Title: SDP / Quantum Lecture Series

Speakers: Jamie Sikora
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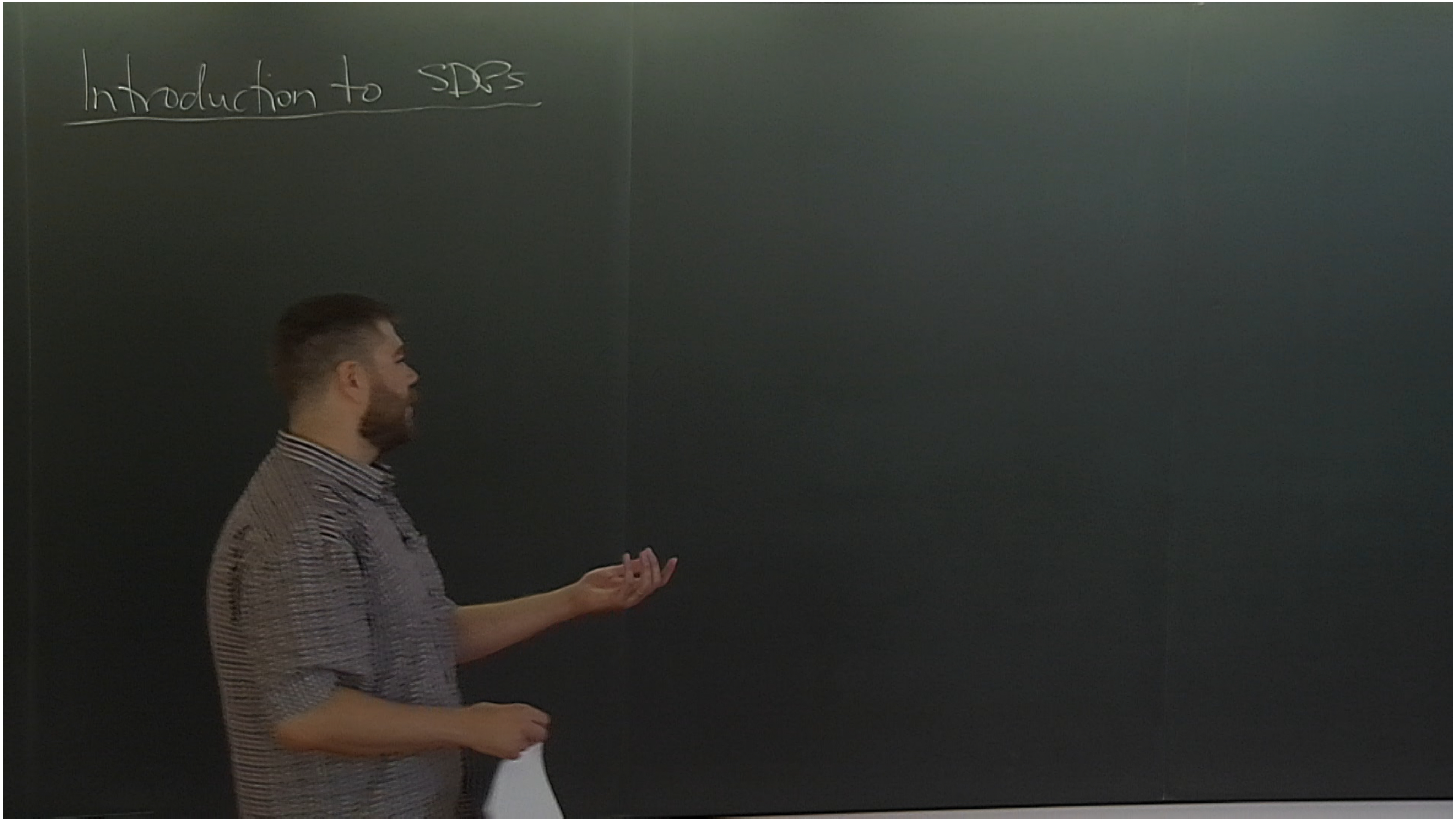

Introduction to SDPs

Introduction to SDPs

A semidefinite program (SDP) is an optimization problem of the form

$$\alpha = \sup \langle A, X\rangle \leftarrow \text{objective function}$$

opt. value s.t. $\Phi(X) = B$ constraints

$X \geq 0$ positive semidefinite

- $\Phi : \mathcal{X} \to \mathcal{Y}$ linear, Hermiticity-preserving
- $B \in \mathrm{Herm}(\mathcal{Y})$, $A \in \mathrm{Herm}(\mathcal{X})$ finite-dim

Hermitian: $X = X^{+}$, $X = X^{*}$ (≥ 0 e-vals)

$\mathcal{X} \cong \mathbb{C}^{k}$

$\mathcal{Y} \cong \mathbb{C}^{p}$

- $\langle A, X\rangle = \mathrm{Tr}(A^{*}X)$

X is feasible if it satisfies $\Phi(X) = B$, $X \geq 0$

X is strictly feasible if it is feasible & positive definite

Ex: $\alpha = \sup \mathrm{Tr}(X)$ $\quad A = \mathbb{1}_5$

$X = \mathbb{1}_5$ $\quad \Phi = \mathbb{1}$

$X \geq 0$ $\quad B = \mathbb{1}_5$

$X = \mathbb{1}_5$ is the only feasible solution $\mathcal{X} = \mathcal{Y} = \mathbb{C}^5$

$X = \mathbb{1}_5$ is the optimal solution

$\alpha = \mathrm{Tr}(\mathbb{1}_5) = 5$

$(X) = B,\ X \succeq 0$

feasible

$= \mathbb{1}_5$

$= 1$

$= \mathbb{1}_5$

$y = \mathbb{R}^5$

Ex: $\alpha = \sup \mathrm{Tr}(X)$

$X = -\mathbb{1}_5$

$X \succeq 0$

B from $\mathbb{1}_5$ to $-\mathbb{1}_5$

No feasible solution

SDP is infeasible

$\alpha = -\infty$

Ex: $\alpha = \sup \mathrm{Tr}(X)$

$X \succeq \mathbb{1}_5$

$X \succeq 0$

Introduce a slack variable

$\alpha = \sup \mathrm{Tr}(X)$

$X - R = \mathbb{1}_5$

$X \succeq 0,\ R \succeq 0$

$$\alpha = \sup \left\langle \begin{pmatrix} \mathbb{1}_s & 0 \\ 0 & 0 \end{pmatrix}, X' \right\rangle \qquad X' = \begin{pmatrix} X & 0 \\ 0 & R \end{pmatrix}$$

$$\Phi(X') = \mathbb{1}_s \qquad \Phi(X') = X - R \; \left(\text{~~top~~ left corner minus bottom-right}\right)$$

$$X' \geq 0$$

$X = t\,\mathbb{1}_s$ for $t \geq 1$ is feasible for all $t \geq 1$.

