Title: SDP / Quantum Lecture Series

Speakers: Jamie Sikora
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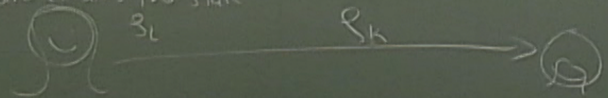
# Lecture 2:

## Quantum State Discrimination Problem (QSD)

Alice chooses $i$
with prob $p_i$
and creates the state

Bob wants to learn
what state he is given.
i.e., guess $i$




What can he do? He can measure $\rho_k$
to guess "k".

Data: The ensemble $\{(p_1, \rho_1), \ldots, (p_n, \rho_n)\}$

Variables: POVM $(M_1, \ldots, M_n)$. $M_k$ corresponds to guess "k"

$$\sum_{k=1}^{n} M_k = \mathbb{1}, \quad M_k \geq 0, \ \forall k$$

## Lecture 2:

### Quantum State Discrimination Problem (QSD)

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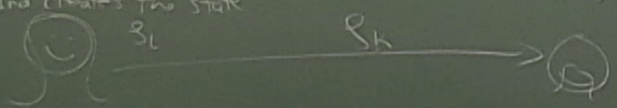
Bob wants to learn what state he is given. i.e., guess $i$



What can he do? He can measure $\rho_k$ to guess "$k$".

Data: The ensemble $\{(p_1, \rho_1), \ldots, (p_n, \rho_n)\}$

Variables: POVM $(M_1, \ldots, M_n)$, $M_k$ corresponds to guess "$k$"

$$\sum_{k=1}^{n} M_k = \mathbb{1}, \quad M_k \succeq 0, \ \forall k$$

Optimize:

Prob[correctly guessing]

$$= \sum_{k=1}^{n} \underbrace{\text{Prob}[\text{getting } \rho_k]}_{p_k} \cdot \underbrace{\text{Prob}[\text{guessing } k \mid \text{getting } \rho_k]}_{\langle M_k, \rho_k \rangle}$$

SDP

$$\alpha = \sup \sum_{k=1}^{n} p_k \langle M_k, \rho_k \rangle$$

$$M_1 + \cdots + M_n = \mathbb{1}$$

$$M_1, \ldots, M_n \succeq 0$$

Dual

$$\inf \ \mathrm{Tr}(Y)$$

$$Y \succeq p_k \rho_k, \quad \forall k \in \{1, \ldots, n\}$$

(QSD)

am

given.

Optimize:

Prob[correctly guessing]

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$$\alpha = \sup \sum_{k=1}^{n} p_k \langle M_k, \rho_k \rangle$$

$$M_1 + \cdots + M_n = \mathbb{1}$$

$$M_1, \ldots, M_n \geq 0$$

Dual

$$\beta = \inf \operatorname{Tr}(Y)$$

$$Y \geq p_k \rho_k, \quad \forall k \in \{1, \ldots, n\}$$

Check:

$(M_1, \ldots, M_n) = (\frac{1}{n}\mathbb{1}, \ldots, \frac{1}{n}\mathbb{1})$ is primal strictly feasible

$Y = 2\mathbb{1}$ is strictly feasible

$\alpha = \beta$ and both are attained.

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SDP

$$\alpha = \sup \sum_{k=1}^{n} p_k \langle M_k, \rho_k \rangle$$

$$M_1 + \cdots + M_n = \mathbb{1}$$

$$M_1, \ldots, M_n \geq 0$$

$$\langle M_k, Y - p_k \rho_k \rangle = 0$$

Dual

$$\beta = \inf \ \mathrm{Tr}(Y)$$

$$Y \geq p_k \rho_k, \quad \forall k \in \{1, \ldots, n\}$$

Check:

$(M_1, \ldots, M_n) = (\frac{1}{n}\mathbb{1}, \ldots, \frac{1}{n}\mathbb{1})$ is primal strictly feasible

$Y = 2\mathbb{1}$ is strictly feasible

$\alpha = \beta$ and both are attained

Let $(M_1, \ldots, M_n)$ be an optimal msmt and $Y$ is dual optimal

By Complementary Slackness,

$$M_k Y = p_k M_k \rho_k \quad \forall k$$

$$\Rightarrow \underbrace{\sum_{k=1}^{n} M_k}_{} Y = \sum_{k=1}^{n} p_k M_k \rho_k$$

$Y$

$Y$ is the unique dual opt sol'n.

$\Rightarrow Y$ is feasible

$Y = \sum_k p_k M_k \rho_k \geq p_\ell \rho_\ell, \; \forall \ell. \quad (1)$

Suppose $(M_1, \dots, M_n)$ is a feasible msmt

If $Y = \sum_k p_k M_k \rho_k \geq p_\ell \rho_\ell, \; \forall \ell$

$Y$ is dual feasible

weak duality

and $\beta \leq Tr(Y) = \sum_k p_k \langle M_k, \rho_k \rangle \leq \alpha \leq \beta$

