Title: Lecture 3: Supersymmetric Field Theory and Topological Twists

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Collection: QFT for Mathematicians

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SUSY Localization

- $N=1 \quad D=10$ SYM

$V = \mathbb{R}^{9,1} \qquad S_+$ Chiral Spinor $(\mathbb{R}^{16})$

Fields $\mathcal{E} = \Omega^1(V, \mathfrak{g}) \oplus \Omega^0(V, S_+ \otimes \mathfrak{g})$

$\qquad\qquad A \qquad\qquad \psi$

$$SYM^{N=1} = \int \frac{1}{4} \langle F_A, F_A \rangle + \frac{1}{2} \langle \psi, \not{D}_A \psi \rangle$$

Berkovits Construction

$\Gamma : \mathrm{Sym}^2(S_+) \longrightarrow V$

Let $Q_\Gamma = \Gamma^{-1}(0) \subset S_+$

Consider $V \times Q_\Gamma \times \Pi S_+$

$$= \int \frac{1}{4}\langle F_A, F_A\rangle + \frac{1}{2}\langle \psi, \not{D}_A \psi\rangle$$

<u>...vits Construction</u>

$$\Gamma : \mathrm{Sym}^2(S_+) \longmapsto V$$

$$Q_\Gamma = \Gamma^{-1}(0) \subset S_+$$

...sider $V \times Q_\Gamma \times \Pi S_+$

Let $B = \mathcal{O}(V \times Q_\Gamma \times \Pi S_+)$

Choose Coordinates

$$\{ x^\mu , \lambda^\alpha , \theta^\alpha \}$$

$$V \qquad S_+ \qquad \Pi S_+$$

$$B = \mathbb{R}\{x^\mu, \lambda^\alpha, \theta^\alpha\} / (\Gamma^\mu_{\alpha\beta}\lambda^\alpha\lambda^\beta)$$

Define $Q = \lambda^\alpha \frac{\partial}{\partial \theta^\alpha} + \Gamma^\mu_{\alpha\beta}\theta^\alpha\lambda^\beta \frac{\partial}{\partial x^\mu}$

$\langle F_A, F_A \rangle + \frac{1}{2}\langle \psi, \not{D}_A \psi \rangle$

...truction

$...) \longmapsto V$

$^{-1}(0) \subset S_+$

$\times Q_\Gamma \times \Pi S_+$

Let $B = \mathcal{O}(V \times Q_\Gamma \times \Pi S_+)$

choose coordinate

$\{ x^\mu , \lambda^\alpha , \theta^\alpha \}$

$V \quad S_+ \quad \Pi S_+$

$B = \mathbb{R}\{ x^\mu, \lambda^\alpha, \theta^\alpha \} \big/ (\Gamma^\mu_{\alpha\beta} \lambda^\alpha \lambda^\beta)$

Define $Q = \lambda^\alpha \frac{\partial}{\partial \theta^\alpha} + \Gamma^\mu_{\alpha\beta} \theta^\alpha \lambda^\beta \frac{\partial}{\partial x^\mu}$

$Q^2 = (\Gamma^\mu_{\alpha\beta} \lambda^\alpha \lambda^\beta) \frac{\partial}{\partial x^\mu}$

$= 0$ on B

$\Rightarrow (B, Q)$ dga

Berkovits:

