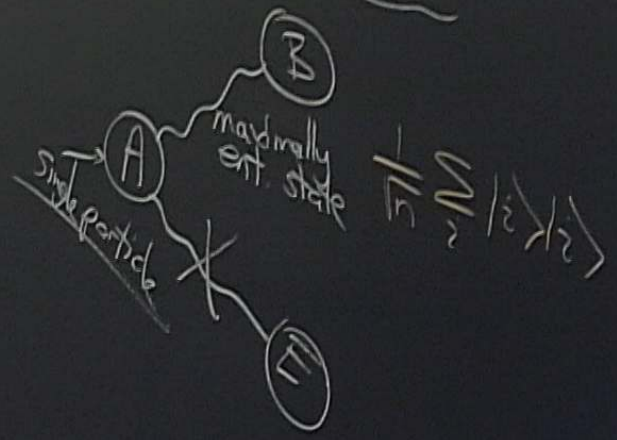
Symmetric Extensions

If Alice is entangled with Bob, can she be entangled with anyone else?

Monogamy of Entanglement



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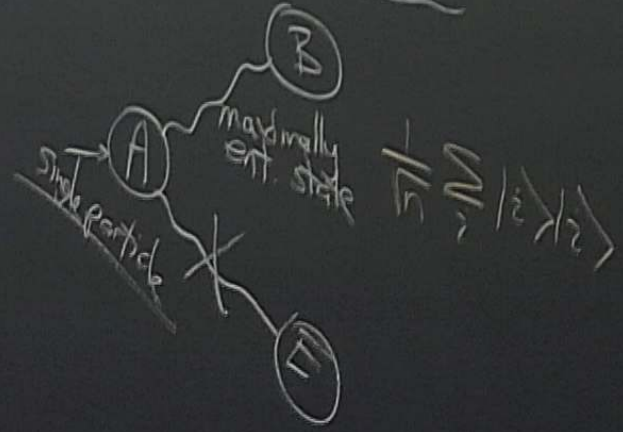
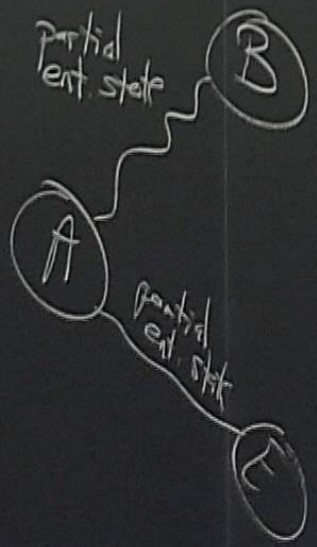


$$\frac{1}{n}\sum_i |i\rangle\langle i|$$



sentation?

she

$\rho \in Pos(\mathcal{X} \otimes \mathcal{Y})$ has a symmetric extension

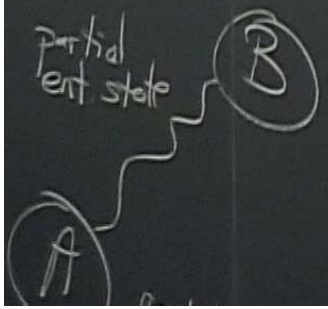
if there exist $\sigma \in Pos(\mathcal{X} \otimes \mathcal{Y}_1 \otimes \mathcal{Y}_2)$

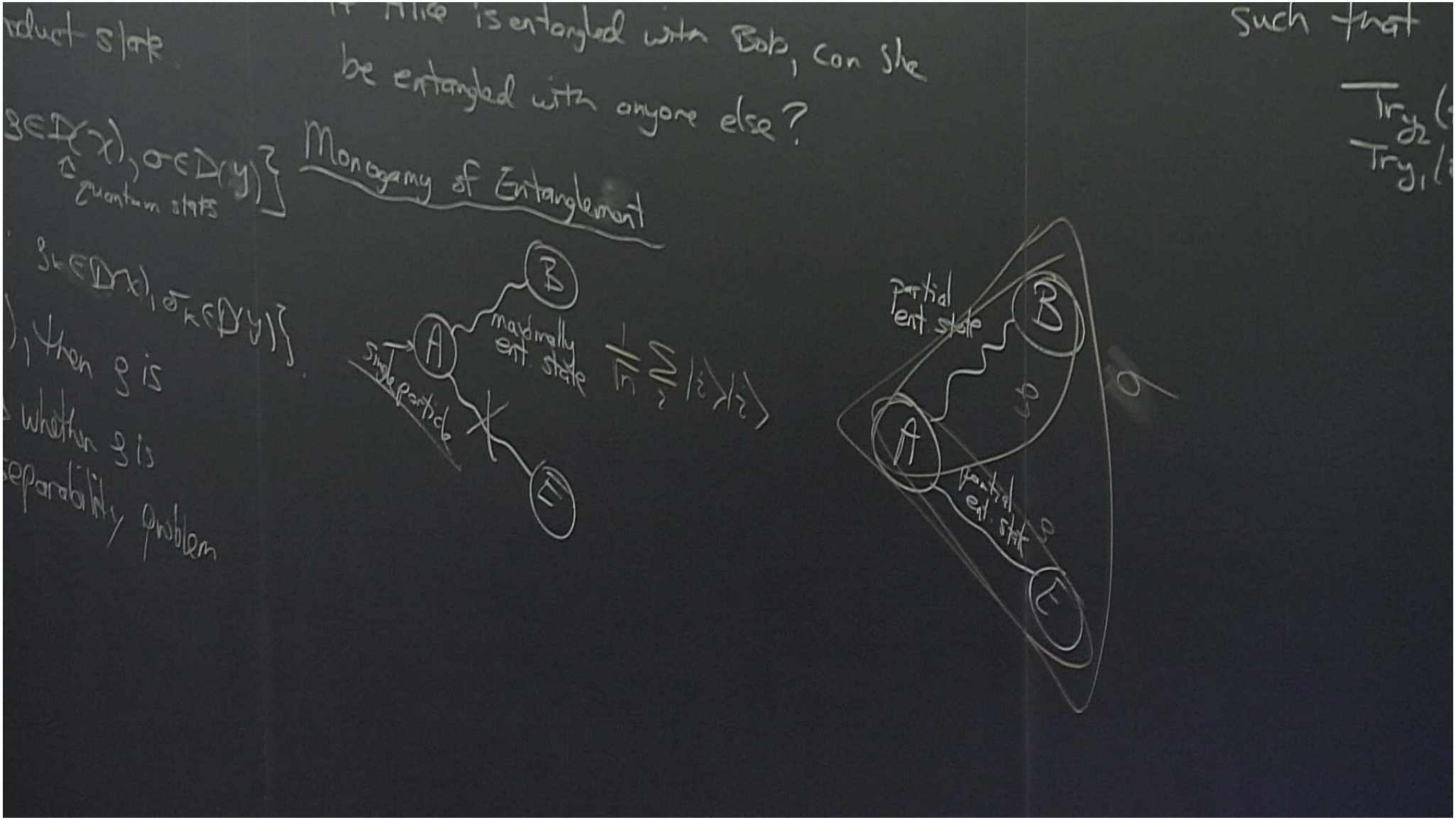
such that

$\mathrm{Tr}_{\mathcal{Y}_2}(\sigma) = \rho$

$\mathrm{Tr}_{\mathcal{Y}_1}(\sigma) = \rho$

two copies of $\mathcal{Y}$.




such that
Alice is entangled with Bob, can she be entangled with anyone else?
Monogamy of Entanglement
quantum states
maximally ent. state
single particle
partial ent state

Two copies of σ.

$\mathrm{Tr}_{\mathcal{Y}_2}(\sigma) = \rho$

$\mathrm{Tr}_{\mathcal{Y}_1}(\sigma) = \rho$

This can be written as an SDP

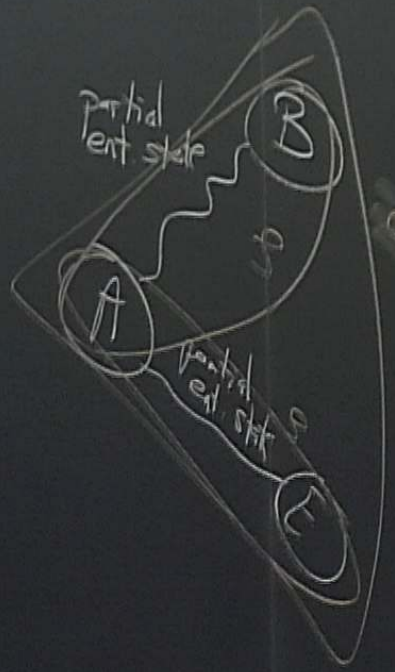
$\alpha = \sup z$

$\mathrm{Tr}_{\mathcal{Y}_2}(\sigma) = \rho$

$\mathrm{Tr}_{\mathcal{Y}_1}(\sigma) = \rho$

$\sigma \geq z\,\Pi$

Sym Ext. exists $\iff$ $\alpha \geq 0$.