$\mathrm{Tr}(X) = st \to +\infty$ as $t \to +\infty$

SDP is unbounded, $\alpha = +\infty$

n to SDPs

nite program (SDP)

ation problem of the

p $\langle A, X\rangle$ ← objective function

t. $\Phi(X) = B$ } constraints

$X \geq 0$ } positive semidefinite

$X = X^{+}$

r, Hermiticity-preserving $X = X^{*}$

(≥ 0 e-vals)

$A \in \mathrm{Herm}(\mathcal{X})$

← finite-dim

Ex: $\alpha = \inf t$

$$\begin{bmatrix} t & 1 \\ 1 & s \end{bmatrix} \geq 0$$

$t, s \in \mathbb{R}$

First, it's OK to minimize

$$\left[-\alpha = \sup -t \quad \begin{bmatrix} t & 1 \\ 1 & s \end{bmatrix} \geq 0 \right]$$

$$\begin{bmatrix} t & 1 \\ 1 & s \end{bmatrix} \geq 0 \iff t, s \geq 0,\ st \geq 1$$

$t > 0$

Ex: $\alpha = \sup \mathrm{Tr}(X)$

$X = -\mathbb{1}_5$

$X \geq 0$

No feasible solution

SDP is infeasible.

$\alpha = -\infty$

Ex: $\alpha = \sup \mathrm{Tr}(X)$

$X \geq \mathbb{1}_5$

$X \geq 0$

Introduce a slack variable

$\alpha = \sup \mathrm{Tr}(X)$

$X - R = \mathbb{1}_5$

$X \geq 0,\ R \geq 0$

t

$\begin{bmatrix} t & 1 \\ 1 & s \end{bmatrix} \geq 0$

$t, s \in \mathbb{R}$

to minimize

$-t$

$\begin{bmatrix} t & 1 \\ 1 & s \end{bmatrix} \geq 0$

$\Leftrightarrow t, s \geq 0, \; st \geq 1$

$(s,t) = \left(k, \frac{1}{k}\right) \quad k > 0$

feasible

$\alpha = 0$ (but, no optimal solution exists

Duality Theory

Every SDP has a dual

Primal SDP	Dual SDP
$\alpha = \sup \langle A, X \rangle$	$\beta = \inf \langle B, Y \rangle$
$\Phi(X) = B$	$\Phi^*(Y) - S = A$
$X \geq 0$	$S \in \mathrm{Pos}(\mathcal{X}) \leftarrow$ PSD
	$Y \in \mathrm{Herm}(\mathcal{Y})$

adjoint of Φ

Adjoint: $\langle \Phi(X), Y \rangle$

$= \langle X, \Phi^*(Y) \rangle$

Optimizing over the same data (A, B, Φ)

Ex: $\alpha = \sup \operatorname{Tr}(X)$

$$X = \mathbb{1}_S$$
$$X \geq 0$$

$\beta = \inf \operatorname{Tr}(Y)$

$$Y - S = \mathbb{1}_S$$
$$S \geq 0$$

Y is "free"

$\alpha = 0$ (but, no optimal solution exists

Duality Theory

Every SDP has a dual

Primal SDP

$\alpha = \sup \langle A, X \rangle$

$\Phi(X) = B$

$X \geq 0$

Dual SDP

$\beta = \inf \langle B, Y \rangle$

$\Phi^*(Y) - S = A$

adjoint of Φ

$S \in \mathrm{Pos}(\mathcal{X}) \leftarrow$ PSD

$Y \in \mathrm{Herm}(\mathcal{Y})$

Adjoint: $\langle \Phi(X), Y \rangle = \langle X, \Phi^*(Y) \rangle$

Optimizing over the same data (A, B, Φ)