So $Tr(Y) = \beta$ so $Y$ is optimal

$(M_1, \dots, M_n)$ is an optimal msmt

Summary: let $(M_1, \dots, M_n)$ be a msmt

$(M_1, \dots, M_n)$ is optimal if and only if

$\sum_k p_k M_k \rho_k$ is Hermitian

& $\sum_k p_k M_k \rho_k \geq p_\ell \rho_\ell, \; \forall \ell.$

$Y$ is the unique dual opt. sol'n

$\Rightarrow$ $Y$ is feasible

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So $Tr(Y) = \beta$ so $Y$ is optimal

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**Summary:** let $(M_1, \dots, M_n)$ be a msmt

$(M_1, \dots, M_n)$ is optimal if and only if

$\sum_k p_k M_k \rho_k$ is Hermitian

& $\sum_k p_k M_k \rho_k \geq p_\ell \rho_\ell, \ \forall \ell$

**Remark:** If instead Bob wants to guess which state he is not given, this Quantum State Exclusion

$$\inf \sum_k p_k \langle M_k, \rho_k \rangle$$
$$M_1 + \dots + M_n = 1$$
$$M_1, \dots, M_n \geq 0$$

# Lecture 2:

## Trace Distance

Consider the ensemble $\left\{\left(\frac{1}{2}, \rho\right), \left(\frac{1}{2}, \sigma\right)\right\}$

SDP $\quad \alpha = \max \frac{1}{2}\langle M, \rho\rangle + \frac{1}{2}\langle M', \sigma\rangle$

$$M + M' = \mathbb{1}$$
$$M, M' \geq 0$$

$$\left.\begin{array}{l} M+M' = \mathbb{1} \\ M, M' \geq 0 \end{array}\right\} \Rightarrow \begin{array}{l} M' = \mathbb{1} - M \\ 0 \leq M \leq \mathbb{1} \end{array}$$

$$\alpha = \max_{0 \leq M \leq \mathbb{1}} \frac{1}{2}\langle M, \rho\rangle + \frac{1}{2}\langle \mathbb{1} - M, \sigma\rangle$$

$$= \max_{0 \leq M \leq \mathbb{1}} \frac{1}{2}\langle M, \rho\rangle + \frac{1}{2} - \frac{1}{2}\langle M, \sigma\rangle$$

$$= \max_{0 \leq M \leq \mathbb{1}} \frac{1}{2} + \frac{1}{2}\langle M, \rho - \sigma\rangle$$

$$= \frac{1}{2} + \frac{1}{2}\sum_{k:\lambda_k > 0} \lambda_k(\rho - \sigma)$$

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$$= \frac{1}{2} + \frac{1}{2}\sum_{k: \lambda_k > 0} \lambda_k(\rho - \sigma)$$

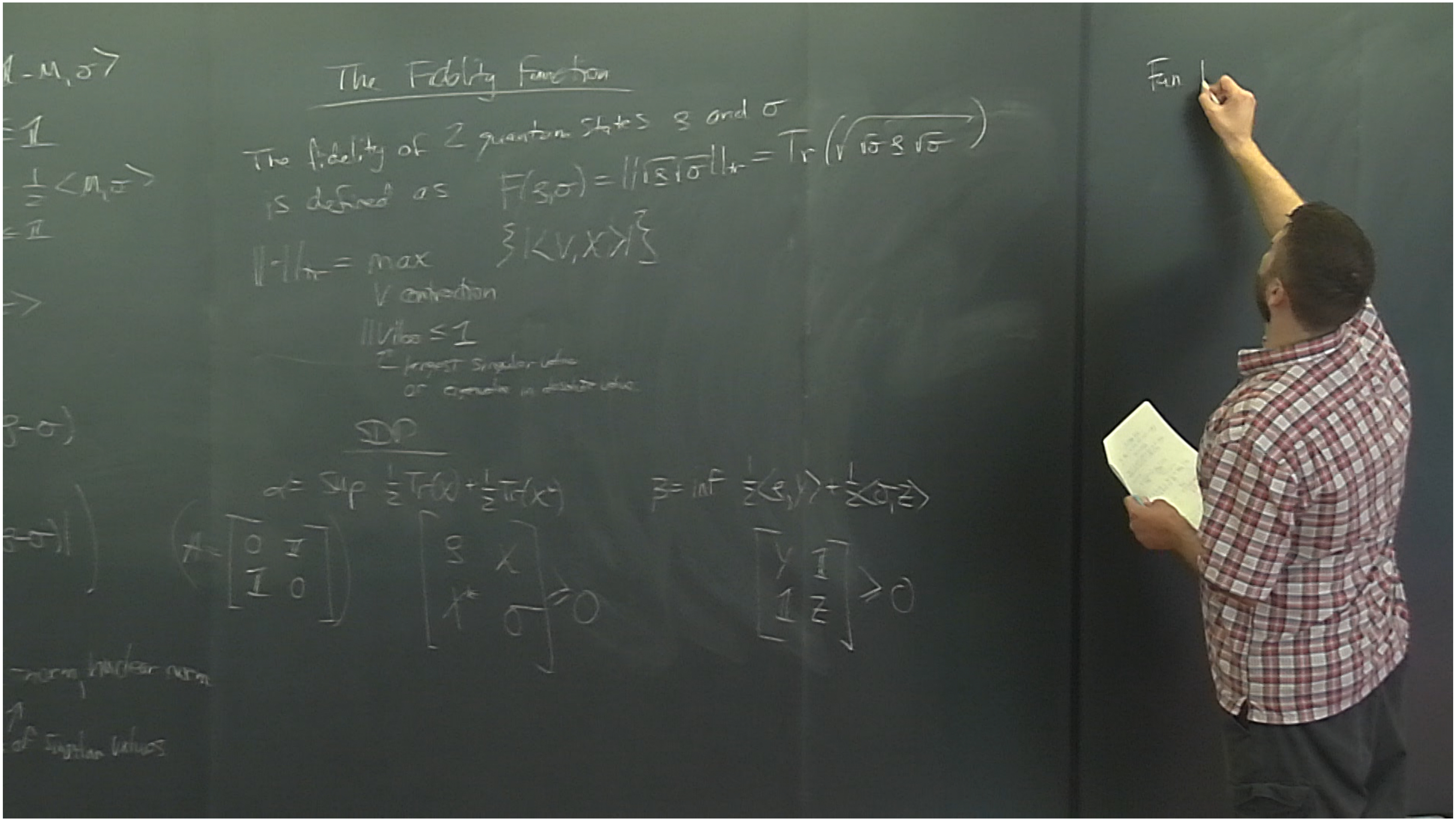
$$= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\sum_k |\lambda_k(\rho - \sigma)|\right)$$

