

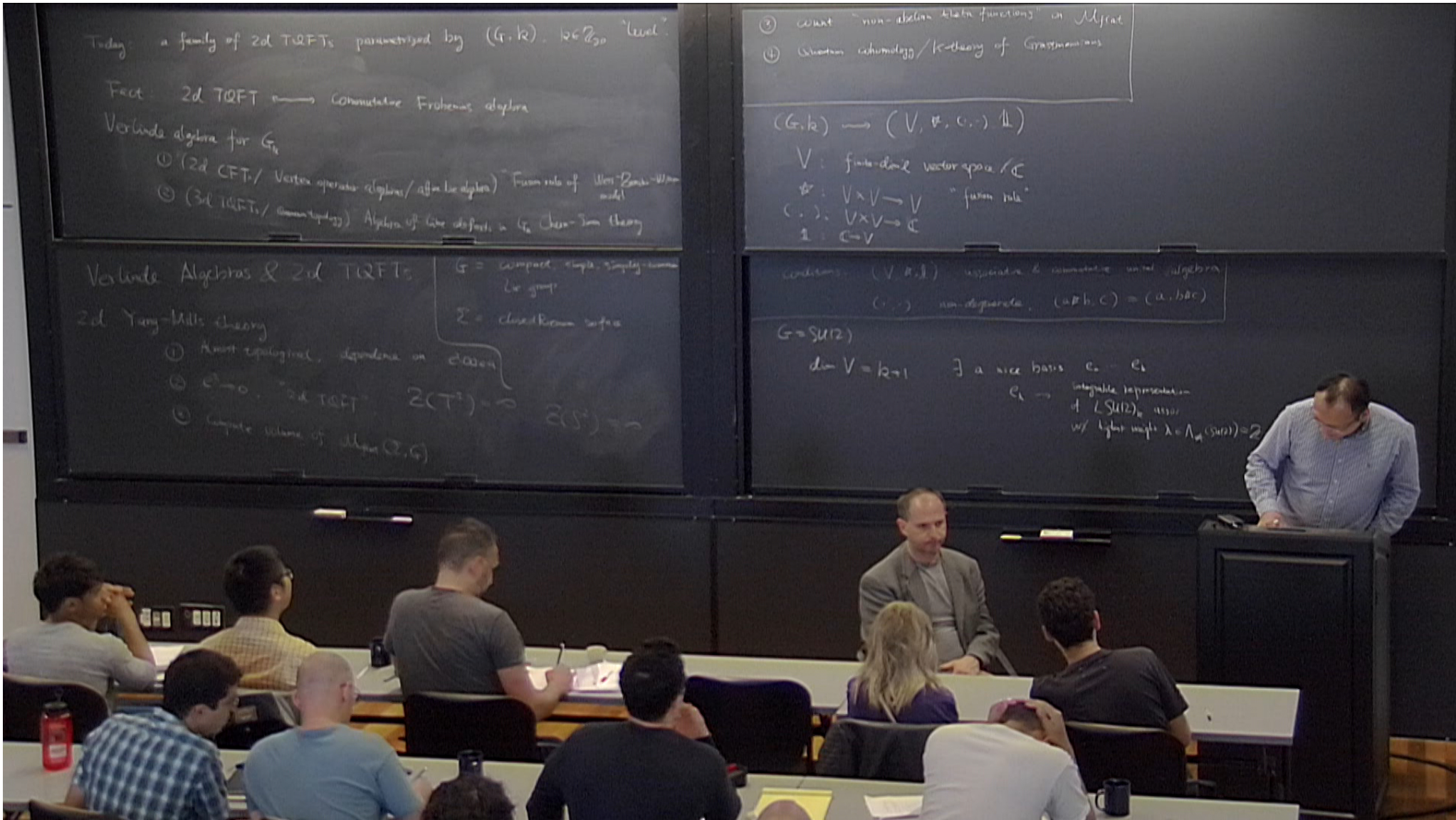
Title: TA Session: Verlinde algebra and 2d TQFT

Speakers: Du Pei

Collection: QFT for Mathematicians

Date: June 21, 2019 - 4:00 PM

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Today: a family of 2d TQFTs parametrized by $(G, k) \in \mathbb{Z}_{>0}$ "level".