Ex: $\alpha = \sup \mathrm{Tr}(X)$

$X = \mathbb{1}_5$

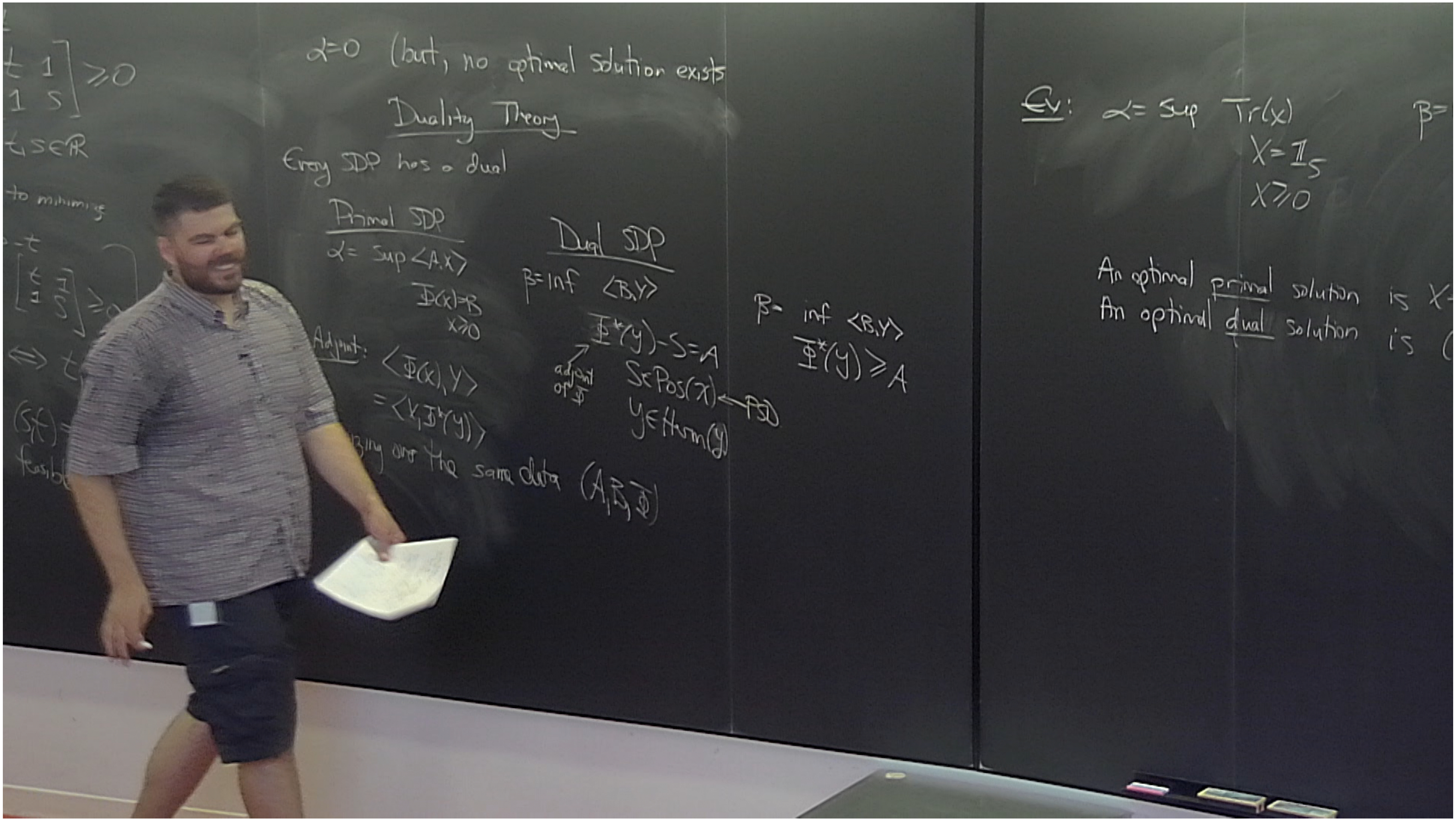
$X \geq 0$

$\beta = \inf \mathrm{Tr}(Y)$

$Y - S = \mathbb{1}_5$

$S \geq 0$

Y is "free"


α=0 (but, no optimal solution exists
Duality Theory
Every SDP has a dual
Primal SDP
α= sup ⟨A,X⟩
Φ(X)=B
X≥0
Adjoint: ⟨Φ(X), Y⟩
= ⟨X, Φ*(Y)⟩
Dual SDP
β= inf ⟨B,Y⟩
Φ*(Y)−S=A
adjoint of Φ
S∈Pos(X) ← PSD
Y∈Herm(Y)
β= inf ⟨B,Y⟩
Φ*(Y) ≥ A
the same data (A,B,Φ)
Ex: α= sup Tr(X)
X=I5
X≥0
An optimal primal solution is
An optimal dual solution is

Ex: $\alpha = \sup \mathrm{Tr}(X)$

$X = \mathbb{1}_5$

$X \succeq 0$

$\beta = \inf \mathrm{Tr}(Y)$

$Y - S = \mathbb{1}_5$ $\quad$ $Y \succeq \mathbb{1}_5$

$S \succeq 0$

Y is "free" Hermitian

An optimal primal solution is $X = \mathbb{1}_5$ so $\alpha = 5$

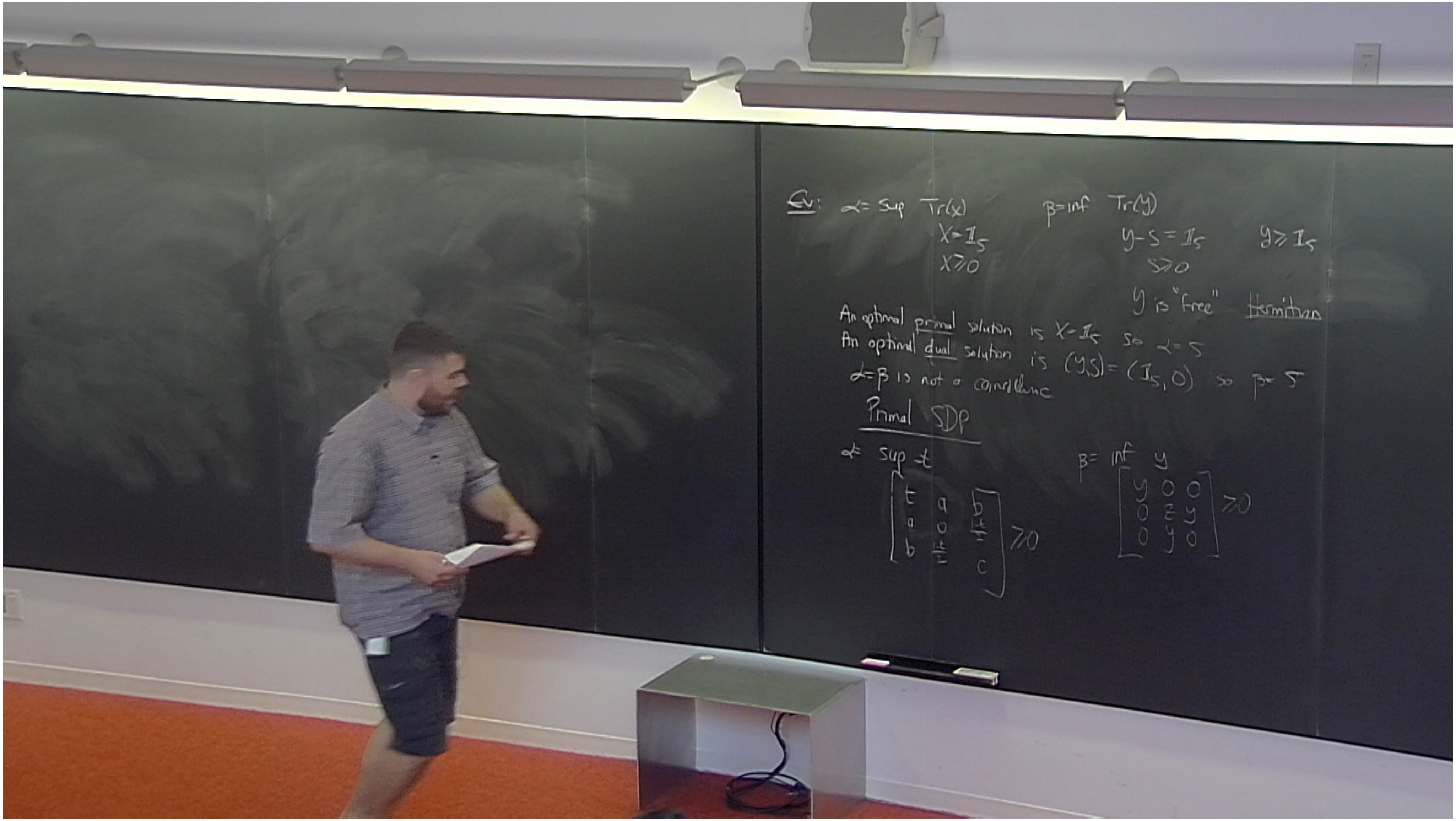
An optimal dual solution is $(Y,S) = (\mathbb{1}_5, 0)$ so $\beta = 5$

$\alpha = \beta$ is not a coincidence

Primal SDP

$$\alpha = \sup t \qquad \begin{bmatrix} t & a & b \\ a & 0 & \frac{1-t}{2} \\ b & \frac{1-t}{2} & c \end{bmatrix} \succeq 0$$

$$\beta = \inf y \qquad \begin{bmatrix} y & 0 & 0 \\ 0 & z & y \\ 0 & y & 0 \end{bmatrix} \succeq 0$$


Ex: α = sup Tr(X)
β = inf Tr(Y)
X = I5
X ≥ 0
Y − S = I5
S ≥ 0
Y ≥ I5
Y is "free" Hermitian
An optimal primal solution is X = I5 so α = 5
An optimal dual solution is (Y,S) = (I5, 0) so β = 5
α = β is not a coincidence
Primal SDP
α = sup t
β = inf y

If a diagonal entry on a PSD matrix is 0

that row & column are all 0's

(a,b,c,t) are feasible $\Rightarrow$ $a=0$

$\frac{1-t}{2}=0 \Rightarrow t=1.$

$(0,0,0,1)$ is feasible

$\alpha=-1$

For the dual: $y=0$ in any dual feasible solution

$(y,z)=(0,0)$ is feasible.

