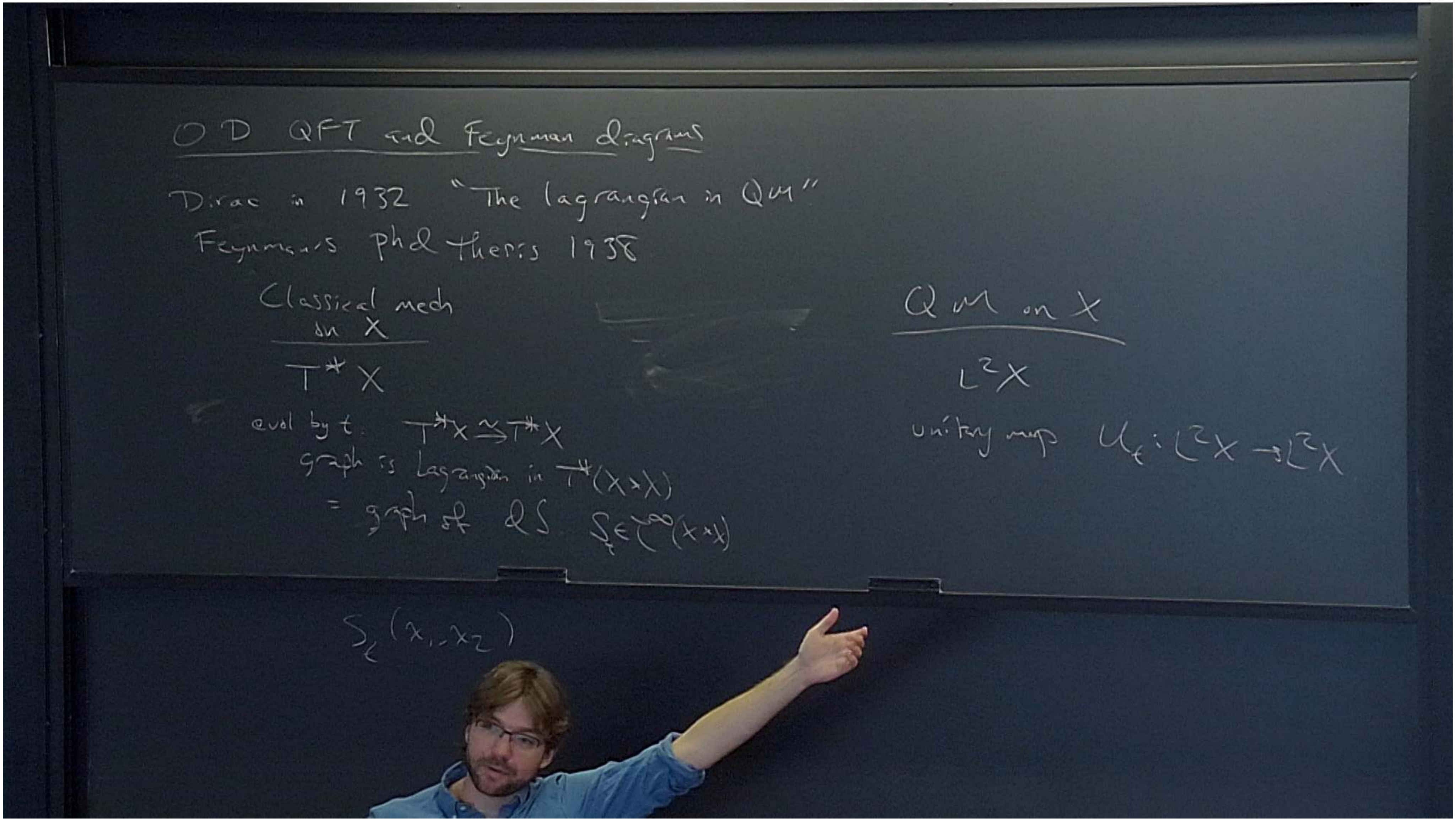
Title: TA Session: 0d QFT and Feynman diagrams

Speakers: Theo Johnson-Freyd

Collection: QFT for Mathematicians

Date: June 17, 2019 - 4:00 PM

URL: http://pirsa.org/19060036


0 D QFT and Feynman diagrams
Dirac in 1932 "The Lagrangian in QM"
Feynman's phd thesis 1938
Classical mech on X
T*X
evol by t: T*X → T*X
graph is Lagrangian in T*(X×X)
= graph of dS S ∈ C^∞(X×X)
QM on X
L²X
unitary map U_t: L²X → L²X
S_t(x_1, x_2)

Feynman's PhD Thesis 1938

Classical mech on X

T^*X

eval by t: $T^*X \xrightarrow{\sim} T^*X$

graph is Lagrangian in $T^*(X \times X)$

= graph of dS_t, $S_t \in C^\infty(X \times X)$

QM on X

$L^2 X$

unitary map $U_t : L^2 X \to L^2 X$

$$(U_t \psi)(x_1) = \int dx_0\, U_t(x_0, x_1)\, \psi(x_0)$$

$S_t(x_0, x_1)$ ← solves some version of H-J

$\exp\left(\frac{\sqrt{-1}}{\hbar} S_t\right) \underset{\hbar \to 0}{\simeq} U_t(x_0, x_1)$ solve Schrodinger eqn.

$$\| \quad \underset{\text{path } \varphi : [0,t] \to X \text{ s.t. } 0 \mapsto x_0,\ t \mapsto x_1}{\min} \overbrace{S}^{\text{action}}(\varphi)$$

?

$$S_{t_1+t_2}(x_0, x_2) = \min_{x_1}\left(S_{t_1}(x_0, x_1) + S_{t_2}(x_1, x_2)\right)$$

$$U_{t_1+t_2}(x_0, x_2) = \int_{x_1} U_{t_1}(x_0, x_1)\, U_{t_2}(x_1, x_2)\, dx_1.$$

Feynman's PhD Thesis 1938

Classical mech on X

T^*X

evol by t: $T^*X \xrightarrow{\sim} T^*X$

graph is Lagrangian in $T^*(X\times X)$

= graph of dS, $S_t \in C^\infty(X\times X)$

QM on X

L^2X

unitary map $U_t : L^2X \to L^2X$

$$(U_t\psi)(x_1) = \int dx_0\, U_t(x_0,x_1)\psi(x_0)$$

$S_t(x_0,x_1)$

↑ solves some version of H-J

$$\exp\left(\frac{\sqrt{-1}}{\hbar} S_t\right) \underset{\hbar\to 0}{\approx} U_t(x_0,x_1)$$ solve Schrödinger eqn.

$\parallel$

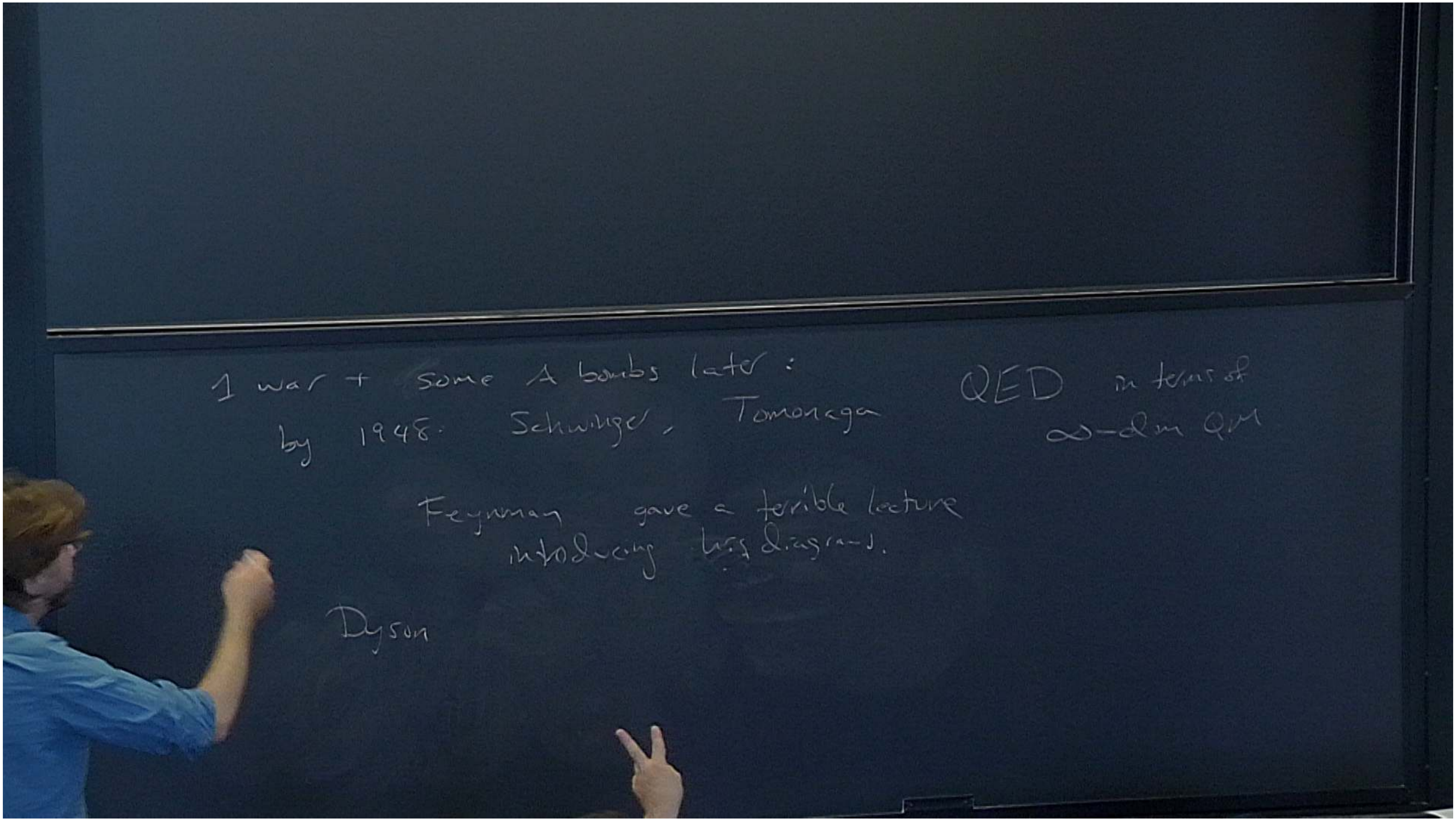
$$\min_{P\left[\substack{\text{paths } \varphi:[0,t]\to X \\ \text{st. } 0\mapsto x_0 \\ t\mapsto x_1}\right]} \overbrace{S}^{\text{action}}(\varphi)$$

