Title: Lecture 3: Supersymmetric Field Theory and Topological Twists

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Collection: QFT for Mathematicians

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SUSY Localization

$$SYM^{N=1} = \int \frac{1}{4}\langle F_A, F_A\rangle + \frac{1}{2}\langle \psi, \not{D}_A \psi\rangle$$

- $N=1$ $D=10$ SYM

$V = \mathbb{R}^{9,1}$ S_+ Chiral Spinor $(\mathbb{R}^{16})$

Fields $\mathcal{E} = \Omega^1(V, \mathfrak{g}) \oplus \Omega^0(V, S_+ \otimes \mathfrak{g})$

A ψ

Berkovits Construction

$\Gamma : Sym^2(S_+) \longrightarrow V$

Let $Q_\Gamma = \Gamma^{-1}(0) \subset S_+$

Consider $V \times Q_\Gamma \times \Pi S_+$

$$= \int \frac{1}{4} \langle F_A, F_A \rangle + \frac{1}{2} \langle \psi, \not{D}_A \psi \rangle$$

uits Construction

$\Gamma : Sym^2(S_+) \longrightarrow V$

$Q_\Gamma = \Gamma^{-1}(0) \subset S_+$

sider $V \times Q_\Gamma \times \Pi S_+$

Let $B = \mathcal{O}(V \times Q_\Gamma \times \Pi S_+)$

Choose Coordinates

$$\{ x^\mu , \lambda^\alpha , \theta^\alpha \}$$

$$V \quad S_+ \quad \Pi S_+$$

$$B = \mathbb{R}\{ x^\mu, \lambda^\alpha, \theta^\alpha \} / \left(\Gamma^\mu_{\alpha\rho} \lambda^\alpha \lambda^\beta \right)$$

Define $Q = \lambda^\alpha \frac{\partial}{\partial \theta^\alpha} + \Gamma^\mu_{\alpha\rho} \theta^\alpha \lambda^\beta \frac{\partial}{\partial x^\mu}$

$\langle F_A, F_A \rangle + \frac{1}{2}\langle \psi, \nabla_A \psi \rangle$

Let $B = \mathcal{O}(V \times Q_\Gamma \times \Pi S_+)$

$Q^2 = \left(\Gamma^\mu_{\alpha\rho} \lambda^\alpha \lambda^\rho\right) \frac{\partial}{\partial x^\mu}$

$= 0$ on B

$\Rightarrow (B, Q)$ dga

choose Coordinates

$\{ x^\mu, \lambda^\alpha, \theta^\alpha \}$

$V \quad S_+ \quad \Pi S_+$

$B = \mathbb{R}\{ x^\mu, \lambda^\alpha, \theta^\alpha \} / \left(\Gamma^\mu_{\alpha\rho} \lambda^\alpha \lambda^\rho\right)$

Berkovits

$Crit(SYM^{N=1}) / \sim$

Define $Q = \lambda^\alpha \frac{\partial}{\partial \theta^\alpha} + \Gamma^\mu_{\alpha\rho} \theta^\alpha \lambda^\rho \frac{\partial}{\partial x^\mu}$

$= MC(B, Q) / \sim$

truction

$\longmapsto V$

$^{-1}(0) \subset S_+$

$\times Q_\Gamma \times \Pi S_+$

SUSY Localization

$$SYM^{N=1} = \int \frac{1}{4} \langle F_A, F_A \rangle + \frac{1}{2} \langle \lambda, \not{D}_A \lambda \rangle$$

- $N=4$ $D=4$ SYM

dim reduction from $N=1$ $D=10$

$$\mathbb{R}^{9,1} = \mathbb{R}^{3,1} \times \mathbb{R}^6$$

Declare A, λ only vary along $\mathbb{R}^{3,1}$

Constant along $\mathbb{R}^6$

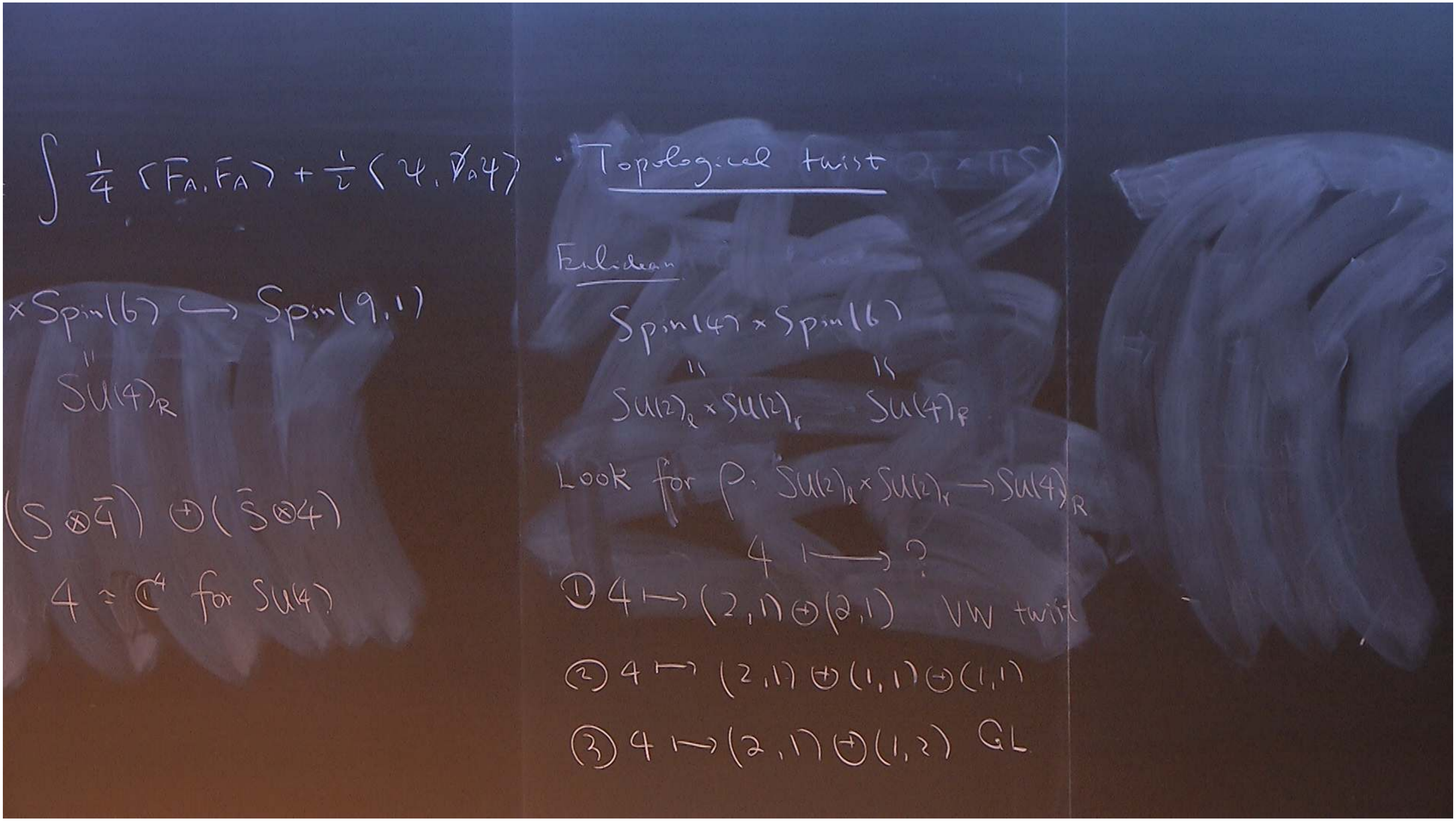
$$Spin(3,1) \times Spin(6) \hookrightarrow Spin(9,1)$$