Ex: $\alpha = \sup \operatorname{Tr}(X)$

$X = \mathbb{1}_5$

$X \succeq 0$

$\beta = \inf \operatorname{Tr}(Y)$

$Y - S = \mathbb{1}_5$ $\quad Y \succeq \mathbb{1}_5$

$S \succeq 0$

Y is "free" Hermitian

An optimal primal solution is $X = \mathbb{1}_5$ so $\alpha = 5$

An optimal dual solution is $(Y,S) = (\mathbb{1}_5, 0)$ so $\beta = 5$

$\alpha = \beta$ is not a coincidence

Ex Primal SDP

$\alpha = \sup t$

$$\begin{bmatrix} t & a & b \\ a & 0 & \frac{1-t}{2} \\ b & \frac{1-t}{2} & c \end{bmatrix} \succeq 0$$

$\beta = \inf y$

$$\begin{bmatrix} y & 0 & 0 \\ 0 & z & y \\ 0 & y & 0 \end{bmatrix} \succeq 0$$

matrix is 0

1 0's

$\Rightarrow a = 0$

$\frac{1-t}{2} = 0 \Rightarrow t = 1.$

any dual feasible solution

(0,0) is feasible

Weak duality

If X is feasible (i.e. $\Phi(X) = B,\ X \geq 0$)

and (y, S) is dual feasible (i.e. $\Phi^*(y) - S = A$, $S \geq 0$)

then $\langle A, X\rangle \leq \langle B, Y\rangle$

In general: $\alpha \leq \beta$

Ex: $\alpha = \sup \operatorname{Tr}(X)$

$X = \mathbb{1}_5$

$X \geq 0$

An optimal primal solution

An optimal dual solution

$\alpha = \beta$ is not a condi...

Ex: Primal SDP

$\alpha = \sup t$

$$\begin{bmatrix} t & a & b \\ a & 0 & \frac{1-t}{2} \\ b & \frac{1-t}{2} & c \end{bmatrix} \geq 0$$

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In general: $\alpha \leq \beta$

Proof: $\langle B, Y\rangle - \langle A, X\rangle$

$= \langle \Phi(X), Y\rangle - \langle A, X\rangle$

$= \langle X, \Phi^*(Y)\rangle - \langle A, X\rangle$

$= \langle X, \Phi^*(Y) - A\rangle$

$= \langle X, S\rangle$

Ex: $\alpha = \sup \operatorname{Tr}(X)$

$X = \mathbb{1}_5$

$X \geq 0$

An optimal primal solution

An optimal dual solution

$\alpha = \beta$ is not a condition

Ex: **Primal SDP**

$\alpha = \sup t$

$$\begin{bmatrix} t & a & b \\ a & 0 & \frac{1-t}{2} \\ b & \frac{1-t}{2} & c \end{bmatrix} \geq 0$$

matrix is 0

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$\Rightarrow a = 0$

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any dual feasible solution

$(0,0)$ is feasible.

Weak duality

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In general: $\alpha \leq \beta$

Proof: $\langle B, Y \rangle - \langle A, X \rangle$

$= \langle \Phi(X), Y \rangle - \langle A, X \rangle$

$= \langle X, \Phi^*(y) \rangle - \langle A, X \rangle$

$= \langle X, \Phi^*(y) - A \rangle$

$= \langle X, S \rangle$

≥ 0 since $X, S \geq 0$

An optimal prim... tion

An optimal dual solution

$\alpha = \beta$ is not a condition...

Ex: Primal SDP

$$\alpha = \sup t$$

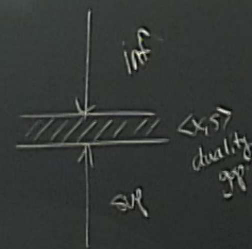
$$\begin{bmatrix} t & a & b \\ a & 0 & \frac{1-t}{2} \\ b & \frac{1-t}{2} & c \end{bmatrix} \geq$$

Weak duality

If X is feasible (i.e. $\Phi(X) = B$, $X \geq 0$)
and (y, S) is dual feasible (i.e. $\Phi^*(y) - S = A$, $S \geq 0$)

then $\langle A, X \rangle \leq \langle B, y \rangle$

$\epsilon = I$. In general: $\alpha \leq \beta$



Proof:
$$
\begin{aligned}
\langle B, y \rangle - \langle A, X \rangle &= \langle \Phi(X), y \rangle - \langle A, X \rangle \\
&= \langle X, \Phi^*(y) \rangle - \langle A, X \rangle \\
&= \langle X, \Phi^*(y) - A \rangle \\
&= \langle X, S \rangle \\
&\geq 0 \quad \text{since } X, S \geq 0
\end{aligned}
$$

sible solution

do

...atrix is 0

...'s

$a = 0$

$\frac{1-t}{2} = 0 \Rightarrow t = 1.$

...ny dual feasible solution

...,0) is feasible

Weak duality

If X is feasible (i.e. $\Phi(X) = B$, $X \succeq 0$)

and (y, S) is dual feasible (i.e. $\Phi^*(y) - S = A$, $S \succeq 0$)