$$= \frac{1}{2} + \frac{1}{4}\|\rho - \sigma\|_1$$

where $\|\cdot\|_{tr}$ is the trace norm, 1-norm, nuclear norm

$$\|X\|_{tr} = tr\left(\sqrt{X^* X}\right)$$

Sum of singular values


The Fidelity Function
The fidelity of 2 quantum states ρ and σ
is defined as F(ρ,σ) = ||√ρ √σ||tr = Tr(√(√σ ρ √σ))
||·||tr = max {|<V,X>|}
V contraction
||V||∞ ≤ 1
SDP
α = sup ½ Tr(X) + ½ Tr(X*)
β = inf ½<ρ,Y> + ½<σ,Z>
of singular values

**Fun Fact**: Given $C > 0$, $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0 \iff A \geq \underbrace{B^* C^{-1} B}_{\text{Schur Complement}}$

$$\begin{bmatrix} \rho & X \\ X^* & \sigma \end{bmatrix} \geq 0 \iff \rho \geq X^* \sigma^{-1} X \qquad (\overset{\text{assume}}{\sigma > 0,\ \rho > 0})$$

$$\iff \rho^{1/2}\rho^{1/2} \geq X^* \sigma^{-1} X$$

$$\iff 1 \geq \rho^{-1/2} X^* \sigma^{-1} X \rho^{-1/2}$$

(Aside: $A \geq B \iff X^* A X \geq X^* B X$
$X$ invertible)

$$\alpha = \sup \tfrac{1}{2}\mathrm{Tr}(X) + \tfrac{1}{2}\mathrm{Tr}(X^*)$$

$$X = \sigma^{1/2} V \rho^{1/2}$$

$V$ contraction

$$\mathrm{Tr}(X) = \mathrm{Tr}(\sigma^{1/2} V \rho^{1/2}) = \langle \rho^{1/2}\sigma^{1/2}, V\rangle$$

$$\mathrm{Tr}(X^*) = \overline{\mathrm{Tr}(X)}$$

$$\tfrac{1}{2}\mathrm{Tr}(X) + \tfrac{1}{2}\mathrm{Tr}(X^*) = \mathrm{Re}(\mathrm{tr}(X))$$

$$= \mathrm{Re}(\langle \rho^{1/2}\sigma^{1/2}, V\rangle)$$

$$\alpha = \sup_{V \text{ contraction}} \mathrm{Re}(\langle \rho^{1/2}\sigma^{1/2}, V\rangle)$$

$$= \sup_{V} |\langle \rho^{1/2}\sigma^{1/2}, V\rangle|$$

$$= F(\rho, \sigma)$$

$$\alpha = \max \tfrac{1}{2}\langle M, \rho\rangle$$

$$0$$

$$= \max \tfrac{1}{2}\langle M, \rho\rangle$$

$$= \max \tfrac{1}{2} + \tfrac{1}{2}\langle$$

$$0 \leq$$

$$= \tfrac{1}{2} + \tfrac{1}{2}\sum_{k: \lambda_k > 0}$$

$$= \tfrac{1}{2} + \tfrac{1}{2}\left(\tfrac{1}{2}\sum_k\right.$$

$$= \tfrac{1}{2} + \tfrac{1}{4}\|\rho - \sigma\|_{tr}$$

where $\|\cdot\|_{tr}$ is the trace norm

$$\|X\|_{tr} = \mathrm{tr}(\sqrt{X^* X})$$

$$\beta = \inf \tfrac{1}{2}\langle \rho, Y\rangle + \tfrac{1}{2}\langle \sigma, Z\rangle$$
$$\begin{bmatrix} Y & \mathbb{1} \\ \mathbb{1} & Z \end{bmatrix} \geq 0$$

$$\begin{bmatrix} Y & \mathbb{1} \\ \mathbb{1} & Z \end{bmatrix} \geq 0 \iff Y \geq Z^{-1} \quad \text{or} \quad Z \geq Y^{-1}$$

By Schur Complements

$$\beta = \inf_{Y>0} \tfrac{1}{2}\langle \rho, Y\rangle + \tfrac{1}{2}\langle \sigma, Y^{-1}\rangle$$

$\uparrow$ $Z = Y^{-1}$ is the best choice.

$$\tfrac{1}{2}\langle \rho, Y\rangle + \tfrac{1}{2}\langle \sigma, Y^{-1}\rangle \geq \sqrt{\langle \rho, Y\rangle \langle \sigma, Y^{-1}\rangle}$$

holds with equality $\iff \langle \rho, Y\rangle = \langle \sigma, Y^{-1}\rangle$

$X^{-*})$

$/_2$

on.