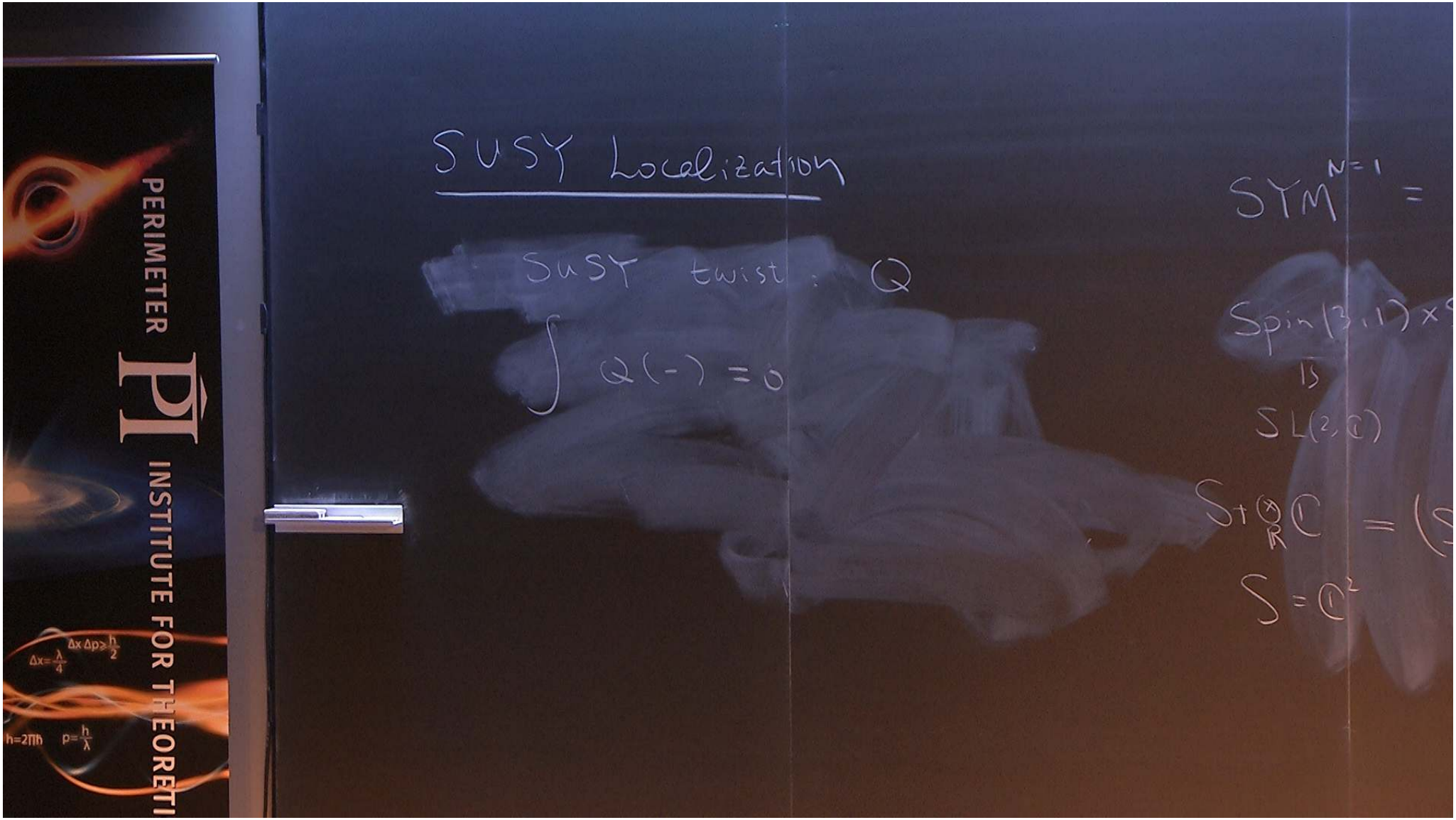
$$\| \qquad\qquad \|$$

$$SL(2,\mathbb{C}) \qquad SU(4)_R$$

$$S_+ \otimes_{\mathbb{R}} \mathbb{C} = (S \otimes \bar{4}) \oplus (\bar{S} \otimes 4)$$

$$S = \mathbb{C}^2 \qquad 4 \simeq \mathbb{C}^4 \text{ for } SU(4)$$


∫ 1/4 ⟨F_A, F_A⟩ + 1/2 ⟨ψ, D̸_A ψ⟩
Topological twist
Euclidean
× Spin(6) ↪ Spin(9,1)
SU(4)_R
Spin(4) × Spin(6)
SU(2)_ℓ × SU(2)_r
SU(4)_R
Look for ρ: SU(2)_ℓ × SU(2)_r → SU(4)_R
(S ⊗ 4̄) ⊕ (S̄ ⊗ 4)
4 ↦ ?
4 ≅ ℂ^4 for SU(4)
① 4 ↦ (2,1) ⊕ (2,1) VW twist
② 4 ↦ (2,1) ⊕ (1,1) ⊕ (1,1)
③ 4 ↦ (2,1) ⊕ (1,2) GL