If ρ is separable, then $\rho = \sum_k p_k \rho_k \otimes \sigma_k$

$$\sigma = \sum_k p_k \rho_k \otimes \sigma_k \otimes \sigma_k$$

$$\mathrm{Tr}_{y_1}(\sigma) = \rho = \mathrm{Tr}_{y_2}(\sigma)$$

$$\mathrm{SEP}(X:Y) \subseteq \mathrm{Sym}(X:Y)$$

[the set of Sym ext states

Example. $\rho = |\Phi^+\rangle\langle\Phi^+|$, $|\Phi^+\rangle = \frac{1}{\sqrt{2}} \sum_{i=0}^{1} |ii\rangle$

$SEP(X:Y) \subseteq Sym(X:Y)$

[the set of Sym ext states

Example.

$\rho = |\Phi^+\rangle\langle\Phi^+|$, $|\Phi^+\rangle = \frac{1}{\sqrt{2}} \sum_{i=0}^{1} |ii\rangle$

$\beta = \inf \langle |\Phi^+\rangle\langle\Phi^+|, Z_1 \rangle + \langle |\Phi^+\rangle\langle\Phi^+|, Z_2 \rangle$

$Z_1 \otimes \mathbb{1}_{y_1} + Z_2 \otimes \mathbb{1}_{y_2} \in D(X \otimes Y_1 \otimes Y_2)$

$$Z_1 = Z_2 = \begin{bmatrix} 1/32 & 0 & 0 & -2/16 \\ 0 & 3/32 & 0 & 0 \\ 0 & 0 & 3/32 & 0 \\ -2/16 & 0 & 0 & 1/32 \end{bmatrix}$$

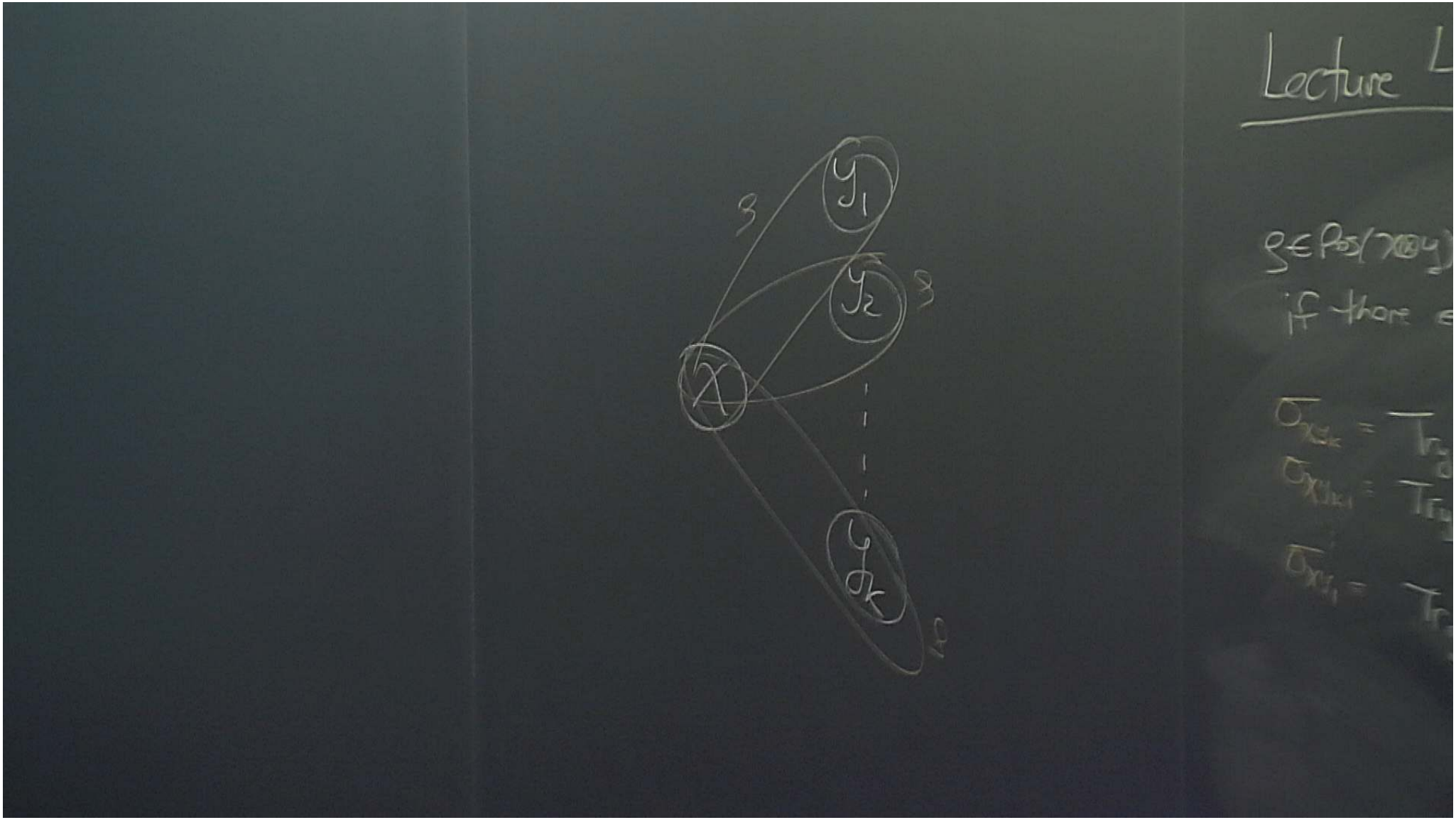
This shows $\beta < 0$. But $\alpha \leq \beta < 0$ (weak duality)

so $\alpha < 0$, no Sym Ext. exists.

Lecture 4: Entanglement & Entropies

$\rho \in Pos(X \otimes Y)$ has a k-th symmetric extension if there exists $\sigma \in Pos(X \otimes Y_1 \otimes \cdots \otimes Y_k)$

$$\sigma_{XY_1\cdots Y_k} = Tr_{Y_1 \cdots Y_{k-1}}(\sigma) = \rho$$
$$\sigma_{XY_1\cdots Y_{k-1}} = Tr_{Y_1 \cdots Y_{k-2} Y_k}(\sigma) = \rho$$
$$\vdots$$
$$\sigma_{XY_1} = Tr_{Y_2 \cdots Y_k}(\sigma) = \rho$$