$\langle \rho^{1/2}\sigma^{1/2}, V\rangle$

$(\mathrm{tr}(X))$

$(\langle \rho^{1/2}\sigma^{1/2}, V\rangle$

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holds with equality $\iff \langle \rho, Y\rangle = \langle \sigma, Y^{-1}\rangle$

$$\beta = \inf_{Y>0} \sqrt{\langle \rho, Y\rangle\langle \sigma, Y^{-1}\rangle}$$

this can be equal by scaling $Y$

$Y' = tY \quad t > 1$

$$\beta^2 = \inf_{Y>0} \langle \rho, Y\rangle\langle \sigma, Y^{-1}\rangle$$

$\beta = \inf \frac{1}{2}\langle \rho, Y\rangle + \frac{1}{2}\langle \sigma, Z\rangle$

$\begin{bmatrix} Y & \mathbb{1} \\ \mathbb{1} & Z \end{bmatrix} \geq 0$

$\begin{bmatrix} Y & \mathbb{1} \\ \mathbb{1} & Z \end{bmatrix} \geq 0 \iff Y \geq Z^{-1}$ or $Z \geq Y^{-1}$

By Schur Complements

$$\beta = \inf_{Y>0} \frac{1}{2}\langle \rho, Y\rangle + \frac{1}{2}\langle \sigma, Y^{-1}\rangle$$

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$$\frac{1}{2}\langle \rho, Y\rangle + \frac{1}{2}\langle \sigma, Y^{-1}\rangle \geq \sqrt{\langle \rho, Y\rangle\langle \sigma, Y^{-1}\rangle}$$

holds with equality $\iff \langle \rho, Y\rangle = \langle \sigma, Y^{-1}\rangle$

$$\beta = \inf_{Y>0} \sqrt{\langle \rho, Y\rangle\langle \sigma, Y^{-1}\rangle}$$

this can be equal by scaling $Y$

$Y' = tY \quad t > 1$

$$\beta^2 = \inf_{Y>0} \langle \rho, Y\rangle\langle \sigma, Y^{-1}\rangle$$

let's check $\alpha = \beta$

$(Y, Z) = (2\mathbb{1}, 2\mathbb{1})$ & $X = \sqrt{\rho}\sqrt{\sigma}$ $\Rightarrow \beta$ is finite

$\Rightarrow \alpha = \beta$ & $\alpha$ is attained

Alberti's theorem

$$\boxed{F(\rho,\sigma)^2 = \inf_{Y>0} \langle \rho, Y\rangle\langle \sigma, Y^{-1}\rangle}$$

$\rho, \sigma \in \mathrm{Pos}(\mathcal{X})$, $\mathcal{Y}$ is a purifying system

(purification of $\rho$)

**SDP**

$\alpha = \sup \langle W, |\psi\rangle\langle\psi| \rangle$

$\mathrm{Tr}_{\mathcal{Y}}(W) = \sigma$

$W \in \mathrm{Pos}(\mathcal{X} \otimes \mathcal{Y})$

$\beta = \inf \langle \sigma, Z \rangle$

$Z \otimes \mathbb{1}_{\mathcal{Y}} \geq |\psi\rangle\langle\psi|$

$Z \in \mathrm{Herm}(\mathcal{X})$

($Z \geq 0$ is implied)

$|\psi\rangle \in \mathcal{X} \otimes \mathcal{Y}$

$|\psi\rangle\langle\psi| \in \mathrm{Pos}(\mathcal{X} \otimes \mathcal{Y})$

$\mathrm{Tr}_{\mathcal{Y}}(|\psi\rangle\langle\psi|) = \rho$

partial trace

$(\mathbb{1}_{\mathcal{X}} \otimes \mathrm{Trace}_{\mathcal{Y}})$

**Let's work with $\beta$**

First, we'll assume $Z > 0$ and this will not change $\beta$ (but it may no longer be attained)

$$Z \otimes \mathbb{1} \geq |\psi\rangle\langle\psi|$$

$$\Leftrightarrow \mathbb{1} \geq (Z^{-1/2} \otimes \mathbb{1})\, |\psi\rangle\langle\psi|\, (Z^{-1/2} \otimes \mathbb{1})$$

$$\Leftrightarrow 1 \geq \lambda_{max}\Big(\underbrace{(Z^{-1/2} \otimes \mathbb{1})\, |\psi\rangle\langle\psi|\, (Z^{-1/2} \otimes \mathbb{1})}_{\text{rank } 1}\Big)$$

