Title: SDP / Quantum Lecture Series

Speakers: Jamie Sikora
Date: July 22, 2019 - 10:00 AM

URL: http://pirsa.org/19070072

Lecture 3-E

Diamond norm

Recall from last lecture, that

$$F(\mathcal{S},\sigma) = \max \frac{1}{2}\mathrm{Tr}(X) + \frac{1}{2}\mathrm{Tr}(X^*)$$

$$\begin{bmatrix} \mathcal{S} & X \\ X^* & \sigma \end{bmatrix} \geq 0$$

Lecture 3-E

Diamond norm

Recall from last lecture, that

$$F(\rho,\sigma) = \max \frac{1}{2}\mathrm{Tr}(X) + \frac{1}{2}\mathrm{Tr}(X^\dagger)$$
$$\begin{bmatrix} \rho & X \\ X^\dagger & \sigma \end{bmatrix} \geq 0$$

The diamond norm is a useful norm for quantum channels. This is also called the completely bounded 1-norm

$\Phi : L(X) \to L(Y)$

operators from $X \to X$

The induced norm

$$\|\Phi\|_1 = \max \left\{ \|\Phi(X)\|_1 : X \in L(X), \|X\|_1 \leq 1 \right\}$$

is the 1-norm

Lecture 3-E

Diamond norm

Recall from last lecture, that

$$F(\rho,\sigma) = \max \tfrac{1}{2}\mathrm{Tr}(X) + \tfrac{1}{2}\mathrm{Tr}(X^*)$$

$$\begin{bmatrix} \rho & X \\ X^* & \sigma \end{bmatrix} \geq 0$$

The diamond norm is a useful norm for quantum channels. This is also called the completely bounded 1-norm.

$$\overline{\Phi}: \underbrace{L(X)}_{\text{operators from } X \to X} \to L(Y)$$

The induced norm

$$\|\overline{\Phi}\|_1 = \max\left\{ \|\overline{\Phi}(X)\|_1 : X \in L(X), \|X\|_1 \leq 1 \right\}$$

is the 1-norm

The completely bounded version is

$$\|\overline{\Phi}\| = \sup_{k \geq 1} \|\overline{\Phi} \otimes 1_{L(\mathbb{C}^k)}\|_1$$

Channel discrimination problem

$\underline{\underline{\Phi}}_0$, $\underline{\underline{\Phi}}_1$

If these are chosen w.p. $1/2$ then the optimal probability of learning which channel is

$$\frac{1}{2} + \frac{1}{4} \left\| \underline{\underline{\Phi}}_0 - \underline{\underline{\Phi}}_1 \right\|_1$$

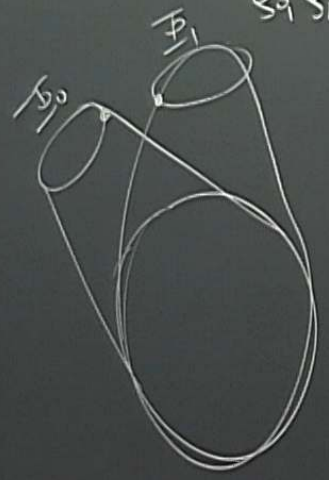
on problem

$\frac{1}{2}$

ity of

Defn: $F_{max}(\mathcal{E}_0, \mathcal{E}_1) = \max F(\mathcal{E}_0(\rho_0), \mathcal{E}_1(\rho_1))$

ρ_0, ρ_1 quantum states



$\Phi_1(\rho_1))$

quantum states

SDP: $F_{max}(\Phi_1, \Phi_1) = \max \frac{1}{2}\mathrm{Tr}(X) + \frac{1}{2}\mathrm{Tr}(X^*)$

$$\begin{bmatrix} \Phi_0(\rho_0) & X \\ X^* & \Phi_1(\rho_1) \end{bmatrix}$$

$\rho_0, \rho_1 \geq 0$

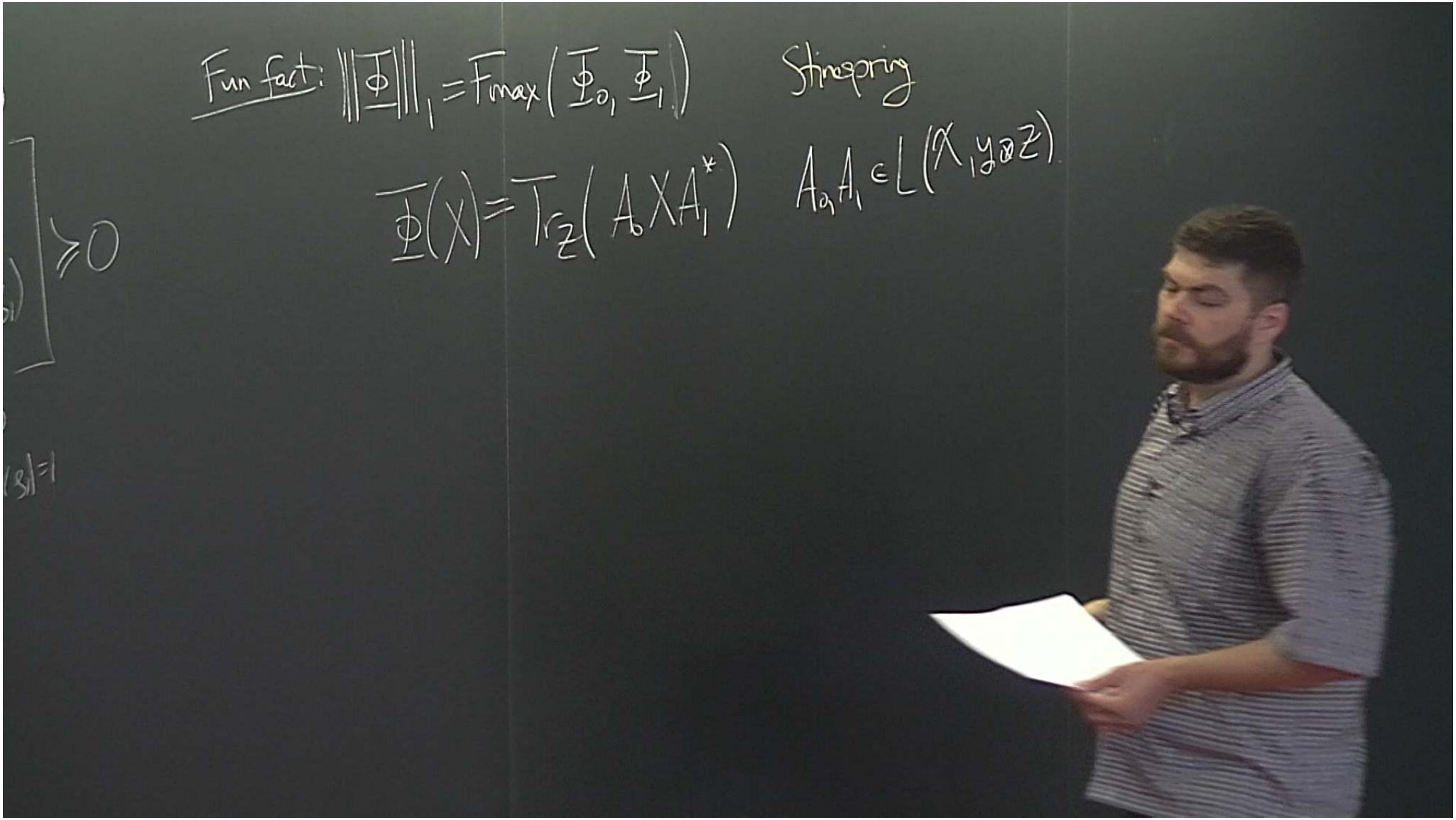
$\mathrm{Tr}(\rho_0) = \mathrm{Tr}(\rho_1) = 1$

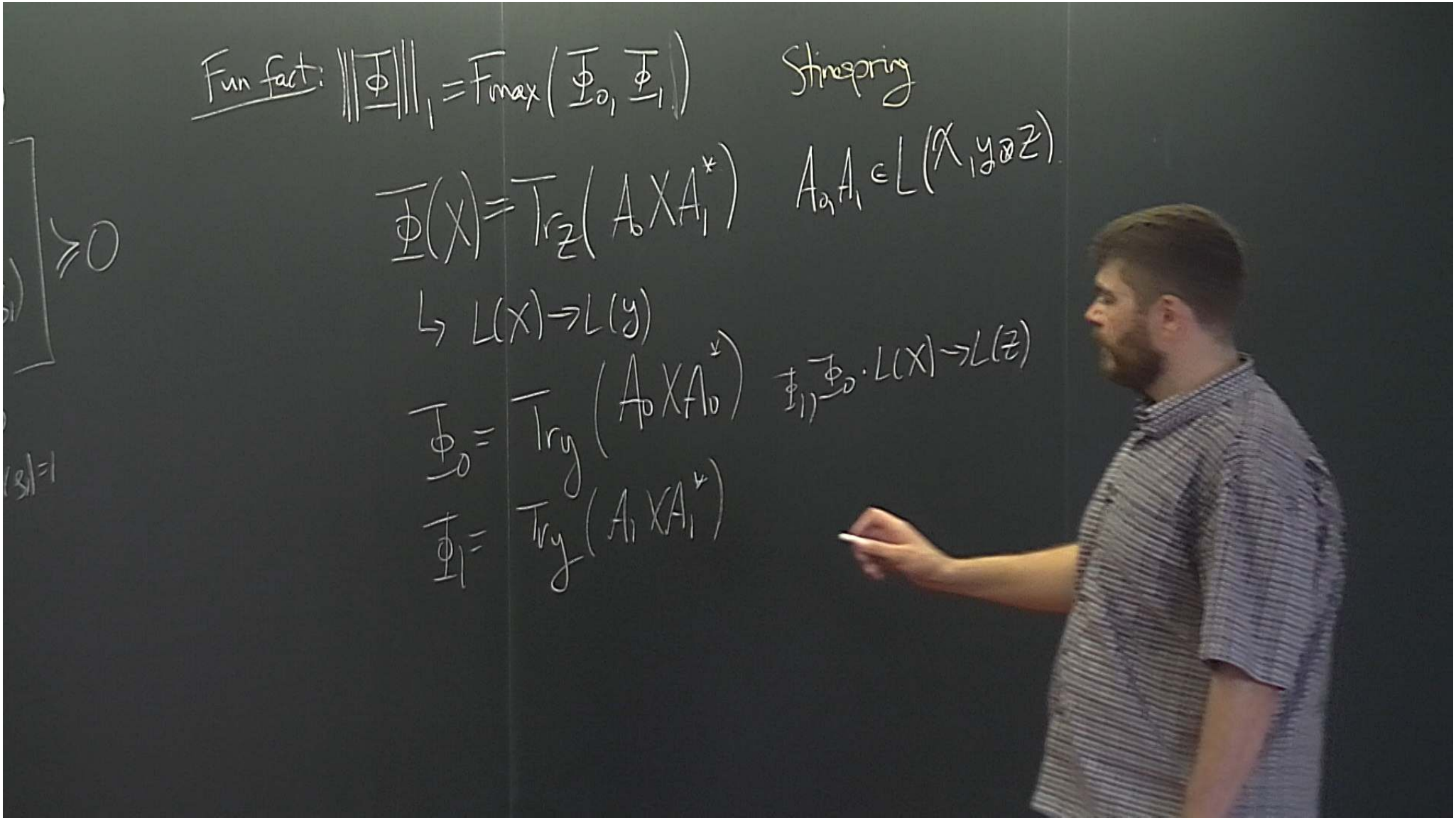
Fun fact: $\|\Phi\|_1 = F_{max}(\Phi_0, \Phi_1)$

$\Phi(X) = Tr_Z(A_0 X A_1^*)$ $A_0, A_1 \in L(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$