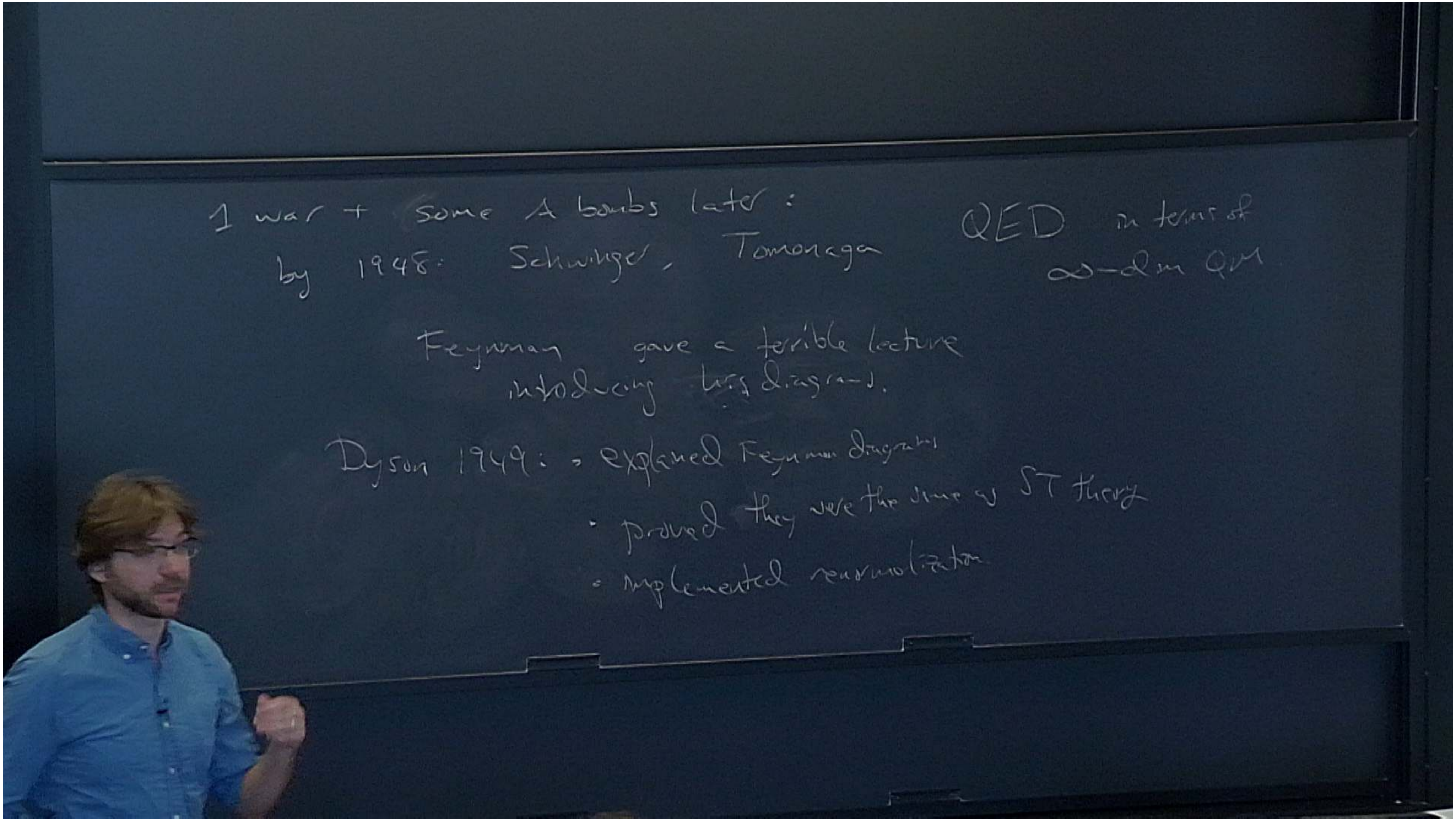
$\min = \lim_{\hbar\to 0} \int$ "Stationary phase"

$$U_t(x_0,x_1) = \int_P \exp\left(\frac{\sqrt{-1}}{\hbar} S(\varphi)\right) d\varphi$$

$$S_{t_1+t_2}(x_0,x_2) = \min_{x_1}\left(S_{t_1}(x_0,x_1) + S_{t_2}(x_1,x_2)\right)$$

$$U_{t_1+t_2}(x_0,x_2) = \int_{x_1} U_{t_1}(x_0,x_1)\,U_{t_2}(x_1,x_2)\,dx_1.$$


1 war + some A bombs later :
by 1948. Schwinger, Tomonaga
QED in terms of
∞-dim QM
Feynman gave a terrible lecture
introducing his diagrams.
Dyson


1 war + some A bombs later:
by 1948: Schwinger, Tomonaga
QED in terms of
∞-dim QM
Feynman gave a terrible lecture
introducing his diagrams.
Dyson 1949: • explained Feynman diagrams
• proved they were the same as ST theory
• implemented renormalization

Dyson 1949: • explained Feynman diagrams

• proved they were the same as ST theory

• implemented renormalization

Goal: compute

$$\langle f \rangle = \frac{1}{Z} \int_{X = \mathbb{R}^n} f(x) \exp\left(\frac{\sqrt{-1}}{\hbar} S(x) \right)$$

Stationary phase. in the

$$Z = \int \exp\left(\frac{\sqrt{-1}}{\hbar} S \right)$$

min S occurs at $x = 0$

Dyson 1949: • explained Feynman diagrams

• proved they were the same as ST theory

• implemented renormalization

$\mathcal{O}(\mathbb{R}^n) = \mathbb{C}[[x_1, \dots, x_n]]$

Goal: compute

$$\langle f \rangle = \frac{1}{Z} \int_{X = \mathbb{R}^n} f(x) \exp\left(\frac{\sqrt{-1}}{\hbar} S(x) \right)$$

$$Z = \int \exp\left(\frac{\sqrt{-1}}{\hbar} S \right)$$

Stationary phase: in the $\hbar \to 0$ limit, $\int$ is supported on a formal nbhd of 0.

$\exists$ Taylor series expansion

min S occurs at $x = 0$

free. Phil called "Q"

$$S(x) = \frac{1}{2} a_{ij} x_i x_j + b(x)$$

cubic and higher "S_{int}"

Ways to compute $\int S$:

· cleverly choose coords ("u-sub")

$\longrightarrow$ · integration by parts. Pick n-many functions $g_1, \dots, g_n$

$$0 = \frac{1}{Z}\int \frac{\partial}{\partial x_i}\left(g_i(x) \cdot \exp\left(\frac{\sqrt{-1}}{\hbar} S(x)\right)\right)$$

$$= \left\langle \frac{\partial g_i}{\partial x_i} \right\rangle + \left\langle g_i \frac{\sqrt{-1}}{\hbar} \frac{\partial S}{\partial x_i}\right\rangle$$

Ways to compute $\int$s:

- cleverly choose coords ("u-sub")
- $\longrightarrow$ integration by parts. Pick n-many functions $g_1, \dots, g_n$

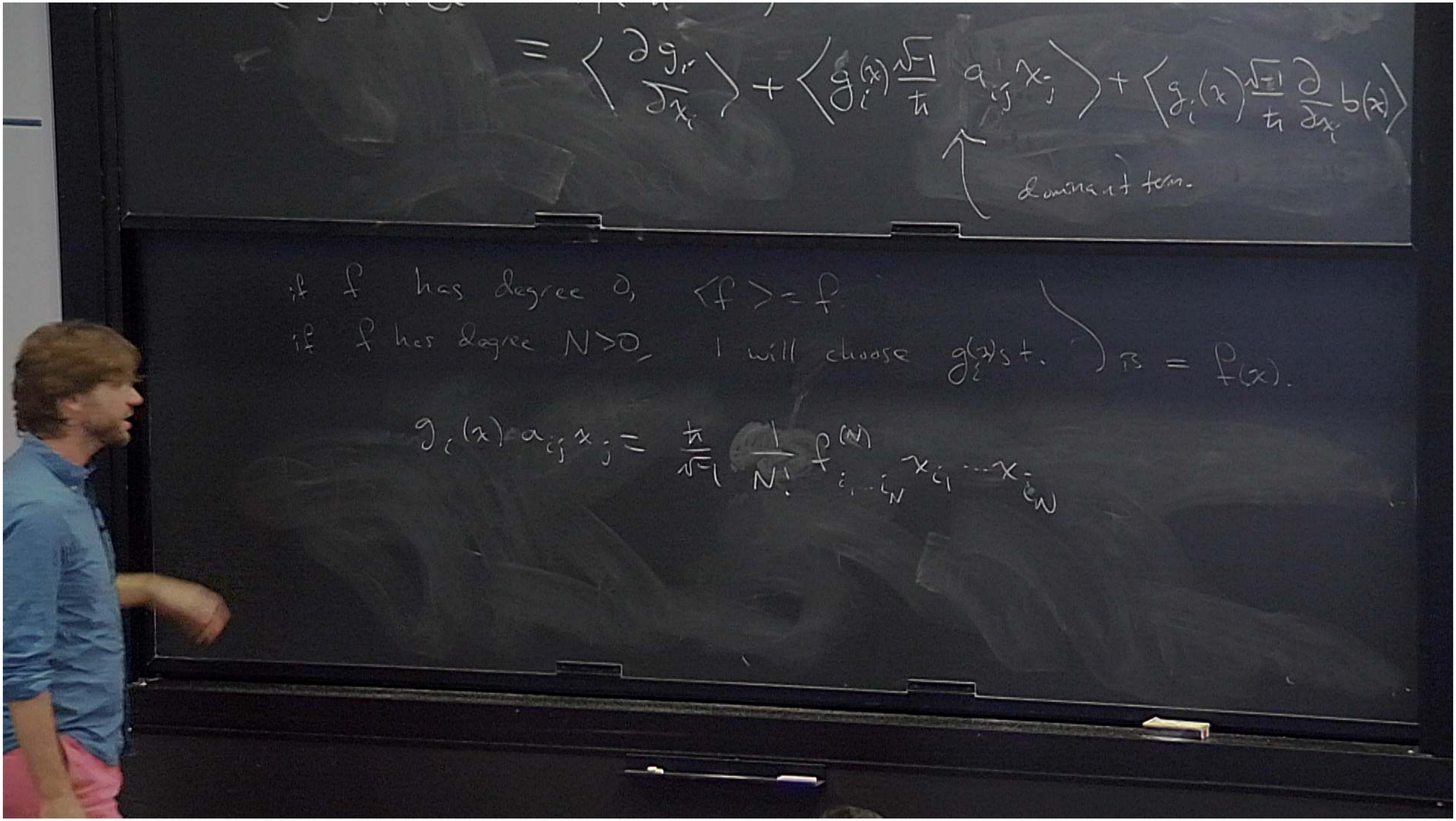
$$0 = \frac{1}{Z}\int \frac{\partial}{\partial x_i}\left(g_i(x) \cdot \exp\left(\frac{\sqrt{-1}}{\hbar} S(x)\right)\right)$$