Fact: 2d TQFT \iff commutative Frobenius algebra

Verdine algebra for G_k

- ① (2d CFT / Vertex operator algebras / affine Lie algebras) "Fusion rules of Wess-Zumino-Witten model"
- ② (3d TQFTs / Chern-Simons theory) Algebra of link invariants in G_k Chern-Simons theory

Verdine Algebras & 2d TQFTs

$G =$ compact, simple, simply-connected Lie group

$\Sigma =$ closed Riemann surface

2d Yang-Mills theory

- ① Almost topological, depends on class
- ② $\epsilon \rightarrow 0$, "2d TQFT" $Z(\mathbb{T}^2) \rightsquigarrow Z(\mathbb{S}^2) \rightsquigarrow$
- ③ Compute volume of $\mathcal{M}_{gen}(G, \Sigma)$

- ③ want "non-abelian Chern functions" in \mathcal{M}_{gen}
- ④ Witten cohomology / K-theory of Grassmannians

$$(G, k) \rightsquigarrow (V, \star, (\cdot, \cdot))$$

V : finite-dim vector space / \mathbb{C}
 $\star: V \times V \rightarrow V$ "fusion rule"
 $(\cdot, \cdot): V \times V \rightarrow \mathbb{C}$
 $1: \mathbb{C} \rightarrow V$

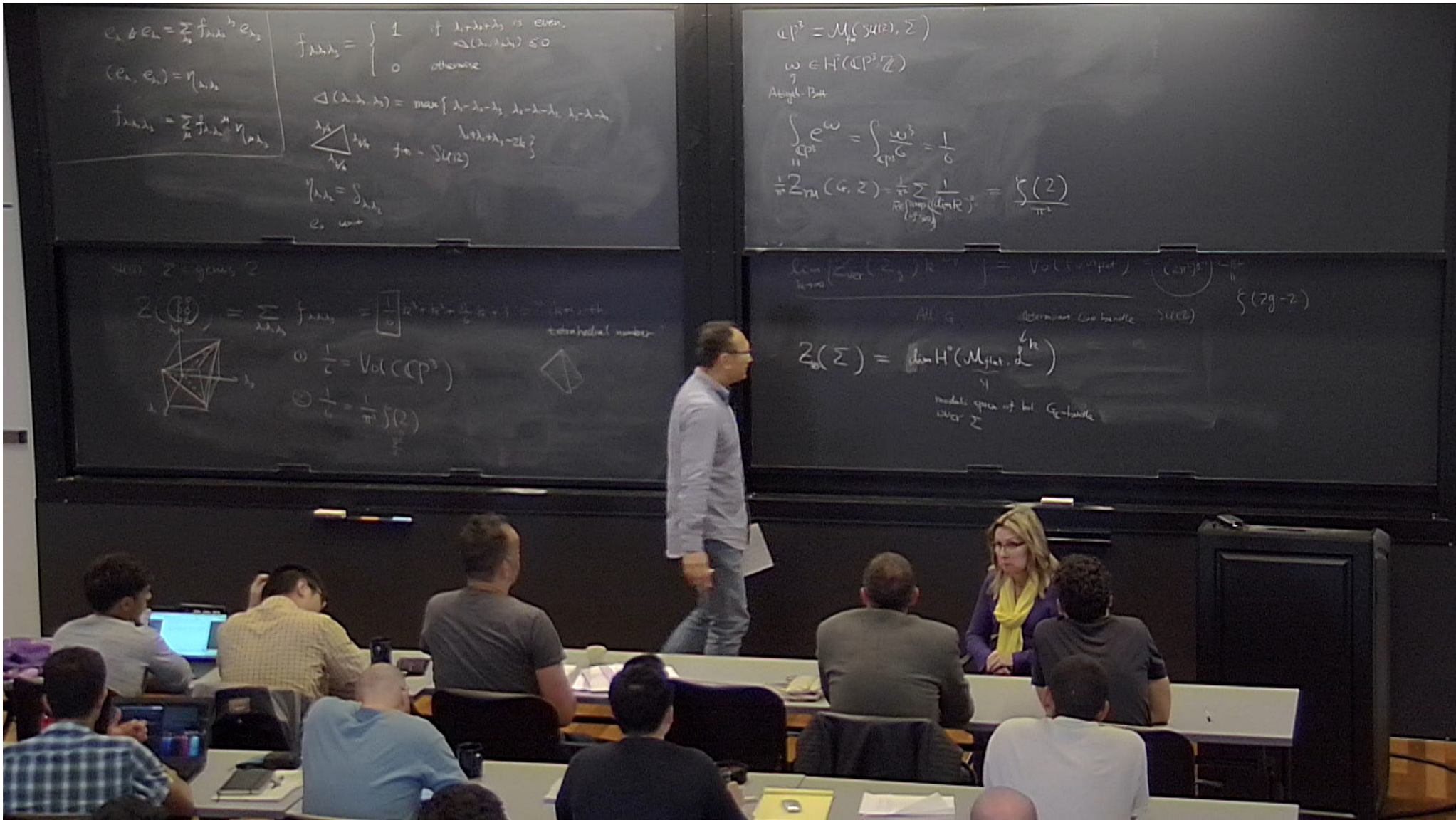
conditions: (V, \star, \cdot) associative & commutative unital algebra
 (\cdot, \cdot) non-degenerate, $(a \star b, c) = (a, b \star c)$