≥ 0

$|\beta_1| = 1$

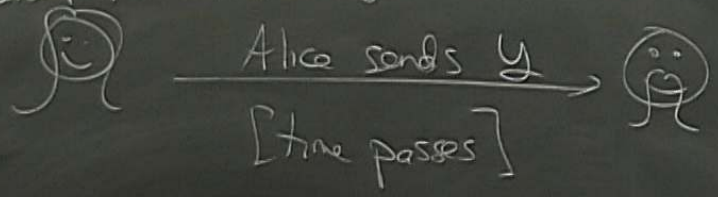

Fun fact: $\|\Phi\|_1 = F_{max}(\Phi_0, \Phi_1)$
Stinespring
$\Phi(X) = Tr_Z(A_0 X A_1^*)$
$A_0, A_1 \in L(X, Y \otimes Z)$
≥ 0


Fun fact: $\|\Phi\|_1 = F\max(\Phi_0, \Phi_1)$
Stinespring
$\Phi(X) = Tr_Z(AXA^*)$
$A_0, A_1 \in L(\mathcal{X}, \mathcal{Y}\otimes\mathcal{Z})$
$\hookrightarrow L(X) \to L(Y)$
$\Phi_0 = Tr_Y(A_0 X A_0^*)$
$\Phi_1 = Tr_Y(A_1 X A_1^*)$
$\Phi_0 \cdot L(X) \to L(Z)$
≥ 0

Lecture 3

Bit-Commitment

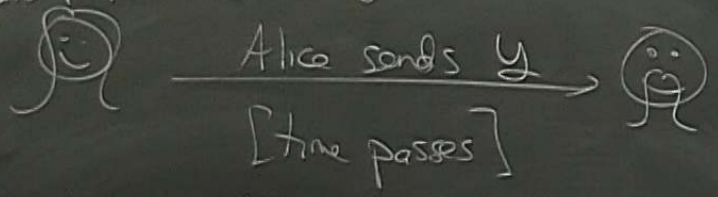
Alice chooses $a \in_R \{0,1\}$
and prepares $|\psi_a\rangle \in X \otimes Y$



Lecture 3

Bit-Commitment

Alice chooses $a \in_R \{0,1\}$
and prepares $|\psi_a\rangle \in X \otimes Y$



Alice sends $a \& X$

Bob checks if $X \otimes Y$ is in the state $|\psi_a\rangle$

$\{M_{accept}, M_{reject}\}$

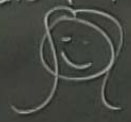
$M_{accept} = |\psi_a\rangle\langle\psi_a|$

$M_{reject} = \mathbb{1} - M_{accept}$

Lecture 3

Bit-Commitment

Alice chooses $a \in_R \{0,1\}$
and prepares $|\psi_a\rangle \in X \otimes Y$



Alice sends Y

[time passes]

Alice sends a & X

Bob checks if $X \otimes Y$ is in the state $|\psi_a\rangle$

$\{M_{accept}, M_{reject}\}$

$M_{accept} = |\psi_a\rangle\langle\psi_a|$

$M_{reject} = \mathbb{1} - M_{accept}$

Objectives

- Bob should not be able to learn a from "Y message"
- Alice should not be able to "change a" before revealing

Lecture 3

Bit-Commitment

Alice chooses $a \in_R \{0,1\}$
and prepares $|\psi_a\rangle \in X_{a,y}$

Alice sends y →

[time passes]

Alice sends $a \& X$ → Bob checks if $X_{a,y}$ is in the state $|\psi_a\rangle$

$\{M_{accept}, M_{reject}\}$

$M_{accept} = |\psi_a\rangle\langle\psi_a|$

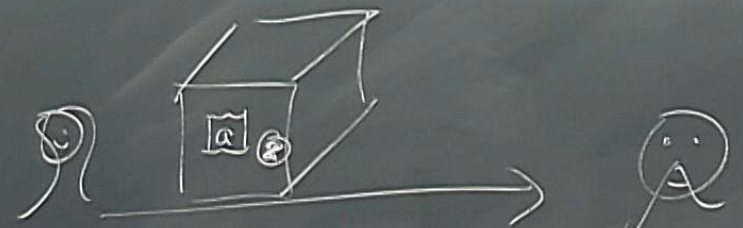
$M_{reject} = \mathbb{1} - M_{accept}$

Objectives

- Bob should not be able to learn a from "y message" hiding
- Alice should not be able to "change a" before revealing binding

be able to learn a "message" hiding

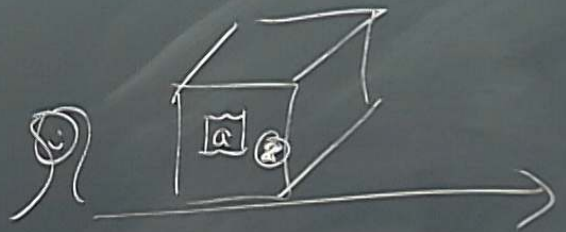
be able to change a "binding"

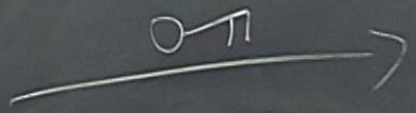


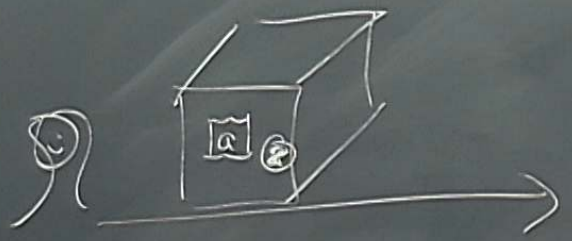
Objectives

- Bob should not be able to learn a from "y message" hiding
- Alice should not be able to "change a" before revealing binding

We will show with SDPs that both of these cannot hold





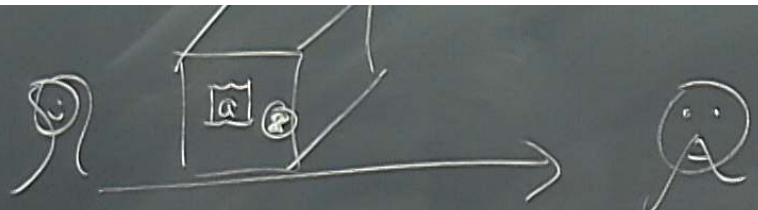


$$0 \to \Pi$$

$$P_B^{\vee} = \sup \frac{1}{2}\langle M, \mathrm{Tr}_X(|\psi_0\rangle\langle\psi_0|)\rangle + \frac{1}{2}\langle M', \mathrm{Tr}_X(|\psi_1\rangle\langle\psi_1|)\rangle$$

$$M + M' = \mathbb{I}$$

$$M, M' \geq 0$$



$$0 \to \pi$$

$$P_B^{\vee} = \max \frac{1}{2}\langle M, \mathrm{Tr}_X(|\psi_0\rangle\langle\psi_0|)\rangle + \frac{1}{2}\langle M', \mathrm{Tr}_X(|\psi_1\rangle\langle\psi_1|)\rangle$$