So $\lambda_{max} = \text{Trace}$

$$\Leftrightarrow 1 \geq \mathrm{Tr}\Big((Z^{-1/2} \otimes \mathbb{1})\, |\psi\rangle\langle\psi|\, (Z^{-1/2} \otimes \mathbb{1})\Big)$$

$$\Leftrightarrow 1 \geq \langle\psi|\, (Z^{-1} \otimes \mathbb{1})\, |\psi\rangle$$

$$\Leftrightarrow 1 \geq \langle Z^{-1} \otimes \mathbb{1},\ |\psi\rangle\langle\psi| \rangle$$

$$\Leftrightarrow 1 \geq \langle Z^{-1},\ \mathrm{Tr}_y(|\psi\rangle\langle\psi|) \rangle$$

$\beta = \inf \frac{1}{2}\langle \rho, Y \rangle + \frac{1}{2}\langle \sigma,$

$\begin{bmatrix} Y & \mathbb{1} \\ \mathbb{1} & Z \end{bmatrix} \geq$

$\begin{bmatrix} Y & \mathbb{1} \\ \mathbb{1} & Z \end{bmatrix} \geq 0 \Leftrightarrow$

By Schur Complem

$\beta = \inf_{Y > 0} \frac{1}{2}\langle \rho, Y \rangle$

$\frac{1}{2}\langle \rho, Y \rangle + \frac{1}{2}\langle \sigma, Y^{-1} \rangle$

holds with equality $\Leftrightarrow$

$$Z \otimes \mathbb{1} \geq |\psi\rangle\langle\psi|$$

$$\Leftrightarrow \quad \mathbb{1} \geq (Z^{-1/2} \otimes \mathbb{1})\, |\psi\rangle\langle\psi|\, (Z^{-1/2} \otimes \mathbb{1})$$

$$\Leftrightarrow \quad 1 \geq \lambda_{max}\Big(\underbrace{(Z^{-1/2} \otimes \mathbb{1})\, |\psi\rangle\langle\psi|\, (Z^{-1/2} \otimes \mathbb{1})}_{\text{rank } 1}\Big)$$

So $\lambda_{max}$ = Trace

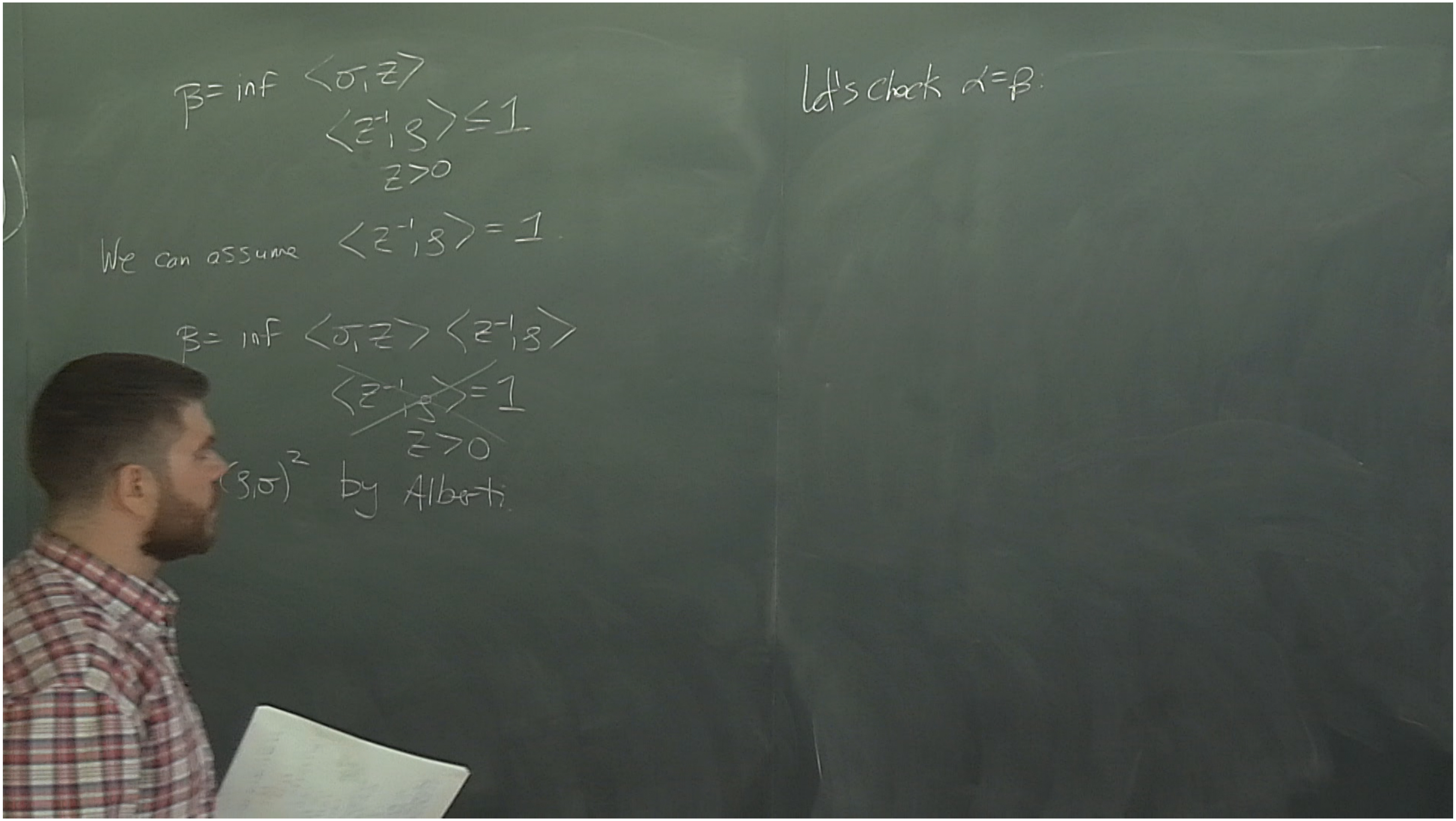
$$\Leftrightarrow 1 \geq \mathrm{Tr}\Big((Z^{-1/2} \otimes \mathbb{1})\, |\psi\rangle\langle\psi|\, (Z^{-1/2} \otimes \mathbb{1})\Big)$$