$\mathrm{Crit}(SYM^{N=1}) /\sim$

$= MC(B, Q) /\sim$

SUSY Localization

- $N=4$ $D=4$ SYM

dim reduction from $N=1$ $D=10$

$$\mathbb{R}^{9,1} = \mathbb{R}^{3,1} \times \mathbb{R}^6$$

Declare A, ψ only vary along $\mathbb{R}^{3,1}$

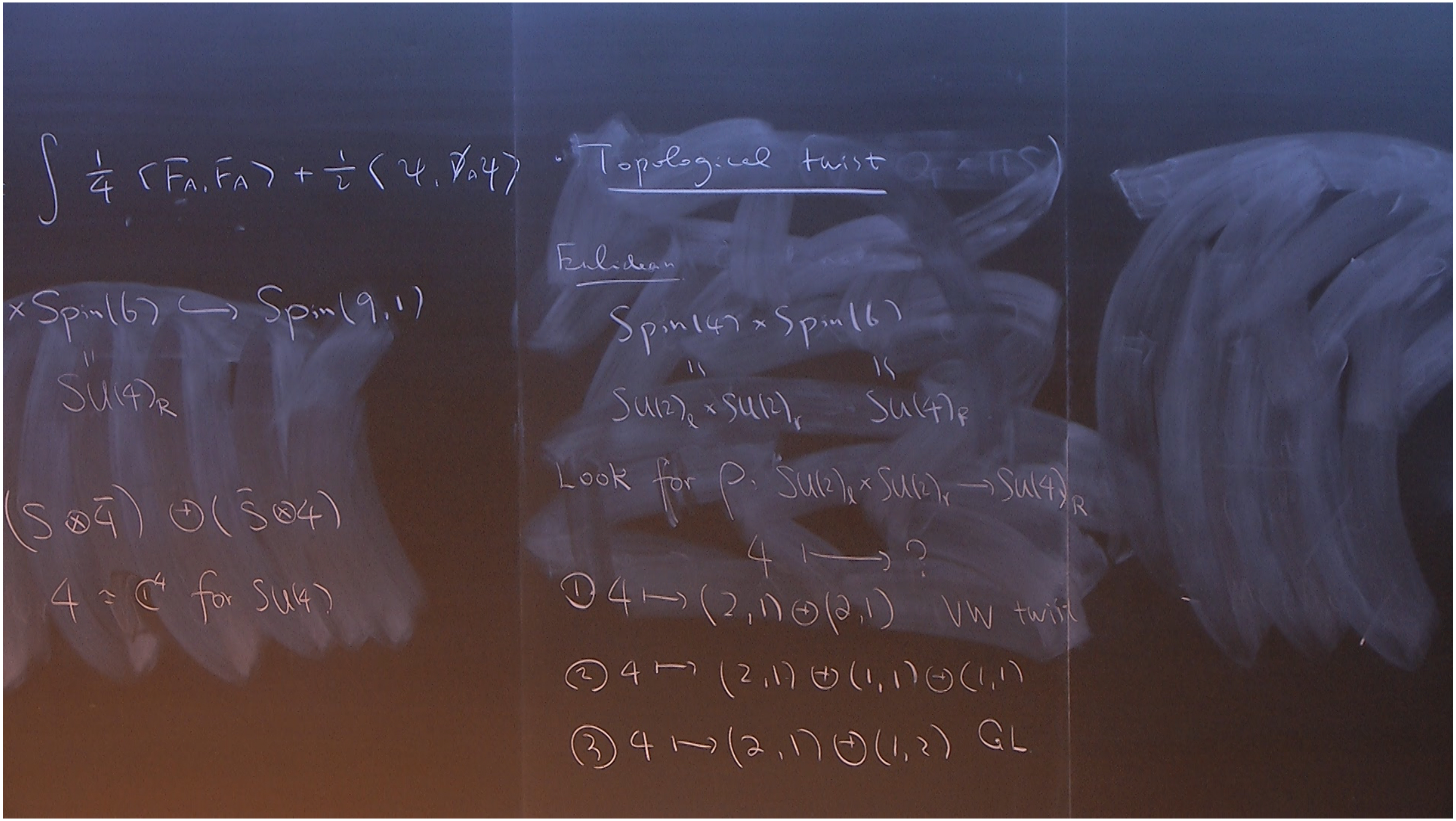
constant along $\mathbb{R}^6$

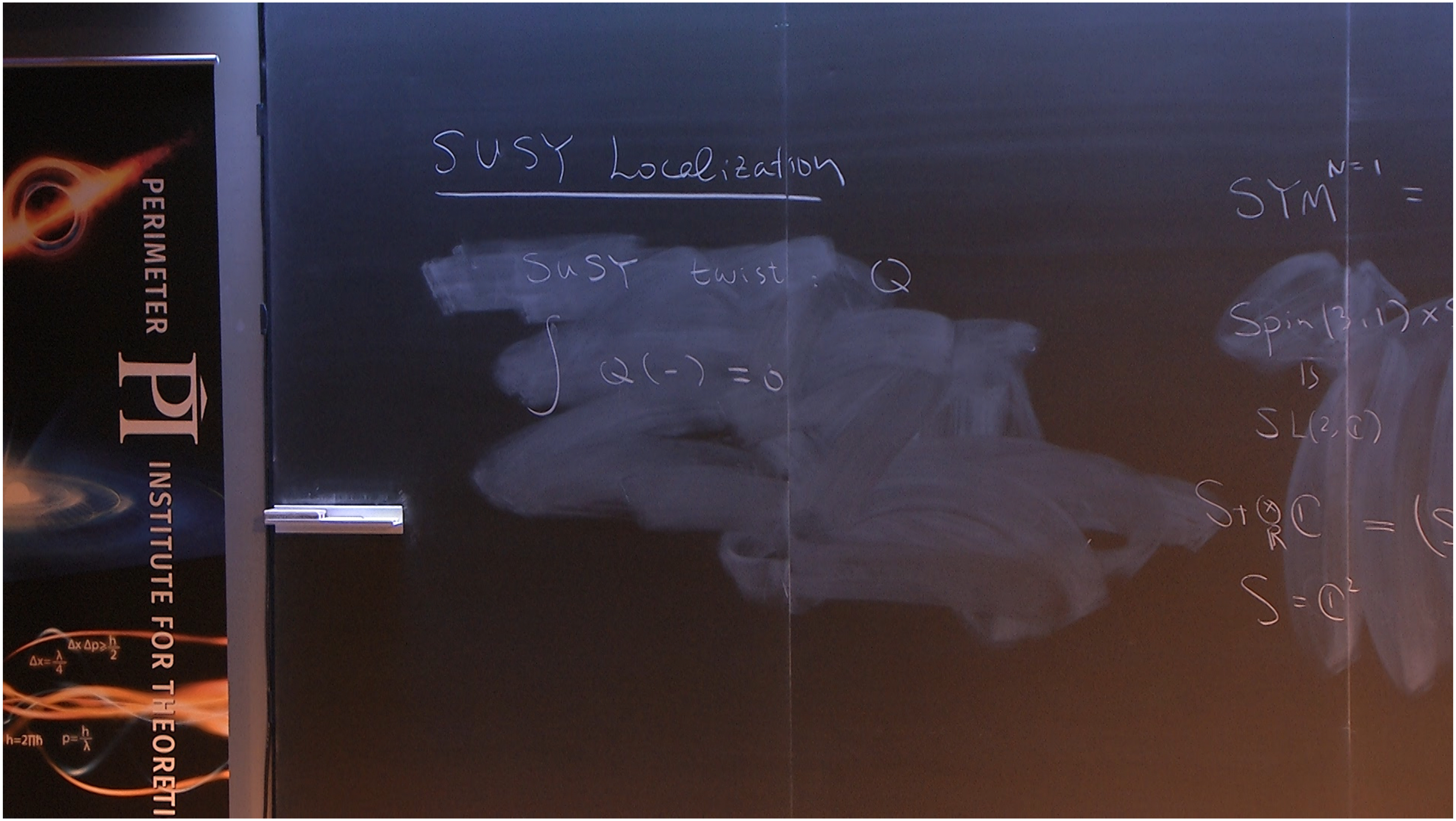
$$SYM^{N=1} = \int \frac{1}{4}\langle F_A, F_A\rangle + \frac{1}{2}\langle \psi, \not{D}_A \psi\rangle$$

$$\underbrace{Spin(3,1)}_{\cong\, SL(2,\mathbb{C})} \times \underbrace{Spin(6)}_{=\, SU(4)_R} \hookrightarrow Spin(9,1)$$

$$S_+ \otimes_{\mathbb{R}} \mathbb{C} = (S \otimes \bar{4}) \oplus (\bar{S} \otimes 4)$$

$$S = \mathbb{C}^2 \qquad 4 = \mathbb{C}^4 \text{ for } SU(4)$$


∫ 1/4 ⟨F_A, F_A⟩ + 1/2 ⟨ψ, D̸_A ψ⟩
× Spin(6) ↪ Spin(9,1)
SU(4)_R
(S ⊗ 4̄) ⊕ (S̄ ⊗ 4)
4 ≅ ℂ^4 for SU(4)
Topological twist
Euclidean
Spin(4) × Spin(6)
SU(2)_l × SU(2)_r
SU(4)_R
Look for ρ: SU(2)_l × SU(2)_r → SU(4)_R
4 ↦ ?
① 4 ↦ (2,1) ⊕ (2,1) VW twist
② 4 ↦ (2,1) ⊕ (1,1) ⊕ (1,1)
③ 4 ↦ (2,1) ⊕ (1,2) GL


SUSY Localization
SUSY twist : Q
∫ Q(−) = 0
SYM^{N=1} =
Spin(3,1) ×
SL(2,ℂ)
S₊ ⊗_ℝ ℂ = (S
S = ℂ²
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Δx Δp ≥ h/2
Δx = λ/4
h=2πħ
p = h/λ

Localization

T twist : Q

$Q(-) = 0$

· A proto example

(Equivariant localization)