Lecture
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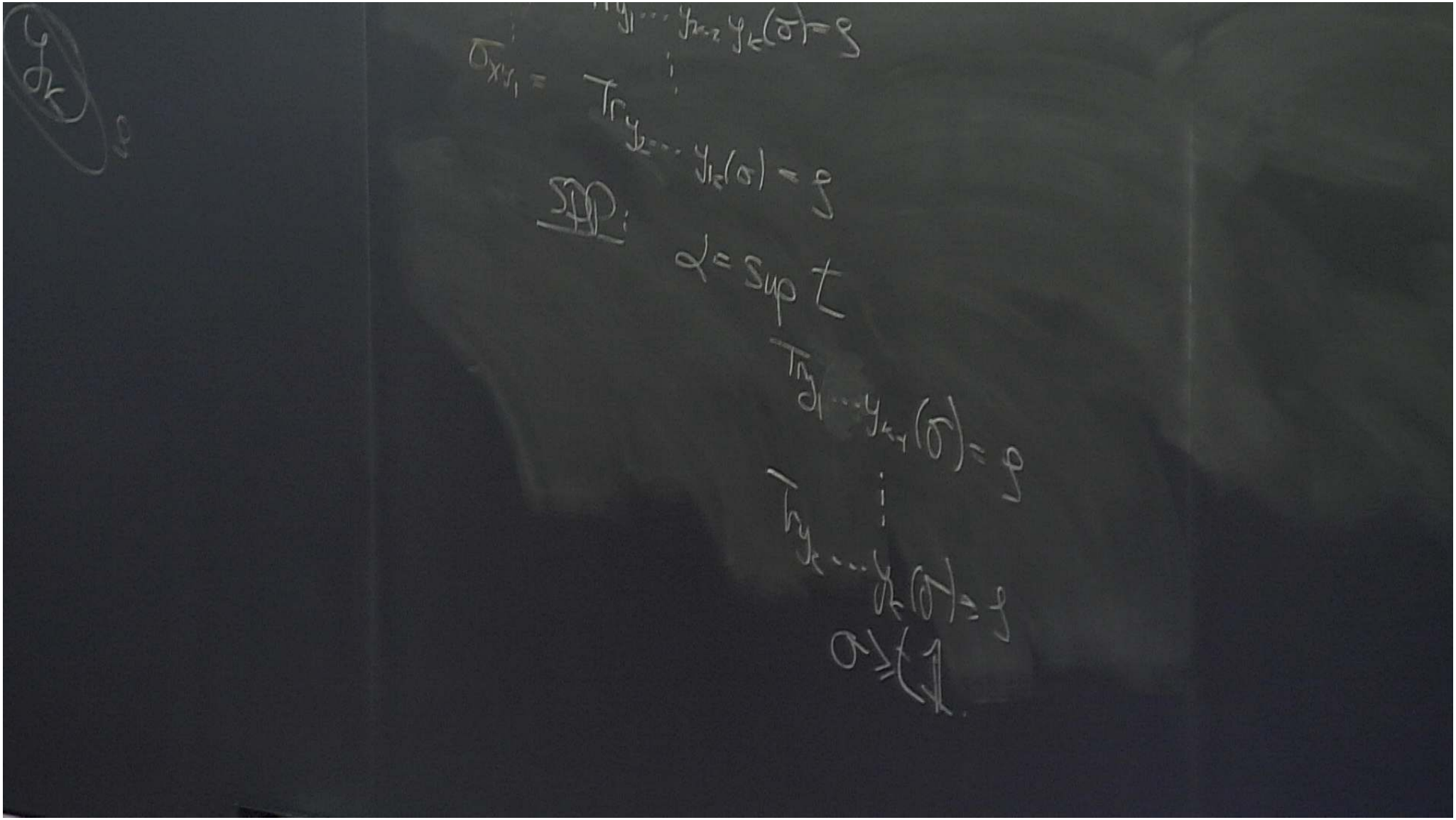


y_k

ρ

$\mathrm{Tr}_{y_1 \cdots y_{k-2}} y_k(\sigma) = \rho$

$\vdots$

$\sigma_{xy_1} = \mathrm{Tr}_{y_2 \cdots y_k}(\sigma) = \rho$

SDP:

$\alpha = \sup t$

$\mathrm{Tr}_{y_1 \cdots y_{k-1}}(\sigma) = \rho$

$\vdots$

$\mathrm{Tr}_{y_k \cdots y_k}(\sigma) = \rho$

$\sigma \geq t\mathbb{1}$

glement & Entropies

k-th symmetric extension

$\in \mathrm{Pos}(\mathcal{X} \otimes \mathcal{Y}, \otimes \cdots \otimes \mathcal{Y}_k)$

$= \rho$

$\sigma = \rho$

$= \rho$

ρt

$\rho \in \mathcal{S}(\mathcal{X} \otimes \mathcal{Y})$

$\rho = \sum_k p_k \rho_k \otimes \sigma_k$

$\sigma = \sum_k p_k \rho_k \otimes \sigma_k \otimes \sigma_k \otimes \cdots \otimes \sigma_k$

Separable states have a k-th sym ext. for all $k \in \mathbb{N}$

Supercharge: Bosonic Sym. Ext.

Defn: Generalized SWAP operator

W_π

$\rho = \sum_k p_k \rho_k \otimes \sigma_k$

ymmetric extension

$\sigma = \sum_k p_k \rho_k \otimes \sigma_k \otimes \sigma_k \otimes \cdots \otimes \sigma_k$

$\sigma_s(\chi \otimes y_1 \otimes \cdots \otimes y_k)$

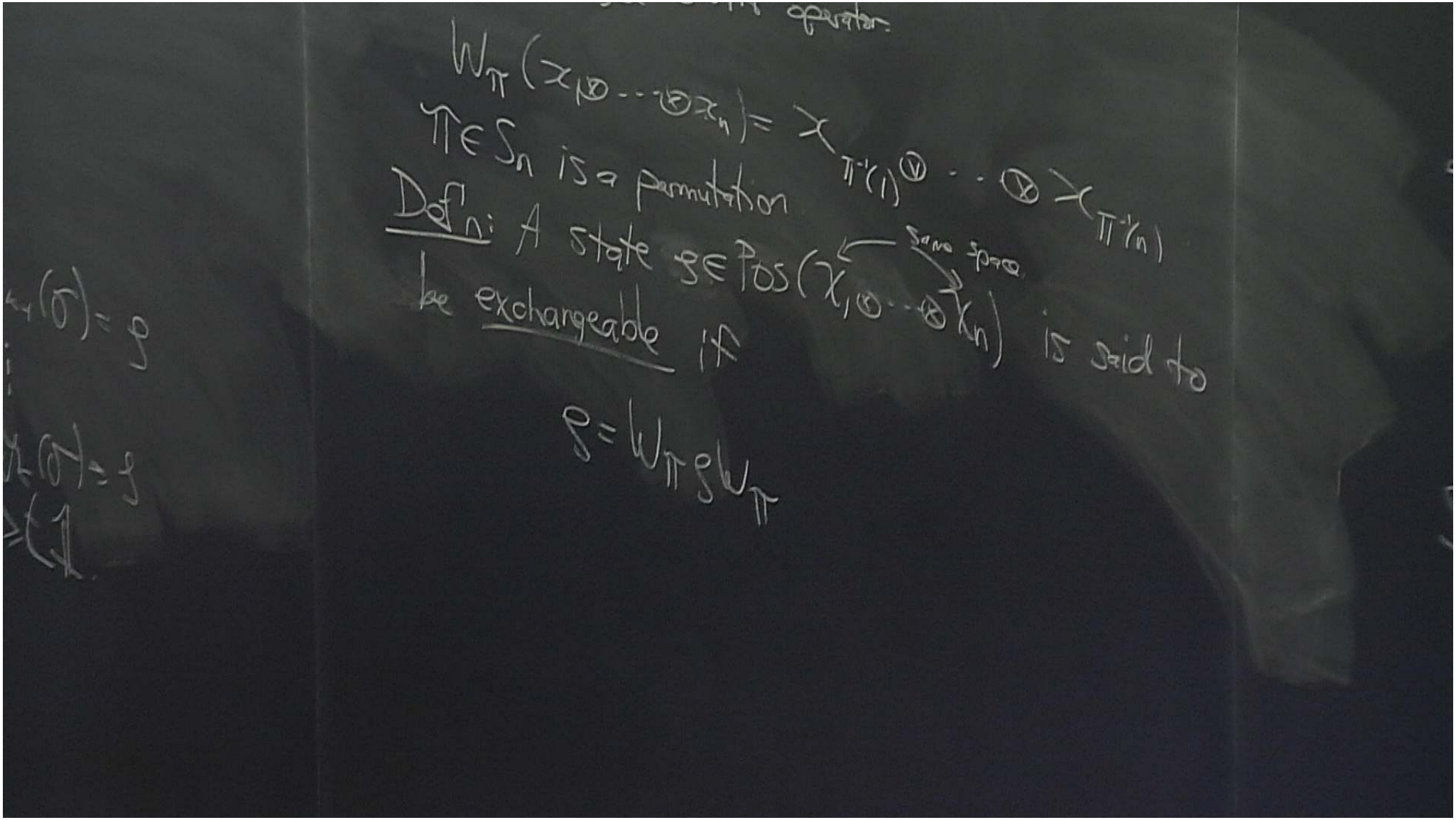
Separable states have a k-th sym ext. for all $k \in \mathbb{N}$