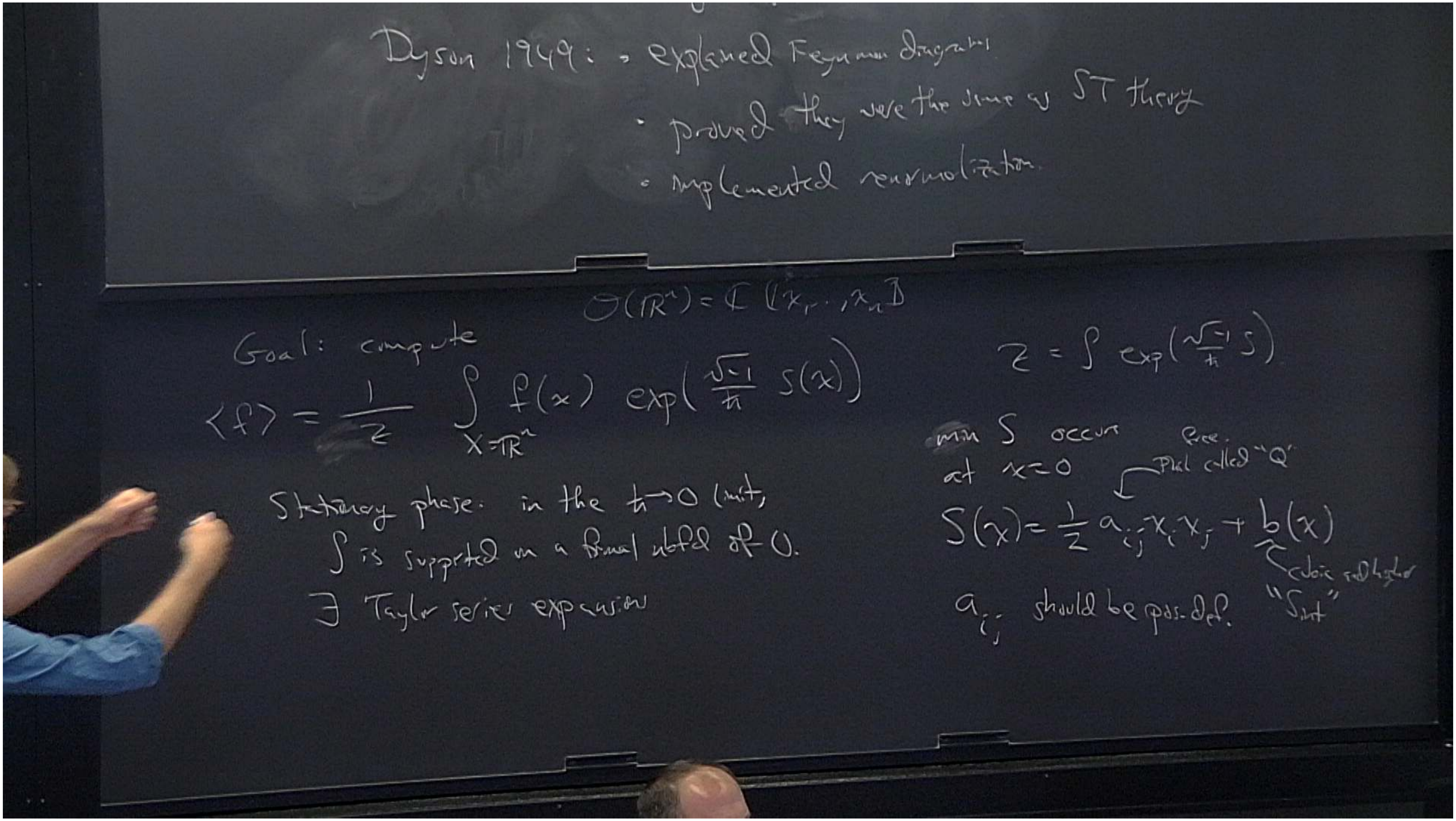
$$= \left\langle \frac{\partial g_i}{\partial x_i} \right\rangle + \left\langle g_i(x) \frac{\sqrt{-1}}{\hbar} a_{ij} x_j \right\rangle + \left\langle g_i(x) \frac{\sqrt{-1}}{\hbar} \frac{\partial}{\partial x_i} b(x) \right\rangle$$

$$= \left\langle \frac{\partial g_i}{\partial x_i} \right\rangle + \left\langle g_i(x) \frac{\sqrt{-1}}{\hbar} a_{ij} x_j \right\rangle + \left\langle g_i(x) \frac{\sqrt{-1}}{\hbar} \frac{\partial}{\partial x_i} b(x) \right\rangle$$

if f has degree 0, $\langle f \rangle = f$.

if f has degree $N>0$, I will choose $g_i(x)$ s.t. $\big)_{B} = f$.


$= \left\langle \frac{\partial g_i}{\partial x_i} \right\rangle + \left\langle g_i(x) \frac{\sqrt{-1}}{\hbar} a_{ij} x_j \right\rangle + \left\langle g_i(x) \frac{\sqrt{-1}}{\hbar} \frac{\partial}{\partial x_i} b(x) \right\rangle$
dominant term.
if f has degree 0, $\langle f \rangle = f$
if f has degree $N>0$, I will choose $g_i(x)$ s.t. $) B = f(x)$.
$g_i(x) a_{ij} x_j = \frac{\hbar}{\sqrt{-1}} \frac{1}{N!} f^{(N)}_{i_1 \cdots i_N} x_{i_1} \cdots x_{i_N}$


Dyson 1949: • explained Feynman diagrams
• proved they were the same as ST theory
• implemented renormalization.
$\mathcal{O}(\mathbb{R}^n) = \mathbb{C}[[x_1, \dots, x_n]]$
Goal: compute
$\langle f \rangle = \frac{1}{Z} \int_{X=\mathbb{R}^n} f(x) \exp\left(\frac{\sqrt{-1}}{\hbar} S(x)\right)$
Stationary phase: in the $\hbar \to 0$ limit,
$\int$ is supported on a formal nbhd of 0.
$\exists$ Taylor series expansion
$Z = \int \exp\left(\frac{\sqrt{-1}}{\hbar} S\right)$
min S occurs at $x=0$
free. Phil called "Q"
$S(x) = \frac{1}{2} a_{ij} x_i x_j + b(x)$
cubic and higher
a_{ij} should be pos. def.
"S_{int}"

$$= \left\langle \frac{\partial g_i}{\partial x_i} \right\rangle + \left\langle g_i(x) \frac{\sqrt{-1}}{\hbar} a_{ij} x_j \right\rangle + \left\langle g_i(x) \frac{\sqrt{-1}}{\hbar} \frac{\partial}{\partial x_i} b(x) \right\rangle$$