SUSY Localization
SUSY twist : Q
∫ Q(-) = 0
SYM^{N=1} =
Spin(3,1) ×
SL(2,ℂ)
S = ℂ^2
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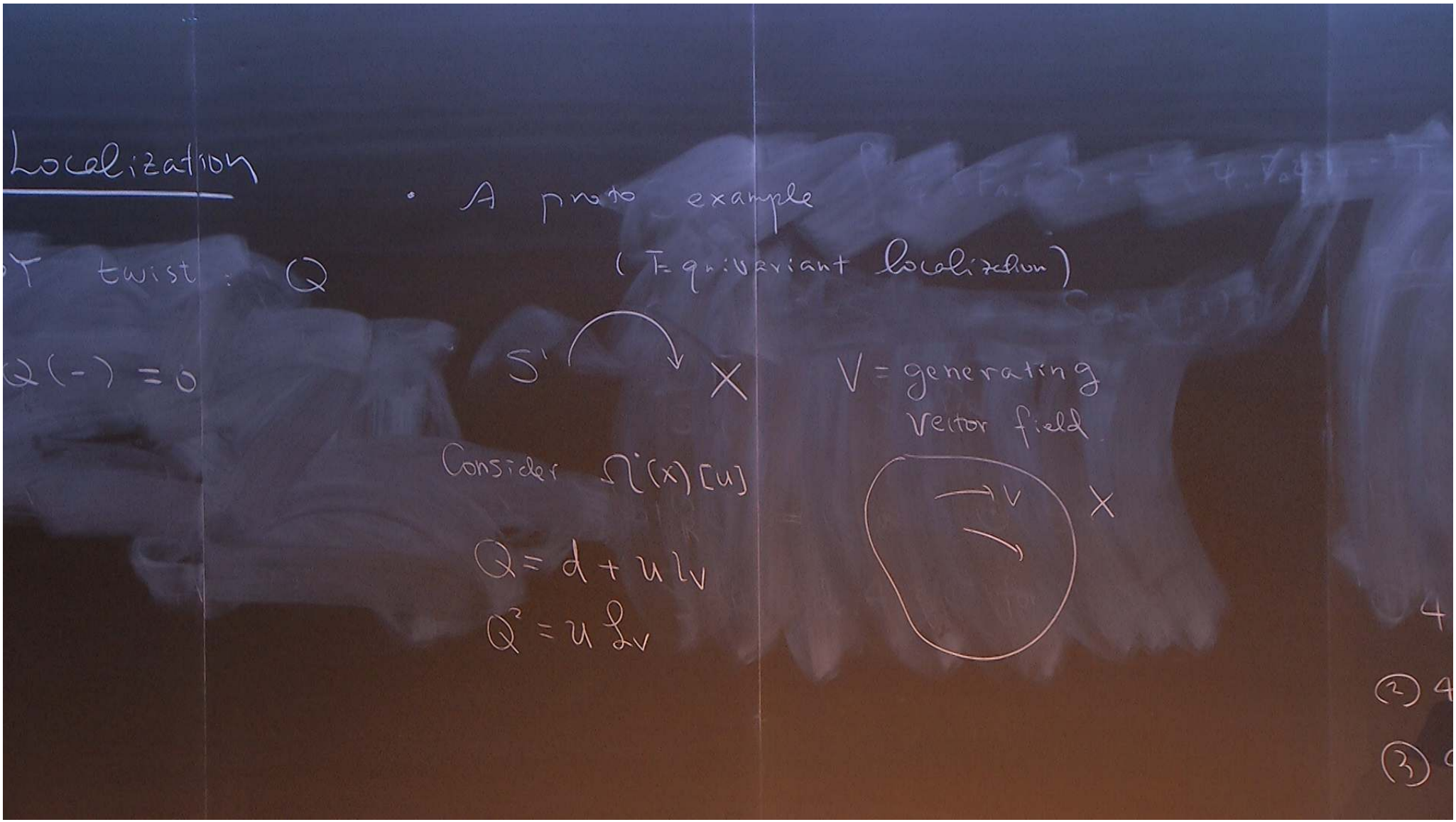
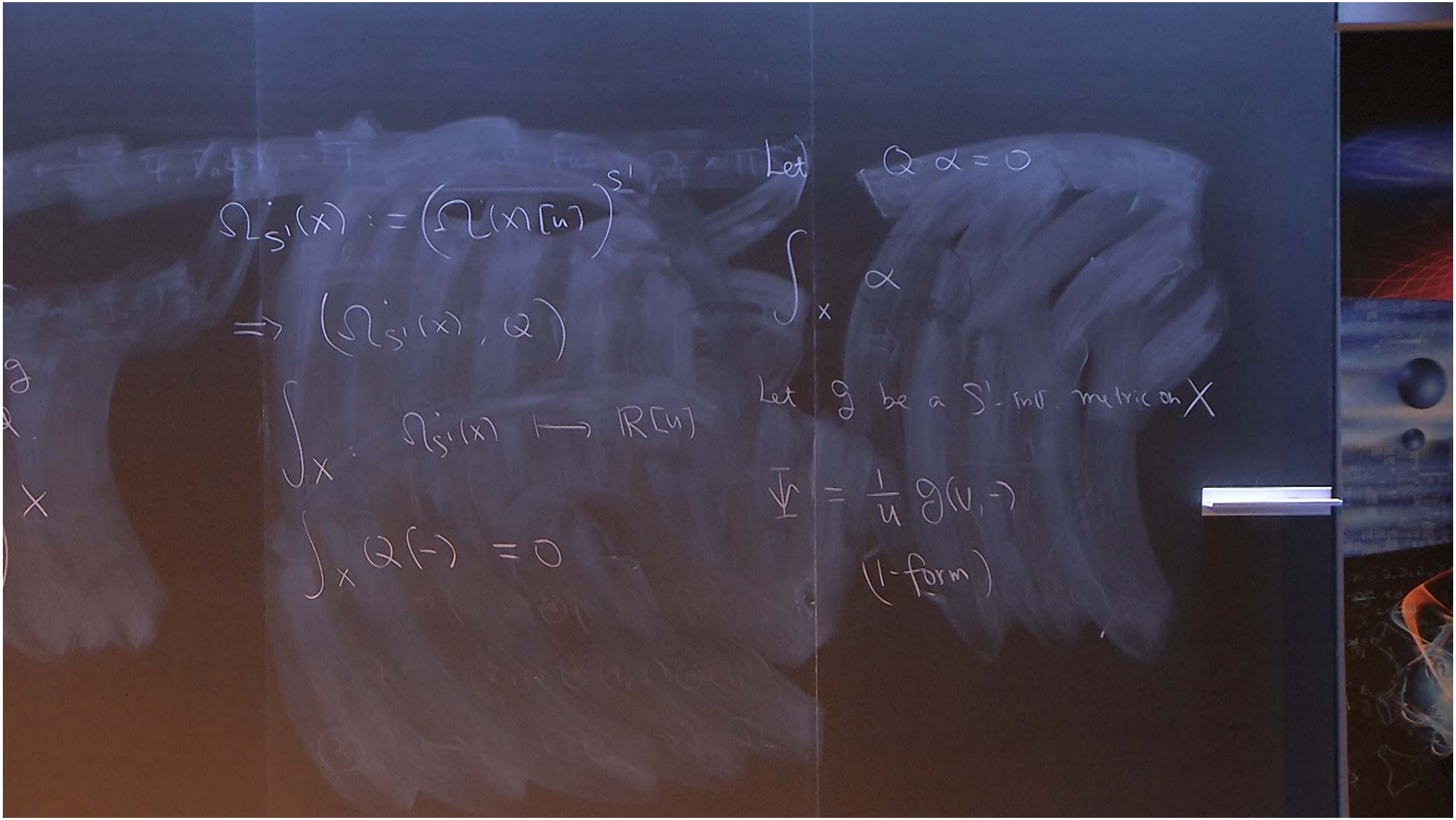
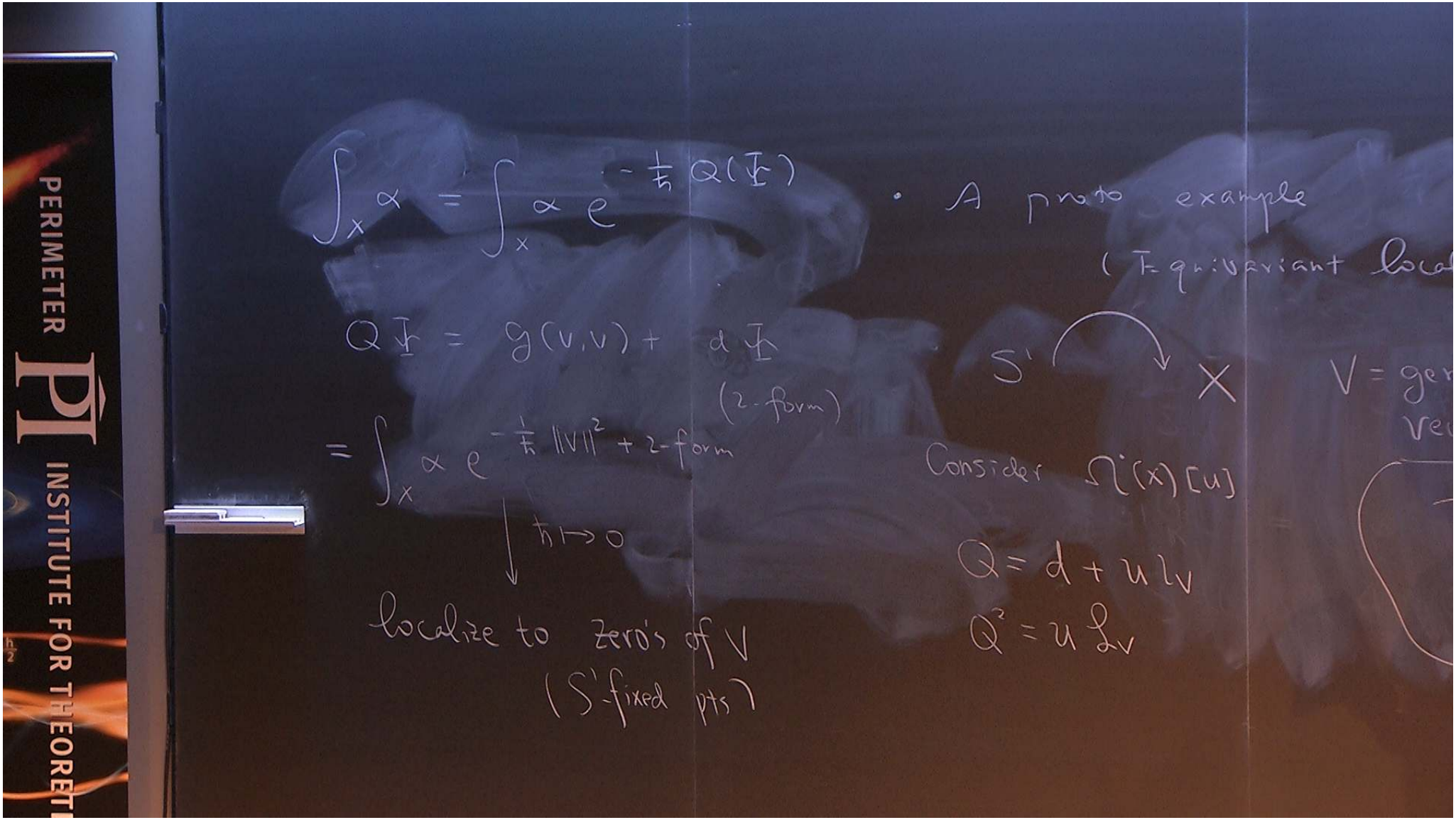

Localization
• A proto example
(Equivariant localization)
Y twist : Q
Q(-) = 0
S¹ X
V = generating vector field
Consider Ω˙(X)[u]
Q = d + u L_V
Q² = u L_V
V
X


$\Omega^i_{S^1}(X) := \left(\Omega^i(X)[u]\right)^{S^1}$
$\Rightarrow \left(\Omega^i_{S^1}(X), Q\right)$
$\int_X : \Omega^i_{S^1}(X) \longmapsto \mathbb{R}[u]$
$\int_X Q(-) = 0$
Let $Q\alpha = 0$
$\int_X \alpha$
Let g be a S^1-Inv. metric on X
$\Psi = \frac{1}{u} g(v, -)$
(1-form)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$\int_X \alpha = \int_X \alpha\, e^{-\frac{1}{\hbar} Q(\Psi)}$
$Q\Psi = g(v,v) + d\Psi$
(2-form)
$= \int_X \alpha\, e^{-\frac{1}{\hbar}\|v\|^2 + 2\text{-form}}$
$\hbar \mapsto 0$
localize to zero's of V
(S^1-fixed pts)
• A proto example
(Equivariant loca
$S^1 \curvearrowright X$
V = ger
ve
Consider $\Omega^\cdot(X)[u]$
$Q = d + u\,\iota_V$
$Q^2 = u\,\mathcal{L}_V$

$\propto e^{-\frac{1}{\hbar} Q(\Psi)}$

$(v,v) + d\Psi$

(2-form)

$\|v\|^2 + 2\text{-form}$

$\hbar \mapsto 0$

o zero's of V

(S^1-fixed pts)

Eg A-model

$$S_A = Q_A \int \; + \text{ top. term}$$

$$\| $$

$$\int |\bar\partial \phi|^2 + \text{fermions}$$

$$(\phi : \Sigma \longmapsto X)$$

$\rightsquigarrow$ localize to $(\bar\partial\phi = 0)$ hol maps

B-model

$+$ top. term $\quad S_B = \int |d\phi|^2 + \cdots$

$+$ fermions $\quad \cdots$ localize to const maps

$\longrightarrow X$)