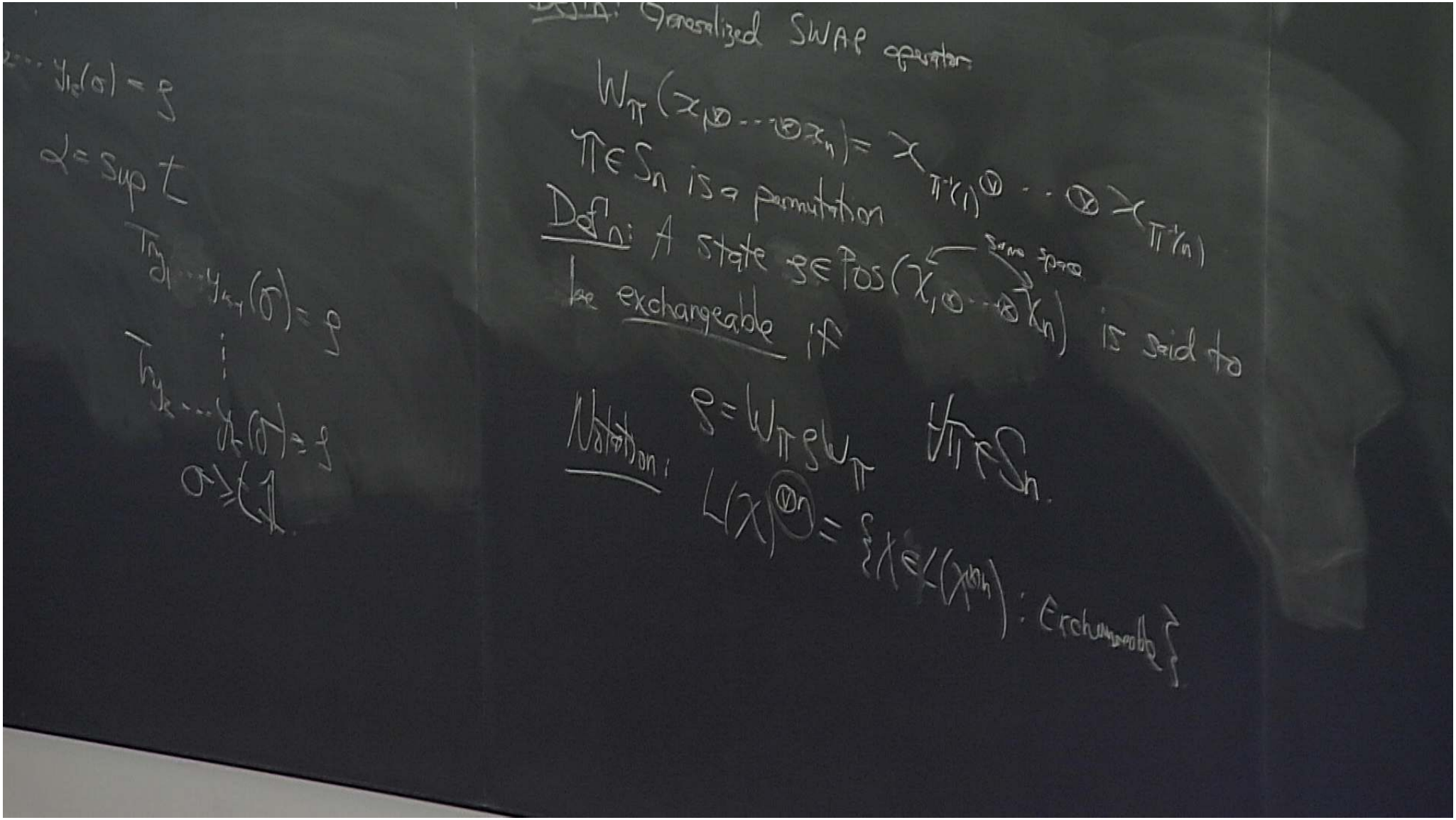
Supercharge: Bosonic Sym. Ext.

Def'n: Generalized SWAP operator

$W_\pi(\chi_1 \otimes \cdots \otimes \chi_n) = \chi_{\pi^{-1}(1)} \otimes \cdots \otimes \chi_{\pi^{-1}(n)}$

$\cdot y_{k+1}(\sigma) = 0$


operator.
$W_\pi(x_1 \otimes \cdots \otimes x_n) = x_{\pi^{-1}(1)} \otimes \cdots \otimes x_{\pi^{-1}(n)}$
$\pi \in S_n$ is a permutation
Def$_n$: A state $\rho \in Pos(\chi_1 \otimes \cdots \otimes \chi_n)$ is said to
same space
be exchangeable if
$\rho = W_\pi \rho W_\pi$



$\cdots y_{1\varepsilon}(\sigma) = \rho$

$\alpha = \sup t$

$\mathrm{Tr}_{y_1 \cdots y_{k-1}}(\sigma) = \rho$

$\vdots$

$\mathrm{Tr}_{y_k \cdots y_\varepsilon}(\sigma) = \rho$

$\sigma \geq t\mathbb{1}$

Defn: Generalized SWAP operator

$$W_\pi(x_1 \otimes \cdots \otimes x_n) = x_{\pi^{-1}(1)} \otimes \cdots \otimes x_{\pi^{-1}(n)}$$

$\pi \in S_n$ is a permutation

Defn: A state $\rho \in \mathrm{Pos}(\mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n)$ is said to be exchangeable if

same space

$$\rho = W_\pi \rho W_\pi \quad \forall \pi \in S_n.$$

Notation: $L(\mathcal{X})^{\otimes n} = \{X \in L(\mathcal{X}^{\otimes n}) : \text{Exchangeable}\}$

$\sigma_k \otimes \cdots \otimes \sigma_k$

k-th sym ext. for all $k \in \mathbb{N}$

Sym Ext

equation

$\chi_{\pi^{-1}(1)} \otimes \cdots \otimes \chi_{\pi^{-1}(n)}$

same types

$(\chi_1 \otimes \cdots \otimes \chi_n)$ is said to

Bosonic Sym Ext

$\rho \in Pos(\mathcal{X} \otimes \mathcal{Y})$ has a k-th Bosonic Sym Ext.

if there exists $\sigma \in Pos(\mathcal{X} \otimes \mathcal{Y}^{\otimes k})$

$\sigma_{\mathcal{Y}_1 \mathcal{X}} = Tr_{\mathcal{Y}_2 \cdots \mathcal{Y}_k}(\sigma) = \rho$

$\parallel$

$\sigma_{\mathcal{Y}_1 \mathcal{X}}$

Only need one partial trace constraint

$\cdots \otimes \chi_{\pi^{-1}(n)}$

space

$\chi_n)$ is said to

$\mathfrak{S}_n$.

: Exchangeable $\}$

$\mathrm{Tr}_{y_2 \cdots y_k}(\sigma) = \rho$

$\parallel$

$\sigma_{y_1 \chi}$

Only need <u>one</u> partial trace constraint

$\alpha = \sup \mathcal{L}$

$\mathrm{Tr}_{y_2 \cdots y_k}(\sigma) = \rho$

$\sigma = W_\pi \sigma W_\pi \quad \forall \pi$ permutations on the y_i's.

Quantum de Finetti Theorem

Exchangeable state $\rho \in Pos(\mathcal{X}^{\otimes n})$, integer $k \in \{1, \ldots, n\}$

Then there exists a probability dist. $p = (p_1, \ldots, p_m)$

and quantum states $(\tau_1, \ldots, \tau_m) \in Pos(\mathcal{X})$ such that $\dim(\mathcal{X})$

$$\left\| Tr_{\mathcal{X}_{k+1} \cdots \mathcal{X}_n}(\rho) - \underbrace{\sum_{i=1}^{m} p_i \tau_i^{\otimes k}}_{\text{SEP state}} \right\|_1 \leq \frac{4 d_{\mathcal{X}} k}{n}$$

Thm: ρ is separable $\iff$ ρ has a k-th Bosonic Sym. Ext. $\forall k$

Pf: Q de Finetti Theorem. (See Doherty-Parrilo-Spedalieri)

and quantum states $(I_1, \dots, I_m) \in \mathrm{Pos}(\mathcal{X})$ such that $\dim(\mathcal{X})$

$$\left\| \mathrm{Tr}_{\mathcal{X}_{k+1} \cdots \mathcal{X}_n}(\rho) - \underbrace{\sum_{i=1}^{m} p_i I_i^{\otimes k}}_{\text{SEP state}} \right\|_1 \leq \frac{4 d_{\mathcal{X}} k}{n}$$

Thm: ρ is separable $\iff$ ρ has a k-th Bosonic Sym. Ext. $\forall k$

Pf: Quantum de Finetti theorem. (See Doherty-Parrilo-Spedalieri)