$$G = \text{SU}(2)$$

$$\dim V = k+1$$

\exists a nice basis e_0, \dots, e_k

$e_k \rightarrow$ integrable representation of $L\text{SU}(2)_k$ asso / highest weight $\lambda = \Lambda_{\text{cl}}(\text{SU}(2)) = 2$



$$e_i \otimes e_j = \sum_k f_{ijk} e_k$$

$$(e_i, e_j) = \eta_{ij}$$

$$f_{ijk} = \sum_l f_{ijkl} \eta_{kl}$$

$$f_{\lambda\mu\nu} = \begin{cases} 1 & \text{if } \lambda_1 + \lambda_2 + \lambda_3 \text{ is even,} \\ & \Delta(\lambda_1, \lambda_2, \lambda_3) \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Delta(\lambda_1, \lambda_2, \lambda_3) = \max\{\lambda_1 - \lambda_2 - \lambda_3, \lambda_2 - \lambda_1 - \lambda_3, \lambda_3 - \lambda_1 - \lambda_2\}$$

$$\Delta(\lambda_1, \lambda_2, \lambda_3) = \max\{\lambda_1 + \lambda_2 - 2\lambda_3, \lambda_1 + \lambda_3 - 2\lambda_2, \lambda_2 + \lambda_3 - 2\lambda_1\}$$

$$\eta_{ij} = \delta_{ij}$$

$$e_i \otimes e_j = \delta_{ij}$$

$$\mathbb{C}P^2 = \mathcal{M}_g(\mathcal{M}(\mathbb{R}^2), \Sigma)$$

$$\omega \in H^2(\mathbb{C}P^2, \mathbb{Z})$$

Atiyah-Bott

$$\int_{\mathbb{C}P^2} e^{\omega} = \int_{\mathbb{C}P^2} \frac{\omega^2}{2} = \frac{1}{6}$$

$$\frac{1}{\pi} Z_{\text{YM}}(G, \Sigma) = \frac{1}{\pi} \sum_{\text{Re}(s) > 2} \frac{1}{(s-2)^2} = \zeta(2)$$