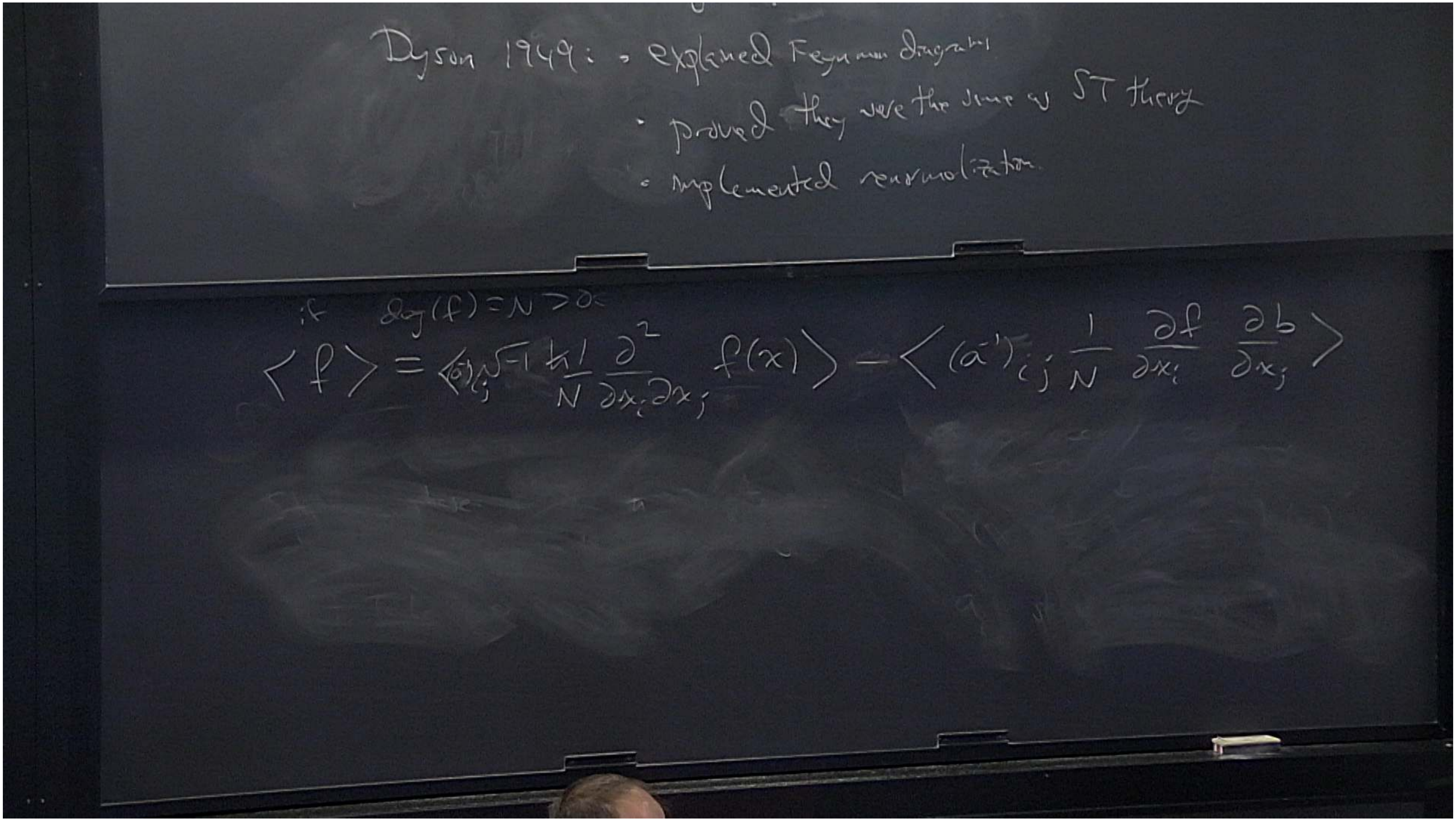
dominant term.

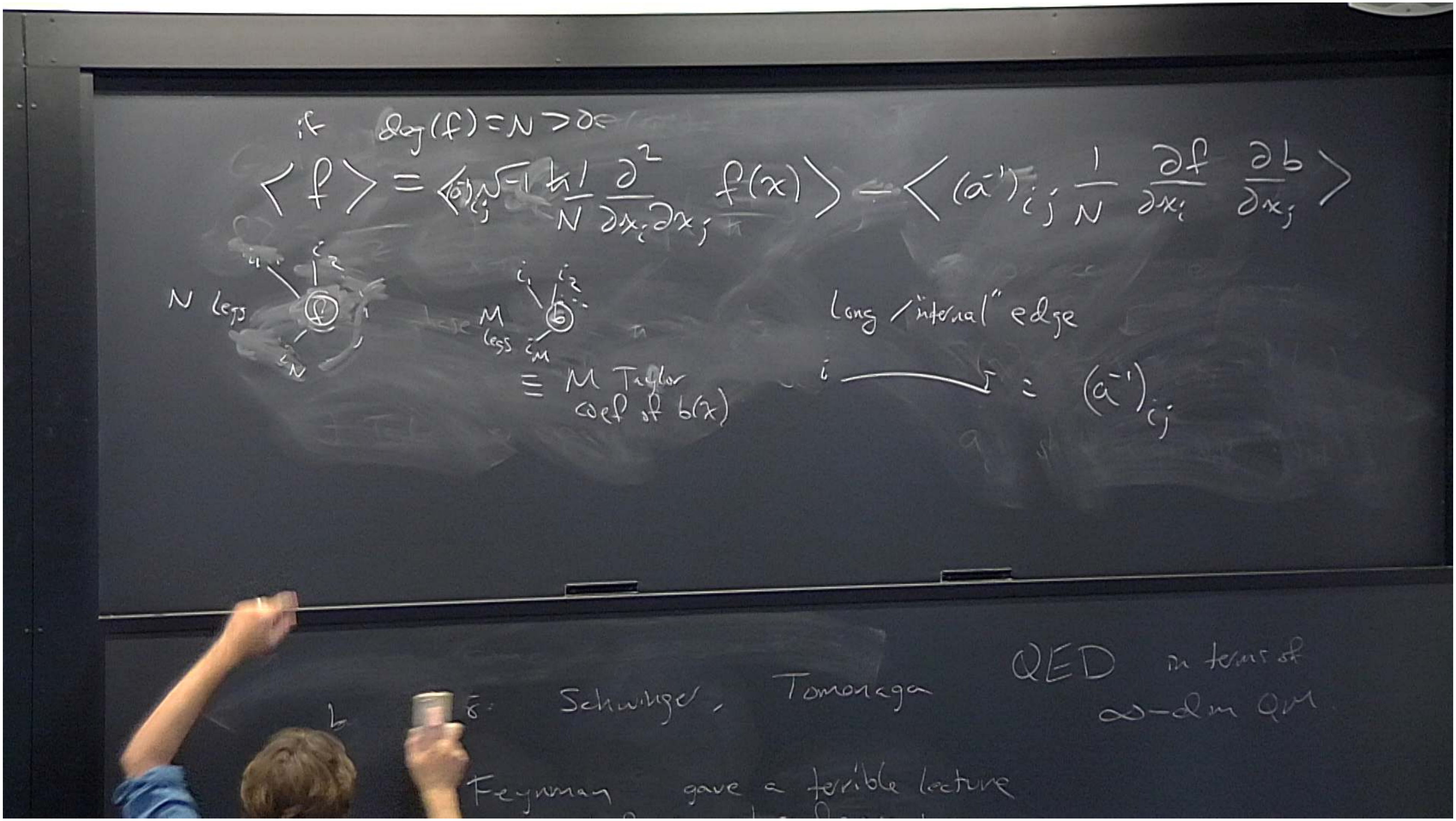
if f has degree 0, $\langle f \rangle = f$.

if f has degree $N > 0$, I will choose $g_i(x)$ s.t. $B = f(x)$.

$$g_i(x)\, a_{ij} x_j = \frac{\hbar}{\sqrt{-1}} \frac{1}{N!} f^{(N)}_{i_1 \ldots i_N} x_{i_1} \cdots x_{i_N}$$

$$g_i(x) = (a^{-1})_{ij} \frac{1}{N} \frac{\partial f}{\partial x_j}$$


Dyson 1949: • explained Feynman diagrams
• proved they were the same as ST theory
• implemented renormalization
if $\deg(f) = N > 0$
$\langle f \rangle = \langle (a^{-1})_{ij} \sqrt{-1}\, \hbar / N \frac{\partial^2}{\partial x_i \partial x_j} f(x) \rangle - \langle (a^{-1})_{ij} \frac{1}{N} \frac{\partial f}{\partial x_i} \frac{\partial b}{\partial x_j} \rangle$