Faithful squashed entanglement (Brandão, Christandl, Yard)

$$\| \cdots \|_{L(\mathcal{X})}$$

Lecture 4: Entanglement & Entropies

Hamza Fawzi showed that there is no finite-sized SDP

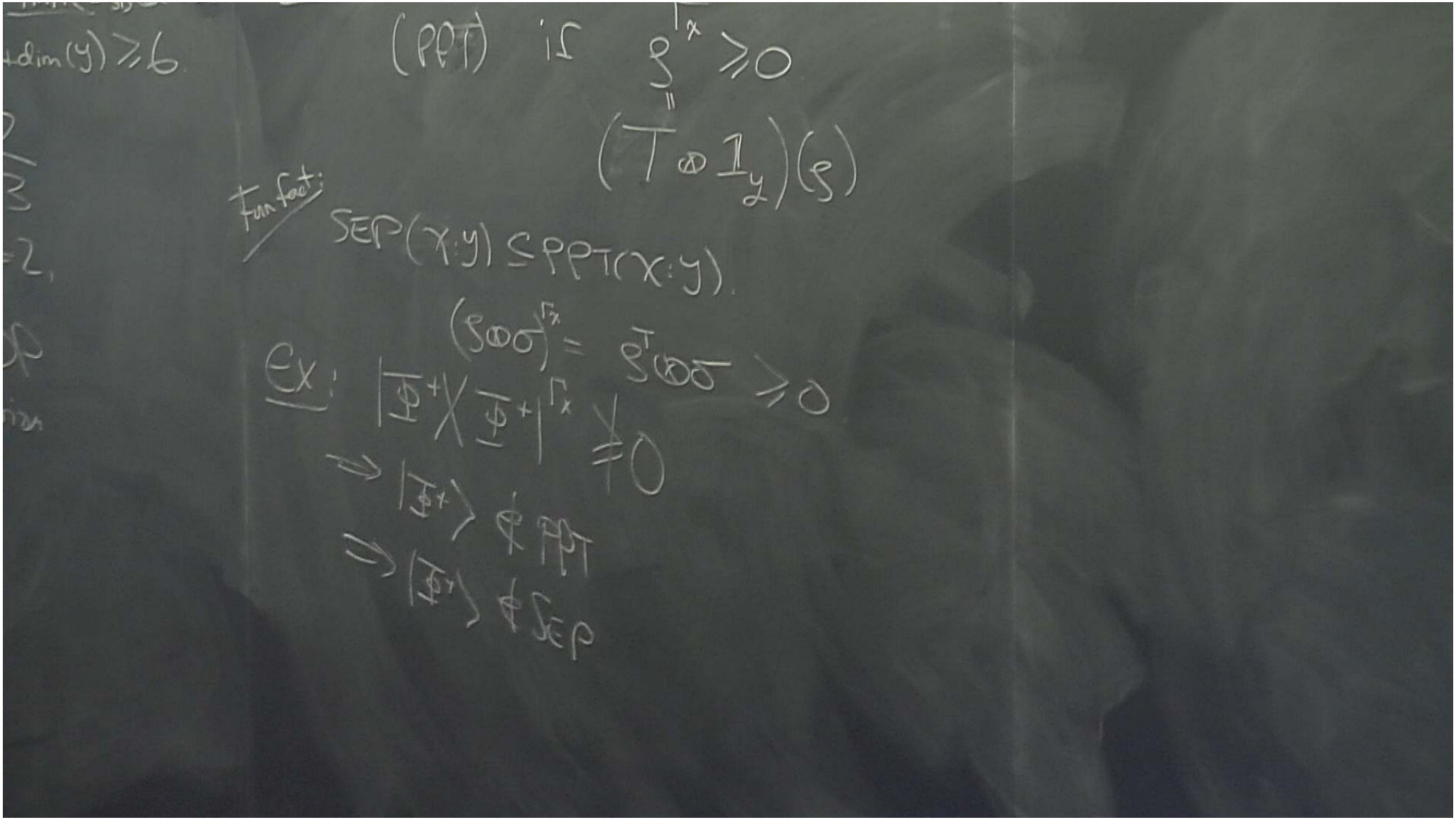
SEP(X,Y), when $\dim(X)+\dim(Y) \geq 6$

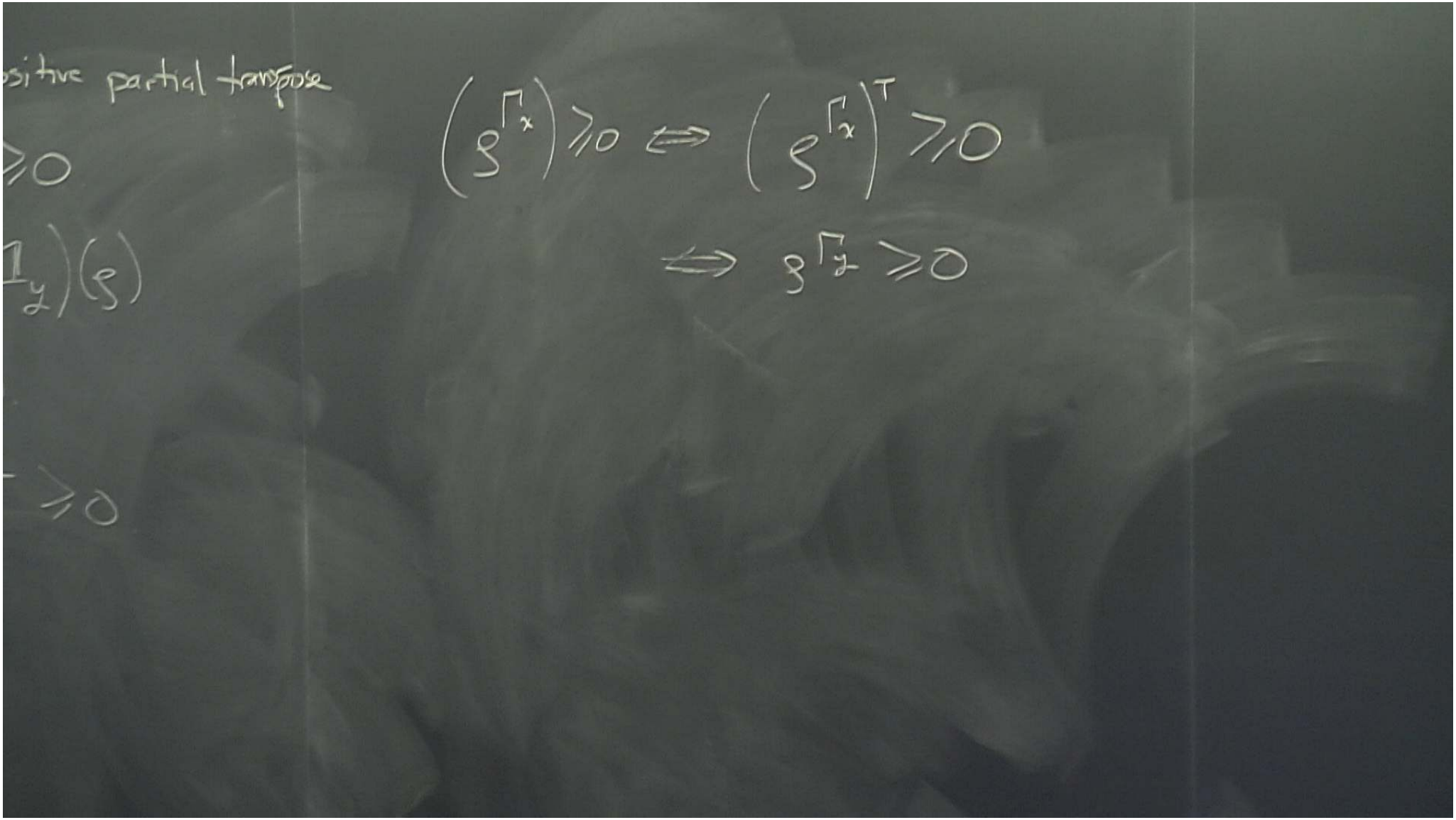
But, when $\dim(X)=\dim(Y)=2$

$2=\dim(X)$, $\dim(Y)=3$

$3=\dim(X)$, $\dim(Y)=2$,

there is a finite-sized SDP given by the PPT criterion


dim(y) ≥ 6
(PPT) iff $\rho^{\Gamma_x} \geq 0$
$= (T \otimes 1_y)(\rho)$
Fun fact:
SEP(X:Y) ⊆ PPT(X:Y)
$(\sigma \otimes \sigma)^{\Gamma_x} = \sigma^{\Gamma} \otimes \sigma \geq 0$
Ex: $|\Phi^+\rangle\langle\Phi^+|^{\Gamma_x} \neq 0$
$\Rightarrow |\Phi^+\rangle \notin$ PPT
$\Rightarrow |\Phi^+\rangle \notin$ SEP


positive partial transpose
≥ 0
$\Gamma_y)(\rho)$
≥ 0
$(\rho^{\Gamma_x}) \geq 0 \Leftrightarrow (\rho^{\Gamma_x})^T \geq 0$
$\Leftrightarrow \rho^{\Gamma_x} \geq 0$