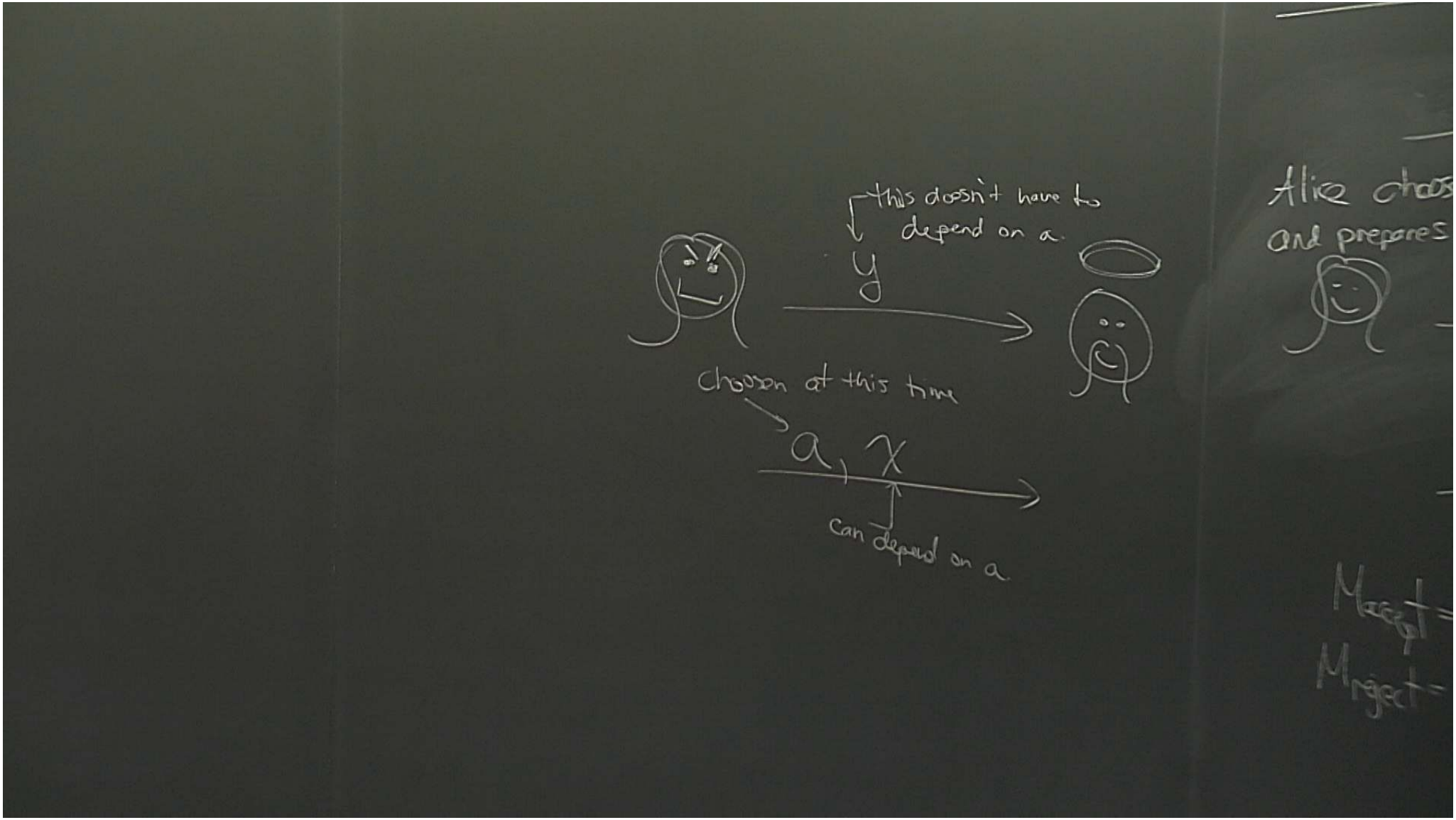
$$M + M' = \mathbb{I}$$

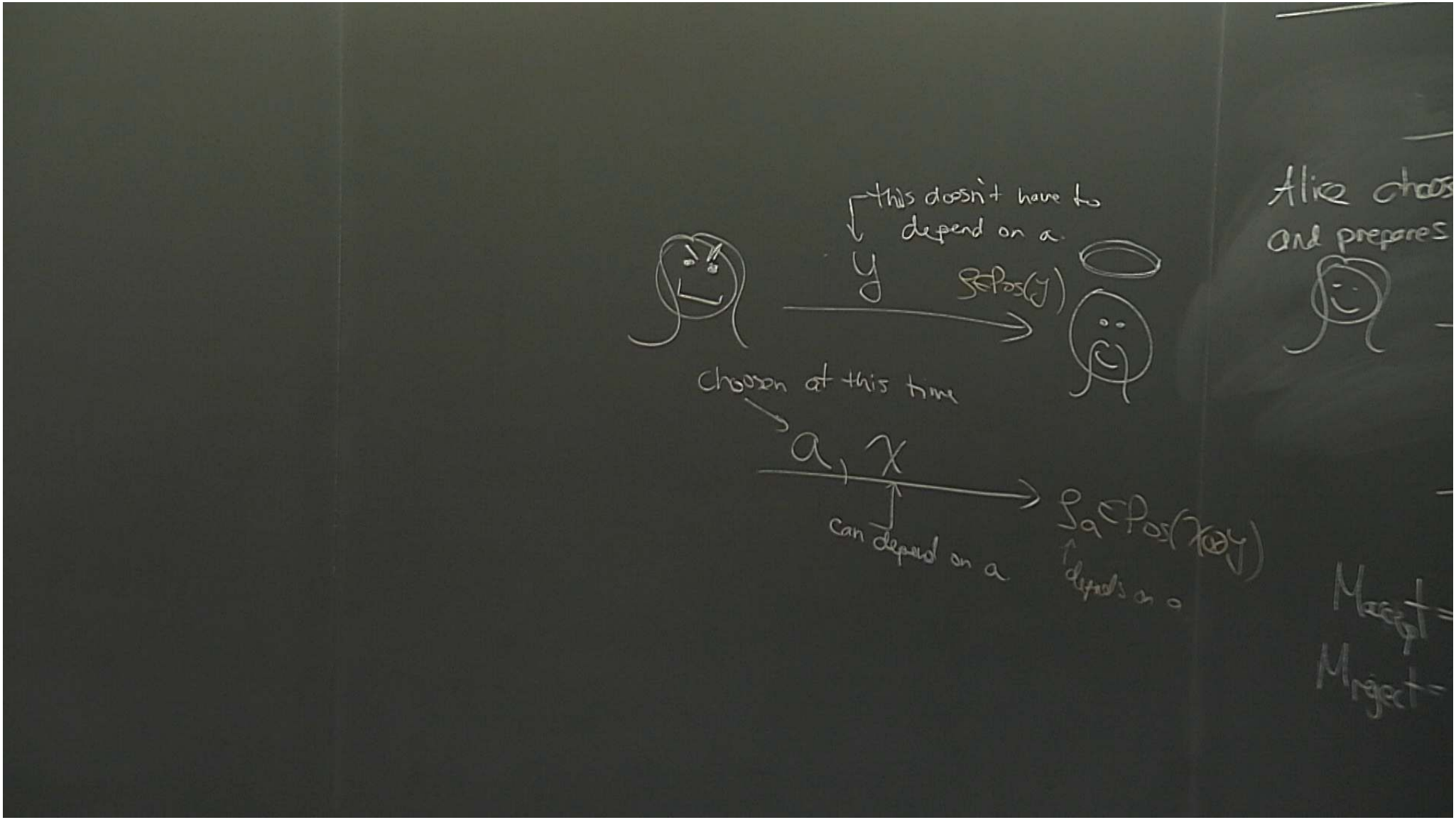
$$M, M' \geq 0$$

$$= \min \mathrm{Tr}(Y)$$

$$Y \geq \frac{1}{2}\mathrm{Tr}_X(|\psi_0\rangle\langle\psi_0|)$$

$$Y \geq \frac{1}{2}\mathrm{Tr}_X(|\psi_1\rangle\langle\psi_1|)$$


this doesn't have to depend on a
y
Alice choos
and prepares
Chosen at this time
a, x
can depend on a
Maccept=
Mreject=


this doesn't have to depend on a.
y
ρ∈Pos(y)
Choosen at this time
a, χ
can depend on a
ρa∈Pos(χ⊗y)
depends on a
Alice choos
and prepares
Maccept=
Mreject=

We have $\mathrm{Tr}_X(\rho_0) = \rho = \mathrm{Tr}_X(\rho_1)$

We have $\mathrm{Tr}_X(\sigma_0) = \sigma = \mathrm{Tr}_X(\sigma_1)$

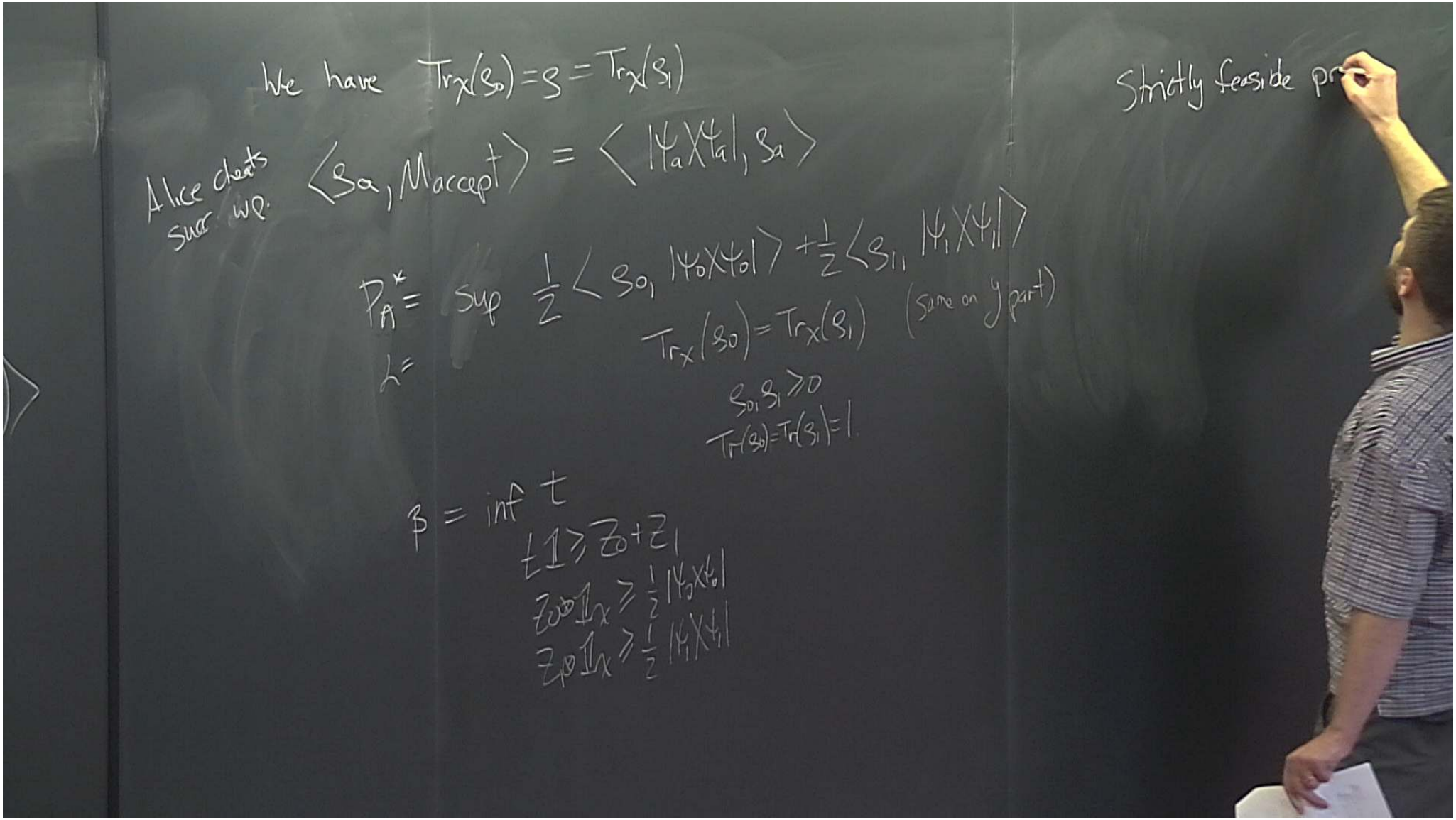
Alice cheats succ wp. $\langle \sigma_a, M_{accept} \rangle = \langle |\psi_a\rangle\langle\psi_a|, \sigma_a \rangle$

$$P_A^* = \sup \frac{1}{2}\langle \sigma_0, |\psi_0\rangle\langle\psi_0| \rangle + \frac{1}{2}\langle \sigma_1, |\psi_1\rangle\langle\psi_1| \rangle$$

$\mathrm{Tr}_X(\sigma_0) = \mathrm{Tr}_X(\sigma_1)$ (same on y part)

$\sigma_0, \sigma_1 \geq 0$

$\mathrm{Tr}(\sigma_0) = \mathrm{Tr}(\sigma_1) = 1.$


We have $\mathrm{Tr}_X(\rho_0) = \sigma = \mathrm{Tr}_X(\rho_1)$
Strictly feasible pr
Alice cheats succ. wp.
$\langle \rho_a, M_{accept} \rangle = \langle |\psi_a\rangle\langle\psi_a|, \rho_a \rangle$
$p_A^* =$ sup $\frac{1}{2}\langle \rho_0, |\psi_0\rangle\langle\psi_0| \rangle + \frac{1}{2}\langle \rho_1, |\psi_1\rangle\langle\psi_1| \rangle$
$\mathrm{Tr}_X(\rho_0) = \mathrm{Tr}_X(\rho_1)$ (same on y part)
$\rho_0, \rho_1 \geq 0$
$\mathrm{Tr}(\rho_0) = \mathrm{Tr}(\rho_1) = 1$
$\beta =$ inf t
$tI \geq Z_0 + Z_1$
$Z_0 \otimes I_X \geq \frac{1}{2}|\psi_0\rangle\langle\psi_0|$
$Z_1 \otimes I_X \geq \frac{1}{2}|\psi_1\rangle\langle\psi_1|$

$\mathrm{Tr}_X(S_1)$

$\langle |\psi_a\rangle\langle\psi_a|, S_a \rangle$

$S_0, |\psi_0\rangle\langle\psi_0| \rangle + \frac{1}{2}\langle S_1, |\psi_1\rangle\langle\psi_1| \rangle$ (same on y part)

$\mathrm{Tr}_X(S_0) = \mathrm{Tr}_X(S_1)$

$S_0, S_1 \geq 0$

$\mathrm{Tr}(S_0) = \mathrm{Tr}(S_1) = 1.$

$Z_0 + Z_1$

$\geq \frac{1}{2}|\psi_0\rangle\langle\psi_0|$

$\geq \frac{1}{2}|\psi_1\rangle\langle\psi_1|$

Strictly feasible primal solution is $S_0 = \frac{1}{4}$ (scaled)
$S_1 = \mathbb{1}$ (scaled)

Strictly feasible dual solution is $Z_0 = \mathbb{1}$
$Z_1 = \mathbb{1}$
$t = 3$

So $\beta = p_A^*$ and the dual is attained.

$$P_B^{\vee} = \max \frac{1}{2}\langle M, \mathrm{Tr}_X(|\psi_0\rangle\langle\psi_0|)\rangle + \frac{1}{2}\langle M', \mathrm{Tr}_X(|\psi_1\rangle\langle\psi_1|)\rangle$$

$$M + M' = \mathbb{I}$$

$$M, M' \geq 0$$

$$= \min \mathrm{Tr}(Y)$$

$$Y \geq \frac{1}{2}\mathrm{Tr}_X(|\psi_0\rangle\langle\psi_0|) \qquad Y \geq 0$$

$$Y \geq \frac{1}{2}\mathrm{Tr}_X(|\psi_1\rangle\langle\psi_1|)$$

Lecture 3

Ideally: $P_B^* = 1/2$ & $P_A^* = 1/2$ (or close to).