$(\bar{\partial}\phi = 0)$ hol maps

B-model

B-model

$$S_D = \int |Q\phi|^2 + \cdots$$

{

localize to const maps

maps

B-model

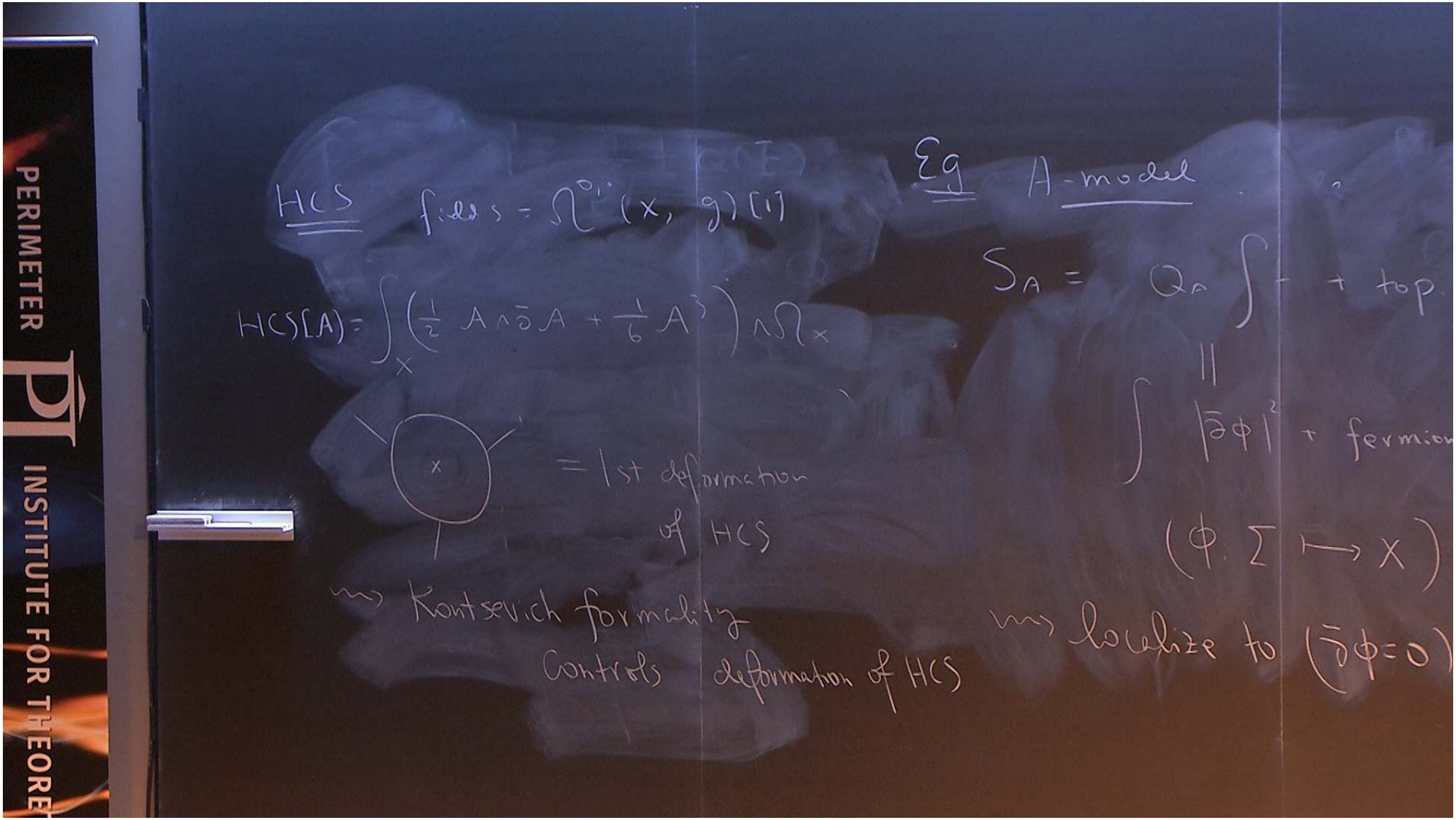
$\rightsquigarrow$ local QFT on X

Closed string

KS theory

Open string

HCS theory


HCS fields = $\Omega^{0,\cdot}(X, g)[1]$
$HCS[A] = \int_X \left(\frac{1}{2} A \wedge \bar{\partial} A + \frac{1}{6} A^3\right) \wedge \Omega_X$
= 1st deformation of HCS
↝ Kontsevich formality controls deformation of HCS
Eg A-model
$S_A = Q_A \int \cdots$ + top.
$= \int |\bar{\partial}\phi|^2$ + fermion
$(\phi : \Sigma \longmapsto X)$
↝ localize to $(\bar{\partial}\phi = 0)$
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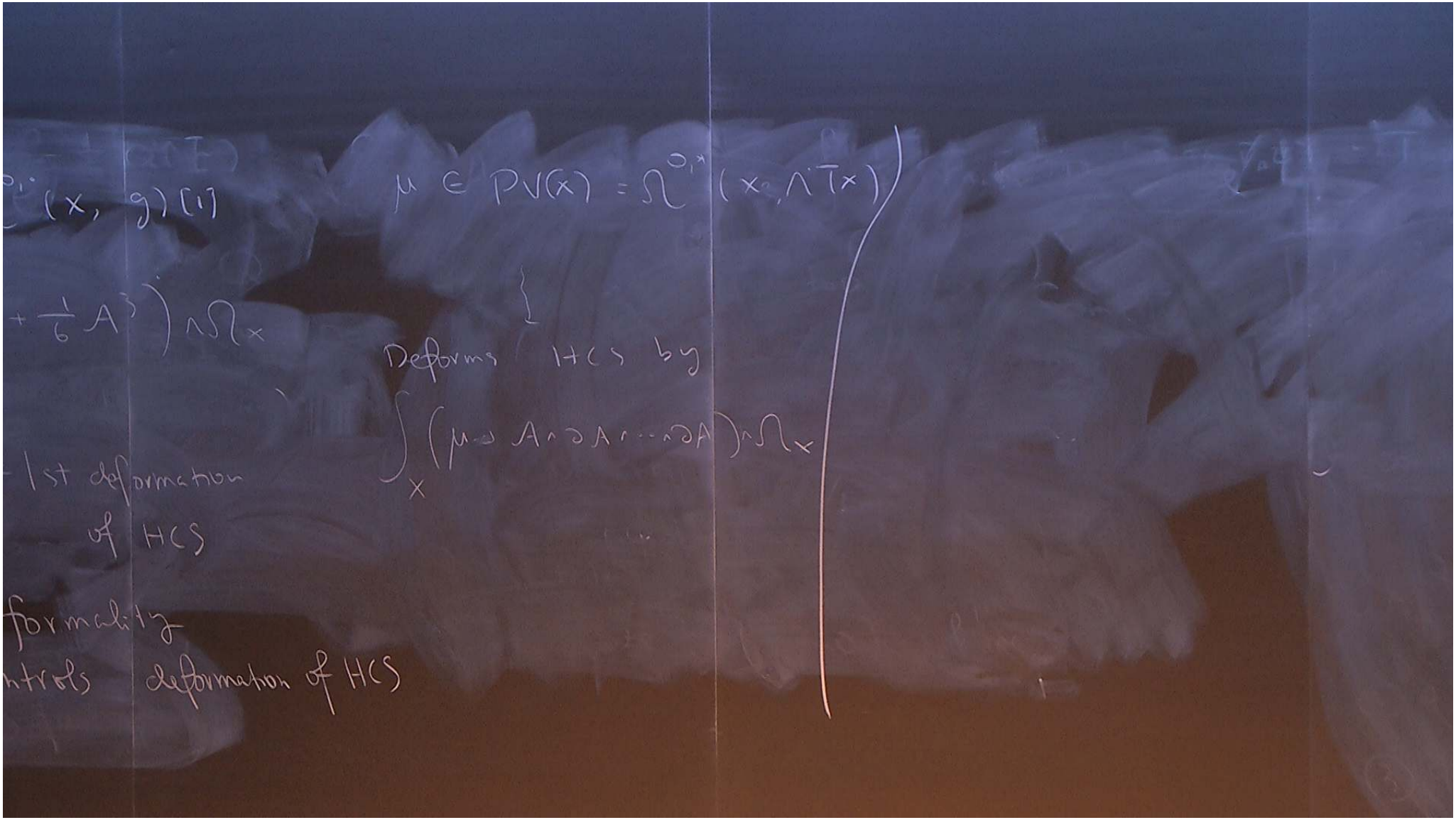

(x, g)[1]
$+\frac{1}{6}A^3)\wedge\Omega_x$
$\mu \in PV(x) = \Omega^{0,*}(x, \Lambda^* \bar{T}x)$
Deforms HCS by
$\int_X (\mu \cdot A \wedge \partial A \wedge \cdots \wedge \partial A)\wedge\Omega_x$
1st deformation
of HCS
formality
ntrols deformation of HCS