4.1.1. Z -genus 2

$$Z(\mathbb{C}P^2) = \sum_{\lambda_1, \lambda_2, \lambda_3} f_{\lambda_1, \lambda_2, \lambda_3} = \left[\frac{1}{6} (2^3 + 1^3) + \frac{2}{3} (2 + 1) \right] = \frac{1}{6} (8 + 1 + 6 + 6) = \frac{21}{6} = \frac{7}{2}$$



$$\omega \otimes \frac{1}{2} = \text{Vol}(\mathbb{C}P^2)$$

$$\omega \otimes \frac{1}{2} = \frac{1}{\pi} \int \mathbb{R}^2$$

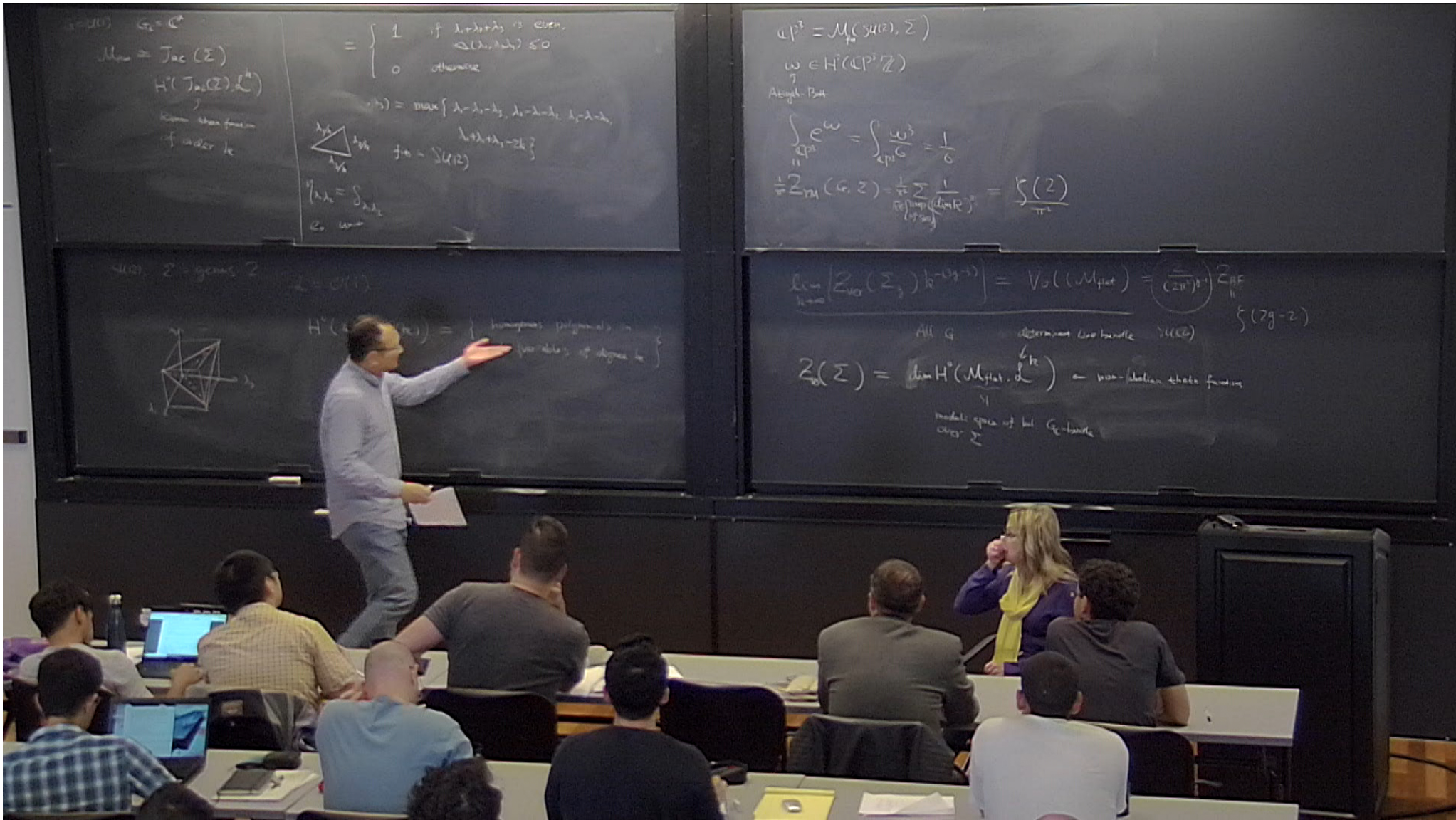


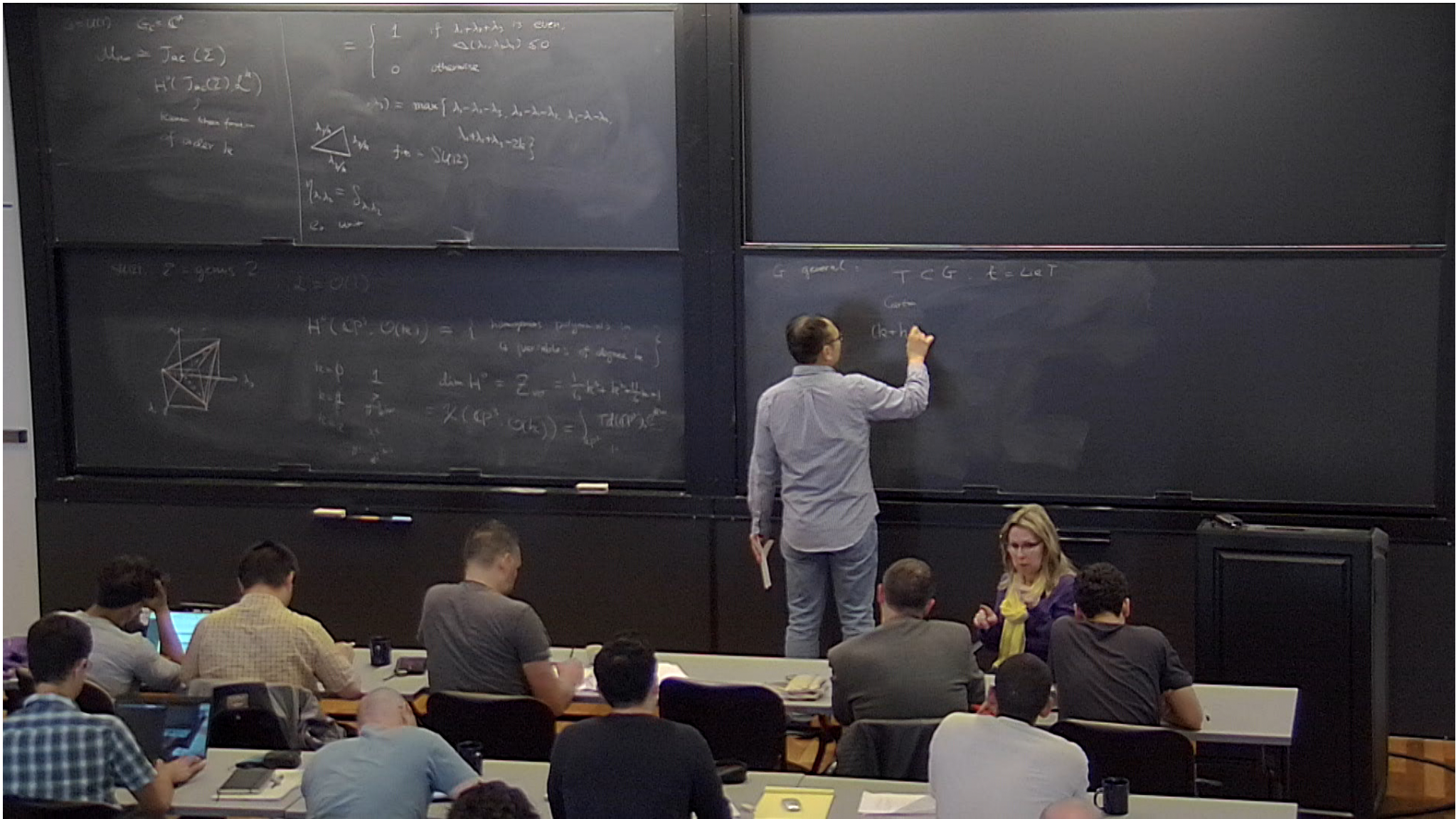
$$\lim_{g \rightarrow \infty} (Z_{\text{ver}}(Z_g))^k = \text{Vol}(\text{hyperboloid}) = \frac{(2\pi)^{2g-2}}{(2\pi)^{2g-2}} = 1$$