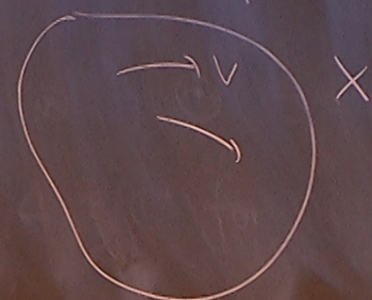
$S^1 \curvearrowright X$

V = generating vector field

Consider $\Omega^\bullet(X)[u]$

$Q = d + u\,\iota_V$

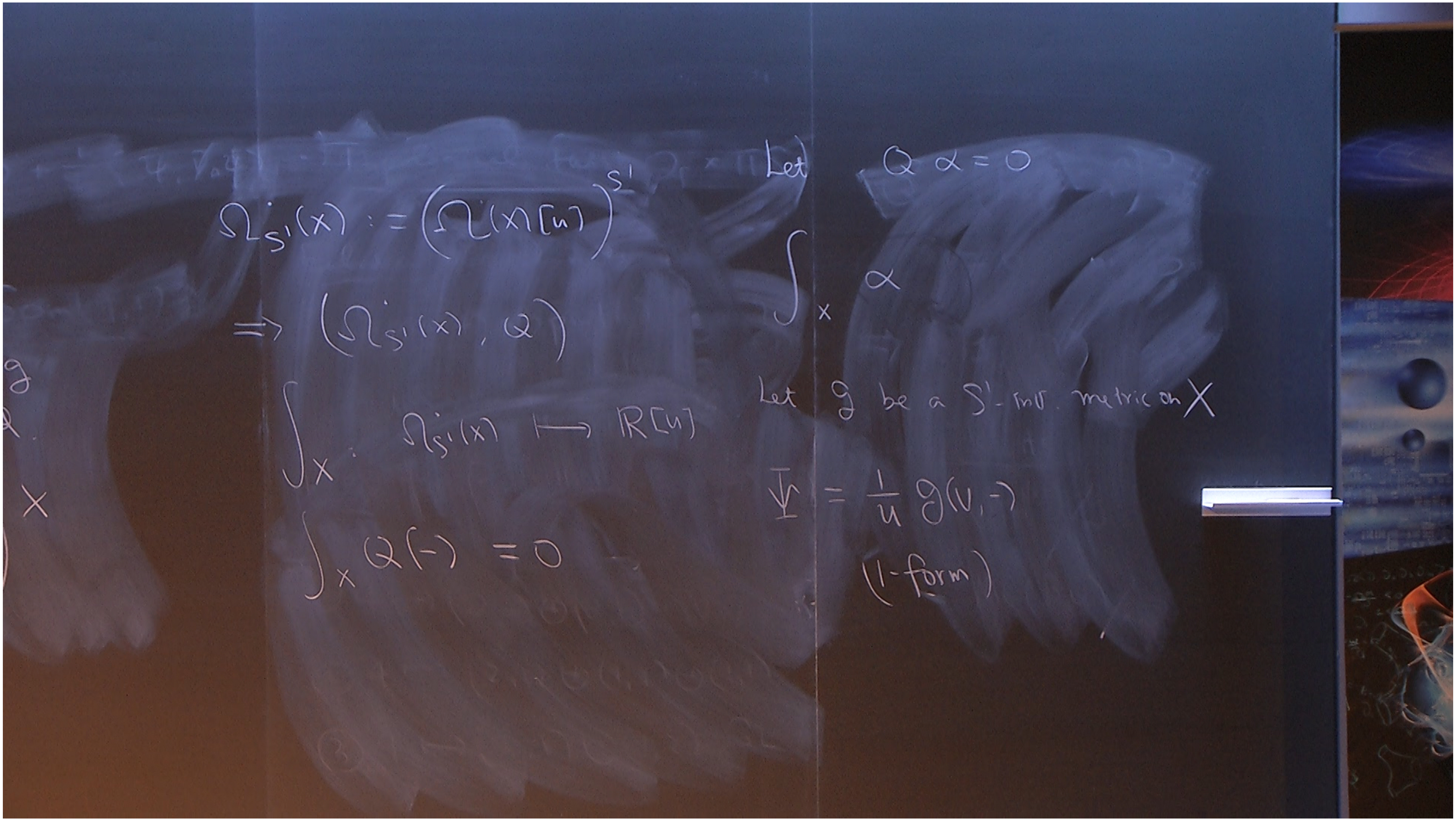
$Q^2 = u\,\mathcal{L}_V$

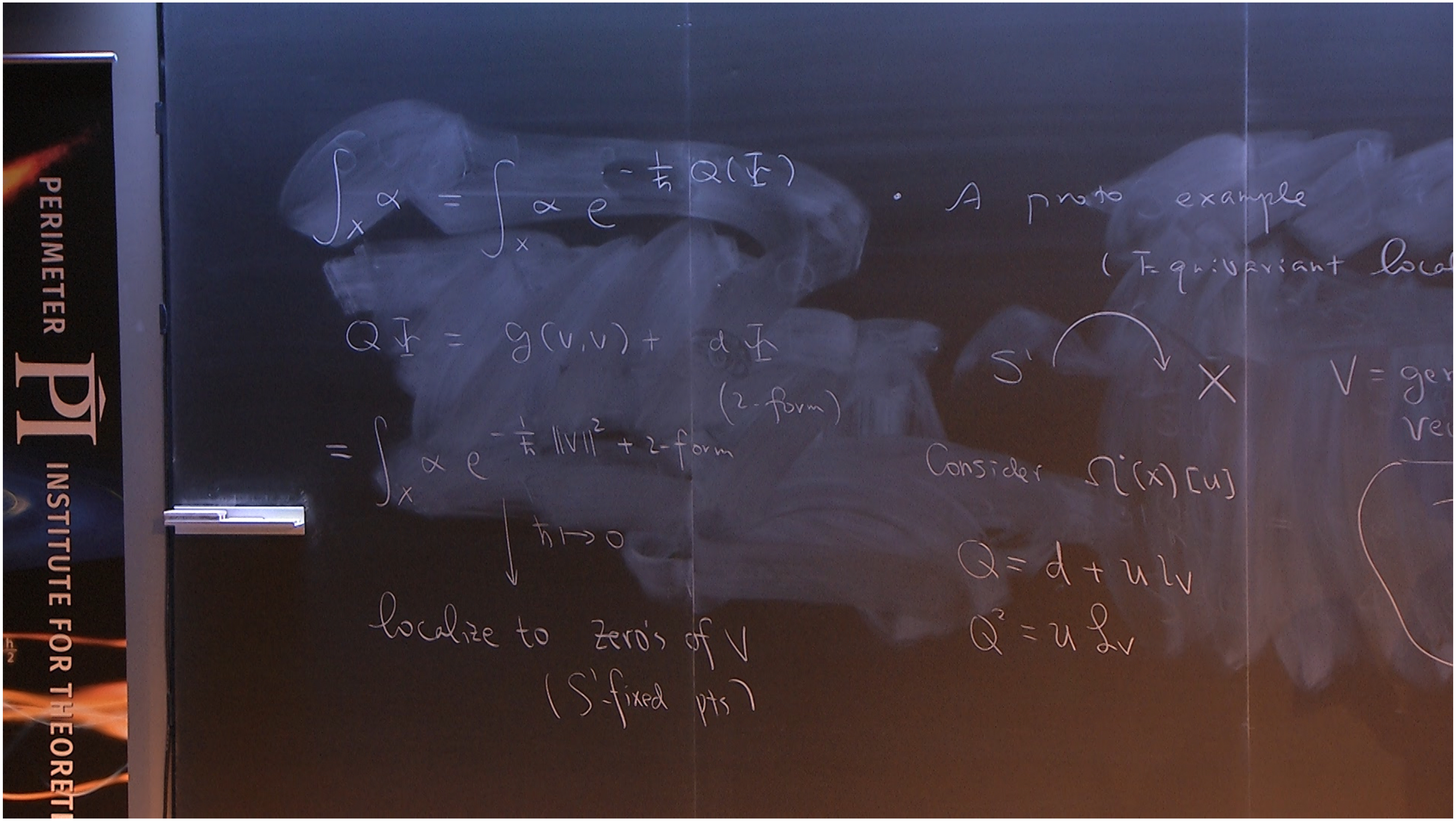


4

(2) 4

(3)


$\Omega_{S^1}(X) := \left(\Omega(X)[u]\right)^{S^1}$
$\Rightarrow \left(\Omega_{S^1}(X), Q\right)$
$\int_X : \Omega_{S^1}(X) \longmapsto \mathbb{R}[u]$
$\int_X Q(-) = 0$
Let $Q\alpha = 0$
$\int_X \alpha$
Let g be a S^1-inv. metric on X
$\Psi = \frac{1}{u} g(v,-)$
(1-form)


$\int_X \alpha = \int_X \alpha \, e^{-\frac{1}{\hbar} Q(\Phi)}$
• A proto example
(Equivariant loca
$Q\Phi = g(v,v) + d\Phi$
(2-form)
$= \int_X \alpha \, e^{-\frac{1}{\hbar}\|v\|^2 + \text{2-form}}$
$\hbar \mapsto 0$
localize to zero's of V
(S^1-fixed pts)
$S^1 \curvearrowright X$
V = ge
ve
Consider $\Omega^\bullet(X)[u]$
$Q = d + u\,\iota_v$
$Q^2 = u\,\mathcal{L}_v$
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$\propto e^{-\frac{1}{\hbar} Q(\Psi)}$

$(V,V) + d\Psi$

(2-form)

$\|V\|^2 + 2$-form

$\hbar \mapsto 0$

o zero's of V

(S^1-fixed pts)