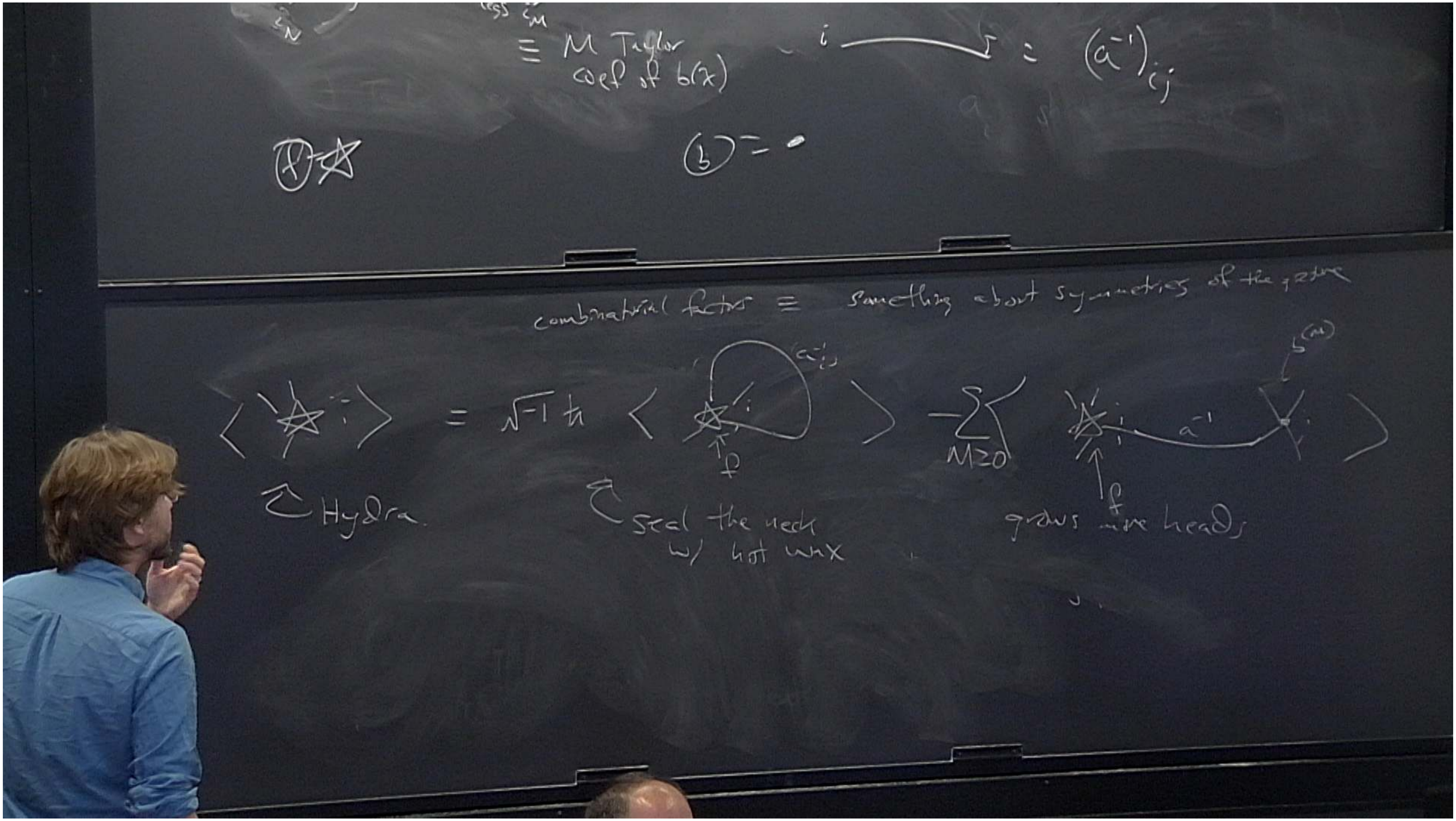

if $\deg(f) = N > 0$
$\langle f \rangle = \langle (a^{-1})_{ij} \sqrt{-1}\, \hbar \frac{1}{N} \frac{\partial^2}{\partial x_i \partial x_j} f(x) \rangle - \langle (a^{-1})_{ij} \frac{1}{N} \frac{\partial f}{\partial x_i} \frac{\partial b}{\partial x_j} \rangle$
N legs
M legs
$= M$ Taylor coef of $b(x)$
Long / "internal" edge
$= (a^{-1})_{ij}$
QED in terms of ∞-dim QM
Schwinger, Tomonaga
Feynman gave a terrible lecture

$\equiv M$ Taylor coef of $b(x)$

$i \longrightarrow j \; : \; (a^{-1})_{ij}$

(f) = ★

(b) = •

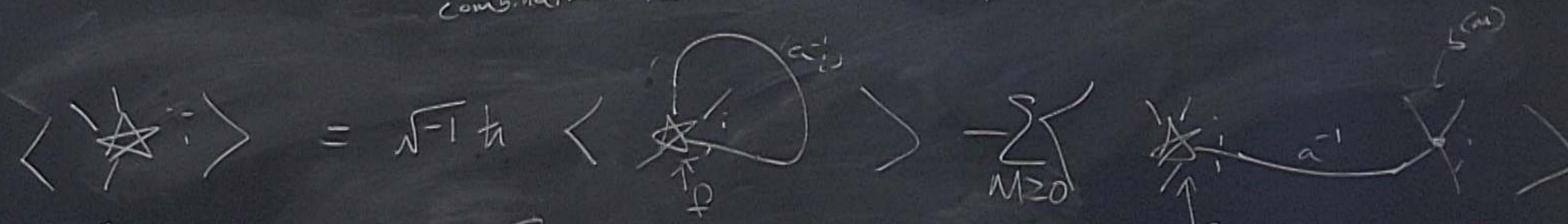


≡ M Taylor coef of b(x)
combinatorial factor ≡ something about symmetries of the graph
Hydra
seal the neck w/ hot wax
grows more heads
M≥0