Let t, Z_0, Z_1 be dual opt for P_A^*
Let Y be dual opt for P_B^*

$$P_A^* P_B^* = t \cdot \mathrm{Tr}(Y)$$

$$= \langle tI, Y \rangle$$

$$\geq \langle Z_0 + Z_1, Y \rangle \qquad Y \geq 0$$

$$= \langle Z_0, Y \rangle + \langle Z_1, Y \rangle$$

$Z_0, Z_1 \geq 0$

$$\geq \left\langle Z_0, \tfrac{1}{2}\mathrm{Tr}_X\left(|\psi_0\rangle\langle\psi_0|\right) \right\rangle + \left\langle Z_1, \tfrac{1}{2}\mathrm{Tr}_X\left(|\psi_1\rangle\langle\psi_1|\right) \right\rangle$$

Lecture 3

Ideally: $P_B^* = 1/2$ & $P_A^* = 1/2$ (or close to)

Let t, Z_0, Z_1 be dual opt for P_A^*
Let Y be dual opt for P_B^*

$$P_A^* P_B^* = t \cdot \mathrm{Tr}(Y)$$

$$= \langle tI, Y \rangle$$

$$\geq \langle Z_0 + Z_1, Y \rangle \qquad Y \geq 0$$

$$= \langle Z_0, Y \rangle + \langle Z_1, Y \rangle$$

$Z_0, Z_1 \geq 0$

$$\geq \left\langle Z_0, \tfrac{1}{2}\mathrm{Tr}_X(|\psi_0\rangle\langle\psi_0|) \right\rangle + \left\langle Z_1, \tfrac{1}{2}\mathrm{Tr}_X(|\psi_1\rangle\langle\psi_1|) \right\rangle$$

$$= \tfrac{1}{2}\langle Z_0 \otimes I_X, |\psi_0\rangle\langle\psi_0| \rangle + \tfrac{1}{2}\langle Z_0 \otimes I_X, |\psi_1\rangle\langle\psi_1| \rangle$$

Lecture 3

Ideally: $P_B^* = 1/2$ & $P_A^* = 1/2$ (or close to).

Let (t, Z_0, Z_1) be dual opt for A^*
Let Y be dual opt for P_B^*

$$P_A^* P_B^* = t \cdot \mathrm{Tr}(Y)$$

$$= \langle tI, Y \rangle$$

$$\geq \langle Z_0 + Z_1, Y \rangle \qquad Y \geq 0$$

$$= \langle Z_0, Y \rangle + \langle Z_1, Y \rangle$$

$Z_0, Z_1 \geq 0$

$$\geq \left\langle Z_0, \tfrac{1}{2}\mathrm{Tr}_X\left(|\psi_0\rangle\langle\psi_0|\right) \right\rangle + \left\langle Z_1, \tfrac{1}{2}\mathrm{Tr}_X\left(|\psi_1\rangle\langle\psi_1|\right) \right\rangle$$

$$= \tfrac{1}{2}\langle Z_0 \otimes I_X, |\psi_0\rangle\langle\psi_0| \rangle + \tfrac{1}{2}\langle Z_0 \otimes I_X, |\psi_1\rangle\langle\psi_1| \rangle$$

$$\geq \tfrac{1}{2}\left\langle \tfrac{1}{2}|\psi_0\rangle\langle\psi_0|, |\psi_0\rangle\langle\psi_0| \right\rangle + \tfrac{1}{2}\left\langle \tfrac{1}{2}|\psi_1\rangle\langle\psi_1|, |\psi_1\rangle\langle\psi_1| \right\rangle$$

Lecture 3

Ideally: $P_B^* = 1/2$ & $P_A^* = 1/2$ (or close to).

$P_A^* P_B^* \geq \frac{1}{2} \Rightarrow \max\{P_A^*, P_B^*\} \geq \frac{1}{\sqrt{2}} \approx 71\%$

Let (t, z_0, z_1) be dual opt for A^*
Let y be dual opt for P_B^*

$$P_A^* P_B^* = t \cdot \mathrm{Tr}(y)$$
$$= \langle tI, y \rangle$$
$$\geq \langle z_0 + z_1, y \rangle \qquad y \geq 0$$
$$= \langle z_0, y \rangle + \langle z_1, y \rangle \qquad z_0, z_1 \geq 0$$
$$\geq \left\langle z_0, \tfrac{1}{2}\mathrm{Tr}_X(|\psi_0\rangle\langle\psi_0|) \right\rangle + \left\langle z_1, \tfrac{1}{2}\mathrm{Tr}_X(|\psi_1\rangle\langle\psi_1|) \right\rangle$$
$$= \frac{1}{2}\langle z_0 \otimes I_X, |\psi_0\rangle\langle\psi_0| \rangle + \frac{1}{2}\langle z_0 \otimes I_X, |\psi_1\rangle\langle\psi_1| \rangle$$
$$\geq \frac{1}{2}\left\langle \tfrac{1}{2}|\psi_0\rangle\langle\psi_0|, |\psi_0\rangle\langle\psi_0| \right\rangle + \frac{1}{2}\left\langle \tfrac{1}{2}|\psi_1\rangle\langle\psi_1|, |\psi_1\rangle\langle\psi_1| \right\rangle$$
$$= \frac{1}{2}$$

P_B^*

Lecture 3

Ideally: $P_B^* = 1/2$ & $P_A^* = 1/2$ (or close to)

Let t, z_0, z_1 be dual opt for A^*

Let y be dual opt for P_B^*

$$P_A^* P_B^* = t \cdot \mathrm{Tr}(y)$$

$$= \langle t\mathbb{I}, y \rangle$$

$$\geq \langle z_0 + z_1, y \rangle \qquad y \geq 0$$

$$= \langle z_0, y \rangle + \langle z_1, y \rangle$$

$z_0, z_1 \geq 0$

$$\geq \left\langle z_0, \tfrac{1}{2}\mathrm{Tr}_X\left(|\psi_0\rangle\langle\psi_0|\right) \right\rangle + \left\langle z_1, \tfrac{1}{2}\mathrm{Tr}_X\left(|\psi_1\rangle\langle\psi_1|\right) \right\rangle$$

$$= \tfrac{1}{2}\langle z_0 \otimes \mathbb{I}_X, |\psi_0\rangle\langle\psi_0| \rangle + \tfrac{1}{2}\langle z_0 \otimes \mathbb{I}_X, |\psi_1\rangle\langle\psi_1| \rangle$$

$$\geq \tfrac{1}{2}\left\langle \tfrac{1}{2}|\psi_0\rangle\langle\psi_0|, |\psi_0\rangle\langle\psi_0| \right\rangle + \tfrac{1}{2}\left\langle \tfrac{1}{2}|\psi_1\rangle\langle\psi_1|, |\psi_1\rangle\langle\psi_1| \right\rangle$$

$$= \frac{1}{2}$$

$$P_A^* P_B^* \geq \tfrac{1}{2} \Rightarrow \max\left\{P_A^*, P_B^*\right\} \geq \frac{1}{\sqrt{2}} \approx 71\%$$

So, quantum BC is impossible. □

Lecture 5

This form

$$p_A^* + p_B^* \geq \frac{3}{2}$$

FvdS ↑

Kitaev CF prof.

Ideally: $p_B^* = 1/2$ & $p_A^* = 1/2$ (or

let t, z_0, z_1 be du

let y be dual

$$p_A^* p_B^* = t \cdot \mathrm{Tr}(y)$$

$$= \langle tI, y \rangle$$

$$\geq \langle z_0 + z_1, y \rangle \qquad y \geq 0$$

$$= \langle z_0, y \rangle + \langle z_1, y \rangle \qquad z_0, z_1 \geq 0$$

$$\geq \left\langle z_0, \tfrac{1}{2}\mathrm{Tr}_X(|\psi_0\rangle\langle\psi_0|) \right\rangle + \langle z_1, \tfrac{1}{2}\mathrm{Tr}$$

$$= \tfrac{1}{2}\langle z_0 \otimes I_X, |\psi_0\rangle\langle\psi_0| \rangle + \tfrac{1}{2}\langle z_0 \otimes I_{X_1}, |\psi$$

$$\geq \tfrac{1}{2}\left\langle \tfrac{1}{2}|\psi_0\rangle\langle\psi_0|, |\psi_0\rangle\langle\psi_0| \right\rangle + \tfrac{1}{2}\left\langle \tfrac{1}{2}|\psi_1\rangle\langle\psi_1| \right.$$

$$= \frac{1}{2}$$

Lecture 3

This form

$$P_A^* + P_B^* \geq \frac{3}{2}$$

FvdS ↑

Kitaev CF prof.

Ideally: $P_B^* = 1/2$ & $P_A^* = 1/2$ (or

let t, z_0, z_1 be dua

let y be dual

$$P_A^* P_B^* = t \cdot \mathrm{Tr}(y)$$

$$= \langle tI, y \rangle$$

$$\geq \langle z_0 + z_1, y \rangle \qquad y \geq 0$$

$$= \langle z_0, y \rangle + \langle z_1, y \rangle \qquad z_0, z_1 \geq 0$$

$$\geq \left\langle z_0, \tfrac{1}{2}\mathrm{Tr}_X(|\psi_0\rangle\langle\psi_0|) \right\rangle + \langle z_1, \tfrac{1}{2}\mathrm{Tr}$$

$$= \tfrac{1}{2}\langle z_0 \otimes I_X, |\psi_0\rangle\langle\psi_0| \rangle + \tfrac{1}{2}\langle z_0 \otimes I_X, |\psi$$

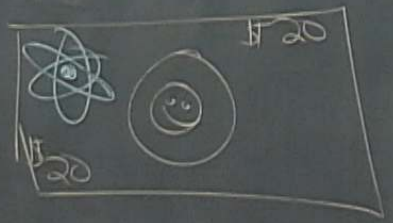
$$= \tfrac{1}{2}\left\langle \tfrac{1}{2}|\psi_0\rangle\langle\psi_0|, |\psi_0\rangle\langle\psi_0| \right\rangle + \tfrac{1}{2}\left\langle \tfrac{1}{2}|\psi_1\rangle\langle\psi_1| \right.$$