All G determinant line bundle $SU(2)$

$$Z_0(\Sigma) = \dim H^0(\mathcal{M}_{\text{flat}}(\Sigma, \mathbb{C}^k))$$

moduli space of hol. G -bundles over Σ





$S = U(1) \times G \subset \mathbb{C}^*$
 $\text{Map} \Rightarrow \text{Jac}(Z)$
 $H^1(\text{Jac}(Z), \mathbb{Z}^*)$
 Known from form of order k

$$= \begin{cases} 1 & \text{if } \lambda_1 + \lambda_2 + \lambda_3 \text{ is even,} \\ & \mathcal{Q}(\lambda_1, \lambda_2, \lambda_3) \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

$h) = \max \{ \lambda_1 - \lambda_2 - \lambda_3, \lambda_2 - \lambda_1 - \lambda_3, \lambda_1 - \lambda_2 - \lambda_3, \lambda_1 + \lambda_2 - 2\lambda_3 \}$
 $f_0 = S(1, 2)$

λ_1
 λ_2
 λ_3

$\lambda_1 = \sum \lambda_i$
 $\mathbb{Z} \rightarrow \mathbb{Z}$

HST: Z - genus 2
 $Z = O(1)$

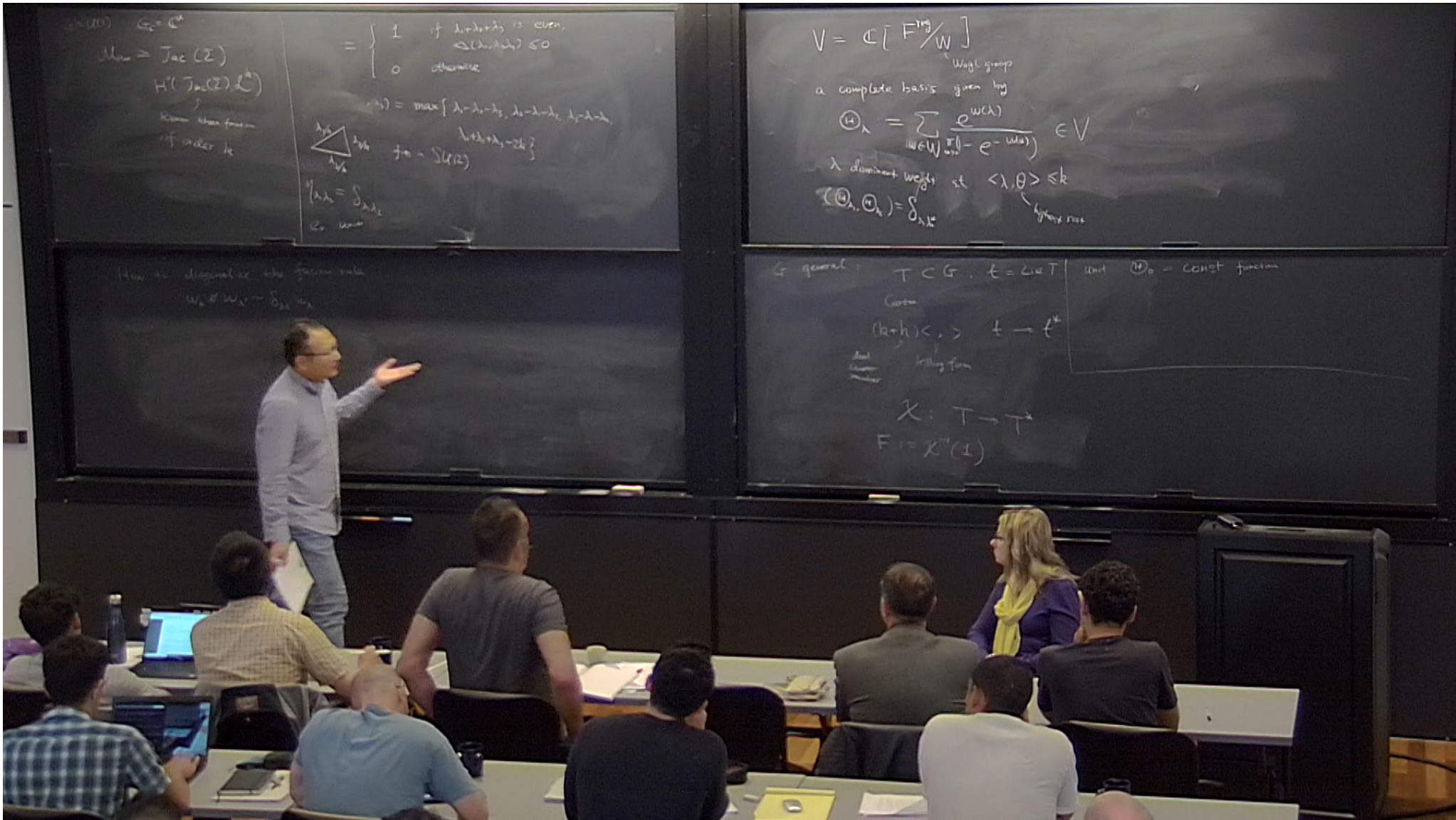
$H^0(\mathbb{C}P^1, \mathcal{O}(k)) = \{ \text{homogeneous polynomials in 4 variables of degree } k \}$