$\equiv M$ Taylor coef of $b(x)$

i —— j $= (a^{-1})_{ij}$

combinatorial factors $\equiv$ something about symmetries of the graph

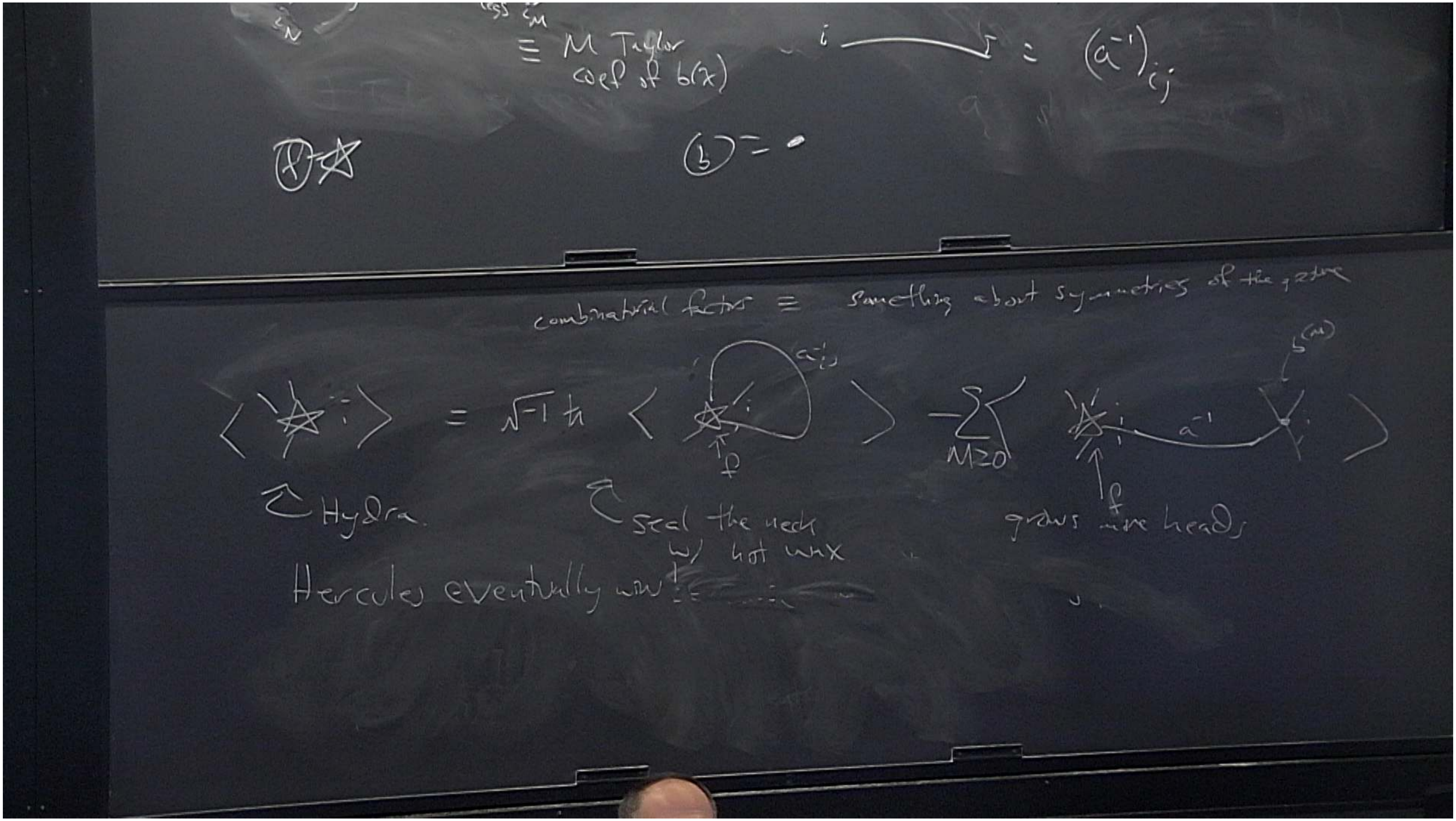


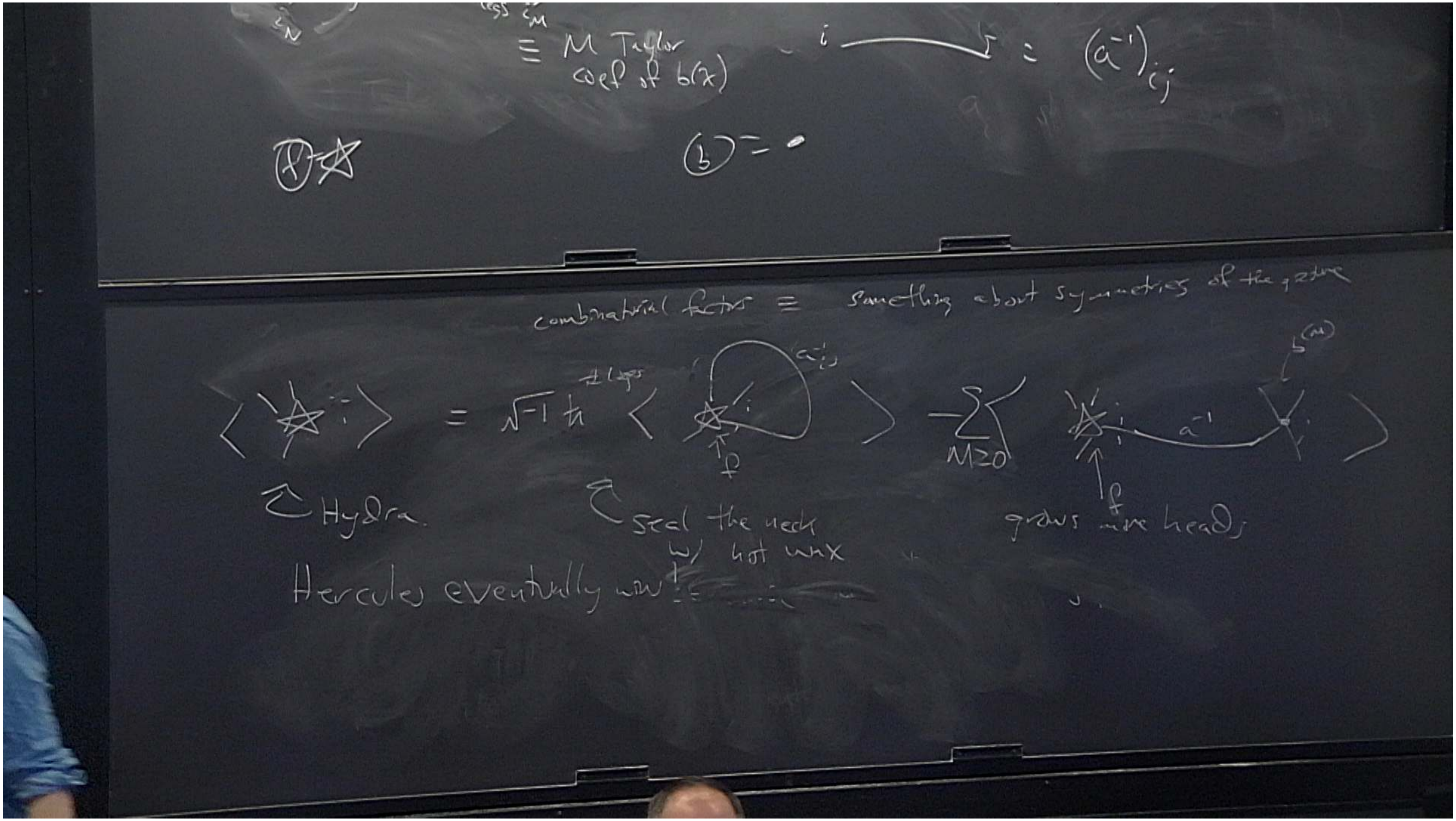
Hydra

seal the neck w/ hot wax

grows more heads

Hercules eventually win

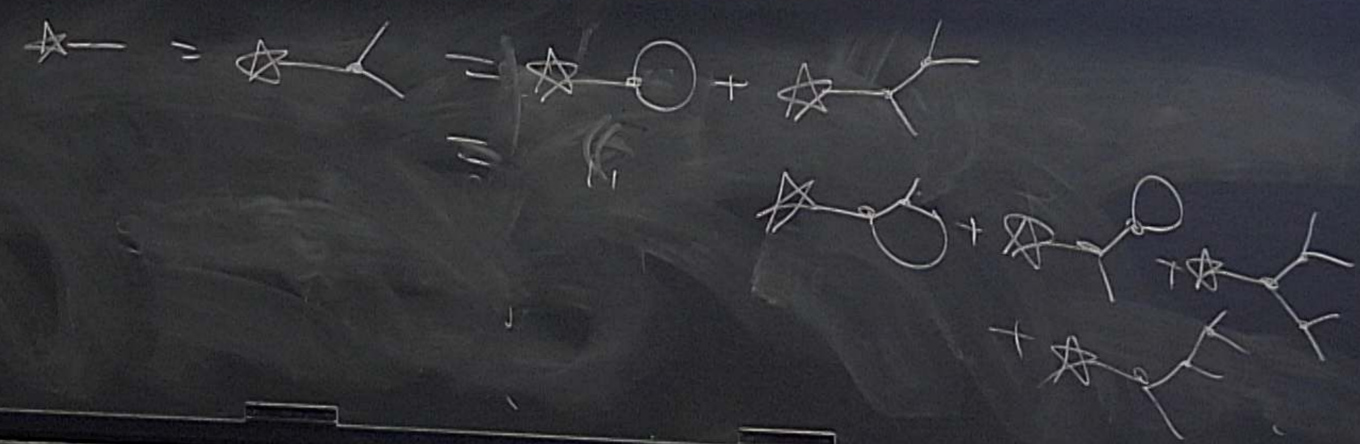

≡ M Taylor coef of b(x)
(a⁻¹)ij
combinatorial factor ≡ something about symmetries of the graph
= √-1 ħ
M≥0
Hydra
seal the neck w/ hot wax
grows more heads
Hercules eventually wins!

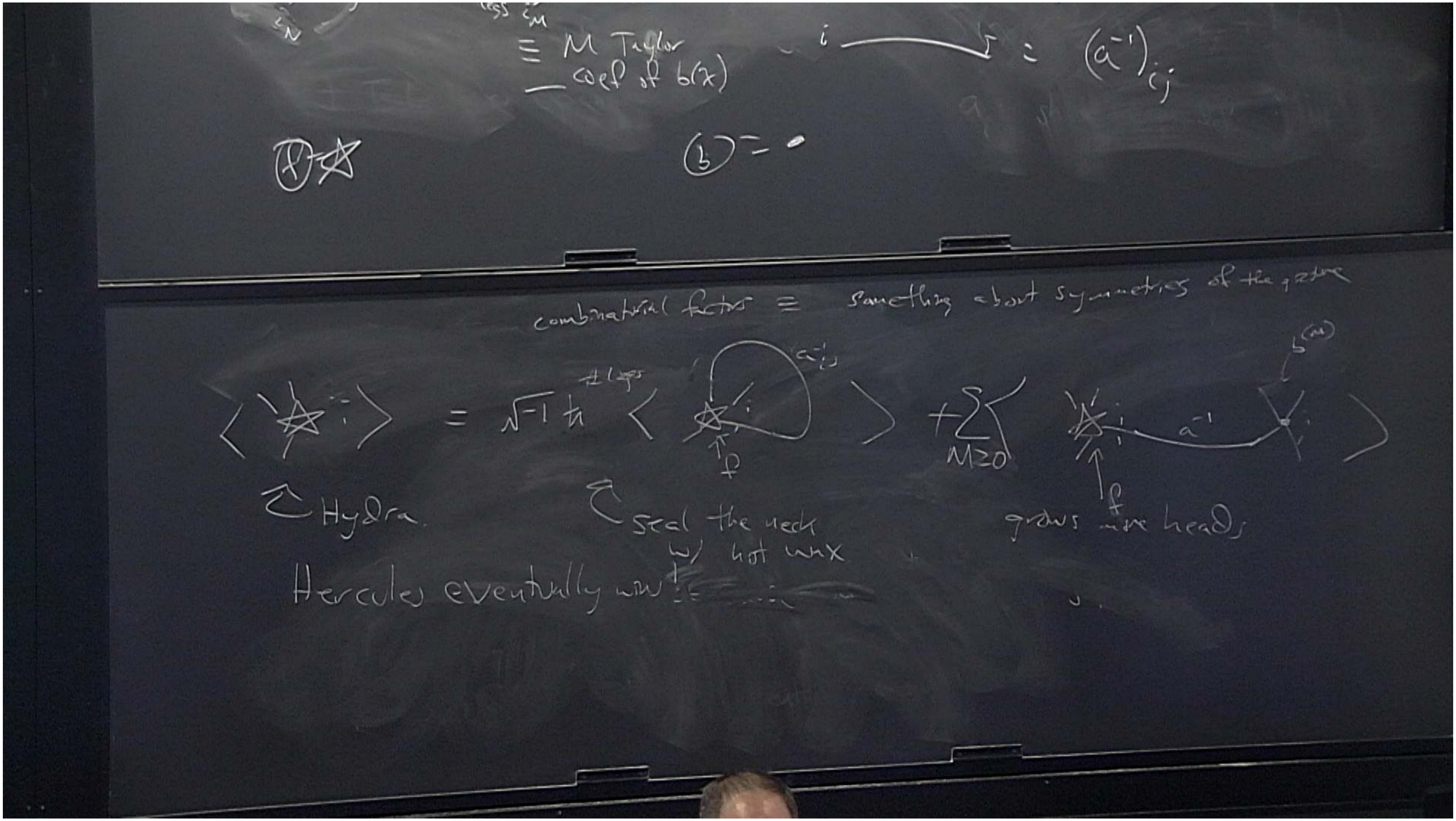

≡ M Taylor coef of b(x)
combinatorial factor ≡ something about symmetries of the graph
loops
M≥0
Hydra
seal the neck w/ hot wax
grows more heads
Hercules eventually wins!

$$= \left\langle \frac{\partial g_i}{\partial x_i} \right\rangle + \left\langle g_i(x) \frac{\sqrt{-1}}{\hbar} a_{ij} x_j \right\rangle + \left\langle g_i(x) \frac{\sqrt{-1}}{\hbar} \frac{\partial}{\partial x_i} b(x) \right\rangle$$

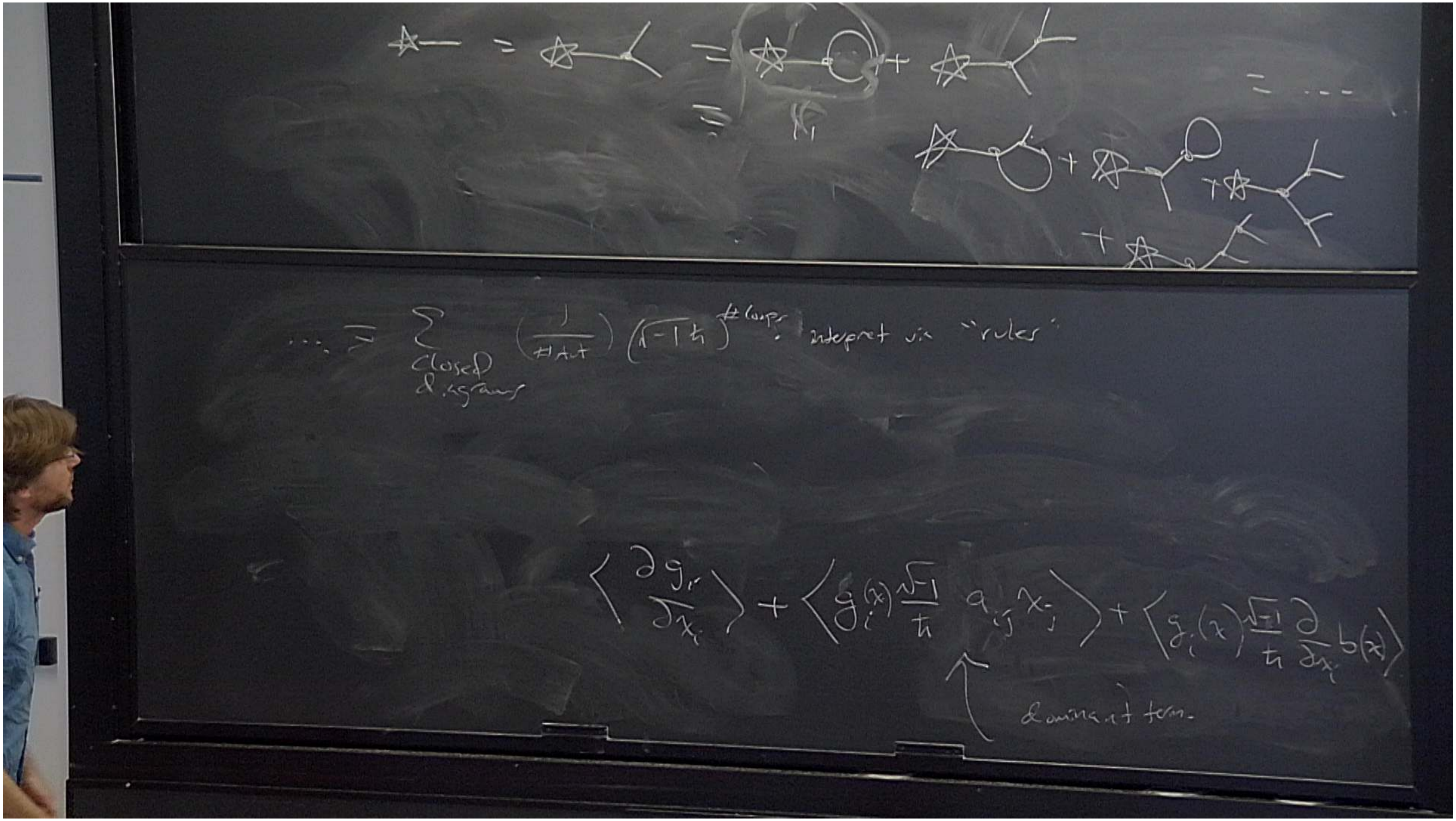
dominant term.