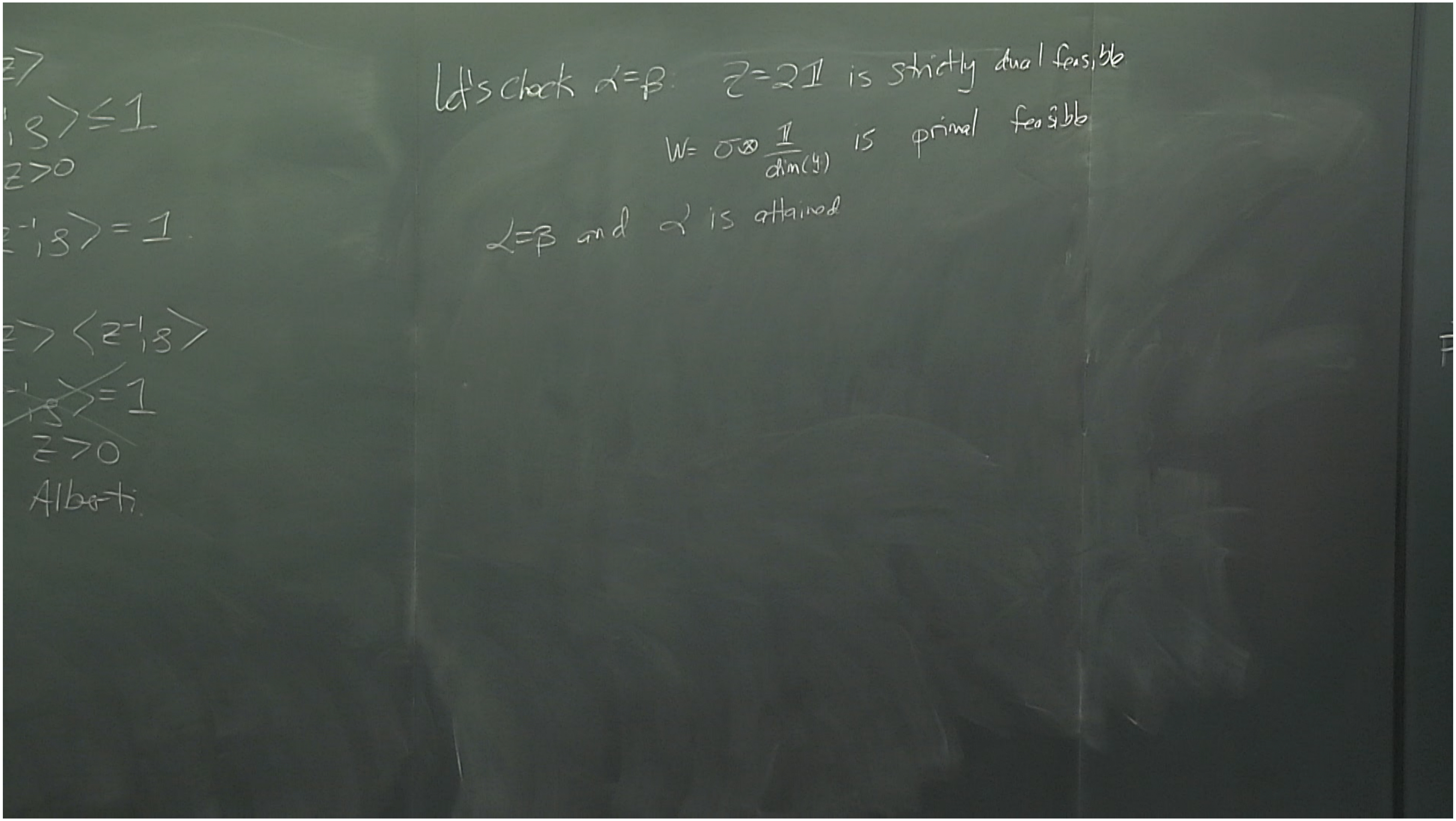
$$\Leftrightarrow 1 \geq \langle\psi| (Z^{-1} \otimes \mathbb{1}) |\psi\rangle$$

$$\Leftrightarrow 1 \geq \langle Z^{-1} \otimes \mathbb{1},\ |\psi\rangle\langle\psi| \rangle$$

$$\Leftrightarrow 1 \geq \langle Z^{-1},\ \mathrm{Tr}_Y(|\psi\rangle\langle\psi|) \rangle$$

$$\Leftrightarrow 1 \geq \langle Z^{-1},\ \rho \rangle$$


$\beta = \inf \langle \sigma, z \rangle$
$\langle z^{-1}, \rho \rangle \leq 1$
$z > 0$
We can assume $\langle z^{-1}, \rho \rangle = 1$.
$\beta = \inf \langle \sigma, z \rangle \langle z^{-1}, \rho \rangle$
$\cancel{\langle z^{-1}, \rho \rangle} = 1$
$z > 0$
$(\rho, \sigma)^2$ by Alberti.
Let's check $\alpha = \beta$:


Let's check $\alpha=\beta$: $Z=2\mathbb{1}$ is strictly dual feasible
$W=\sigma\otimes\frac{\mathbb{1}}{\dim(Y)}$ is primal feasible
$\alpha=\beta$ and $\alpha$ is attained
$Z>0$
Alberti

## Interesting Properties

completely positive & trace preserving

$\mathbb{1}_{L(\mathbb{C}^k)} \otimes \Phi$ is positive for all $k$.

### Data Processing:

① $\|\rho - \sigma\|_{tr} \geq \|\Phi(\rho) - \Phi(\sigma)\|_{tr}$

② $F(\Phi(\rho), \Phi(\sigma)) \geq F(\rho, \sigma)$.

(1) $\frac{1}{2}+\frac{1}{4}\|\Phi(\rho)-\Phi(\sigma)\|_{tr} = \max_{\substack{M+M'=\mathbb{1}\\ M,M'\geq 0}} \frac{1}{2}\langle M,\Phi(\rho)\rangle+\frac{1}{2}\langle M',\Phi(\sigma)\rangle$

$= \max_{\substack{M+M'=\mathbb{1}\\ M,M'\geq 0}} \frac{1}{2}\langle \Phi^*(M),\rho\rangle+\frac{1}{2}\langle \Phi^*(M'),\sigma\rangle$

$(\Phi^*(M),\Phi^*(M'))$ is a POVM

$\leq \max_{\substack{N+N'=\mathbb{1}\\ N,N'\geq 0}} \frac{1}{2}\langle N,\rho\rangle+\frac{1}{2}\langle N',\sigma\rangle$

$\Phi^*$ is unital

$\Phi^*$ is comp. pos.

$\Phi^*(M)+\Phi^*(M')$

$=\Phi^*(\mathbb{1})=\mathbb{1}$

$=\frac{1}{2}+\frac{1}{4}\|\rho-\sigma\|_{tr}$

$\square$

(2) $F(\rho\sigma)=$

$\Phi(\sigma)\rangle$

$\langle \Phi^*(M'), \sigma\rangle$

$\rho\rangle + \frac{1}{2}\langle N', \sigma\rangle$

$N + N' = \mathbb{I}$

$N, N' \geq 0$

$\square$.

(2) $F(\rho,\sigma) = \max \frac{1}{2}\mathrm{Tr}(X) + \frac{1}{2}\mathrm{Tr}(X^*)$

$$\begin{bmatrix} \rho & X \\ X^* & \sigma \end{bmatrix} \geq 0 \quad (*)$$

$$\begin{bmatrix} \rho & X \\ X^* & \sigma \end{bmatrix} \geq 0 \Rightarrow \begin{bmatrix} \Phi(\rho) & \Phi(X) \\ \Phi(X)^* & \Phi(\sigma) \end{bmatrix} \geq 0$$

(2-positive)

let $X$ be an optimal sol'n for $(*)$

$F(\rho,\sigma) = \frac{1}{2}\mathrm{Tr}(X) + \frac{1}{2}\mathrm{Tr}(X^*)$

$W = \Phi(X)$ is feasible in

$F(\Phi(\rho), \Phi(\sigma)) = \max \frac{1}{2}\mathrm{Tr}(W) + \frac{1}{2}\mathrm{Tr}(W^*)$