$$= \frac{1}{2}$$

≈71%

Quantum Money

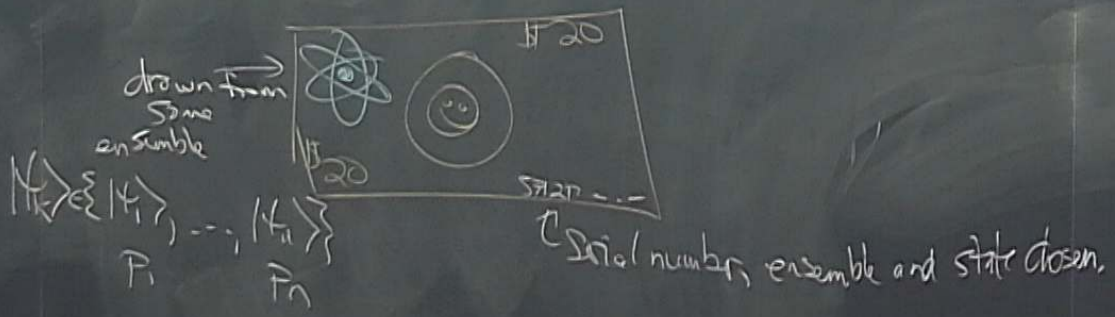
mint some money
verifies it later



≈ 71%

Quantum Money

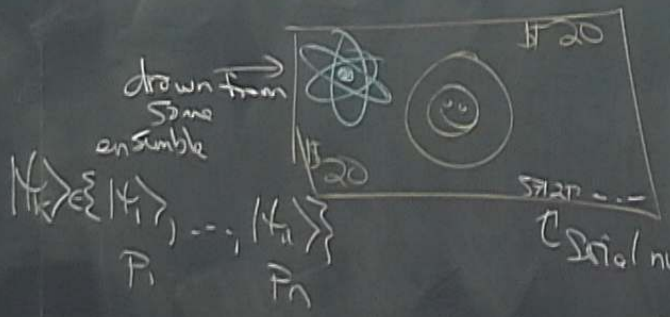
mint some money
verifies it later



≈ 71%

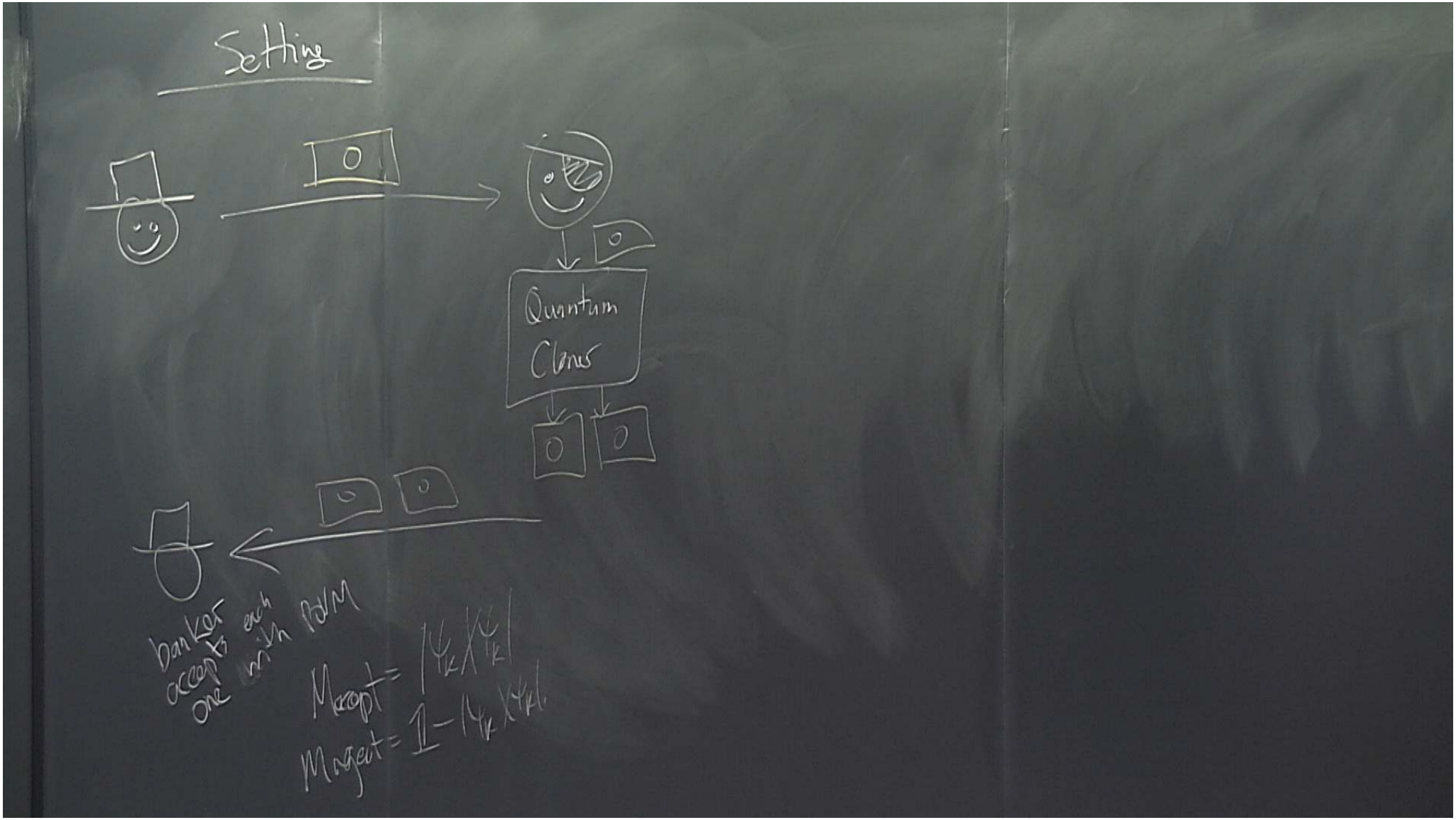
Quantum Money

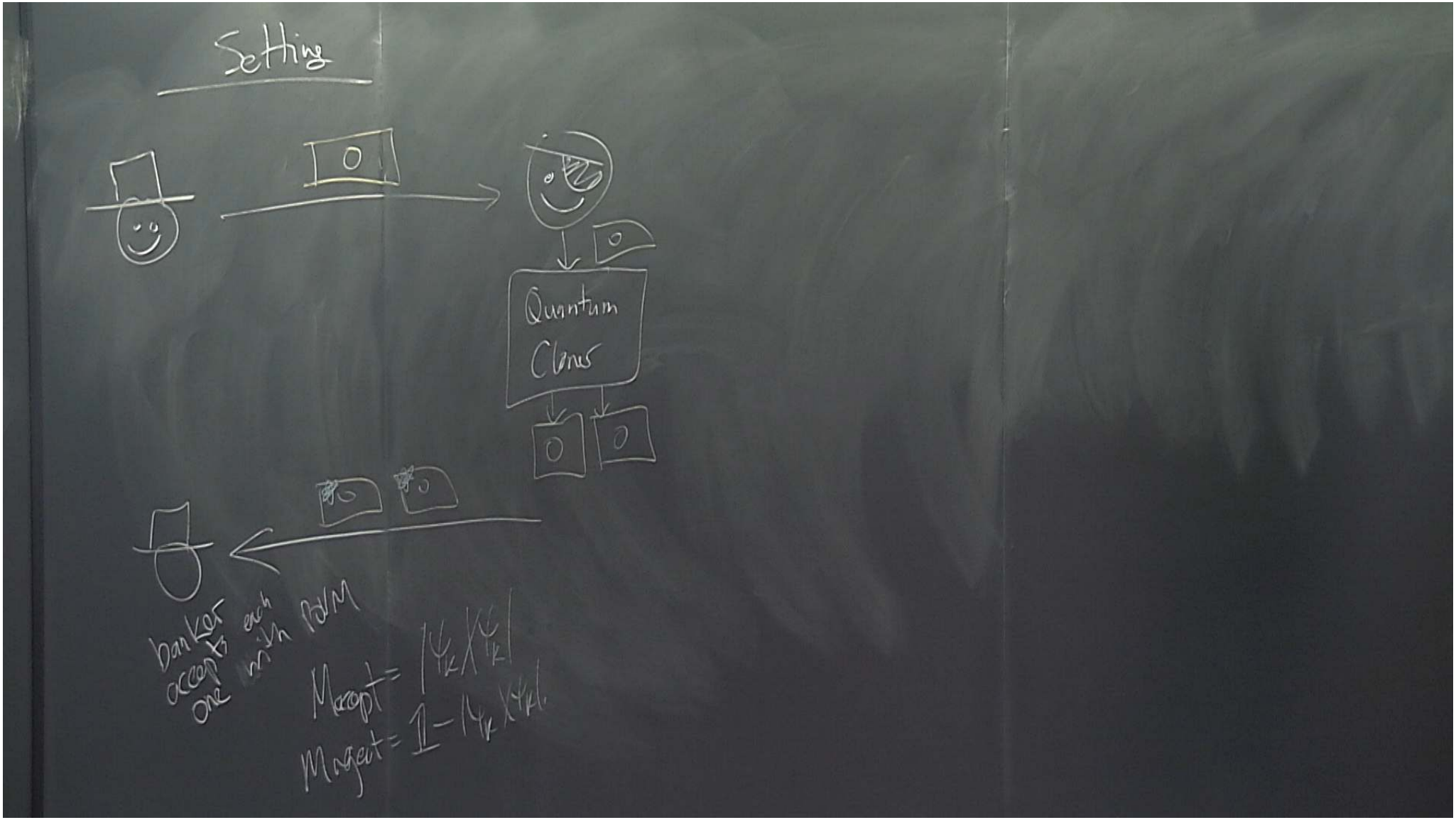
mint some money
verifies it later



(Serial number, ensemble and state chosen.

A forger wants to copy the money.


Setting
Quantum Cloner
banker accepts each with p≥M
$M_{accept} = |\psi_k\rangle\langle\psi_k|$
$M_{reject} = \mathbb{1} - |\psi_k\rangle\langle\psi_k|$


Setting
Quantum Cloner
banker accepts each one with POVM
$M_{accept} = |\psi_k\rangle\langle\psi_k|$
$M_{reject} = \mathbb{1} - |\psi_k\rangle\langle\psi_k|$

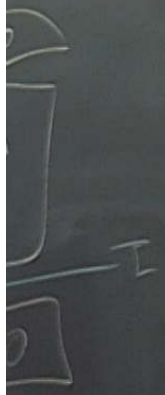
If an honest person returns a banknote, it will be accepted with certainty

Will the banker accept a forgery?

Let $\Sigma \in Pos(\mathcal{X} \otimes \mathcal{X})$ $\quad |\psi_k\rangle \in \mathcal{X}$.

The forger succeeds w.p. $\langle \Sigma, M_{accept} \rangle$

$= \langle \psi_k | \langle \psi_k | \Sigma | \psi_k \rangle | \psi_k \rangle$



If an honest person returns a banknote, it will be accepted with certainty

Will the banks accept a forgery?

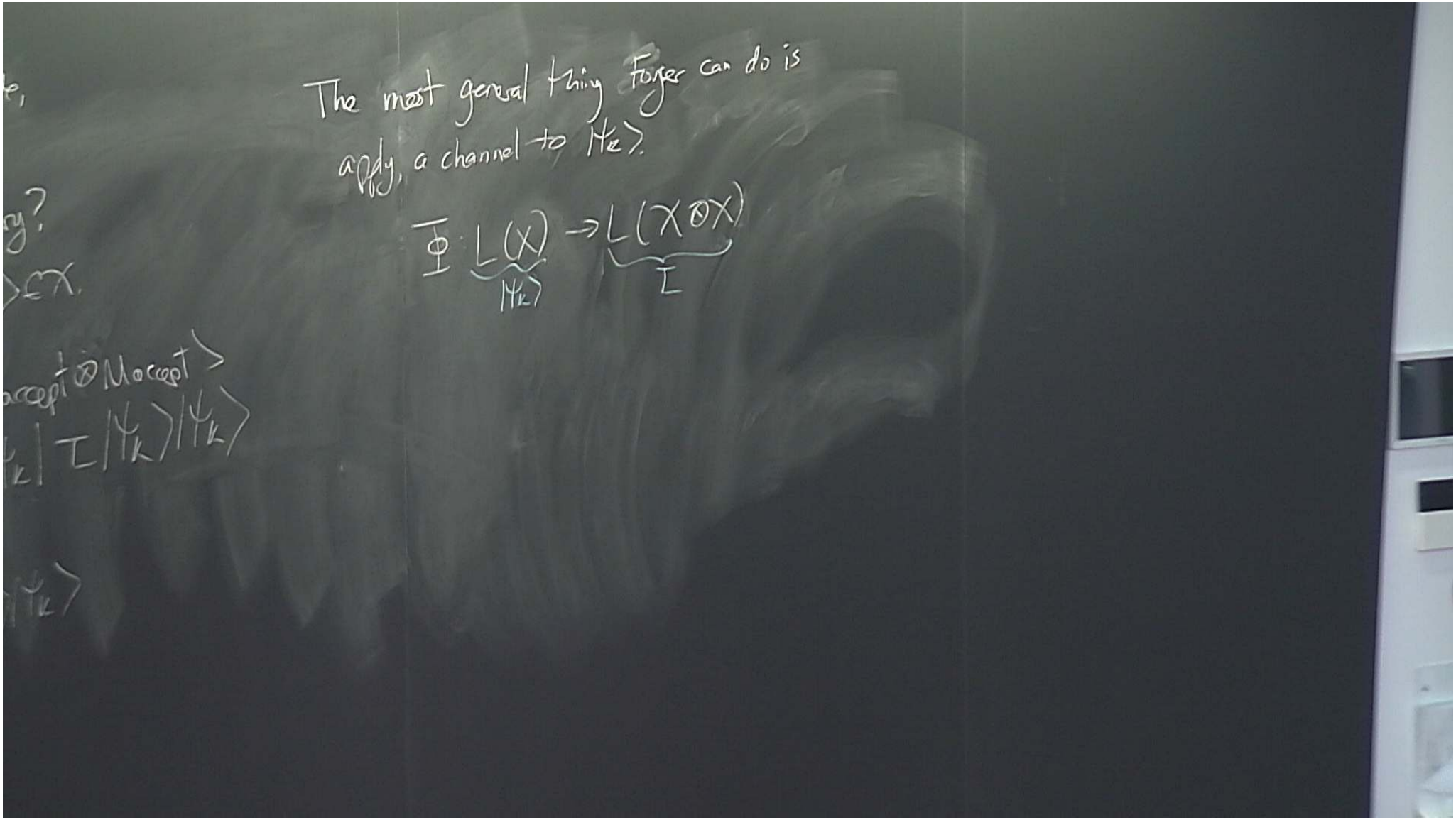
Let $\Sigma \in \mathrm{Pos}(\mathcal{X} \otimes \mathcal{X})$ $\quad |\psi_k\rangle \in \mathcal{X}$.