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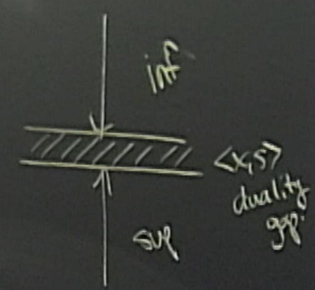
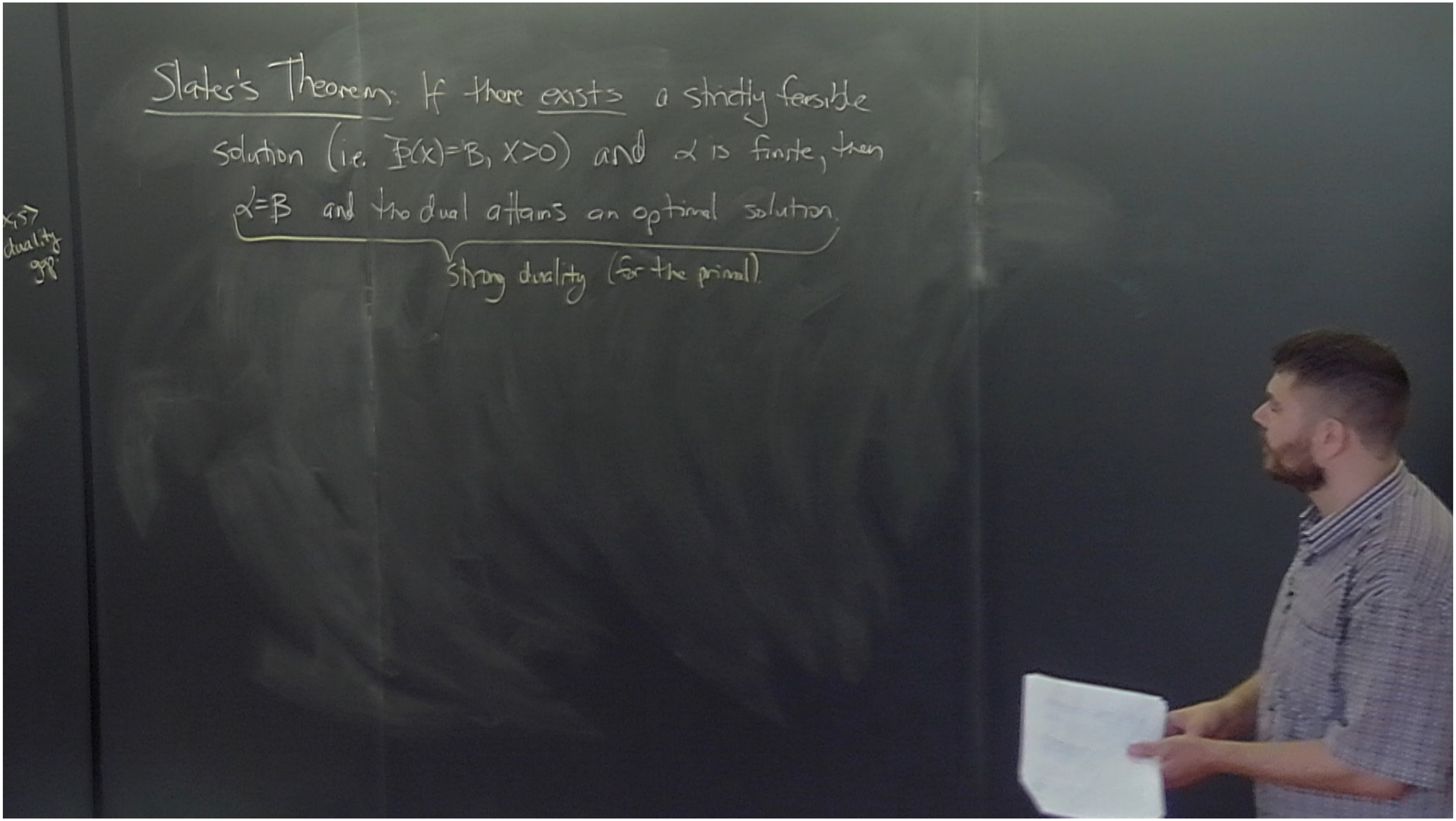
In general: $\alpha \leq \beta$



Proof:

$$\begin{aligned} &\langle B, y \rangle - \langle A, X \rangle \\ &= \langle \Phi(X), y \rangle - \langle A, X \rangle \\ &= \langle X, \Phi^*(y) \rangle - \langle A, X \rangle \\ &= \langle X, \Phi^*(y) - A \rangle \\ &= \langle X, S \rangle \\ &\geq 0 \quad \text{since } X, S \succeq 0 \end{aligned}$$


Slater's Theorem: If there exists a strictly feasible
solution (i.e. Φ(X)=B, X>0) and α is finite, then
α=β and the dual attains an optimal solution.
Strong duality (for the primal)
duality
gap

If X is feasible (i.e. $\Phi(X)=B,\ X\geq 0$)

and (y,S) is dual feasible (i.e. $\Phi^*(y)-S=A$)
$S\geq 0$

then $\langle A,X\rangle \leq \langle B,y\rangle$.

I. In general: $\alpha\leq\beta$

Proof: $\langle B,y\rangle-\langle A,X\rangle$

$$= \langle \Phi(X),y\rangle-\langle A,X\rangle$$

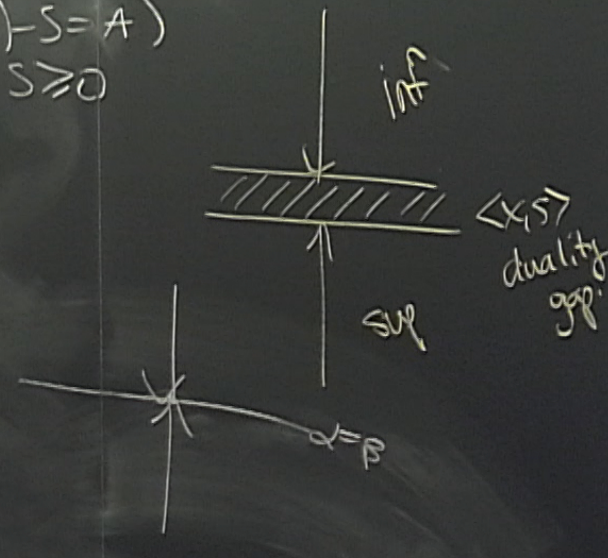
solution

$$= \langle X,\Phi^*(y)\rangle-\langle A,X\rangle$$

$$= \langle X,\Phi^*(y)-A\rangle$$

$$= \langle X,S\rangle$$

$$\geq 0 \quad \text{since } X,S\geq 0$$



Slater's The

solution (

$\alpha=\beta$ a

Slater's Theorem: If there exists a strictly feasible solution (i.e. $\Phi(X) = B$, $X > 0$) and α is finite, then $\underbrace{\alpha = \beta \text{ and the dual attains an optimal solution.}}_{\text{Strong duality (for the primal)}}$

Proof: Define the following two convex sets

$$M = \{ (S, V, t) \in (\mathrm{Herm}(\mathcal{X}), \overset{\mathrm{Herm}(\mathcal{Y})}{Q}, \mathbb{R}) : \exists X \geq S,\ B - \Phi(X) = V,\ \langle A, X \rangle \geq t \}$$

$$N = \{ (0, 0, s) \in (\mathrm{Herm}(\mathcal{X}), Q, \mathbb{R}) : s > \alpha \}$$ not empty as α finite.