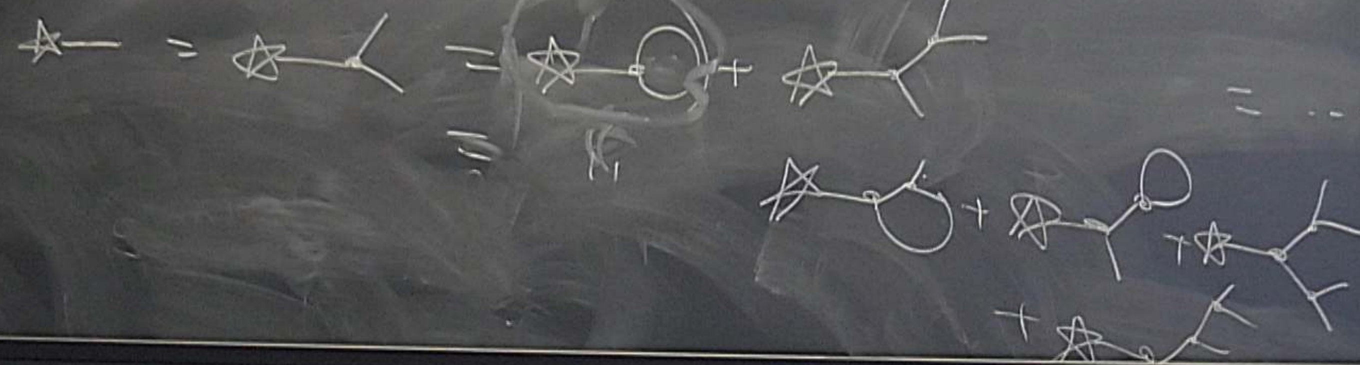
ex. f is linear, b is cubic.





$\equiv$ M Taylor coef of $b(x)$
i ——— j : $(a^{-1})_{ij}$
combinatorial factors $\equiv$ something about symmetries of the graph
$= \sqrt{-1}\,\hbar^{\#\text{loops}}$
$+\sum_{M\geq 0}$
a^{-1}
$b^{(M)}$
f
Hydra
seal the neck w/ hot wax
grows more heads
Hercules eventually wins!


... = $\sum_{\text{closed diagrams}} \left(\frac{1}{\# Aut}\right) (\sqrt{-1}\hbar)^{\# loops}$: interpret via "rules"
$\left\langle \frac{\partial g_i}{\partial x_i} \right\rangle + \left\langle g_i(x) \frac{\sqrt{-1}}{\hbar} a_{ij} x_j \right\rangle + \left\langle g_i(x) \frac{\sqrt{-1}}{\hbar} \frac{\partial}{\partial x_i} b(x) \right\rangle$
dominant term.