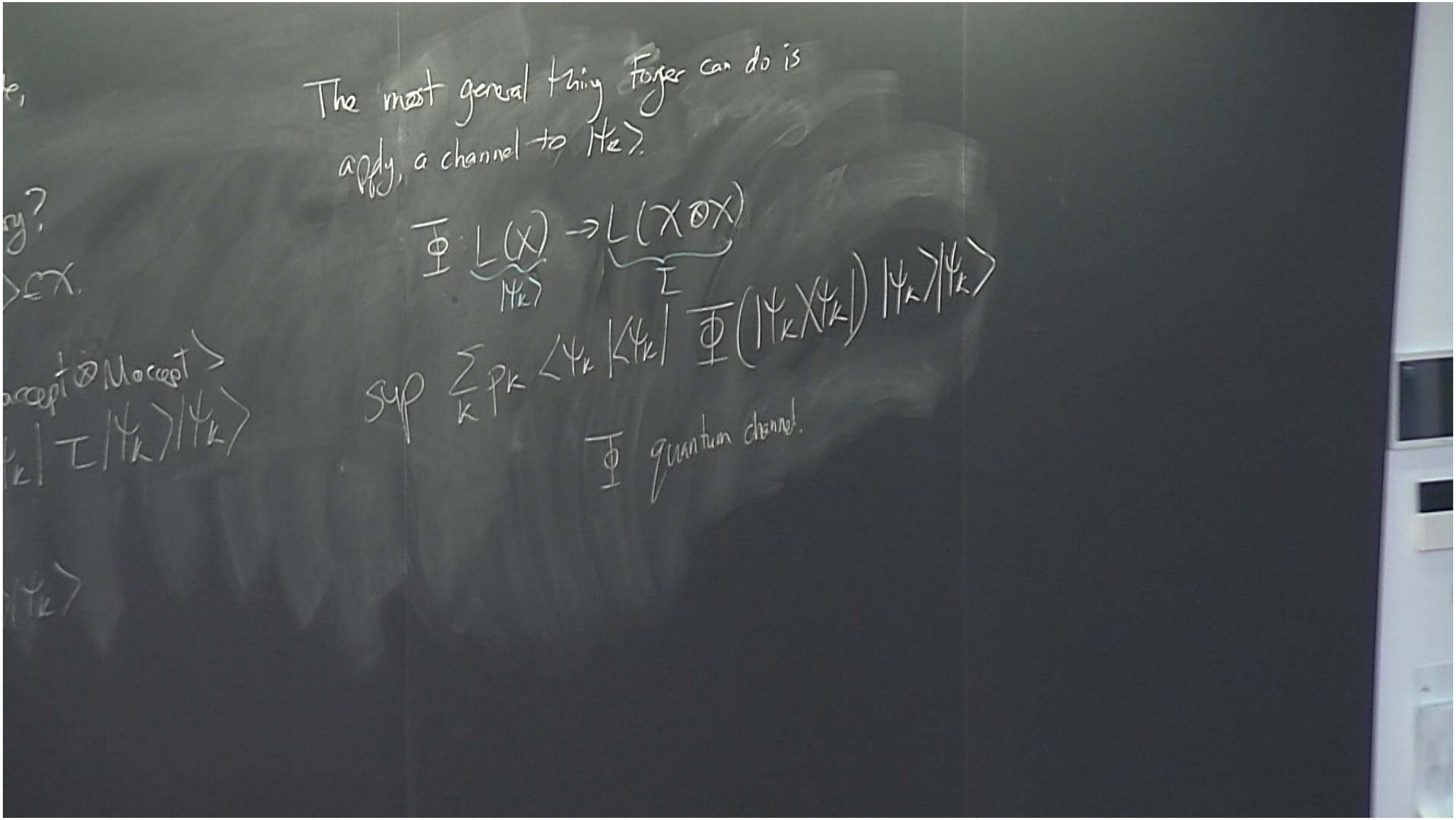
The forger succeeds w.p. $\langle \Sigma, M_{\mathrm{accept}} \otimes M_{\mathrm{accept}} \rangle$

$= \langle \psi_k | \langle \psi_k | \Sigma | \psi_k \rangle | \psi_k \rangle$

Average over k.

$\sum_k p_k \langle \psi_k | \langle \psi_k | \Sigma | \psi_k \rangle | \psi_k \rangle$


The most general thing Forger can do is
apply a channel to |ψ_k⟩.
Φ : L(X) → L(X⊗X)
|ψ_k⟩
I


The most general thing Forger can do is
apply a channel to $|\psi_k\rangle$.
$\Phi : L(X) \to L(X \otimes X)$
$|\psi_k\rangle$
$\sup \sum_k p_k \langle \psi_k | \psi_k | \Phi(|\psi_k\rangle\langle\psi_k|) |\psi_k\rangle |\psi_k\rangle$
Φ quantum channel.

Lecture 3

Choi-Jamiołkowski representation of quantum channels

$\Phi \in L(X) \to L(Y)$ $\longleftrightarrow$ $J(\Phi)$

$$J(\Phi) = \sum_{k,l} \Phi(|k\rangle\langle l|) \otimes |k\rangle\langle l| \in L(Y \otimes X)$$

Φ is trace-preserving $\iff$ $\mathrm{Tr}_Y(J(\Phi)) = \mathbb{1}_X$

Φ is completely-positive $\iff$ $J(\Phi) \geq 0$

Lecture 3

Choi–Jamiołkowski representation of quantum channels

$\Phi \in L(X) \to L(Y)$ $\longleftrightarrow$ $J(\Phi)$

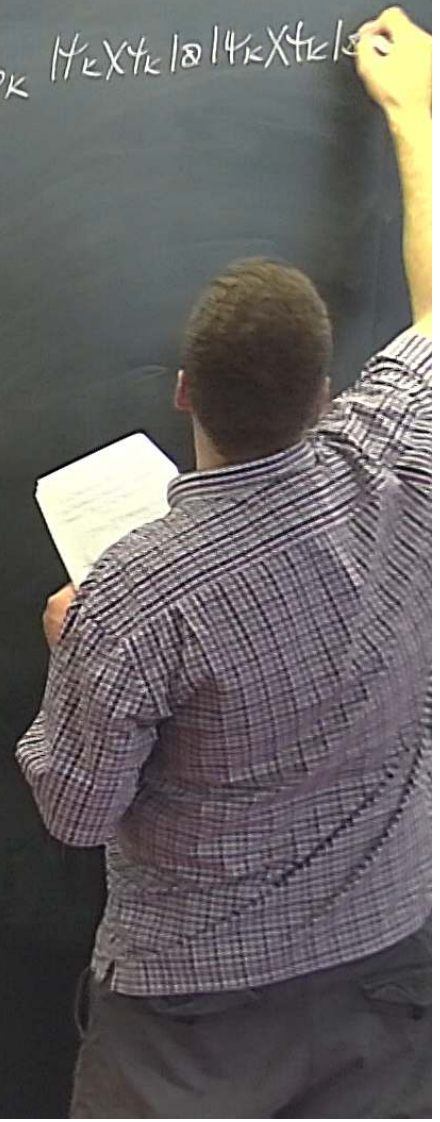
$$J(\Phi) = \sum_{k,\ell} \Phi(|k\rangle\langle\ell|) \otimes |k\rangle\langle\ell| \in L(Y \otimes X)$$

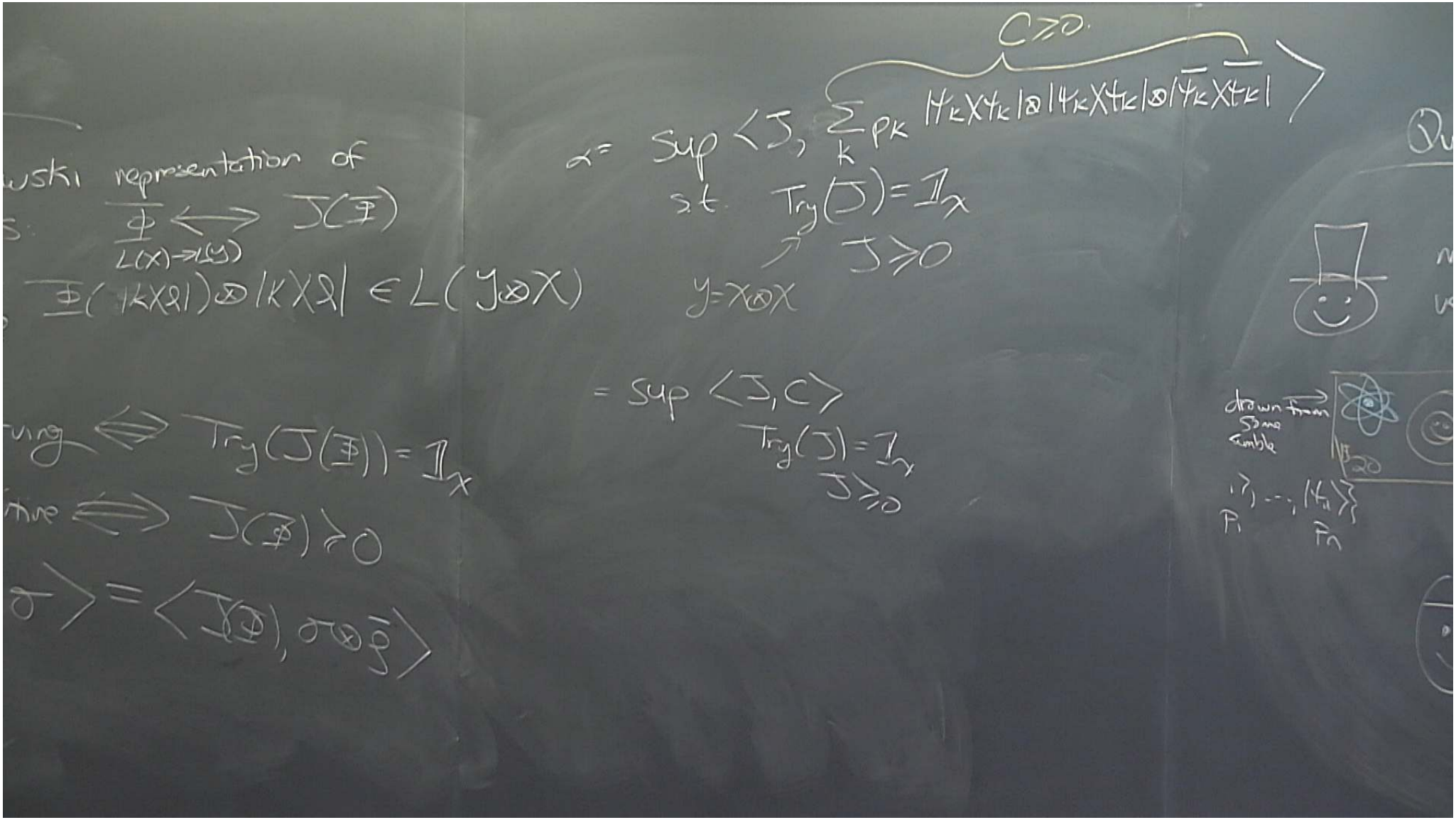
Φ is trace-preserving $\iff$ $\mathrm{Tr}_Y(J(\Phi)) = \mathbb{1}_X$