Slater's Theorem: If there exists a strictly feasible solution (i.e. $\Phi(X)=B,\ X>0$) and α is finite, then $\underbrace{\alpha=\beta \text{ and the dual attains an optimal solution.}}_{\text{Strong duality (for the primal)}}$

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not empty as α finite.

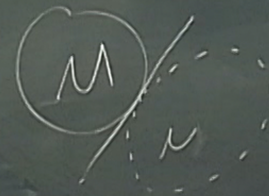
$$N=\{(0,0,s)\in \mathrm{Herm}(\mathcal{X}),\ Q,\ \mathbb{R}) : s>\alpha\}$$

Image of Φ

$Q=\{\Phi(T) : T\in \mathrm{Herm}(\mathcal{X})\}\subseteq \mathrm{Herm}(\mathcal{Y})$

$B\in Q$

We defined M, N such that $M \cap N = \emptyset$.



By the separating hyperplane theorem

$$\exists\ (\underbrace{\Lambda, Y, \lambda}_{\text{not all } 0}) \in (\mathrm{Hom}(X), Q, \mathbb{R}) \quad \text{and } c \in \mathbb{R}$$

$$(s,v,t) \in M \Rightarrow \langle \Lambda, s\rangle + \langle Y, v\rangle + \lambda t \leq c \quad (1)$$

$$(s,v,t) \in N \Rightarrow \langle \Lambda, s\rangle + \langle Y, v\rangle + \lambda t \geq c \quad (2)$$

(2) can be written $\lambda s \geq c \quad \forall\ s > \alpha$

$$\Rightarrow \boxed{\lambda \geq 0} \ \& \ \boxed{\lambda \alpha \geq c}$$

$M \cap N = \emptyset$.

theorem

$x), Q, \mathbb{R})$.

and $C \in \mathbb{R}$

$y, v\rangle + \lambda t \leq C$ (1)

$y, v\rangle + \lambda t \geq C$ (2)

$\geq C \quad \forall\, s > \alpha$

$\boxed{\alpha \geq C}$

M

$s < 0$

<u>Case 1: $\lambda > 0$</u>: Rescale:

$\Lambda' = \frac{\Lambda}{\lambda}, \quad y' = \frac{y}{\lambda}$

$(s, v, t) \in M \Rightarrow \langle \Lambda', s\rangle + \langle y', v\rangle + t \leq \frac{1}{\lambda} C \leq \alpha$

<u>Slater's Theorem</u>:

solution (i.e. $\exists$

$\alpha = \beta$ and the

<u>Proof</u>: Define the f

$M = \{(s, v, t) \in \mathrm{Her}$

$N = \{(0, 0, s) \in \mathrm{Her}$

$M \cap N = \emptyset$.

theorem

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$y, v \rangle + \lambda t \leq C$ (1)

$y, v \rangle + \lambda t \geq C$ (2)

$\geq C \quad \forall\, s > \alpha$

$\boxed{\alpha \geq C}$

M

$S \leq 0$

Case 1: $\lambda > 0$: Rescale:

$$\Lambda' = \frac{\Lambda}{\lambda}, \quad y' = \frac{y}{\lambda}$$

$$(S, v, t) \in M \Rightarrow \langle \Lambda', S \rangle + \langle y', v \rangle + t \leq \frac{1}{\lambda} C \leq \alpha$$

Since $(S, B - \Phi(S), \langle A, S \rangle) \in M$

$S \in \mathrm{Herm}(\mathcal{X})$

$$\langle S, \Lambda' - \Phi^*(y') + A \rangle + \langle B, y' \rangle \leq \alpha$$

Slater's Theorem:

solution (i.e. $\exists$

$\alpha = \beta$ and the

Proof: Define the f

$M = \{ (S, v, t) \in \mathrm{Her}$

$N = \{ (0, 0, s) \in \mathrm{Herm}$

N such that $M \cap N = \emptyset$.

...ting hyperplane theorem

$) \in (\mathrm{Hom}(\mathcal{X}), Q, \mathbb{R})$

all O and $C \in \mathbb{R}$

$\langle \Lambda, S \rangle + \langle y, V \rangle + \lambda t \leq C$ (1)

$\langle \Lambda, S \rangle + \langle y, V \rangle + \lambda t \geq C$ (2)

...itten $\lambda s \geq C \quad \forall\, s > \alpha$

$\boxed{\lambda \alpha \geq C}$

$) \in M$

and $S \leq 0$

$\boxed{\geq 0}$

Case 1: $\lambda > 0$: Rescale:

$$\Lambda' = \frac{\Lambda}{\lambda}, \quad y' = \frac{y}{\lambda}$$

$$(S, V, t) \in M \Rightarrow \langle \Lambda', S \rangle + \langle y', V \rangle + t \leq \frac{1}{\lambda} C \leq \alpha$$

Since $(S, B - \Phi(S), \langle A, S \rangle) \in M$

$S \in \mathrm{Herm}(\mathcal{X})$

$$\langle S, \underbrace{\Lambda' - \Phi^*(y') + A}\rangle + \langle B, y' \rangle \leq \alpha$$

$$\Rightarrow \Lambda' - \Phi^*(y') + A = 0$$

(y', Λ') is dual feasible

$$\beta \leq \langle B, y' \rangle \leq \alpha \leq \beta$$

defn of β — weak duality

$\Rightarrow \alpha = \beta$ & (y', Λ') is dual optimal.

Slate...

Proof:

$M = \{$

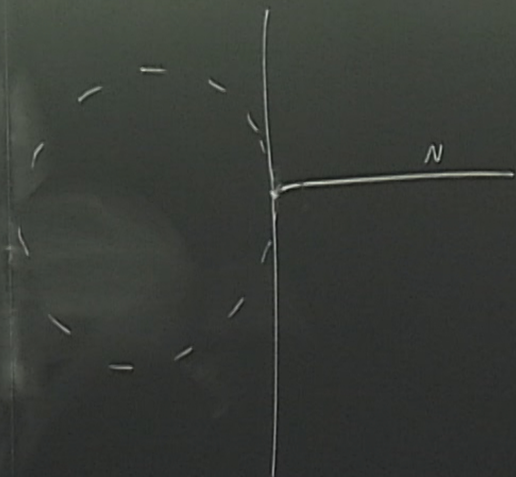
$N = \{$

a strictly feasible

α is finite, then

optimal solution.

the primal).



vex sets

$X \geq S,\ B - \Phi(X) = V,\ \langle A, X \rangle \geq t \}$

$> \alpha \}$ not empty as α finite.