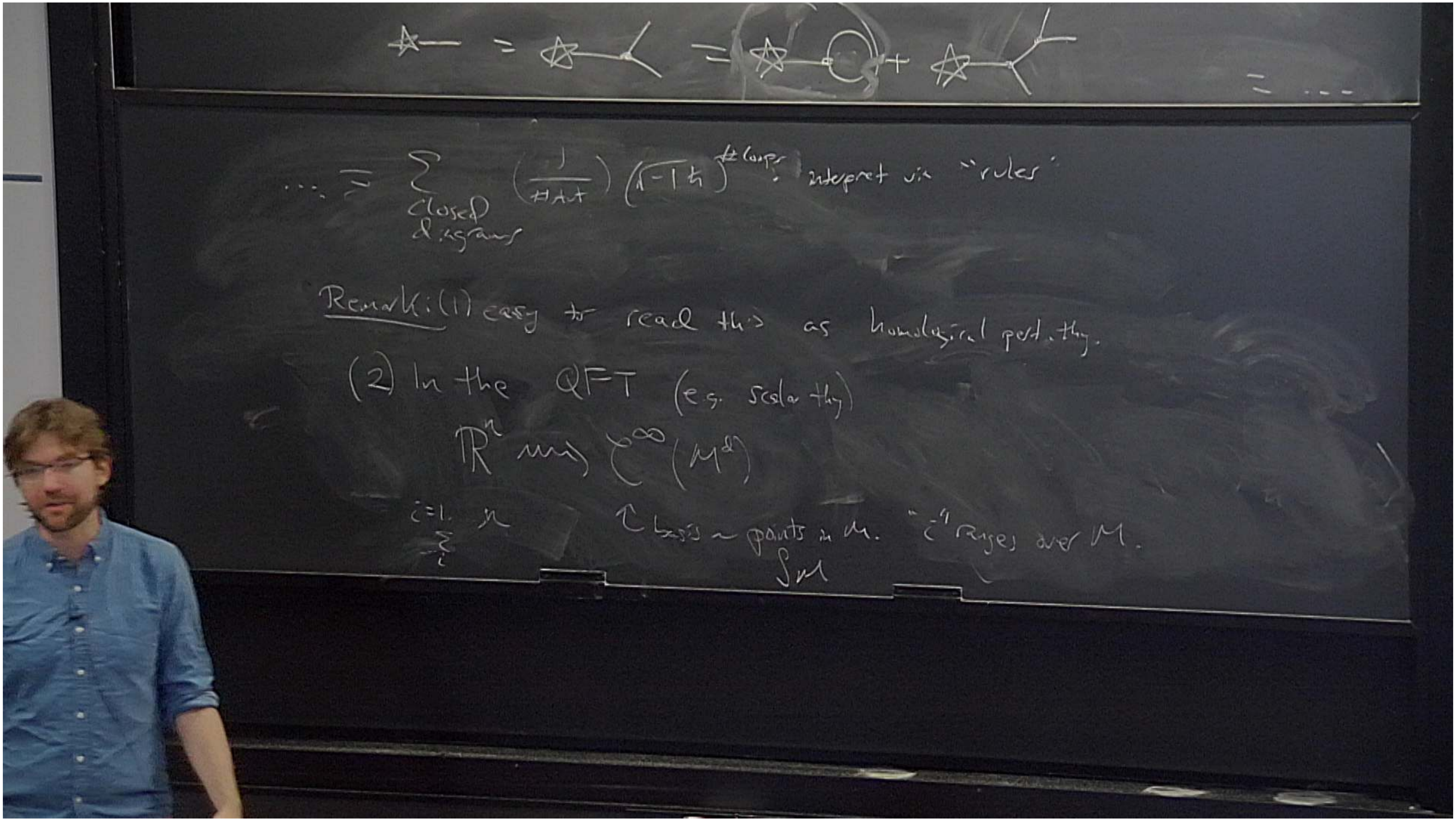


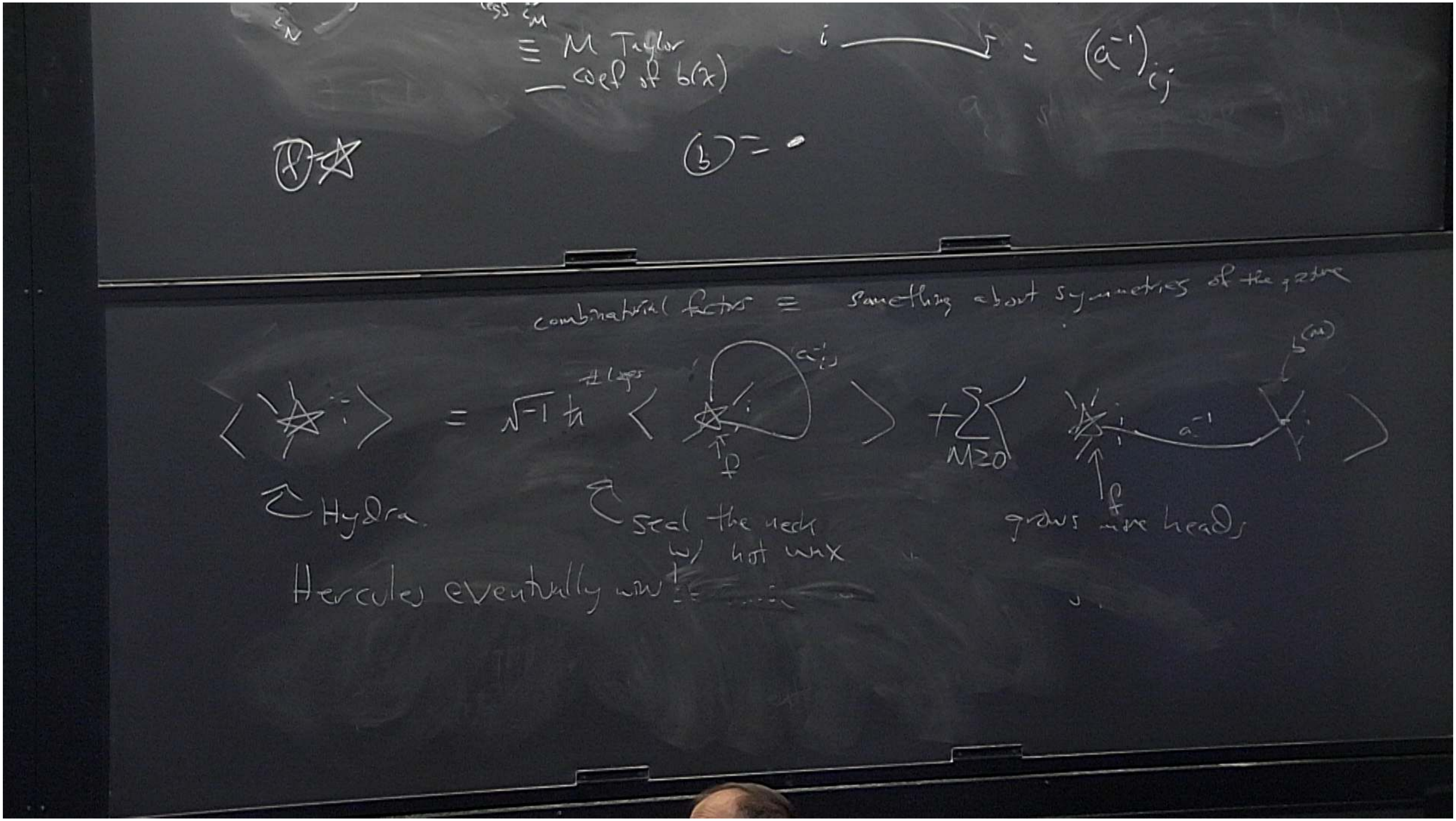
$$\cdots = \sum_{\substack{\text{closed} \\ \text{diagrams}}} \left(\frac{1}{\#\mathrm{Aut}}\right) \left(\sqrt{-1}\,\hbar\right)^{\#\text{loops}}$$: interpret via "rules"

Remark: (1) easy to read this as homological pert. thy.

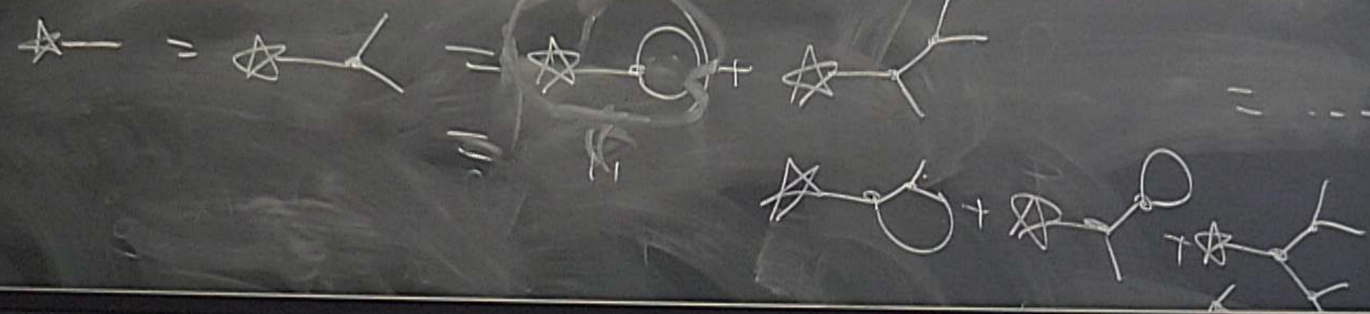
(2) In the QFT (e.g. scalar thy)

$$\mathbb{R}^n \rightsquigarrow C^\infty(M^d)$$


... = Σ closed diagrams (1/#Aut) (√-1 ħ)^#loops : interpret via "rules"
Remark: (1) easy to read this as homological pert. thy.
(2) In the QFT (e.g. scalar thy)
R^n ⇝ C^∞(M^d)
i=1..n
basis ~ points in M. "i" ranges over M.
Σ_i
∫_M



≡ M Taylor coef of b(x)
$i \longrightarrow j = (a^{-1})_{ij}$
combinatorial factors ≡ something about symmetries of the graph
a^{-1}_{ij}
loops
$= \sqrt{-1}\,\hbar$
$+\sum_{M \geq 0}$
a^{-1}
$b^{(M)}$
f
Hydra
seal the neck w/ hot wax
grows more heads
Hercules eventually wins!



$$\cdots = \sum_{\substack{\text{closed} \\ \text{diagrams}}} \left(\frac{1}{\#\mathrm{Aut}}\right) \left(\sqrt{-1}\,\hbar\right)^{\#\text{loops}}$$: interpret via "rules"

Remark: (1) easy to read this as homological pert. thy.

(2) In the QFT (e.g. scalar thy)

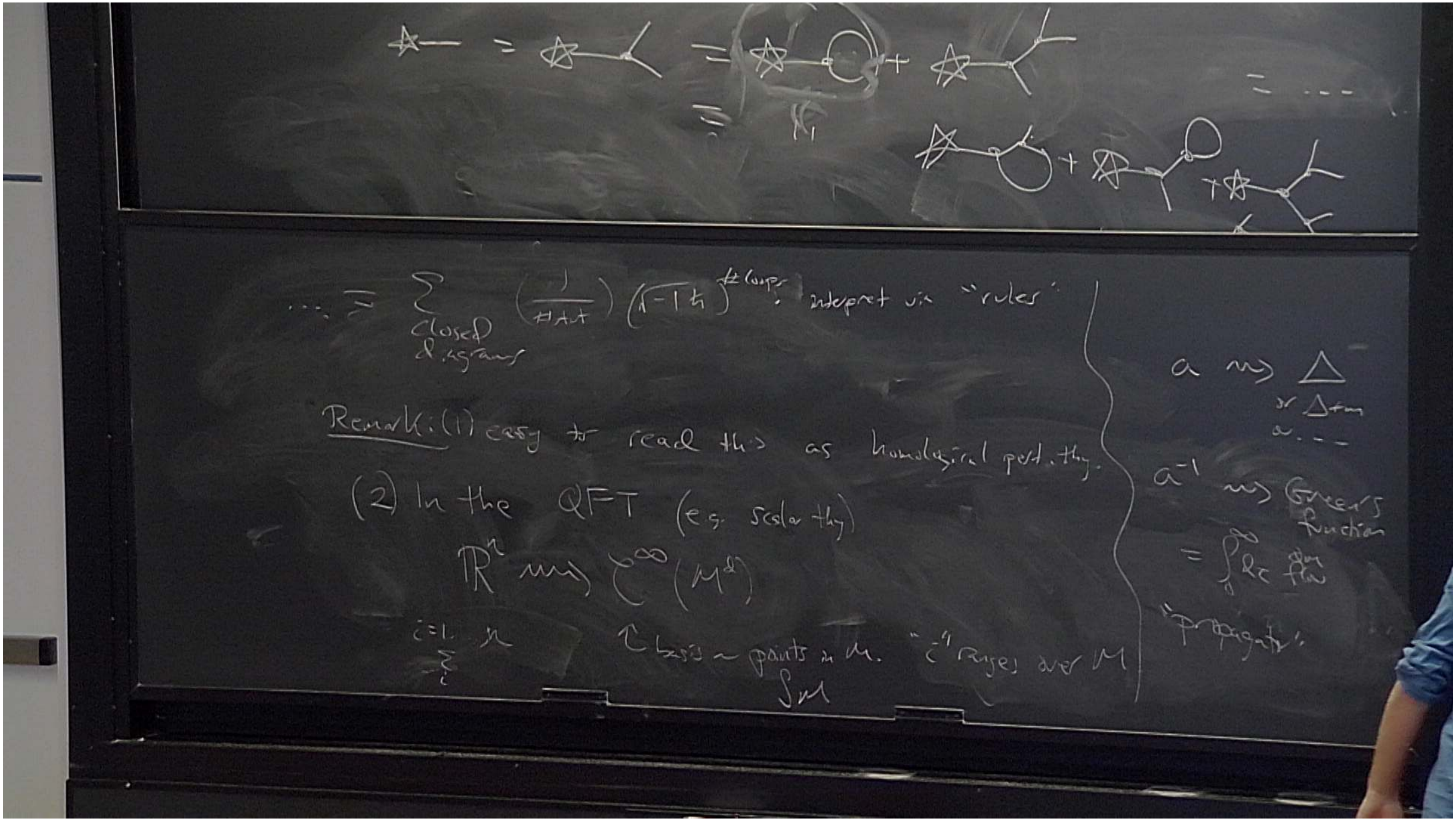
$$\mathbb{R}^n \rightsquigarrow C^\infty(M^d)$$

$i = 1..n$ ↑ basis ~ points in M. "i" ranges over M

$\sum_i \rightsquigarrow \int_M$

$a \rightsquigarrow \Delta$ or $\Delta + m$ or ...

$a^{-1} \rightsquigarrow$ Green's function $= \int_0^\infty dt$ heat flow


... = Σ closed diagrams (1/#Aut) (√-1 ħ)^#loops : interpret via "rules"
Remark: (1) easy to read this as homological pert. thy.
(2) In the QFT (e.g. scalar thy)
ℝ^n ⇝ C^∞(M^d)
i=1..n
Σ_i
∫_M
basis ~ points in M. "i" ranges over M
a ⇝ Δ
or Δ+m
a^-1 ⇝ Green's function
"propagator"