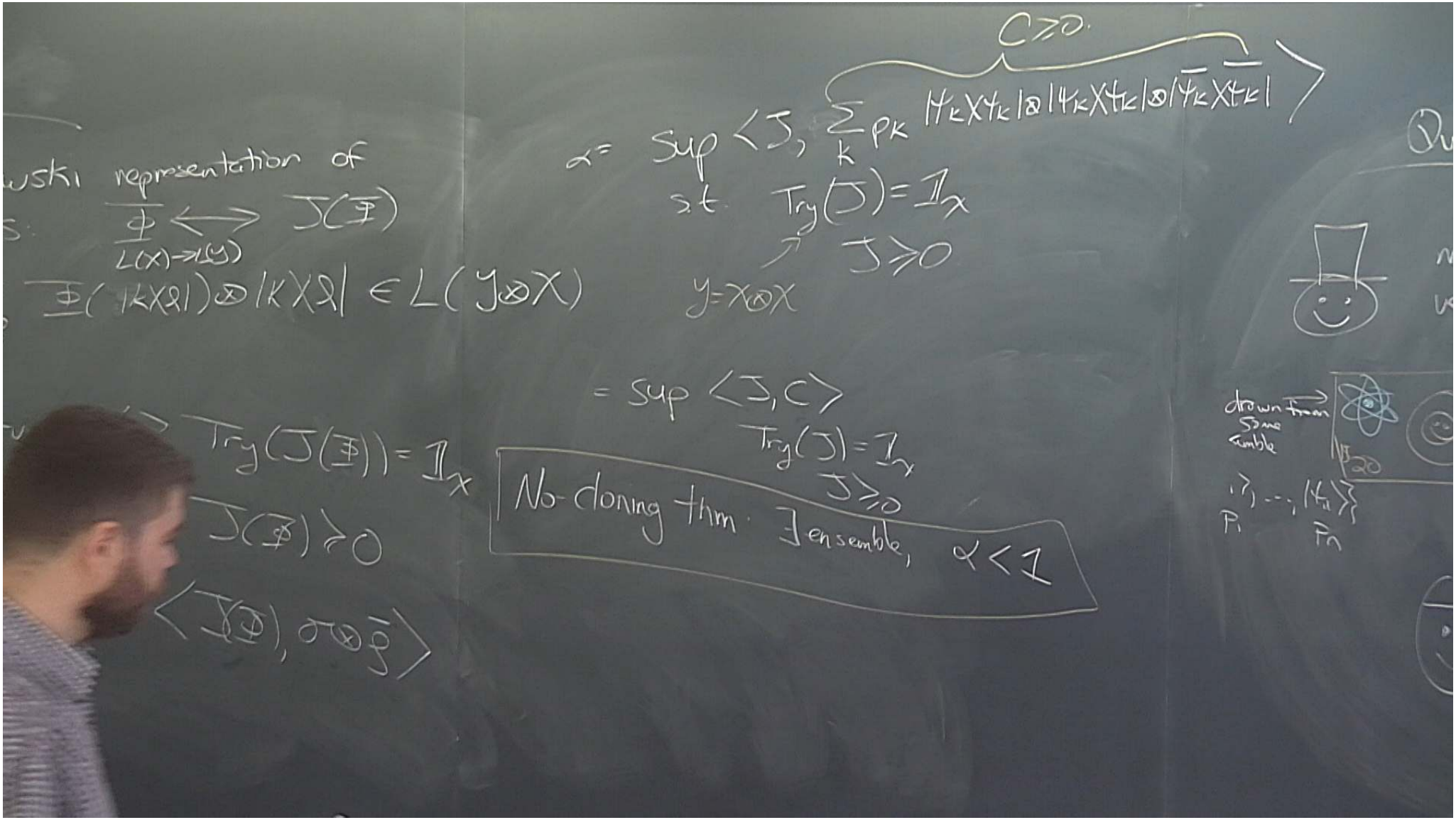
Φ is completely-positive $\iff$ $J(\Phi) \geq 0$

$$\langle \Phi(\rho), \sigma \rangle = \langle J(\Phi), \sigma \otimes \bar{\rho} \rangle$$

$$\sup \langle J, \sum_k p_k |\psi_k\rangle\langle\psi_k| \otimes |\phi_k\rangle\langle\phi_k| \rangle$$




wski representation of
s: $\Phi \longleftrightarrow J(\Phi)$
$L(X) \to L(Y)$
$\Phi(|k\rangle\langle l|) \otimes |k\rangle\langle l| \in L(Y \otimes X)$
ving $\Longleftrightarrow$ $\mathrm{Tr}_Y(J(\Phi)) = \mathbb{1}_X$
tive $\Longleftrightarrow$ $J(\Phi) \geq 0$
$\sigma\rangle = \langle J(\Phi), \sigma \otimes \bar{\rho}\rangle$
$\alpha = \sup \langle J, \overbrace{\sum_k p_k |\psi_k\rangle\langle\psi_k| \otimes |\psi_k\rangle\langle\psi_k| \otimes |\bar{\psi}_k\rangle\langle\bar{\psi}_k|}^{C \geq 0} \rangle$
s.t. $\mathrm{Tr}_Y(J) = \mathbb{1}_X$
$J \geq 0$
$Y = X \otimes X$
$= \sup \langle J, C \rangle$
$\mathrm{Tr}_Y(J) = \mathbb{1}_X$
$J \geq 0$
drawn from some ensemble
$\{p_1, \ldots, p_n\}$



wski representation of
$\Phi \longleftrightarrow J(\Phi)$
$L(X) \to L(Y)$
$\Phi(|k\rangle\langle l|) \otimes |k\rangle\langle l| \in L(Y \otimes X)$
$\mathrm{Tr}_Y(J(\Phi)) = \mathbb{1}_X$
$J(\Phi) \geq 0$
$\langle J(\Phi), \sigma \otimes \bar{\rho} \rangle$
$\alpha = \sup \langle J, \sum_k p_k |\psi_k\rangle\langle\psi_k| \otimes |\psi_k\rangle\langle\psi_k| \otimes |\bar{\psi}_k\rangle\langle\bar{\psi}_k| \rangle$
$C \geq 0$
s.t. $\mathrm{Tr}_Y(J) = \mathbb{1}_X$
$J \geq 0$
$Y = X \otimes X$
$= \sup \langle J, C \rangle$
$\mathrm{Tr}_Y(J) = \mathbb{1}_X$
$J \geq 0$
No-cloning thm: $\exists$ ensemble, $\alpha < 1$
drawn from some ensemble
p_1
p_n

C≥0

What if the banker is unhappy with d?

Can we boost security?

ensemble 1: $p_k \to |\psi_k\rangle$

ensemble 2: $q_k \to |\phi_k\rangle$

$C \geq 0$

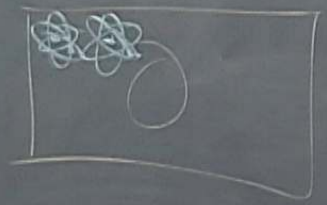
$|\psi_k\rangle\langle\psi_k| \otimes |\psi_k\rangle\langle\psi_k| \otimes |\bar{\psi}_k\rangle\langle\bar{\psi}_k|$

What if the banker is unhappy with d?

Can we boost security?

ensemble 1: $p_k \to |\psi_k\rangle$

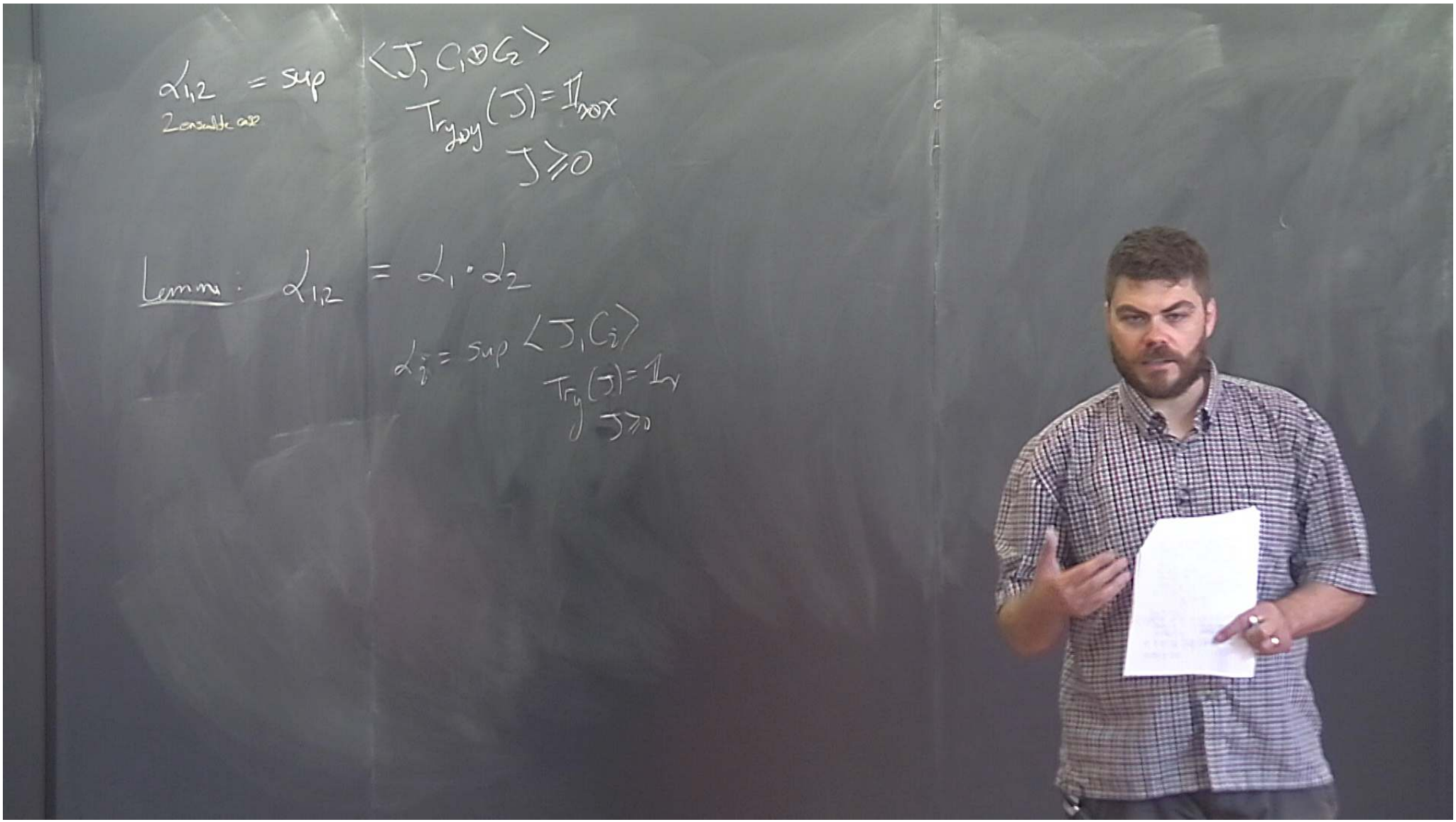
ensemble 2: $q_k \to |\phi_k\rangle$



$F < 1$

$M_{accept} = |\psi_k\rangle\langle\psi_k| \otimes |\phi_k\rangle\langle\phi_k|$

$= M_{accept}^{(1)} \otimes M_{accept}^{(2)}$


$\alpha_{1,2} = \sup \langle J, C_1 \otimes C_2 \rangle$
2 ensemble case
$\mathrm{Tr}_{y \otimes y}(J) = \mathbb{1}_{x \otimes x}$
$J \geq 0$
Lemma: $\alpha_{1,2} = \alpha_1 \cdot \alpha_2$
$\alpha_i = \sup \langle J, C_i \rangle$
$\mathrm{Tr}_y(J) = \mathbb{1}_x$
$J \geq 0$