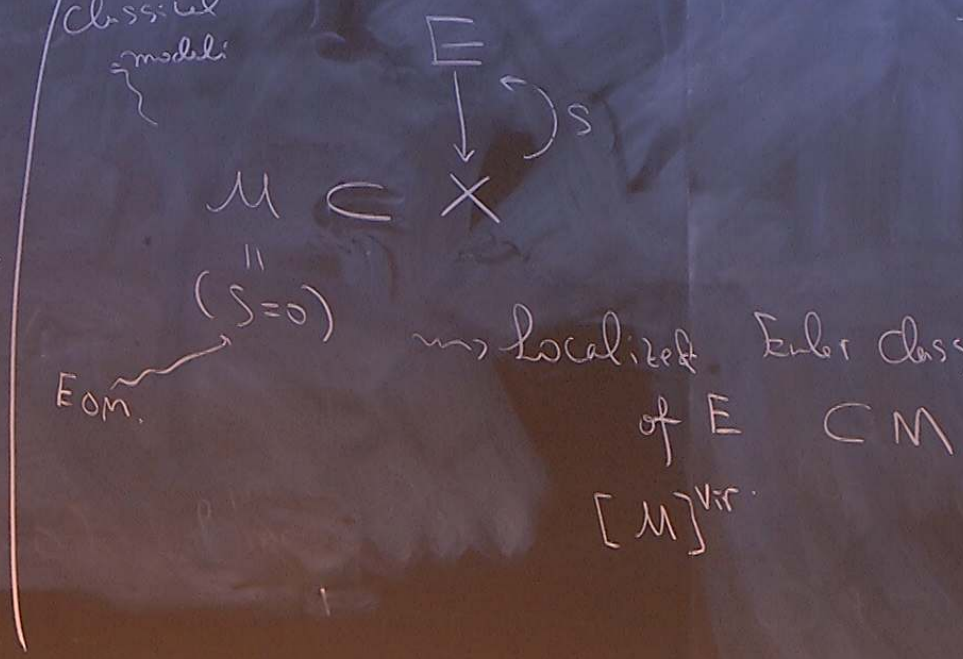
$PV(x) = \Omega^{0,*}(x, \wedge^{\cdot} Tx)$

ms HCS by

$\ldots \to A \wedge \partial A \wedge \cdots \wedge \partial A) \wedge \Omega_x$

Classical moduli

Kuranishi model



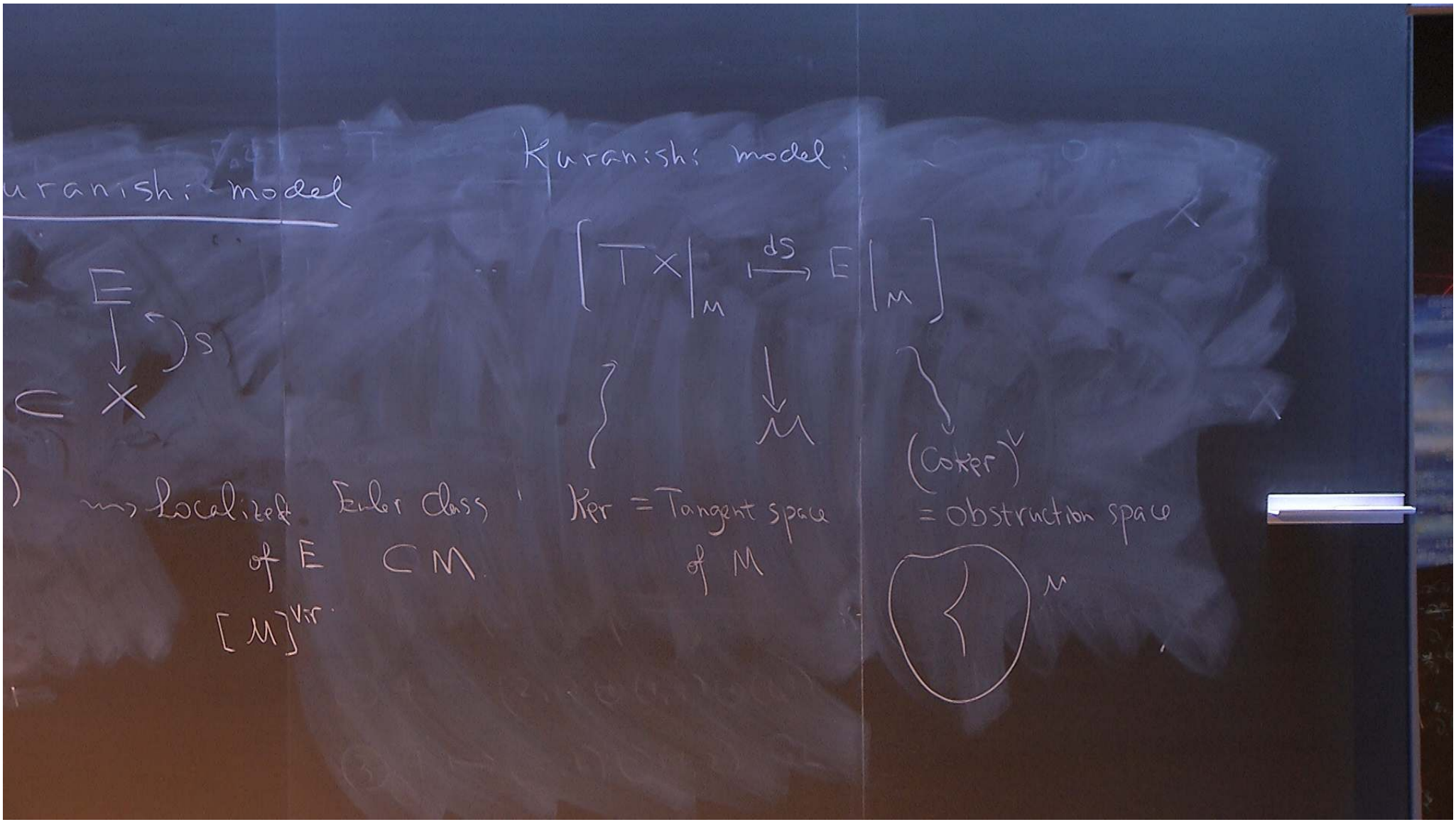
$$\begin{array}{ccc} & & E \\ & & \downarrow \circlearrowleft s \\ \mathcal{M} & \subset & X \end{array}$$

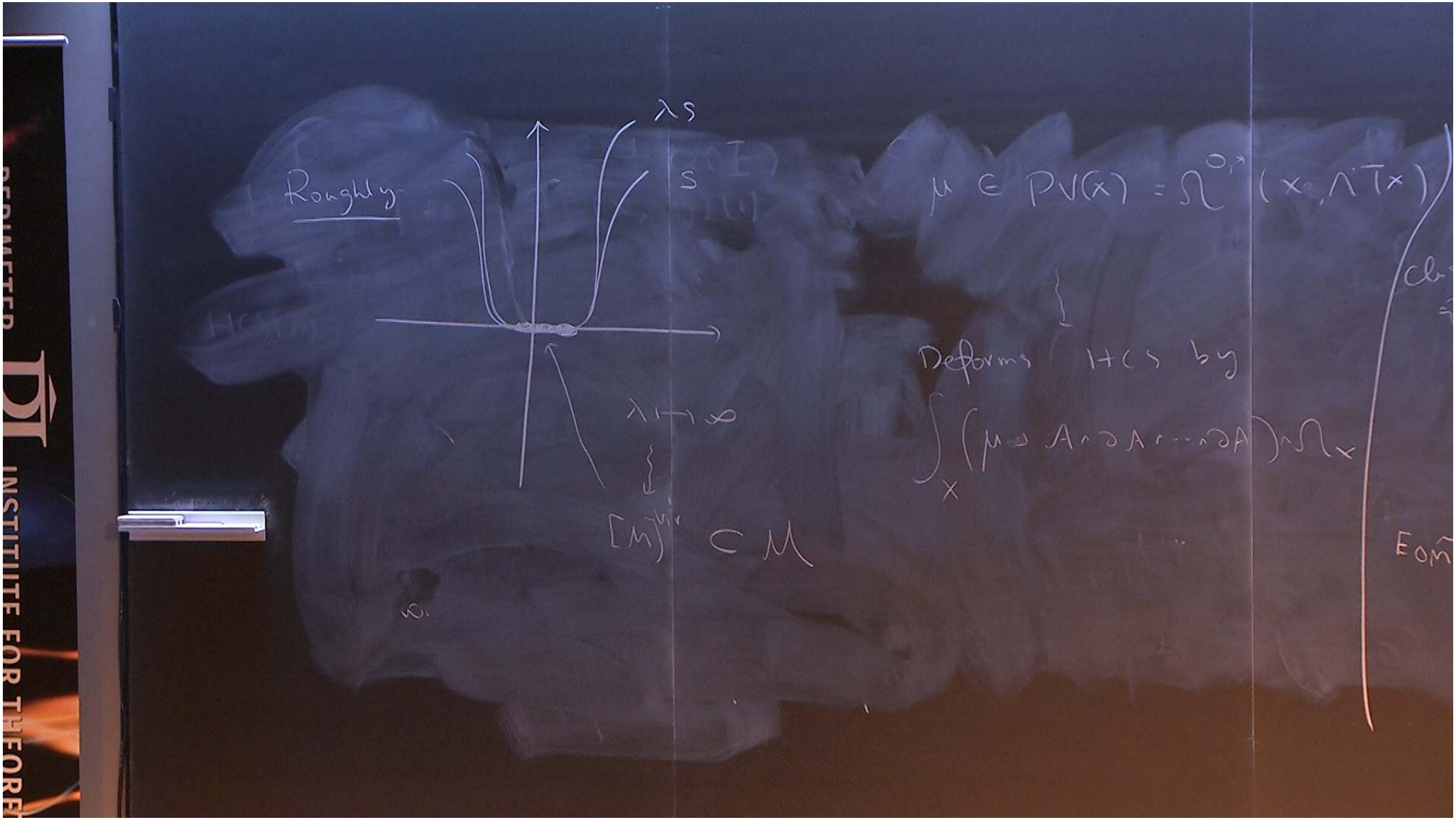
$\mathcal{M} = (s=0)$

EOM.