$\{1, \ldots, n\}$

$, \rho_m)$

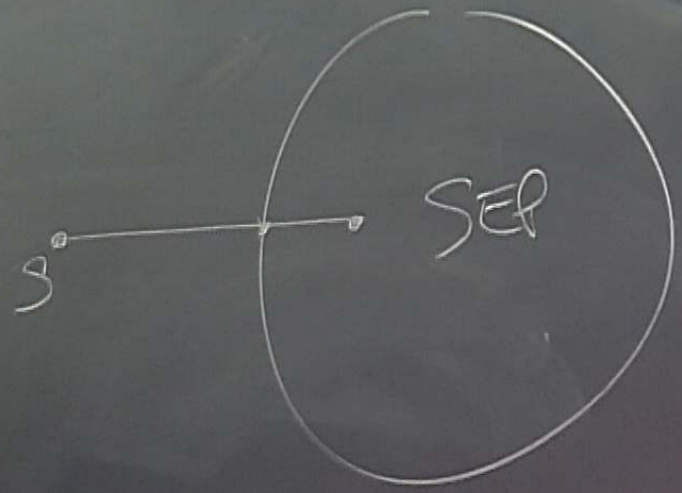
such that $\dim(\mathcal{X})$

$$\leq \frac{4 d_{\mathcal{X}} K}{n}$$

Bosonic Sym. Ext. $\forall K$

herty - Parrilo - Spedalieri)

nglement (Brandão, Christandl, Yard)



Lecture 4: Entanglement & Entropies

Von Neumann entropy of ρ

$$S(\rho) = -\langle \rho, \log_2(\rho) \rangle$$

Quantum relative entropy

$$S(\rho \| \sigma) = \langle \rho, \log_2(\rho) - \log_2(\sigma) \rangle$$

We'll show SDP hierarchies to compute those functions

Theorem: For $P,Q \in \mathrm{Pos}(\mathcal{X})$, we have

$$\mathrm{Tr}\left(P^{1-z^{-m}} Q^{z^{-m}}\right) = \sup \; \mathrm{vec}\left(\sum_x |xx\rangle\right)^* W_m \, \mathrm{vec}\left(\sum_x |xx\rangle\right)$$

$$\begin{bmatrix} P \otimes \mathbb{1}_{\mathcal{X}} & W_k \\ W_k & W_{k-1} \end{bmatrix} \geq 0 \quad \forall k \in \{1, \ldots, m\}$$

$$W_0 = \mathbb{1}_{\mathcal{X}} \otimes Q^{\mathsf{T}}.$$

(X), we have

$\sum_x |x\rangle\langle x|$ $\qquad$ $\sum_{x \in \mathrm{CONJ}} |x\rangle\langle x|$

$$\sup \mathrm{vec}(\mathbb{1}_{x \otimes x})^\dagger W_m \mathrm{vec}(\mathbb{1}_{x \otimes x})$$

$$\begin{bmatrix} P_0 \mathbb{1}_x & W_k \\ W_k & W_{k-1} \end{bmatrix} \geq 0 \quad \forall k \in \{1, \ldots, m\}$$

$$W_0 = \mathbb{1}_x \otimes Q^T.$$

...anglement & Entropies

... of ρ

$= \langle \rho, \log_2(\rho) \rangle$

...ropy

$) = \langle \rho, \log_2(\rho) - \log_2(\sigma) \rangle$

... hierarchies to

...ions.

Theorem: For $P, Q \in \mathrm{Pos}(\mathcal{X})$, we have

$$\mathrm{Tr}\left(P^{1-2^{-m}} Q^{2^{-m}}\right) = \sup \ \mathrm{vec}(\mathbb{1}_{\mathcal{X}\otimes\mathcal{X}})^* W_m \boxed{\mathrm{vec}(\mathbb{1}_{\mathcal{X}\otimes\mathcal{X}})}$$

$$\begin{bmatrix} P \otimes \mathbb{1}_{\mathcal{X}} & W_k \\ W_k & W_{k-1} \end{bmatrix} \geq 0 \quad \forall k \in \{1, \ldots$$

$$W_0 = \mathbb{1}_{\mathcal{X}} \otimes Q^T.$$

$$F_H(P, Q) = \mathrm{Tr}\left(P^{1/2} Q^{1/2}\right)$$

$\sum_x |x\rangle|x\rangle$

$\sum_{x \in \ldots}$

Def'n: For $X, Y > 0$, we define their t-weighted operator geometric mean as

$$X \#_t Y = X^{1/2} \left(X^{-1/2} Y X^{-1/2} \right)^t X^{1/2}$$

$$X \#_t Y = \lim_{\varepsilon \to 0} (X + \varepsilon \mathbb{I}) \#_t (Y + \varepsilon \mathbb{I})$$

$\forall k \in \{1, \ldots, m\}$

Properties:

1. $X \#_{t_1} (X \#_{t_2} Y) = X \#_{t_1 t_2} Y$
2. For $t \in (0,1)$, if $X_1 \geq Y_1 \geq 0$
   $X \#_t Y_1 \geq X \#_t Y_2$

$k \in \{1, \dots, m\}$

$$X \#_t Y = X^{1/2} \left(X^{-1/2} Y X^{-1/2} \right)^t X^{1/2}$$

$$X \#_t Y = \lim_{\varepsilon \to 0} (X + \varepsilon \mathbb{I}) \#_t (Y + \varepsilon \mathbb{I})$$

Properties:

① $X \#_{t_1} (X \#_{t_2} Y) = X \#_{t_1 t_2} Y$

② For $t \in [0,1]$, if $X_1 \geq X_2 \geq 0$

$$X \#_t Y_1 \geq X \#_t Y_2$$

③ $X \#_t Y = X^{1-t} Y^t$ if X, Y commute.

$$\mathrm{Tr}\left(P^{1-2^{-m}} Q^{2^{-m}}\right) = \sup \; \mathrm{vec}(\mathbb{1}_{x\otimes x})^\dagger \, W_m \, \mathrm{vec}(\mathbb{1}_{x\otimes x})$$

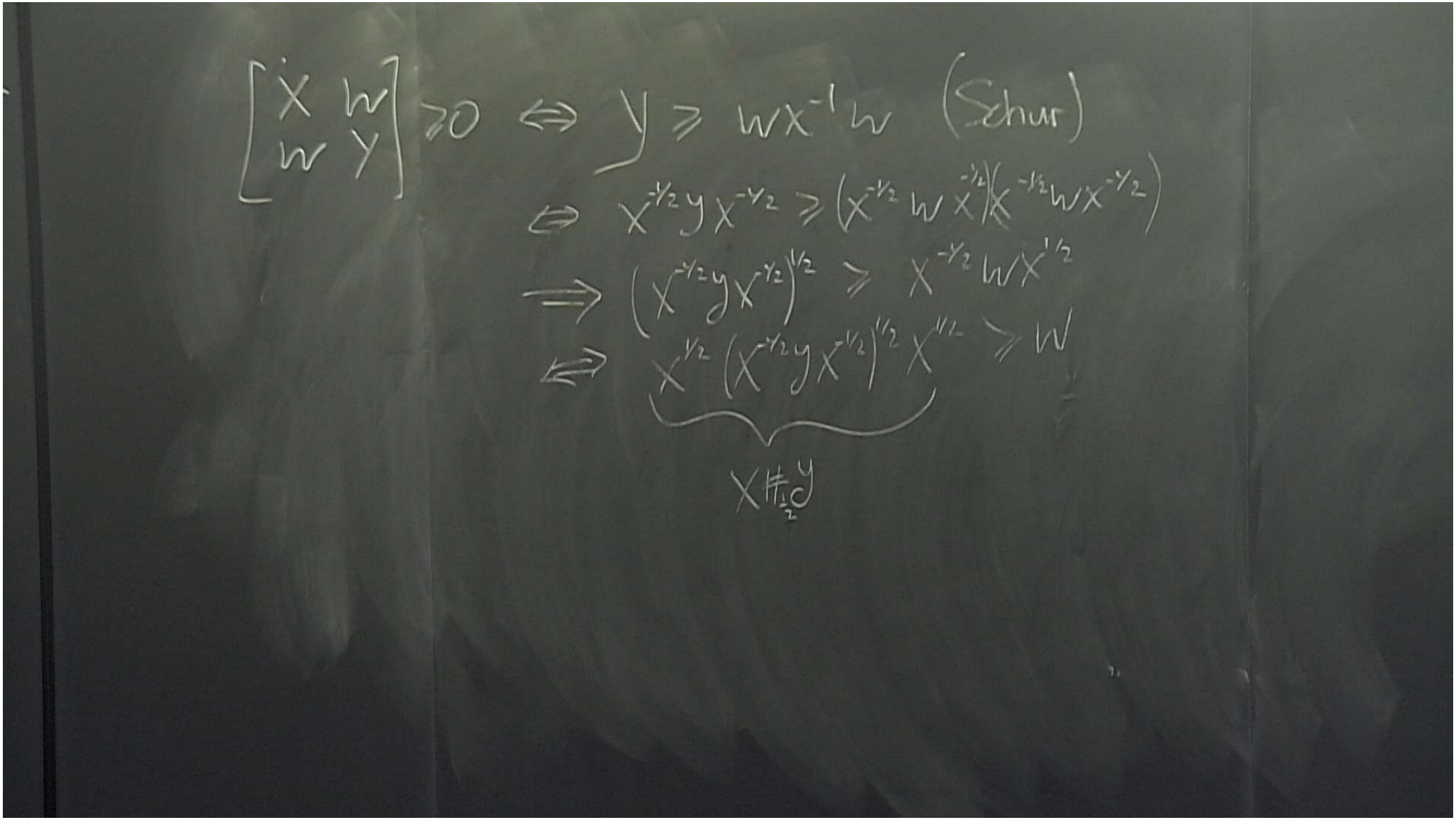
$$\begin{bmatrix} P\otimes \mathbb{1}_x & W_k \\ W_k & W_{k-1} \end{bmatrix} \geq 0 \quad \forall k \in \{1, \ldots, m\}$$