Eg A-model

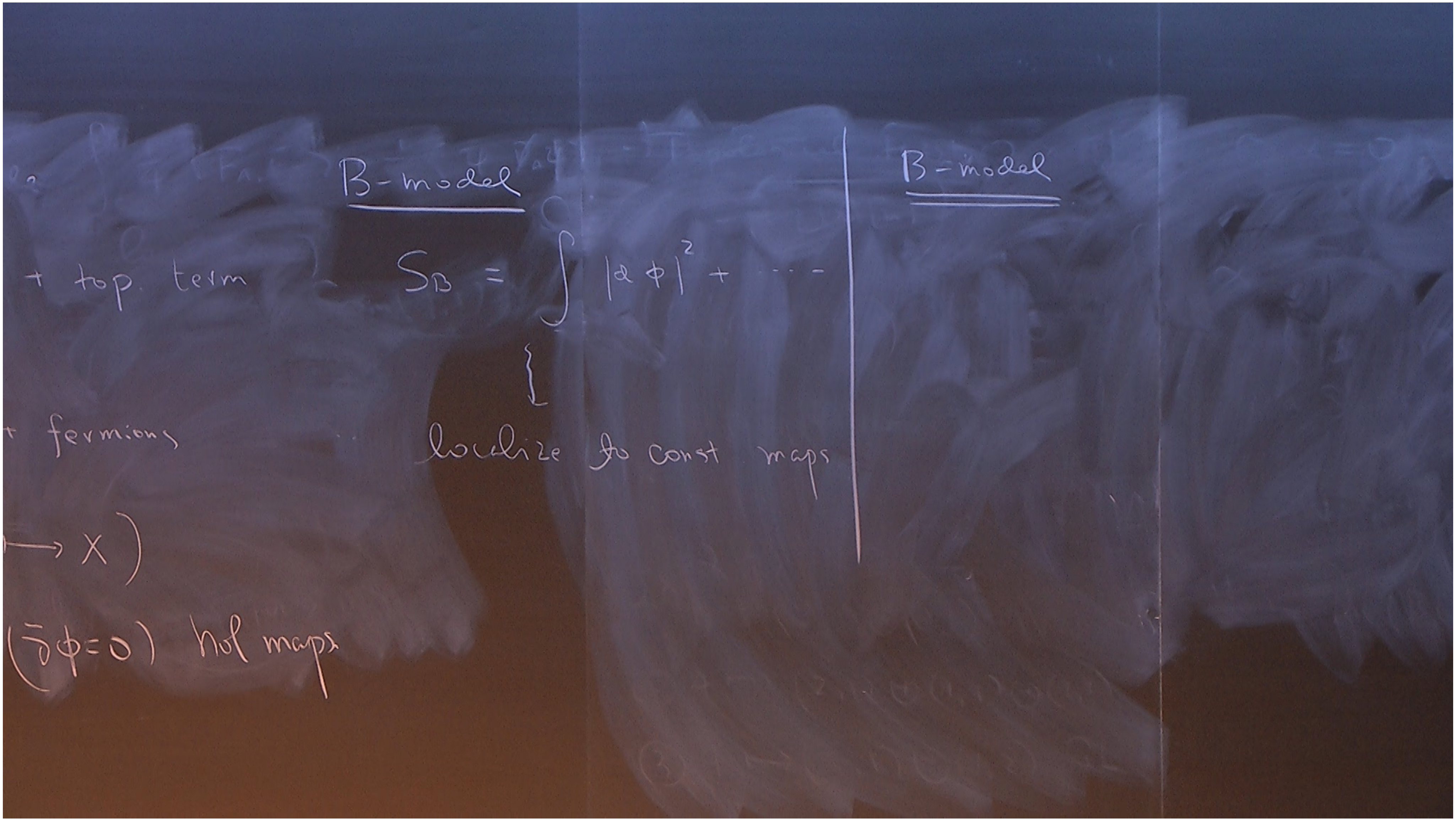
$S_A = Q_A \int - \; +$ top. term

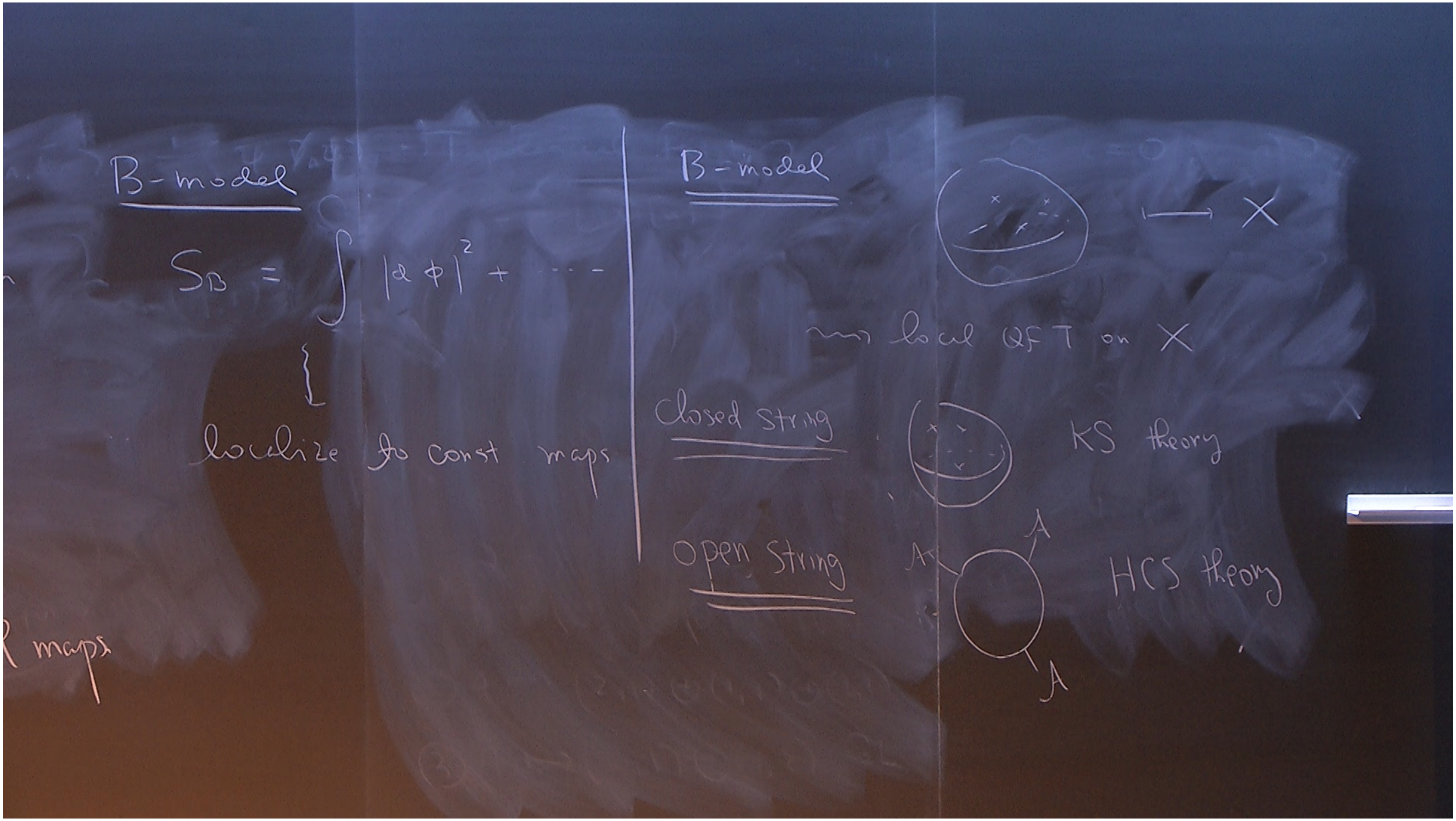
$\| $

$\int |\bar\partial \phi|^2 +$ fermions

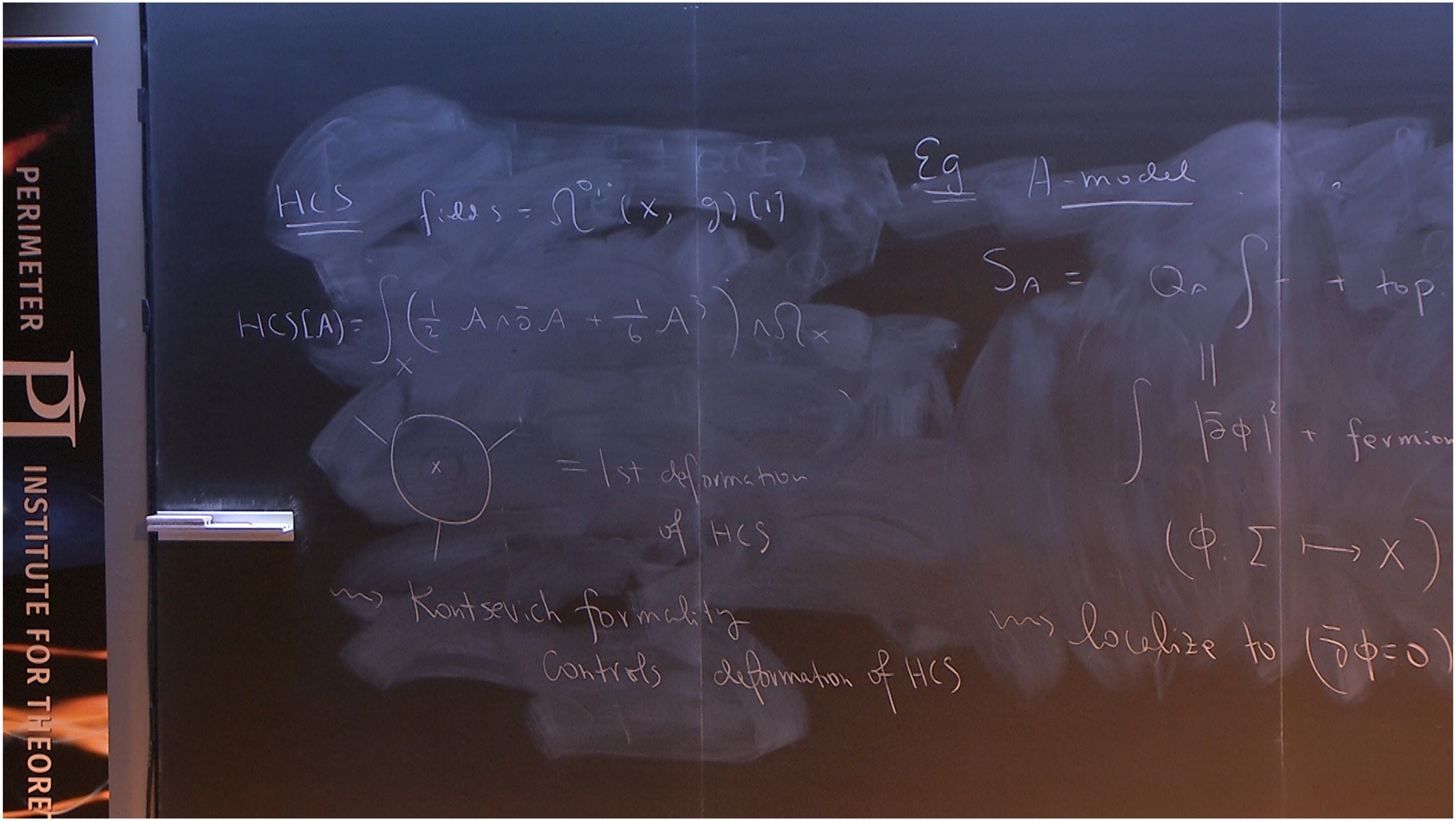
$(\phi : \Sigma \longmapsto X)$

$\rightsquigarrow$ localize to $(\bar\partial\phi = 0)$ hol maps


B-model
+ top. term
S_B = ∫ |∂φ|² + ...
+ fermions
localize to const maps
→ X)
(∂̄φ=0) hol maps
B-model