Φ

$Q = \{ \Phi(T) : T \in \mathrm{Herm}(X) \} \subseteq \mathrm{Herm}(Y)$

Case: $\lambda = 0$

Argue that $\Lambda = Y = 0$ as well.

- Since $(X, 0, \langle A, X\rangle) \in M$ for strictly feasible X

$$\langle A, X\rangle \leq C \leq \lambda\alpha = 0$$

But, $X > 0$, $\Lambda \geq 0$

$\Rightarrow \boxed{\Lambda = 0}$

$\langle X, \Lambda\rangle \geq 0$ if $X, \Lambda \geq 0$

$\langle X, \Lambda\rangle > 0$ if $X > 0, \Lambda \neq 0$

$(S, V, t) \in M \Rightarrow$

$(S, V, t) \in N \Rightarrow \langle \Lambda, S$

(2) Can be

$\Rightarrow \boxed{\lambda}$

Notice

for any

$\Rightarrow \boxed{\Lambda}$

well.

...or

$\alpha = 0$

Since $(S, B - \Phi(S), \langle A, S\rangle) \in \mathcal{M}$

$S \in \mathrm{Herm}(\mathcal{X})$

$$\langle \underset{Q}{Y}, \underbrace{\underset{Q}{B} - \underset{Q}{\Phi(S)}}\rangle \le c \le 0.$$

$$\langle Y, B\rangle \le \langle Y, \Phi(S)\rangle$$

$\Downarrow$

$Y = 0$

Case 1: $\lambda > 0$: Rescale:

$$\Lambda' = \frac{\Lambda}{\lambda}, \quad Y' = \frac{Y}{\lambda}$$

$$(S, V, t) \in \mathcal{M} \Rightarrow \langle \Lambda', S\rangle + \langle Y', V\rangle + t \le \frac{1}{\lambda} c$$

Since $(S, B - \Phi(S), \langle A, S\rangle) \in \mathcal{M}$

$S \in \mathrm{Herm}(\mathcal{X})$

$$\langle S, \underbrace{\Lambda' - \Phi^*(Y') + A}\rangle + \langle B, Y'\rangle$$

$$\Rightarrow \Lambda' - \Phi^*(Y') + A = 0$$

(Y', Λ') is dual feasible

$$\beta \underset{\text{defn of } \beta}{\le} \langle B, Y'\rangle \le \alpha \overset{\text{weak duality}}{\le} \beta$$

$\Rightarrow \alpha = \beta$ & (Y', Λ') is dual optimal.

Since $(S, B - \Phi(S), \langle A, S \rangle) \in \mathcal{M}$
$S \in \mathrm{Herm}(\mathcal{X})$

$$\langle \underset{\substack{\uparrow\\ Q}}{Y}, \underbrace{\underset{Q}{B} - \underset{Q}{\Phi(S)}} \rangle \leq c \leq 0$$

$$\langle Y, B \rangle \leq \langle Y, \Phi(S) \rangle$$

$$\Downarrow$$

$$Y = 0$$

Contradiction

$\square$

Case 1: $\lambda > 0$: Rescale:

$$\Lambda' = \frac{\Lambda}{\lambda}, \quad Y' = \frac{Y}{\lambda}$$

$(S, V, t) \in \mathcal{M} \Rightarrow \langle \Lambda', S \rangle + \langle Y', V \rangle + t \leq \frac{1}{\lambda} c \leq \alpha$

Since $(S, B - \Phi(S), \langle A, S \rangle) \in \mathcal{M}$
$S \in \mathrm{Herm}(\mathcal{X})$

$$\langle S, \underbrace{\Lambda' - \Phi^*(Y') + A} \rangle + \langle B, Y' \rangle \leq \alpha$$

$$\Rightarrow \Lambda' - \Phi^*(Y') + A = 0$$

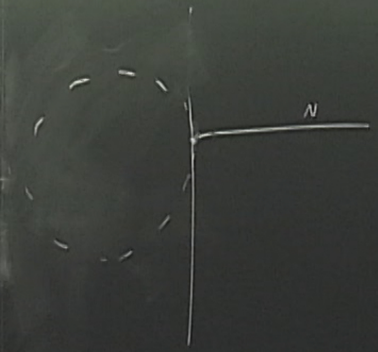
(Y', Λ') is dual feasible

$$\beta \underset{\substack{\uparrow\\ \text{defn of } \beta}}{\leq} \langle B, Y' \rangle \leq \alpha \overset{\substack{\text{weak duality}\\ \downarrow}}{\leq} \beta$$

$\Rightarrow \alpha = \beta$ & (Y', Λ') is dual optimal

Slater's Theorem: If there exists a strictly feasible solution (i.e. $\Phi(X)=B$, $X>0$) and α is finite, then

$\underbrace{\alpha=\beta \text{ and the dual attains an optimal solution.}}_{\text{strong duality (for the primal)}}$



Proof: Define the following two convex sets

$M=\{(S,V,t)\in(\mathrm{Herm}(\mathcal{X}),Q,\mathbb{R}): \exists X\geq S,\ B-\Phi(X),\ \langle A,X\rangle\geq t\}$