$$\begin{bmatrix} \Phi(\rho) & W \\ W^* & \Phi(\sigma) \end{bmatrix} \geq 0$$

$F(\Phi(\rho), \Phi(\sigma))$

$\geq \frac{1}{2}\mathrm{Tr}(\Phi(X)) + \frac{1}{2}\mathrm{Tr}(\Phi(X$

$= \frac{1}{2}\mathrm{Tr}(X) + \frac{1}{2}\mathrm{Tr}(X^*)$

$= F(\rho,\sigma)$

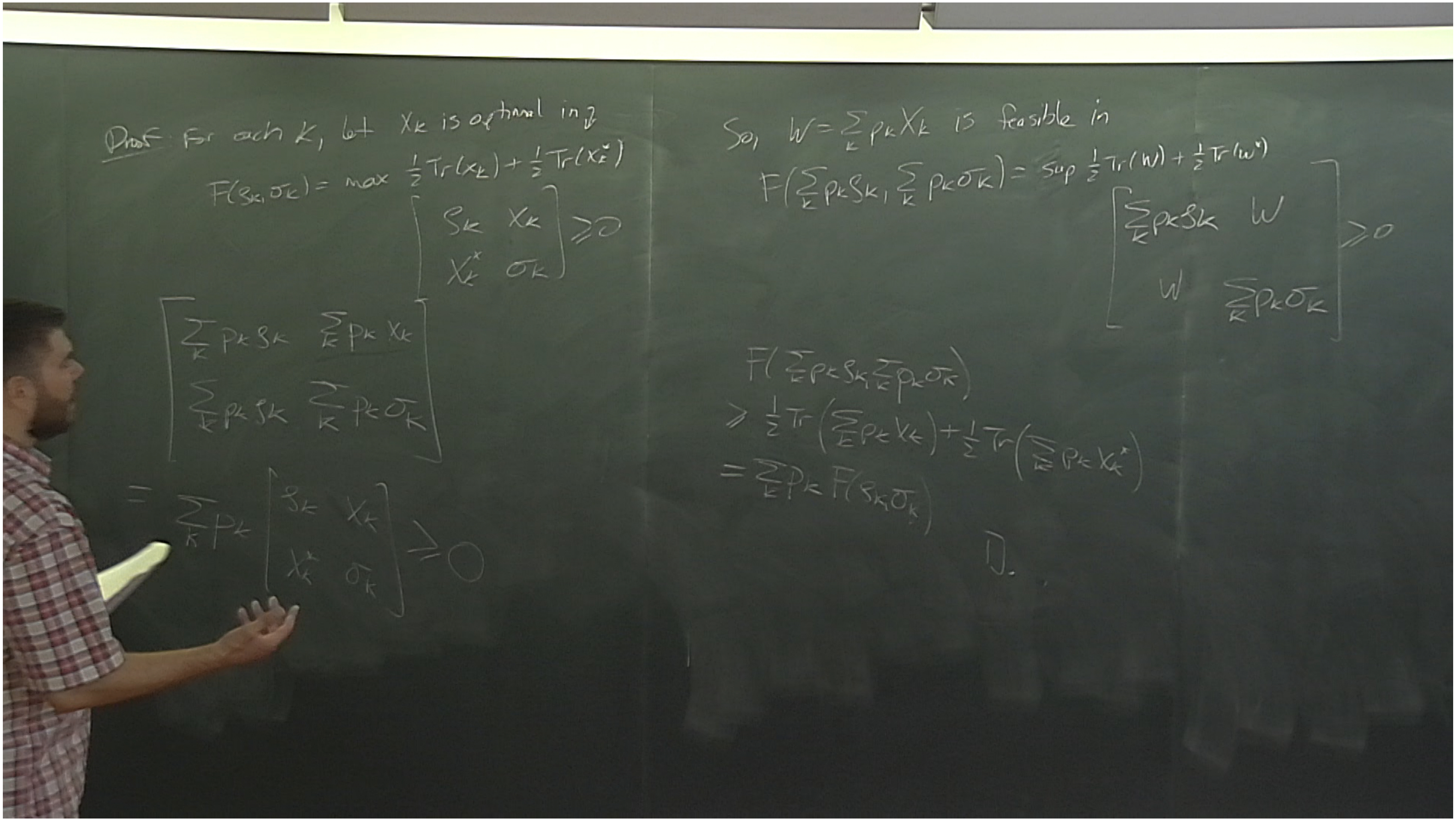
Proof: For each $k$, let

$$F(\rho_k, \sigma_k) = \max \tfrac{1}{2}\mathrm{Tr}(X_k) + \tfrac{1}{2}\mathrm{Tr}(X_k^*)$$

$$\begin{bmatrix} \rho_k & X_k \\ X_k^* & \sigma_k \end{bmatrix} \geq 0$$

$$\begin{bmatrix} \sum_k p_k \rho_k & \sum_k p_k X_k \\ \sum_k p_k \rho_k & \sum_k p_k \sigma_k \end{bmatrix}$$

$$= \sum_k p_k \begin{bmatrix} \rho_k & X_k \\ X_k^* & \sigma_k \end{bmatrix} \geq 0$$


Proof: For each $k$, let $X_k$ is optimal in
$F(\rho_k,\sigma_k) = \max \frac{1}{2}\mathrm{Tr}(X_k) + \frac{1}{2}\mathrm{Tr}(X_k^*)$
$\begin{bmatrix} \rho_k & X_k \\ X_k^* & \sigma_k \end{bmatrix} \geq 0$
$\begin{bmatrix} \sum_k p_k \rho_k & \sum_k p_k X_k \\ \sum_k p_k \rho_k & \sum_k p_k \sigma_k \end{bmatrix}$
$= \sum_k p_k \begin{bmatrix} \rho_k & X_k \\ X_k^* & \sigma_k \end{bmatrix} \geq 0$
So, $W = \sum_k p_k X_k$ is feasible in
$F\left(\sum_k p_k \rho_k, \sum_k p_k \sigma_k\right) = \sup \frac{1}{2}\mathrm{Tr}(W) + \frac{1}{2}\mathrm{Tr}(W^*)$
$\begin{bmatrix} \sum_k p_k \rho_k & W \\ W & \sum_k p_k \sigma_k \end{bmatrix} \geq 0$
$F\left(\sum_k p_k \rho_k, \sum_k p_k \sigma_k\right)$
$\geq \frac{1}{2}\mathrm{Tr}\left(\sum_k p_k X_k\right) + \frac{1}{2}\mathrm{Tr}\left(\sum_k p_k X_k^*\right)$
$= \sum_k p_k F(\rho_k, \sigma_k)$
$\square$.