$$W_0 = \mathbb{1}_x \otimes Q^T.$$

(W_k is Hermitian)

$$F_H(P,Q) = \mathrm{Tr}(P^{1/2} Q^{1/2})$$

$\ldots(S) - \log_2(\sigma)\rangle$


$\begin{bmatrix} X & W \\ W & Y \end{bmatrix} \geq 0 \iff Y \geq WX^{-1}W$ (Schur)
$\iff X^{-1/2} Y X^{-1/2} \geq (X^{-1/2} W X^{-1/2})(X^{-1/2} W X^{-1/2})$
$\implies (X^{-1/2} Y X^{-1/2})^{1/2} \geq X^{-1/2} W X^{1/2}$
$\iff X^{1/2} (X^{-1/2} Y X^{-1/2})^{1/2} X^{1/2} \geq W$
$X \#_{\frac{1}{2}} Y$

SDP

$$\begin{bmatrix} P\otimes I & W_1 \\ W_1 & W_0 \end{bmatrix} = \begin{bmatrix} P\otimes I & W_1 \\ W_1 & I\otimes Q^T \end{bmatrix} \Rightarrow W_1 \leq (P\otimes I)\#_{1/2}(I\otimes Q^T)$$

$$= P^{1/2}\otimes (Q^{1/2})^T$$

$$\begin{bmatrix} P\otimes I & W_1 \\ W_1 & W_0 \end{bmatrix} = \begin{bmatrix} P\otimes I & W_1 \\ W_1 & I\otimes Q^T \end{bmatrix} \succeq 0 \Rightarrow W_1 \le (P\otimes I)\#_{1/2}(I\otimes Q^T)$$

$$= P^{1/2}\otimes (Q^{1/2})^T$$

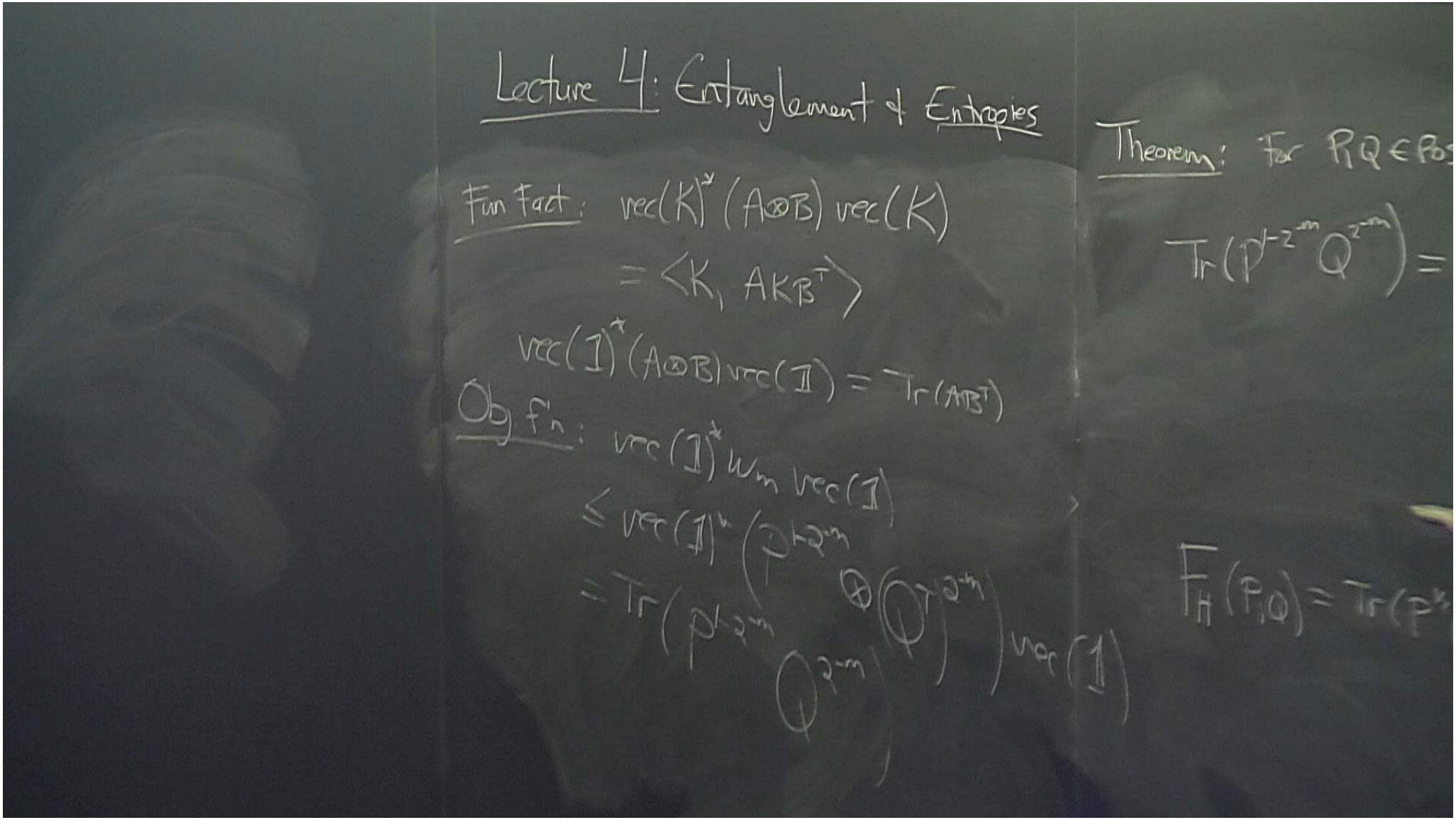
$$\begin{bmatrix} P\otimes I & W_2 \\ W_2 & W_1 \end{bmatrix} \succeq 0 \Rightarrow W_2 \le (P\otimes I)\#_{\frac{1}{2}} W_1$$