$k=0 \quad 1$
 $k=1 \quad 4$
 $k=2 \quad 10$

$\dim H^0 = \sum_{k=0}^{\infty} \binom{k+3}{3} = \frac{1}{6} k^3 + \frac{1}{2} k^2 + \frac{1}{2} k + 1$
 $= \chi(\mathbb{C}P^1, \mathcal{O}(k)) = \int_{\mathbb{C}P^1} c_1(\mathcal{O}(k))$

G general: $T \subset G, t = \text{Lie } T$

Cartan
 $(k+h)$



$\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}(n, \mathbb{R})$
 $\mathfrak{h}_{\mathbb{R}} \supseteq \text{Jac}(\mathbb{Z})$
 $H^1(\text{Jac}(\mathbb{Z}), \mathbb{Z}^n)$
 Root system form of order kn
 $\Delta = \{ \alpha_1, \alpha_2, \dots, \alpha_n \}$
 $\alpha_i = \delta_{i,1} - \delta_{i,2}$
 $\alpha_n = \delta_{n,1}$
 $\mathbb{R} \oplus \mathbb{R} \oplus \dots$
 $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$
 $\rho = \frac{1}{2} (\alpha_1 + \alpha_2 + \dots + \alpha_n)$
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How to diagonalize the form ρ
 $\rho = \frac{1}{2} (\alpha_1 + \alpha_2 + \dots + \alpha_n)$
 $\rho = \frac{1}{2} (\alpha_1 + \alpha_2 + \dots + \alpha_n)$

$V = \mathbb{C}[F/W]$
 Weyl group
 a complete basis given by
 $\Theta_{\lambda} = \sum_{w \in W} \frac{e^{w(\lambda)}}{w(\lambda) - \lambda} \in V$
 λ dominant weight st $\langle \lambda, \theta \rangle \leq k$
 $(\Theta_{\lambda}, \Theta_{\mu}) = \delta_{\lambda, \mu}$

G general: $T \subset G$, $t = \text{Lie } T$ unit $\Theta_0 = \text{const function}$
 Cartan
 $(k+h) \langle, \rangle$ $t \rightarrow t^k$
 Killing form
 $\chi: T \rightarrow T$
 $F := \chi^{-1}(1)$