$\rightsquigarrow$ Localized Euler class of $E \subset M$

$[\mathcal{M}]^{vir}$


uranishi model
E
s
X
⇝ Localized Euler class
of E ⊂ M
[M]^vir
Kuranishi model:
[TX|_M ↦^ds E|_M]
M
Ker = Tangent space
of M
(Coker)^∨
= obstruction space
M


Roughly
λs
s
λ ↦ ∞
[M]^vir ⊂ M
μ ∈ PV(X) = Ω^{0,*}(X, ∧^{·} TX)
Deforms HCS by
∫_X (μ ⊣ A ∧ ∂A ∧ ··· ∧ ∂A) ∧ Ω_X
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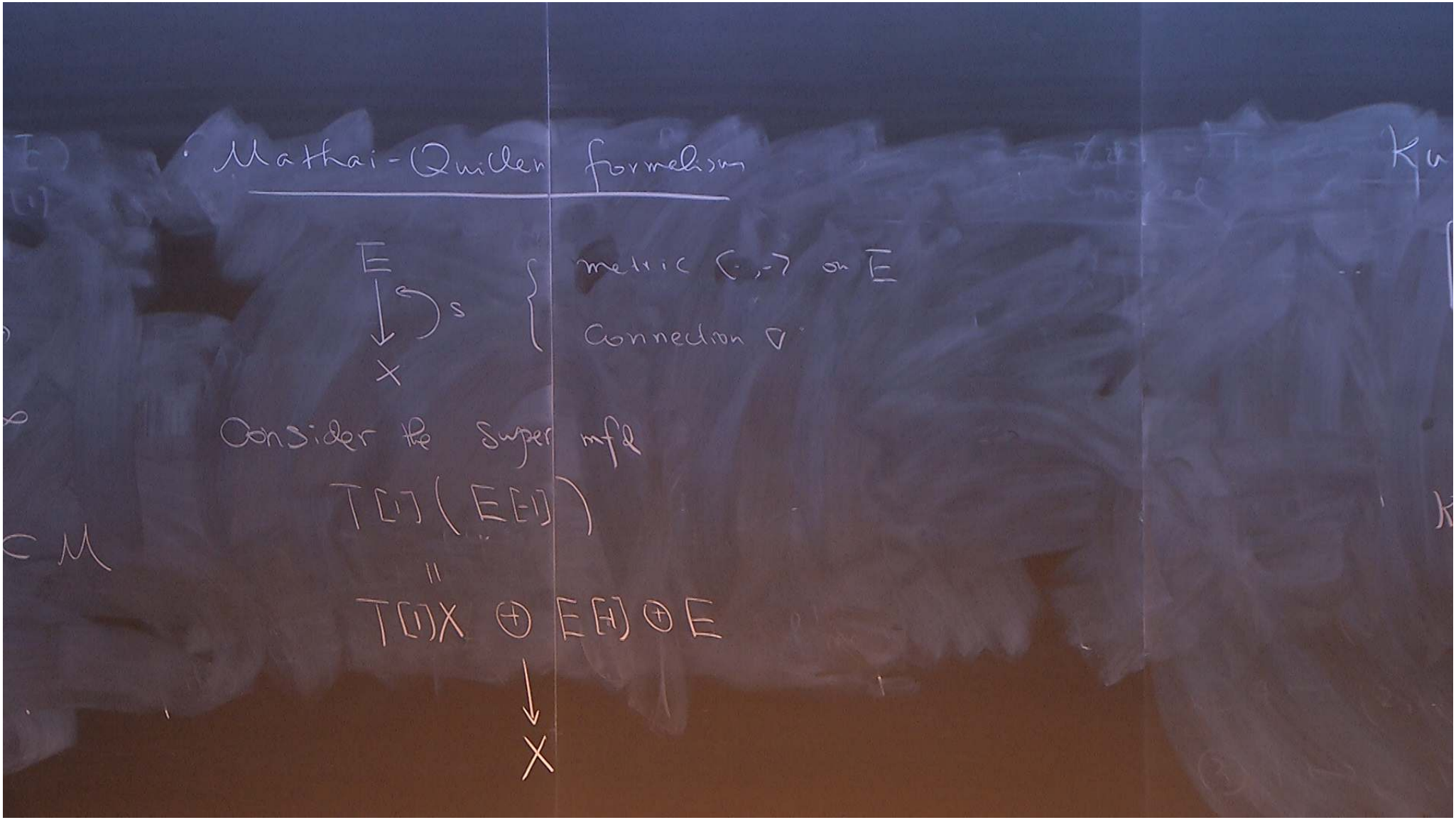

Mathai-Quillen formalism
E
X
s
metric ⟨·,·⟩ on E
Connection ∇
Consider the super mfd
T[1](E[-1])
=
T[1]X ⊕ E[1] ⊕ E
X

Kai-Quillen formalism

$E \xrightarrow{s} X$ { metric $\langle \cdot, \cdot \rangle$ on E ; Connection ∇

der the super mfd $T[1](E[-1])$

$= T[1]X \oplus E[1] \oplus E$

$\downarrow$

X

Choose local basis $\{e_a\}$ for E Kuranishi model:

and local coordinates

	x^μ	η^μ	χ^α	B^α
deg	0	1	1	0
	X	$T[1]X$	$E[1]$	E

Q = de Rham diff.

$Q x^\mu = \eta^\mu \quad Q \chi^\alpha = B^\alpha$

$\left[TX|_M \xrightarrow{ds} E \right]$ $\downarrow$ M

Ker = Tangent space of M

Quillen formalism

Choose local basis $\{e_\alpha\}$ for E and local coordinates

Kuranishi model:

$$\left[TX\big|_M \xrightarrow{ds} E\big|_M \right]$$

$$\downarrow$$

$$M$$

$\mathcal{O}_s$ $\begin{cases} \text{metric } \langle \cdot, \cdot \rangle \text{ on } E \\ \text{Connection } \nabla \end{cases}$