$$\alpha_{1,2} = \sup \langle J, C_1 \otimes C_2 \rangle$$

2 ensemble case

$$\mathrm{Tr}_{y \otimes y}(J) = \mathbb{1}_{x \otimes x}$$

$$J \geq 0$$

<u>Lemma</u>: $\alpha_{1,2} = \alpha_1 \cdot \alpha_2$

$$\alpha_i = \sup \langle J, C_i \rangle$$

$$\mathrm{Tr}_y(J) = \mathbb{1}_x$$

$$J \geq 0$$

$$\beta_{1,2} = \inf \mathrm{Tr}(Z)$$

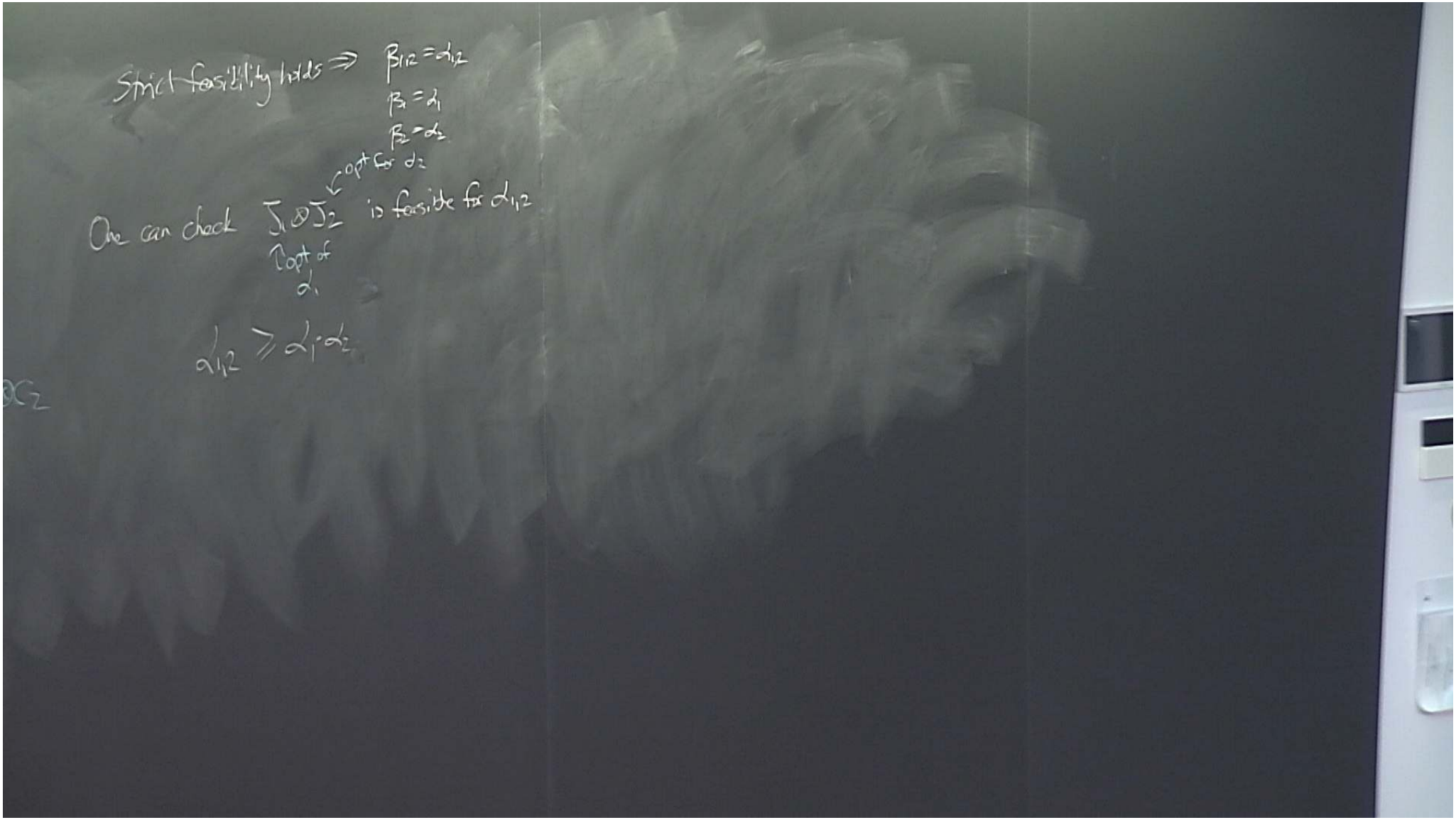
$$Z \otimes \mathbb{1}_{y \otimes y} \geq C_1 \otimes C_2$$

$$\beta_1 = \inf \mathrm{Tr}(Z_1)$$

$$Z_1 \otimes \mathbb{1}_y \geq C_1$$

$$\beta_2 = \inf \mathrm{Tr}(Z_2)$$

$$Z_2 \otimes \mathbb{1}_y \geq C_2$$


Strict feasibility holds ⇒ $\beta_{1,2} = \alpha_{1,2}$
$\beta_1 = \alpha_1$
$\beta_2 = \alpha_2$
← opt for α_2
One can check $J_1 \otimes J_2$ is feasible for $\alpha_{1,2}$
↑ opt of α_1
$\alpha_{1,2} \geq \alpha_1 \cdot \alpha_2$

Lecture 3

We now $\beta_{1,2} \leq \beta_1 + \beta_2$

$\alpha_{1,2} \quad \alpha_1 \quad \alpha_2$

Let Z_1 & Z_2 be dual opt.

$\mathrm{Tr}(Z_1) = \beta_1 \qquad \mathrm{Tr}(Z_2) = \beta_2$

$Z_1 \otimes I_y \geq C_1 \qquad Z_2 \otimes I_y \geq C_2$

Lecture 3

We now $\beta_{1,2} \leq \beta_1 \cdot \beta_2$

$\alpha_{1,2} \quad \alpha_1 \quad \alpha_2$

Let Z_1 & Z_2 be dual opt.

$\mathrm{Tr}(Z_1) = \beta_1 \qquad \mathrm{Tr}(Z_2) = \beta_2$

$Z_1 \otimes I_y \geq C_1 \qquad Z_2 \otimes I_y \geq C_2$

$Z_1 \otimes I - C_1 \geq 0 \qquad Z_2 \otimes I - C_2 \geq 0$

Since $C_1 \geq 0$ $Z_1 \otimes I + C_1 \geq 0$ $Z_2 \otimes I + C_2 \geq 0$

$(Z_1 \otimes I - C_1) \otimes (Z_2 \otimes I + C_2) \geq 0$

Lecture 3

We now $\beta_{1,2} \leq \beta_1 \cdot \beta_2$

$\alpha_{1,2} \quad \alpha_1 \quad \alpha_2$

Let Z_1 & Z_2 be dual opt.

$\mathrm{Tr}(Z_1) = \beta_1$ $\qquad$ $\mathrm{Tr}(Z_2) = \beta_2$

$Z_1 \otimes I_y \geq C_1$ $\qquad$ $Z_2 \otimes I_y \geq C_2$

$Z_1 \otimes I - C_1 \geq 0$ $\qquad$ $Z_1 \otimes I - C_2 \geq 0$

Since $C_1 \geq 0$ $\quad Z_1 \otimes I + C_1 \geq 0$ $\qquad$ $Z_2 \otimes I + C_2 \geq 0$

$(Z_1 \otimes I - C_1) \otimes (Z_2 \otimes I + C_2) \geq 0$

$(Z_1 \otimes I + C_1) \otimes (Z_2 \otimes I - C_2) \geq 0$

Add to get

$$2\left[Z_1 \otimes Z_2 \otimes 1_{y \times y} - C_1 \otimes C_2 \right] \geq 0$$

Lecture 3

We now $\beta_{1,2} \leq \beta_1 \cdot \beta_2$

$\alpha_{1,2} \quad \alpha_1 \quad \alpha_2$

Let Z_1 & Z_2 be dual opt.

$\mathrm{Tr}(Z_1) = \beta_1$ $\quad$ $\mathrm{Tr}(Z_2) = \beta_2$

$Z_1 \otimes I_y \geq C_1$ $\quad$ $Z_2 \otimes I_y \geq C_2$

$Z_1 \otimes I - C_1 \geq 0$ $\quad$ $Z_1 \otimes I - C_2 \geq 0$

Since $C_1 \geq 0$ $\quad$ $Z_1 \otimes I + C_1 \geq 0$ $\quad$ $Z_2 \otimes I + C_2 \geq 0$

$(Z_1 \otimes I - C_1) \otimes (Z_2 \otimes I + C_2) \geq 0$

$(Z_1 \otimes I + C_1) \otimes (Z_2 \otimes I - C_2) \geq 0$

Add to get

$$2\left[Z_1 \otimes Z_2 \otimes 1_{yy} - C_1 \otimes C_2 \right] \geq 0$$

So, $Z_1 \otimes Z_2$ is feasible for $\beta_{1,2}$

$\beta_{1,2} \leq \mathrm{Tr}(Z_1 \otimes Z_2) = \mathrm{Tr}(Z_1) \cdot \mathrm{Tr}(Z_2) = \beta_1 \cdot \beta_2$

□

$\beta_{1,2} \leq \beta_1 \cdot \beta_2$

$\alpha_{1,2} \quad \alpha_1 \quad \alpha_2$

be dual opt.

$\mathrm{Tr}(Z_2) = \beta_2$

$Z_2 \otimes \mathbb{I}_y \geq C_2$

$Z_1 \otimes \mathbb{I} - C_2 \geq 0$

$Z_2 \otimes \mathbb{I} + C_2 \geq 0$

$(Z_1 \otimes \mathbb{I} - C_1) \otimes (Z_2 \otimes \mathbb{I} + C_2) \geq 0$

$(Z_1 \otimes \mathbb{I} + C_1) \otimes (Z_2 \otimes \mathbb{I} - C_2) \geq 0$

Add to get

$$2\left[Z_1 \otimes Z_2 \otimes 1_{yay} - C_1 \otimes C_2 \right] \geq 0$$

So, $Z_1 \otimes Z_2$ is feasible for $\beta_{1,2}$

$\beta_{1,2} \leq \mathrm{Tr}(Z_1 \otimes Z_2) = \mathrm{Tr}(Z_1) \cdot \mathrm{Tr}(Z_2) = \beta_1 \cdot \beta_2$

If ensemble with $q < 1$ □

Repeat n times to get q^n as new cheating of forging.

Quantum money is possible.

What if the bo

Can we boost s

ensemble 1: p

ensemble 2: q

$M_{accept} = |\psi_k\rangle\langle\psi_k| \otimes |\phi_k$

$= M_{accept}^{(1)} \otimes M_{accept}^{(2)}$