maps
B-model
$S_B = \int |\partial \phi|^2 + \cdots$
localize to const maps
B-model
$\longmapsto X$
$\rightsquigarrow$ local QFT on X
Closed String
KS theory
Open String
A
A
A
HCS theory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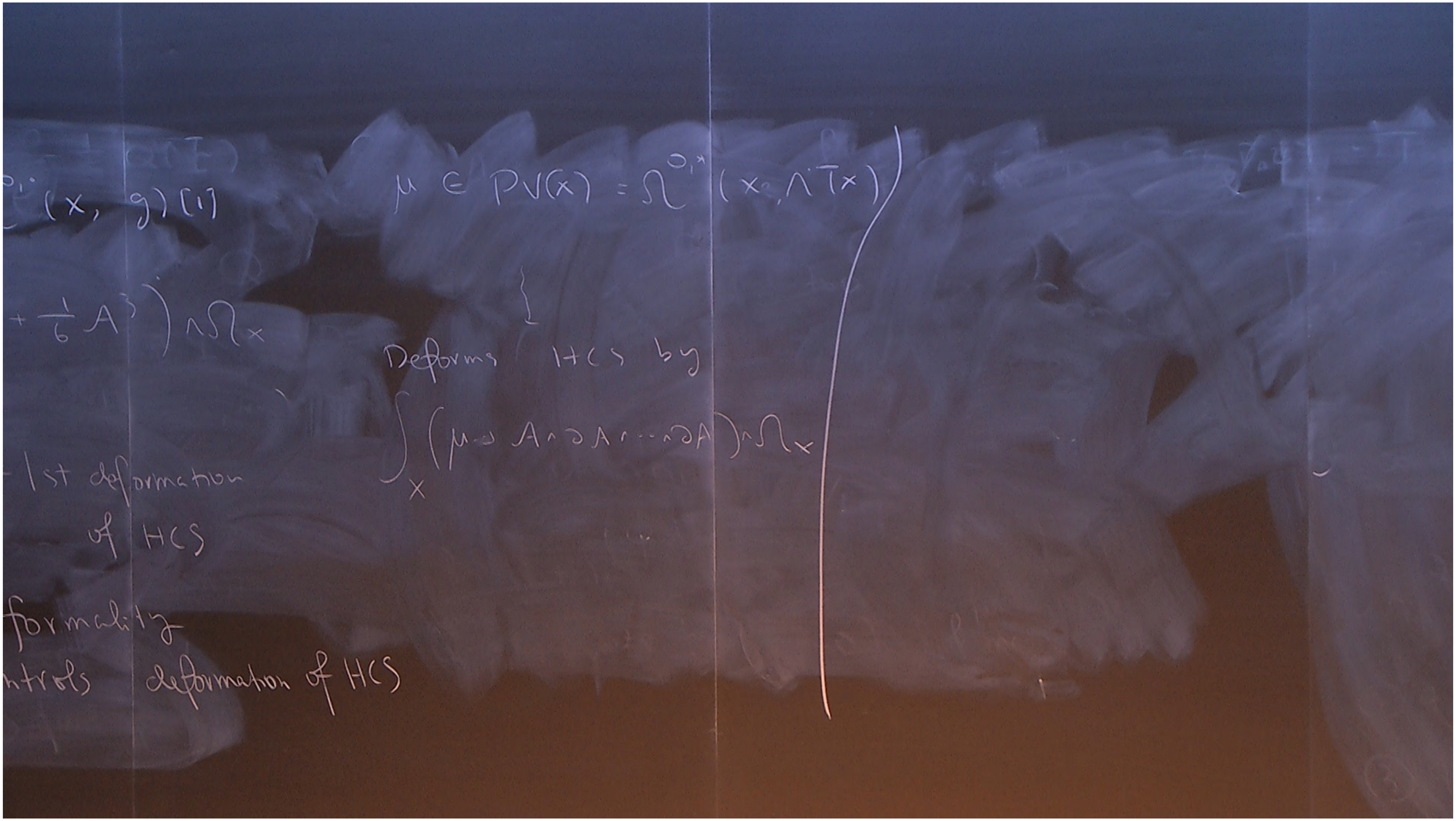
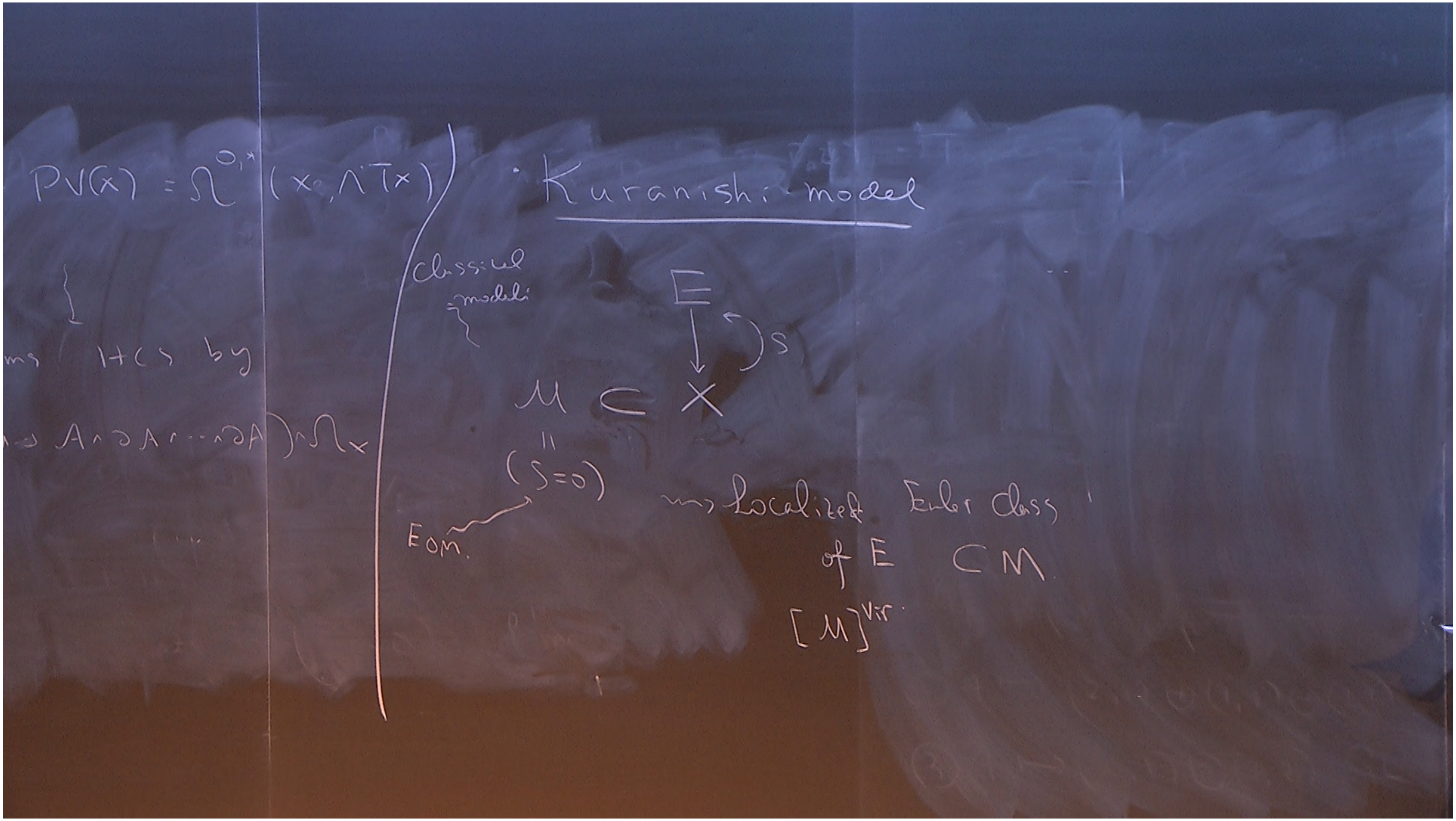
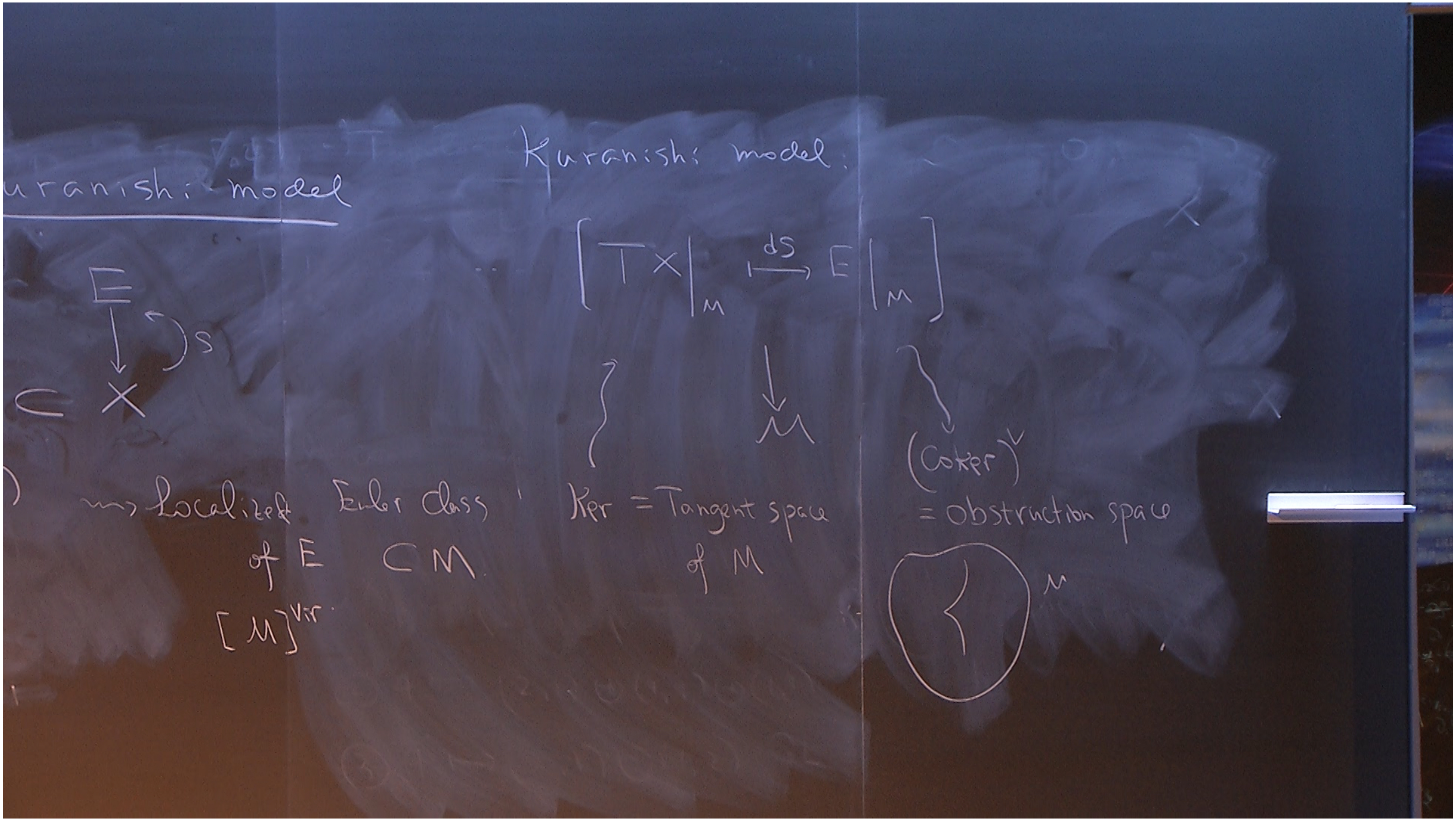
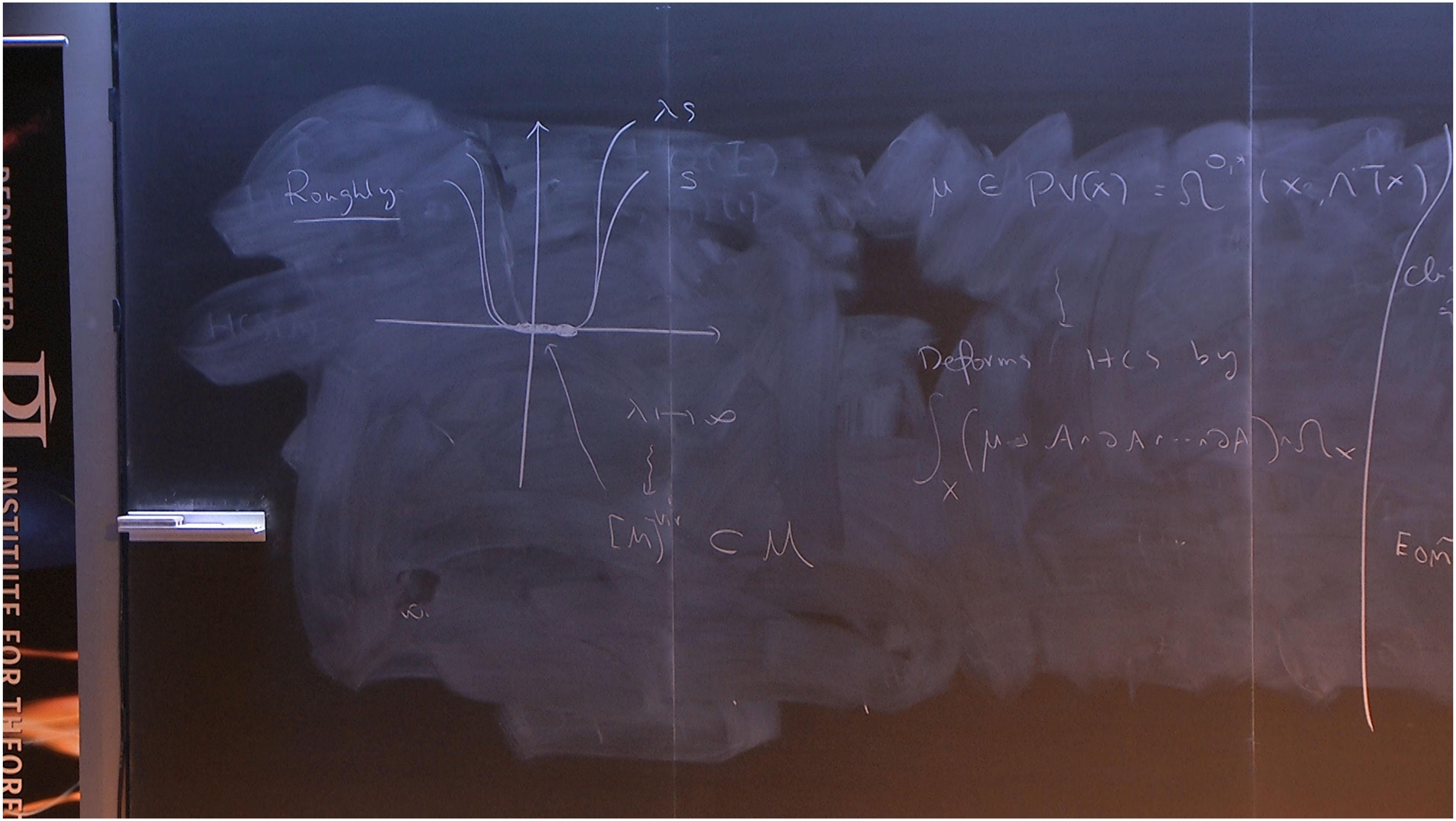
HCS
fields = $\Omega^{0,*}(X, \mathfrak{g})[1]$
$HCS[A] = \int_X \left(\frac{1}{2} A \wedge \bar\partial A + \frac{1}{6} A^3\right) \wedge \Omega_X$
= 1st deformation of HCS
⇝ Kontsevich formality
controls deformation of HCS
Eg A-model
$S_A = Q_A \int - + \text{top.}$
$\int |\bar\partial \phi|^2 + \text{fermion}$
$(\phi : \Sigma \longmapsto X)$
⇝ localize to $(\bar\partial\phi = 0)$