	X^μ	ψ^μ	χ^α	B^α
deg	0	1	-1	0
	X	$T[1]X$	$E[1]$	E

the super mfd

$T[1](E[-1])$

$\|$

$T[1]X \oplus E[-1] \oplus E$

$\downarrow$

X

Ker = Tangent space of M

$(\text{Coker})^\vee$ = obstructi

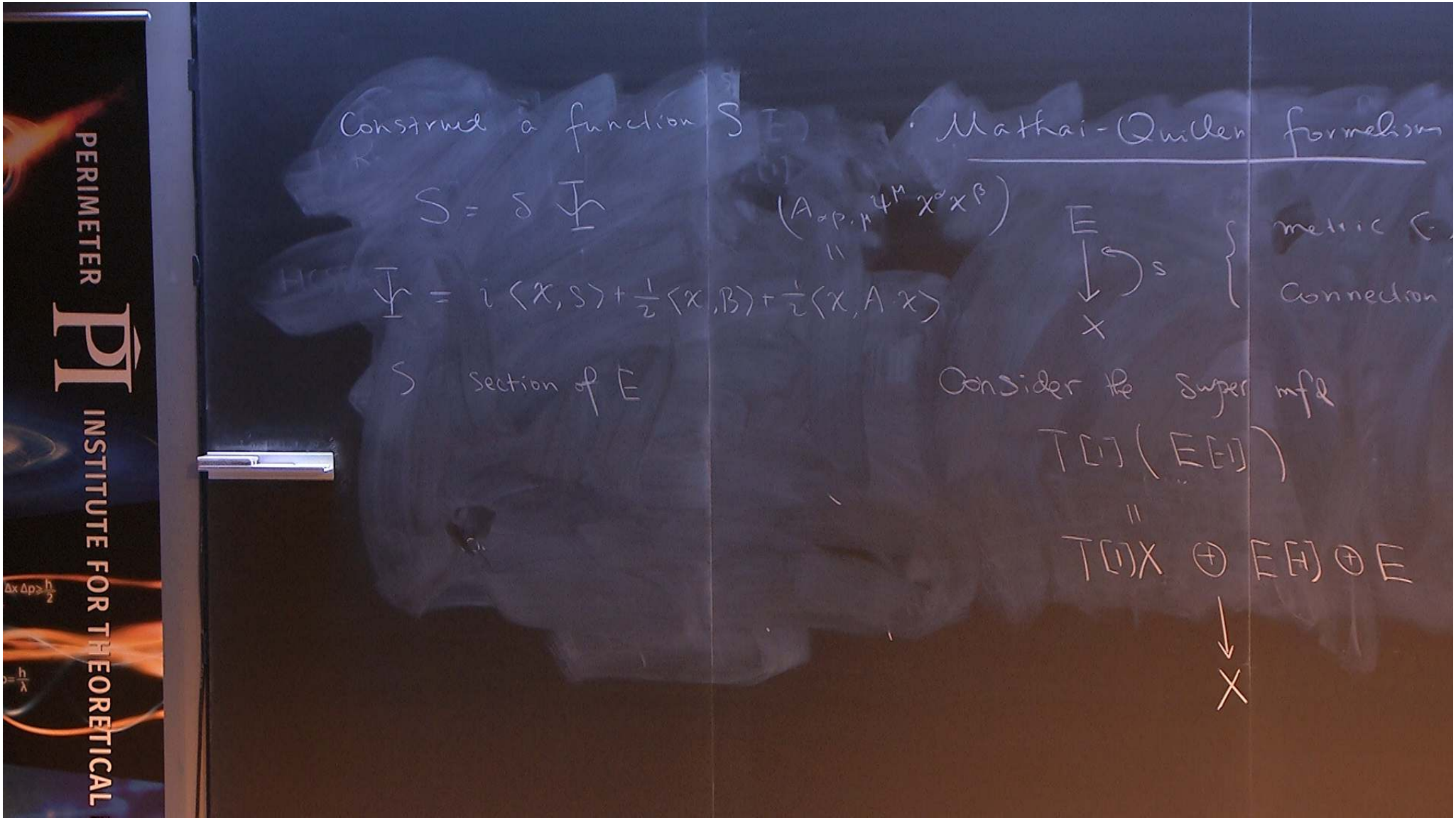
M

δ = de Rham diff.

$\delta X^\mu = \psi^\mu \quad \delta \chi^\alpha = B^\alpha$

$\nabla e_\alpha = A_\alpha^\beta e_\beta$

Connection 1-form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Construct a function S
S = δ Ψ
(A_{αβ,μ} ψ^μ χ^α χ^β)
Ψ = i ⟨χ, S⟩ + ½ ⟨χ, B⟩ + ½ ⟨χ, A·χ⟩
S section of E
Mathai-Quillen formalism
E
s
X
metric
Connection
Consider the super mfd
T[1](E[-1])
T[1]X ⊕ E[-1] ⊕ E
X

Construct a function S

$\delta \Psi$

$$\Psi = i\langle x, S\rangle + \frac{1}{2}\langle x, B\rangle + \frac{1}{2}\langle x, A\cdot x\rangle$$

$$\left(A_{\alpha\beta,\mu}\psi^\mu x^\alpha x^\beta\right)$$

S section of E

$e^{-\delta\Psi}$ is δ-closed

MQ construction

$$e_{S,\nabla}(\xi) = \int dx\, dB\, e^{-\delta\Psi}$$

is a closed diff form on X

whose de Rham class is indep. of the choice of S, ∇

deg

$x^\alpha x^\beta)$

MQ construction

$$e_{s,\nabla}(\mathcal{E}) = \int dx\, dB\, e^{-\delta \Psi}$$

is a closed diff form on X

whose de Rham class is indep. of the choice of s, ∇

$$= \frac{1}{(2\pi)^{\dim(X)}} \int dx\, e^{-\frac{1}{2}\langle s,s\rangle + \frac{1}{2}\langle x, Rx\rangle + i\langle x, \nabla s\rangle}$$

R: curvature 2-form

$$s \longmapsto \lambda s$$

$\lambda \mapsto 0$: $\frac{1}{(2\pi)} \int dx\, e^{\frac{1}{2}\langle x, Rx\rangle} = \mathrm{Pf}(R)$

$\lambda \mapsto \infty$ localize to $(s=0)$

Kuranishi model:

$$\left[TX\big|_M \xrightarrow{ds} E \right] \longrightarrow M$$

Ker = Tangent space of M

$$= \frac{1}{(2\pi)^{\dim(X)}} \int dx\, e^{-\frac{1}{2}\langle s,s\rangle + \frac{1}{2}\langle x, Rx\rangle + i\langle x, ps\rangle}$$

R: Curvature 2-form

$s \longmapsto \lambda s$

$\lambda \to 0$: $\frac{1}{(2\pi)} \cdot \int dx\, e^{\frac{1}{2}\langle x, Rx\rangle} = Pf(R)$

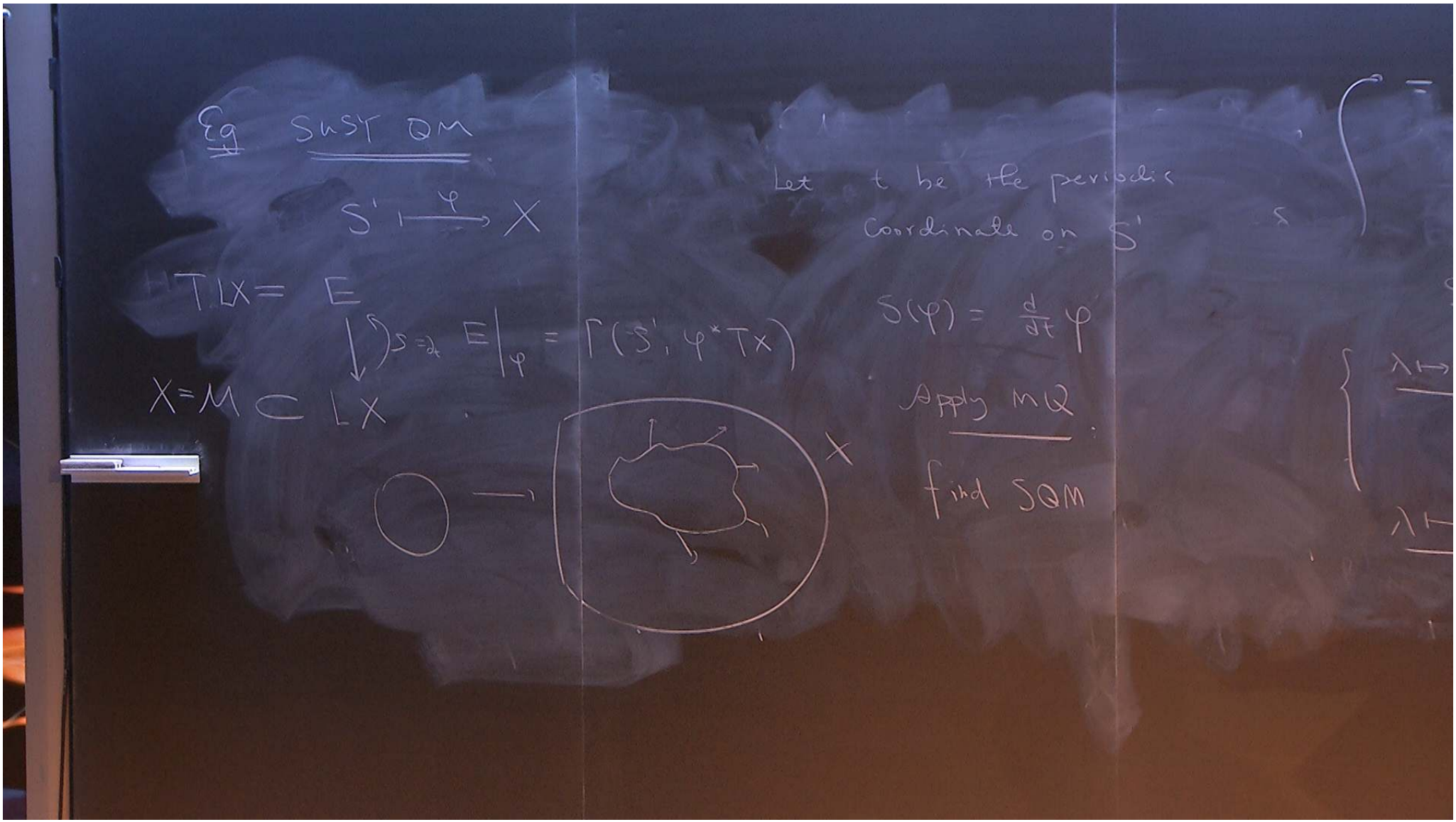
$\lambda \to \infty$ localize to $(s=0)$