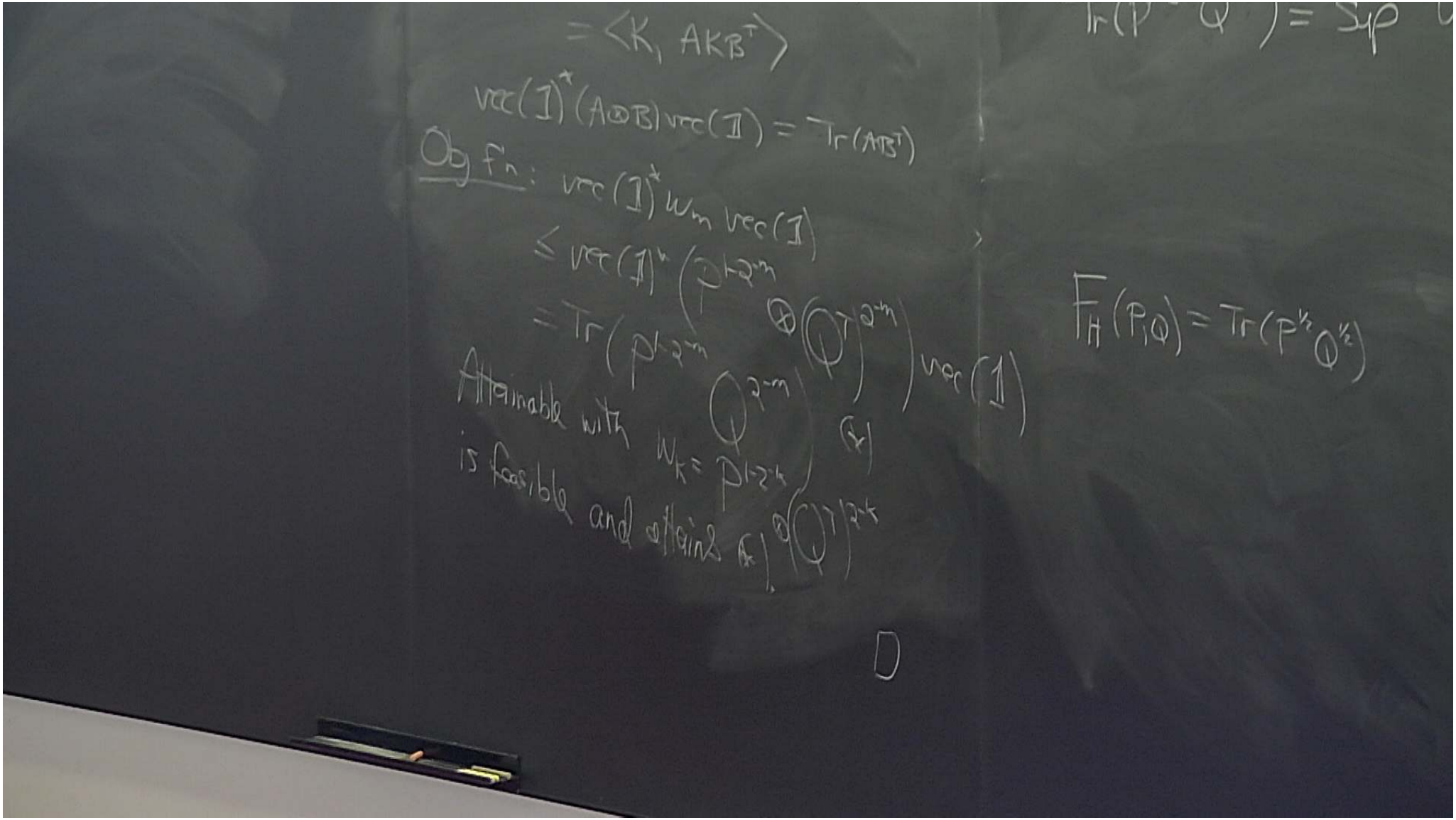
$$\le (P\otimes I)\#_{1/2}\left((P\otimes I)\#_{1/2}(I\otimes Q^T)\right)$$

$$= (P\otimes I)\#_{1/4}(I\otimes Q^T)$$

$$= P^{1/4}\otimes (Q^{1/4})^T$$

$$\begin{bmatrix} P\otimes I & W_k \\ W_k & W_{k-1} \end{bmatrix} \succeq 0 \Rightarrow W_k \le P^{1-2^{-k}}\otimes (Q^{2^{-k}})^T$$


Lecture 4: Entanglement & Entropies
Fun fact: $\mathrm{vec}(K)^\dagger (A \otimes B)\, \mathrm{vec}(K) = \langle K, AKB^\dagger \rangle$
$\mathrm{vec}(\mathbb{I})^\dagger (A \otimes B)\, \mathrm{vec}(\mathbb{I}) = \mathrm{Tr}(AB^\dagger)$
Obj fn: $\mathrm{vec}(\mathbb{I})^\dagger W_m \mathrm{vec}(\mathbb{I})$
$\leq \mathrm{vec}(\mathbb{I})^\dagger (P^{1-2^{-m}} \otimes Q^{2^{-m}})$
$= \mathrm{Tr}(P^{1-2^{-m}} Q^{2^{-m}})$
Theorem: For $P, Q \in Pos$
$\mathrm{Tr}(P^{1-2^{-m}} Q^{2^{-m}}) =$
$F_H(P,Q) = \mathrm{Tr}(P^{\frac{1}{2}}$



$$= \langle K, AKB^T \rangle$$

$$\mathrm{vec}(\mathbb{1})^* (A \otimes B)\, \mathrm{vec}(\mathbb{1}) = \mathrm{Tr}(AB^T)$$

Obj fn: $\mathrm{vec}(\mathbb{1})^* W_m \, \mathrm{vec}(\mathbb{1})$

$$\leq \mathrm{vec}(\mathbb{1})^* \left(P^{1-2^{-m}} \otimes (Q^T)^{2^{-m}} \right) \mathrm{vec}(\mathbb{1})$$

$$= \mathrm{Tr}\left(P^{1-2^{-m}} Q^{2^{-m}} \right) \quad (*)$$

Attainable with $W_k = P^{1-2^{-k}} \otimes (Q^T)^{2^{-k}}$ is feasible and attains $(*)$.

$\square$

$$\mathrm{Tr}(P^{\cdots} Q^{\cdots}) = \sup \ldots$$

$$F_H(P,Q) = \mathrm{Tr}(P^{1/2} Q^{1/2})$$

Tsallis entropy

$$S_t(P\|Q) = \frac{1}{t}\langle \mathbb{I}, P - P^{1-t}Q^t \rangle$$

$$S_t(P) = \frac{1}{t}\langle \mathbb{I}, P^{1-t} - P \rangle$$

SDPs for $t = 2^{-m}$.

Fun Facts:

$$S(\rho\|\sigma) = \lim_{t\to 0} S_t(\rho\|\sigma) = \lim_{m\to\infty} S_{2^{-m}}(\rho\|\sigma)$$

$$S(\rho) = \lim_{t\to 0} S_t(\rho) = \lim_{m\to\infty} S_{2^{-m}}(\rho)$$

Lemma: Joint convexity of quantum relative entropy.

Let $P_1, \ldots, P_n, Q_1, \ldots, Q_n \in \mathrm{Pos}(\mathcal{X})$

$$S\left(\sum_k P_k \middle\| \sum_k Q_k\right) \leq \sum_k S(P_k \| Q_k)$$

$S_{\max}(\rho\|\sigma)$

$S_{\max}(\rho)$

Lemma: Joint convexity of quantum relative entropy.

Let $P_1, \ldots, P_n, Q_1, \ldots, Q_n \in Pos(\mathcal{X})$

$$S\left(\sum_k P_k \middle\| \sum_k Q_k\right) \leq \sum_k S(P_k \| Q_k).$$

Let $(W_0^\ell, \ldots, W_m^\ell)$ be an optimal solution for $Tr(P_\ell^{1-2^{-m}} Q_\ell^{2^{-m}})$

Then $\left(\sum_\ell W_0^\ell, \ldots, \sum_\ell W_m^\ell\right)$ this is feasible for Tr

α)

$S(P_k \| Q_k)$.

inal solution for $\mathrm{Tr}(P_\ell^{1-\alpha-m} Q_\ell^{\alpha-m})$

) this is feasible for $\mathrm{Tr}\left(\left(\sum_\ell P_\ell\right)^{1-\alpha-m} \left(\sum_\ell Q_\ell\right)^{\alpha-m} \right)$

(exercise)

Let $(W_0^{\ell}, \ldots, W_m^{\ell})$ be an optimal solution for $\mathrm{Tr}(P_\ell \; Q_\ell)$

Then $\left(\sum_\ell W_0^\ell, \ldots, \sum_\ell W_m^\ell\right)$ this is feasible for $\mathrm{Tr}\left(\left(\sum_\ell P_\ell\right)^{1-2^{-m}} \left(\sum_\ell Q_\ell\right)^{2^{-m}}\right) \geq \mathrm{vec}(\mathbb{I})^\dagger \left(\sum_\ell W_m^\ell\right) \mathrm{vec}(\mathbb{I})$

$$= \sum_\ell \mathrm{vec}(\mathbb{I}) W_m^\ell \mathrm{vec}(\mathbb{I})$$

$$= \sum_\ell \mathrm{Tr}\left(P_\ell^{1-2^{-m}} Q_\ell^{2^{-m}}\right)$$

Lecture 4: Entanglement & Entropies

Theorem: For $P, Q \in Pos(\mathcal{X})$, $\alpha \in (1, \infty)$

$$S_{2^{-m}}\left(\sum_x P_x \middle\| \sum_x Q_x\right) \leq \sum_x S_{2^{-m}}(P_x \| Q_x) \operatorname{Tr}\left(P_x^{1-2^{-m}} Q_x^{2^{-m}}\right) = \exp$$

$\lim_{m \to \infty}$ implies $S\left(\sum_x P_x \middle\| \sum_x Q_x\right)$

$$\leq \sum_x S(P_x \| Q_x)$$

$\square$

$$F_H(P, Q) = \operatorname{Tr}(P^{\frac{1}{2}} Q^{\frac{1}{2}})$$