(x, g)[1]
μ ∈ PV(X) = Ω^{0,*}(X, ∧^• TX)
+ 1/6 A^3) ∧ Ω_X
Deforms HCS by
∫_X (μ ⌟ A∧∂A∧⋯∧∂A)∧Ω_X
1st deformation
of HCS
formality
ntrols deformation of HCS


PV(X) = Ω^{0,*}(X, ∧TX)
Kuranishi model
Classical moduli
E
s
M ⊂ X
(s=0)
EOM.
localized Euler class
of E ⊂ M
[M]^{vir}


uranishi model
E
S
X
Localized Euler class
of E ⊂ M
[M]^vir
Kuranishi model:
[TX|_M ⟶dS E|_M]
M
Ker = Tangent space
of M
(Coker)^∨
= obstruction space


Roughly
λs
S
λ ↦ ∞
[M]^vir ⊂ M
μ ∈ PV(X) = Ω^{0,*}(X, ∧^• TX)
Deforms HCS by
∫_X (μ ⌟ A∧∂A∧⋯∧∂A)∧Ω_X
EOM

Mathai-Quillen formalism

$$\begin{array}{c} E \\ \downarrow \circlearrowleft s \\ X \end{array} \qquad \begin{cases} \text{metric } \langle \cdot , \cdot \rangle \text{ on } E \\ \text{Connection } \nabla \end{cases}$$

Consider the super mfd

$$T[1](E[-1])$$
$$\|$$
$$T[1]X \oplus E[-1] \oplus E$$
$$\downarrow$$
$$X$$

hai-Quillen formalism

$E \xrightarrow{} X$, section s

$\left\{ \begin{array}{l} \text{metric } \langle -,- \rangle \text{ on } E \\ \text{Connection } \nabla \end{array} \right.$

der the super mfld

$$T[1](E[-1]) = T[1]X \oplus E[-1] \oplus E \to X$$

Choose local basis $\{e_\alpha\}$ for E and local coordinates

	x^μ	ψ^μ	χ^α	B^α
deg	0	1	-1	0
	X	$T[1]X$	$E[-1]$	E

δ = de Rham diff.

$$\delta x^\mu = \psi^\mu \qquad \delta \chi^\alpha = B^\alpha$$

Kuranishi model:

$$\left[TX|_M \xrightarrow{ds} E \right] \to M$$

Ker = Tangent space of M

Quillen formalism

Choose local basis $\{e_\alpha\}$ (for E) and local coordinates

Kuranishi model:

$$\left[\, TX\big|_M \xrightarrow{dS} E\big|_M \,\right]$$

$)s$ $\left\{ \begin{array}{l} \text{metric } \langle \cdot , \cdot \rangle \text{ on } E \\ \text{Connection } \nabla \end{array} \right.$

the super mfd

$[1](E[-1])$

$=$

$T[1]X \oplus E[-1] \oplus E$

$\downarrow$

X

	X^μ	ψ^μ	χ^α	B^α
deg	0	1	-1	0
	X	$T[1]X$	$E[1]$	E

δ = de Rham diff.