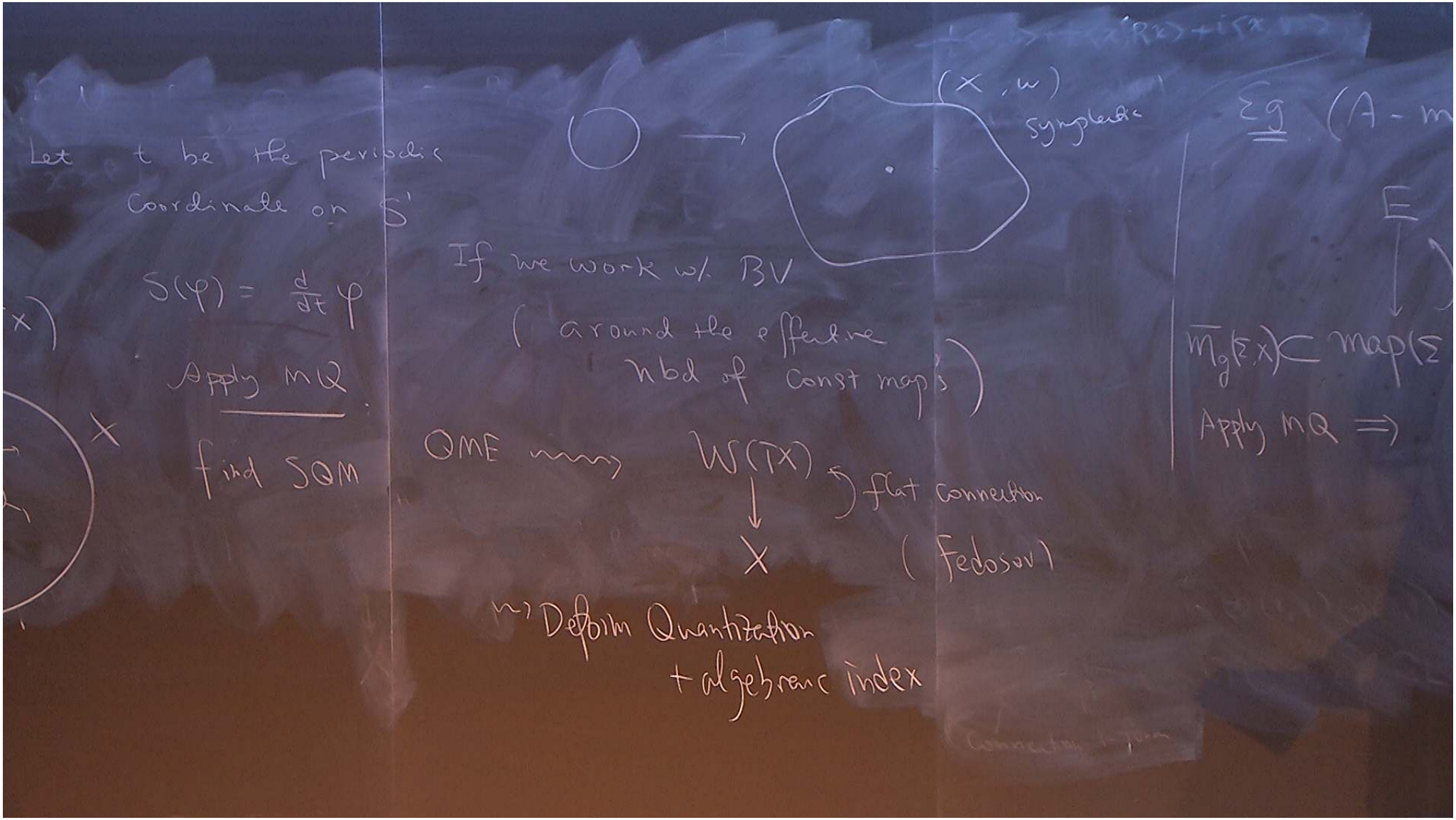
Eg (A-model) $\Sigma \xrightarrow{\varphi} X$

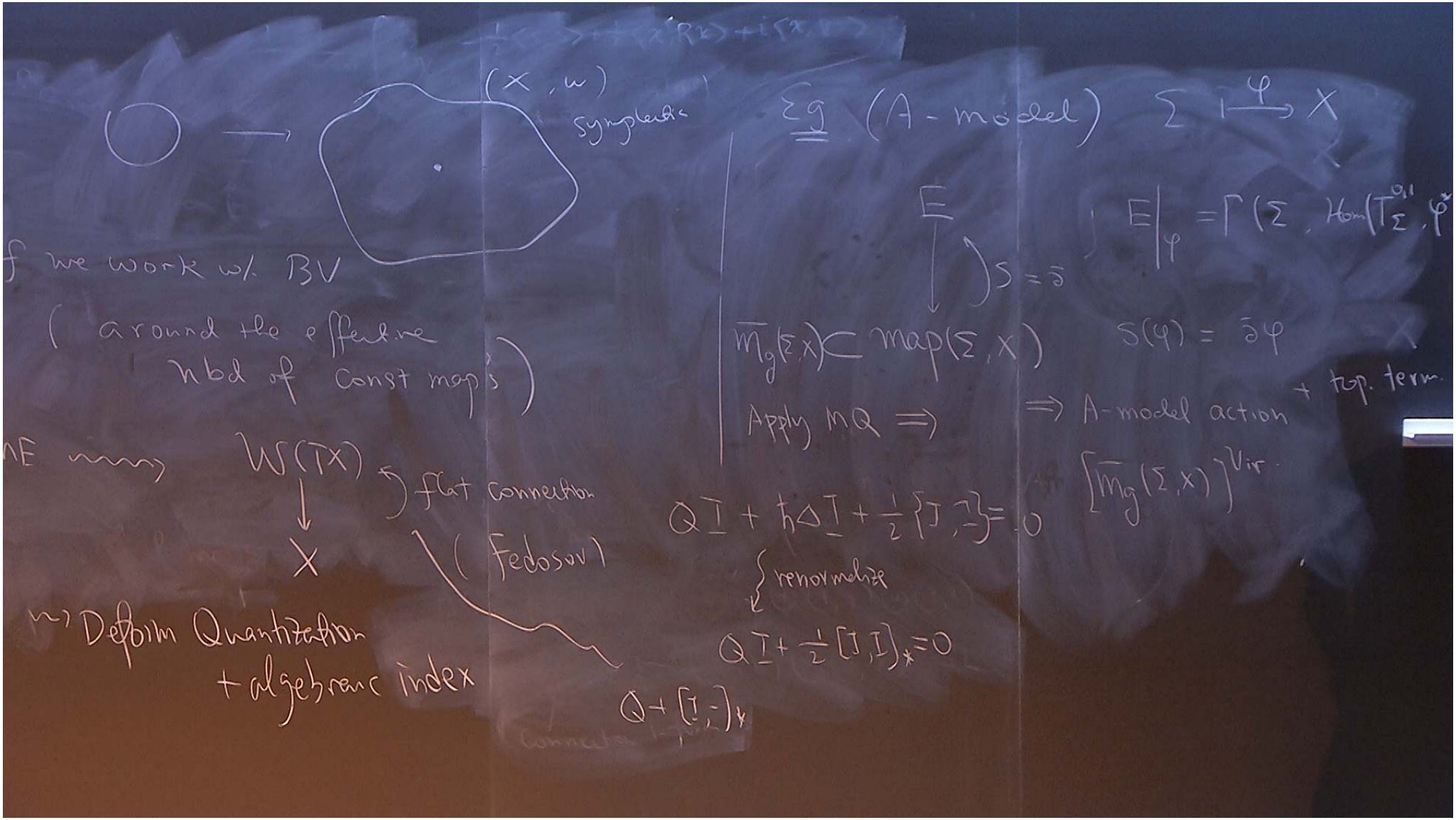
$$\begin{array}{c} E \\ \downarrow s=\bar\partial \\ \overline{M}_g(\Sigma, X) \subset map(\Sigma, X) \end{array} \qquad E\big|_\varphi = \Gamma\left(\Sigma, Hom\left(T^{0,1}_\Sigma, \varphi^* T^{1,0}_X\right)\right)$$

$S(\varphi) = \bar\partial\varphi$

Apply HQ $\Rightarrow$ $\Rightarrow$ A-model action + $\left[\overline{M}_g(\Sigma, X)\right]^{vir}$


Eg SUSY QM
S' ↦ φ X
T LX = E
s = ∂t
E|φ = Γ(S', φ*TX)
X = M ⊂ LX
X
Let t be the periodic
Coordinate on S'
S(φ) = d/dt φ
Apply MQ
find SQM


Let t be the periodic coordinate on S¹
S(φ) = d/dt φ
Apply MQ
find SQM
(X, ω) symplectic
If we work w/ BV
(around the effective nbd of Const map's)
QME
W(TX)
X
flat connection
(Fedosov)
Deform Quantization + algebraic index
Eg (A - m
E
$\overline{M}_g(\Sigma, X) \subset$ map(Σ
Apply MQ ⇒


(X, ω)
symplectic
f we work w/ BV
(around the effective
nbd of const maps)
NE ⇝ W(TX)
X
flat connection
(Fedosov)
⇝ Deform Quantization
+ algebraic index
Eg (A-model) Σ ↦ φ X
E
S = ∂̄
M̄g(Σ,X) ⊂ map(Σ,X)
Apply MQ ⇒
E|φ = Γ(Σ, Hom(T^{0,1}_Σ, φ*
S(φ) = ∂̄φ
⇒ A-model action + top. term.
[M̄g(Σ,X)]^{Vir}
QI + ħΔI + ½{I,I} = 0
renormalize
QI + ½[I,I]* = 0
Q + [I,-]*