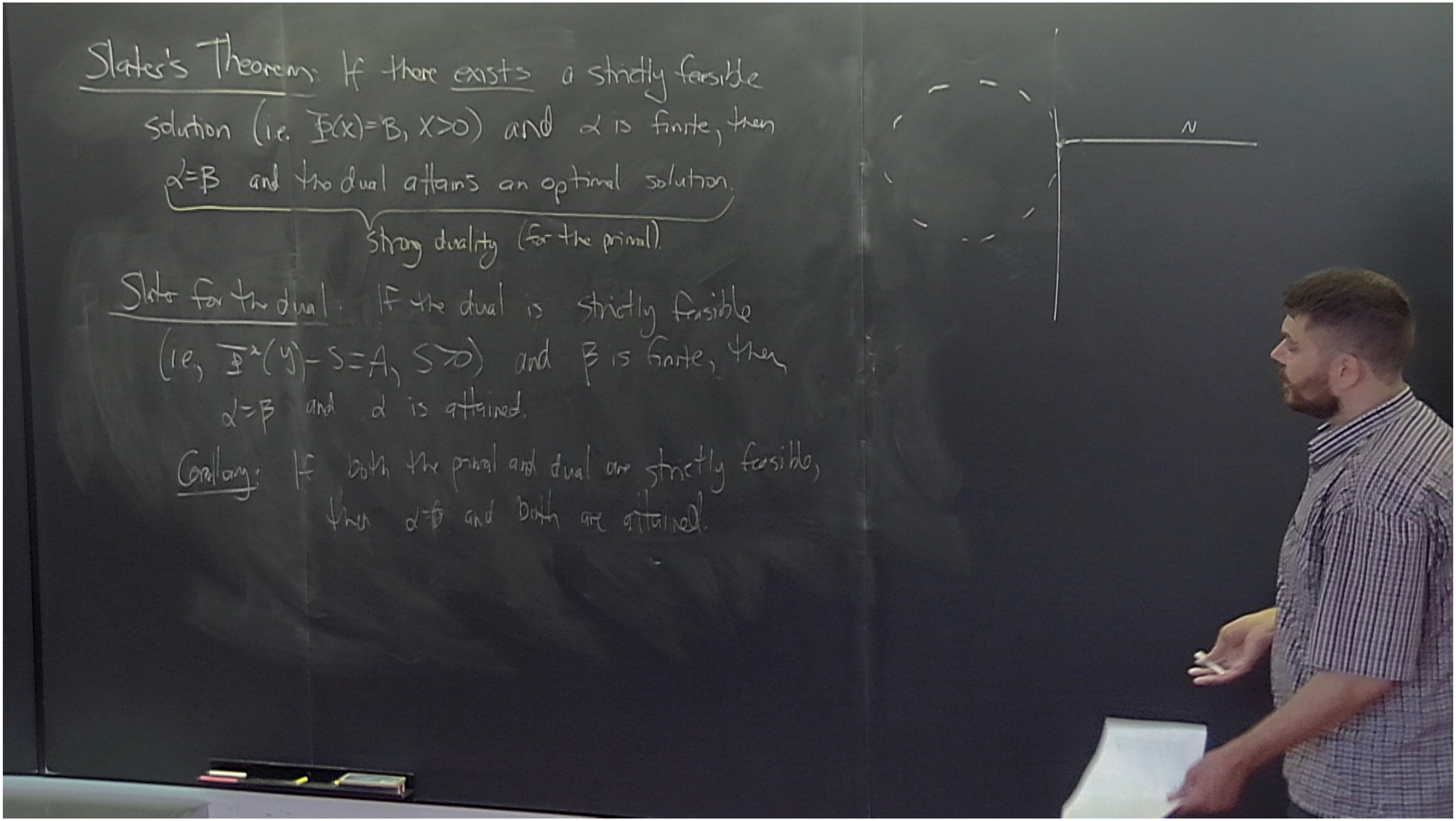
$N=\{(0,0,s)\in(\mathrm{Herm}(\mathcal{X}),Q,\mathbb{R}): s>\alpha\}$ not

Herm($\mathcal{Y}$)

Image of Φ

$Q=\{\Phi(T):T\in\mathrm{Herm}(\mathcal{X})\}\subseteq\mathrm{Herm}(\mathcal{Y})$

$B\in Q$


Slater's Theorem: If there exists a strictly feasible
solution (i.e. $\Phi(X)=B$, $X>0$) and α is finite, then
$\alpha=\beta$ and the dual attains an optimal solution.
Strong duality (for the primal)
Slater for the dual: If the dual is strictly feasible
(ie, $\Phi^*(Y)-S=A$, $S>0$) and β is finite, then
$\alpha=\beta$ and α is attained.
Corollary: If both the primal and dual are strictly feasible,
then $\alpha=\beta$ and both are attained.
N

Ex: The ground energy of a Hamiltonian H can be computed as

$$\alpha = \inf_{\substack{Tr(\rho)=1 \\ \rho \geq 0}} \langle \rho, H \rangle \leftarrow \text{energy}$$

$Tr(\rho)=1$, $\rho \geq 0$ } quantum states

Since $(S, B - \Phi(S), \langle A, S \rangle) \in M$

$S \in Herm(\mathcal{X})$

$\langle Y, B - \Phi(S) \rangle \leq C \leq 0$

$\Phi(X) = B$

$\langle Y, B \rangle \leq \langle Y, \Phi(S) \rangle$

$y = 0$

Contradiction.

$\square$

Case 1: λ

$\Lambda' =$

$(S, V, c) \in M$

Herm(X)

Since

$S \in Herm$

$\Rightarrow \Lambda'$

$B \leq$

defn of

$\Rightarrow \alpha = \beta$ &

dual optimal

Ex: The ground energy of a Hamiltonian H can be computed as

$$\alpha = \inf \langle \rho, H \rangle \leftarrow \text{energy}$$
$$\left.\begin{array}{l} Tr(\rho)=1 \\ \rho \geq 0 \end{array}\right\} \text{quantum states}$$

$\rho = \frac{1}{\dim} \mathbb{1}$ is strictly feasible.

The dual is

$$\beta = \sup t$$
$$t\mathbb{1} \leq H$$

$t < \lambda_{min}(H)$ is strictly feasible

$\rho = |\psi\rangle\langle\psi|$ $\quad H|\psi\rangle = \lambda_{min}(H)|\psi\rangle$

$\langle H, \rho \rangle = \lambda_{min}(H)$

Optimal t:

Case 1: λ

$\Lambda' =$

$(S, V, t) \in M$

Since

$S \in Herm$

$<$

$\Rightarrow \Lambda' -$

$\beta \leq <$

$\Rightarrow \alpha = \beta$ &

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Ex: The ground energy of a Hamiltonian H can be computed as

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$\langle H, \rho \rangle = \lambda_{min}(H)$

Optimal $t = \lambda_{min}(H)$

$$\alpha \leq \lambda_{min}(H)$$

$$\rotatebox[origin=c]{-90}{$\geq$}$$

$$\beta \geq \lambda_{min}(H)$$

Weak duality certifies optimality.

Optimal $t = \lambda_{min}(H)$

$$\alpha \leq \lambda_{min}(H)$$
$$\rotatebox{90}{$\leq$}$$
$$\beta \geq \lambda_{min}(H)$$

Weak duality certifies optimality

Complementary Slackness:

$$\beta - \alpha' = \langle X, S \rangle \geq 0$$

optimal solutions

If $\alpha = \beta$ then $\langle X, S \rangle = 0$

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$\alpha = \beta$

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Optimal $t = \lambda_{min}(H)$

$\alpha \leq \lambda_{min}(H)$

$\vee\!|$

$\beta \geq \lambda_{min}(H)$

Weak duality certifies optimality

Complementary Slackness:

$\beta - \alpha = \langle X, S \rangle \geq 0$

optimal solutions

If $\alpha = \beta$ then $\langle X, S \rangle = 0$

If $X, S \succeq 0$ and $\langle X, S \rangle = 0$

$\Longrightarrow XS = 0$ &

$SX = 0$

If $\alpha = \beta$ & X is primal opt.

(y, S) is dual opt.

$X \Phi^*(y) = XA$ & $\Phi^*(y) X = AX$

Slater's

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