$\delta X^\mu = \psi^\mu \quad \delta \chi^\alpha = B^\alpha$

$\nabla e_\alpha = A_\alpha^\beta e_\beta$

Connection 1-form

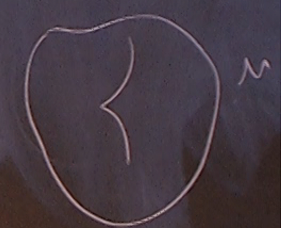
$\downarrow$

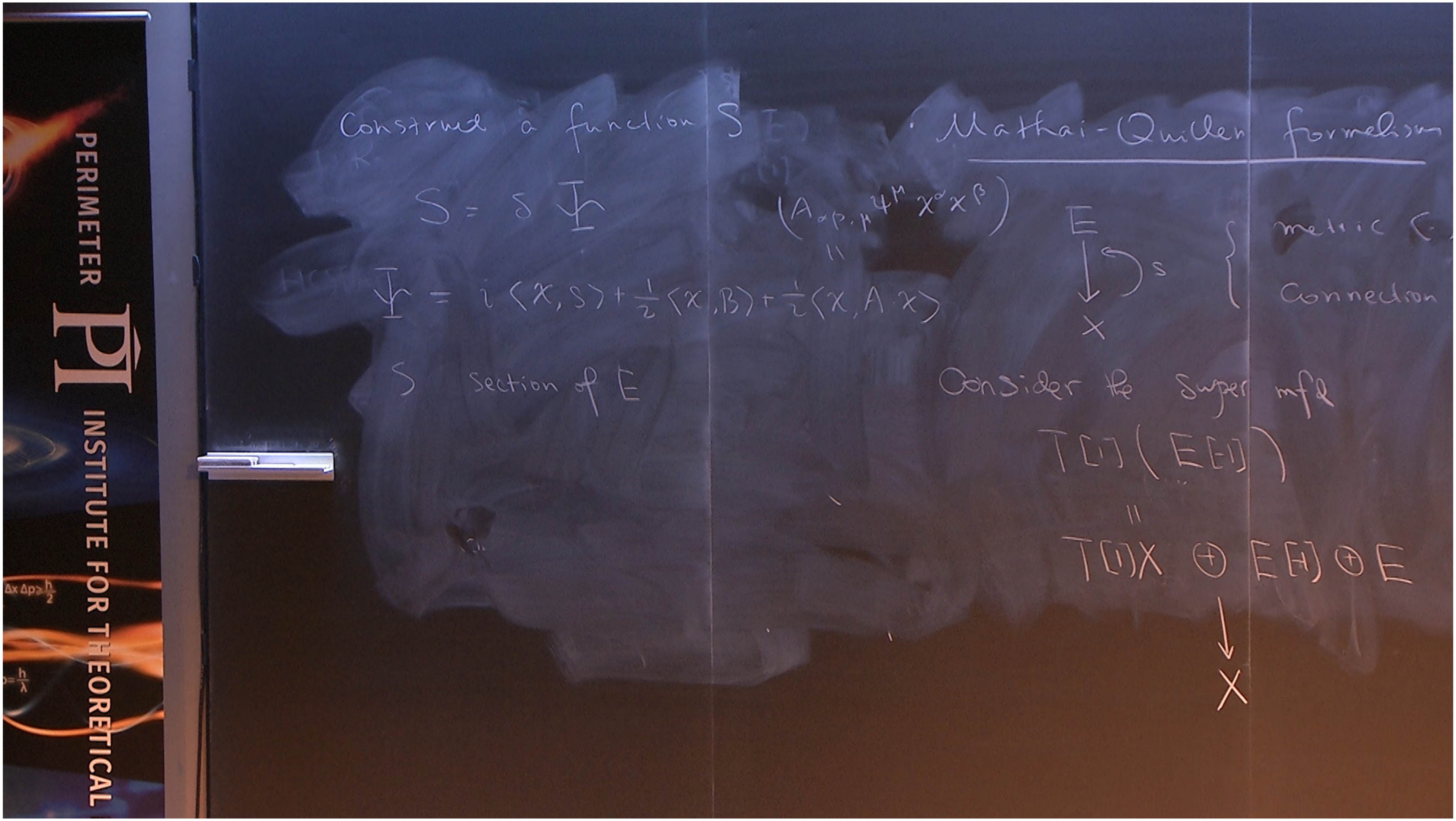
M

Ker = Tangent space of M

$(\text{Coker})^\vee$

= obstructi

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Construct a function S
S = δ Ψ
(A_{αβ,μ} ψ^μ χ^α χ^β)
Ψ = i⟨χ, s⟩ + 1/2 ⟨χ, B⟩ + 1/2 ⟨χ, A·χ⟩
s section of E
Mathai-Quillen formalism
E ↓ X s
metric
connection
Consider the super mfd
T[1](E[-1])
= T[1]X ⊕ E[-1] ⊕ E
↓
X

Construct a function S

$\delta \Psi$

$$\left(A_{\alpha\beta,\mu} \psi^{\mu} \chi^{\alpha} \chi^{\beta}\right)$$

$$\Psi = i\langle \chi, s\rangle + \frac{1}{2}\langle \chi, B\rangle + \frac{1}{2}\langle \chi, A\cdot\chi\rangle$$

s section of E

$e^{-\delta\Psi}$ is δ-closed

MQ Construction

$$e_{s,\nabla}(\xi) = \int d\chi\, dB\, e^{-\delta\Psi}$$

is a closed diff form on X

whose de Rham class is indep. of the choice of s, ∇

deg

$$= \frac{1}{(2\pi)^{\dim(x)}} \int dx\, e^{-\frac{1}{2}\langle s,s\rangle + \frac{1}{2}\langle x, Rx\rangle + i\langle x, \nabla s\rangle}$$

MQ Construction

$$e_{s,\nabla}(\xi) = \int dx\, dB\, e^{-\delta \Psi}$$

is a closed diff form on X

whose de Rham class is indep. of the choice of s, ∇

R: curvature 2-form

$$s \longmapsto \lambda s$$

$\lambda \mapsto 0$: $\frac{1}{(2\pi)} \int dx\, e^{\frac{1}{2}\langle x, Rx\rangle} = Pf(R)$

$\lambda \mapsto \infty$: localize to $(s=0)$

Kuranishi model:

$$TX|_M \xrightarrow{ds} E$$

$$\downarrow$$

$$M$$

Ker = Tangent space of M

$$= \frac{1}{(2\pi)^{\dim(X)}} \int dx\, e^{-\frac{1}{2}\langle s,s\rangle + \frac{1}{2}\langle x, Rx\rangle + i\langle x, \nabla s\rangle}$$

R: Curvature 2-form

$$S \longmapsto \lambda S$$

$\underline{\lambda \mapsto 0}$: $\frac{1}{(2\pi)^{\cdot}} \int dx\, e^{\frac{1}{2}\langle x, Rx\rangle}$

$$= Pf(R)$$

$\underline{\lambda \mapsto \infty}$ localize to $(s=0)$

$\underline{\underline{Eg}}$ (A-model) $\Sigma \xrightarrow{\varphi} X$

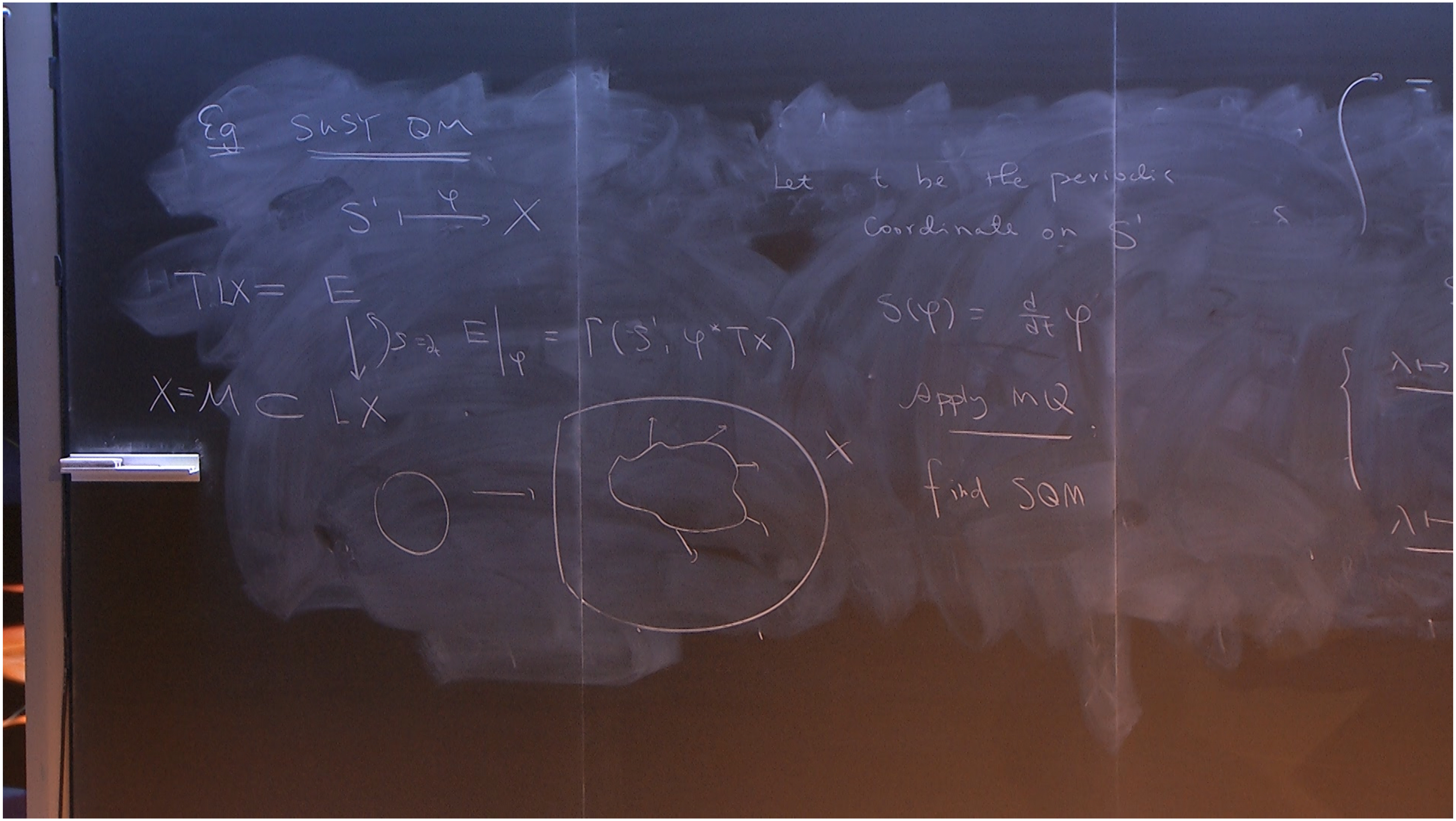
$$E \quad E|_{\varphi} = \Gamma(\Sigma, Hom(T^{0,1}_{\Sigma}, \varphi^* T^{1,0}_X))$$

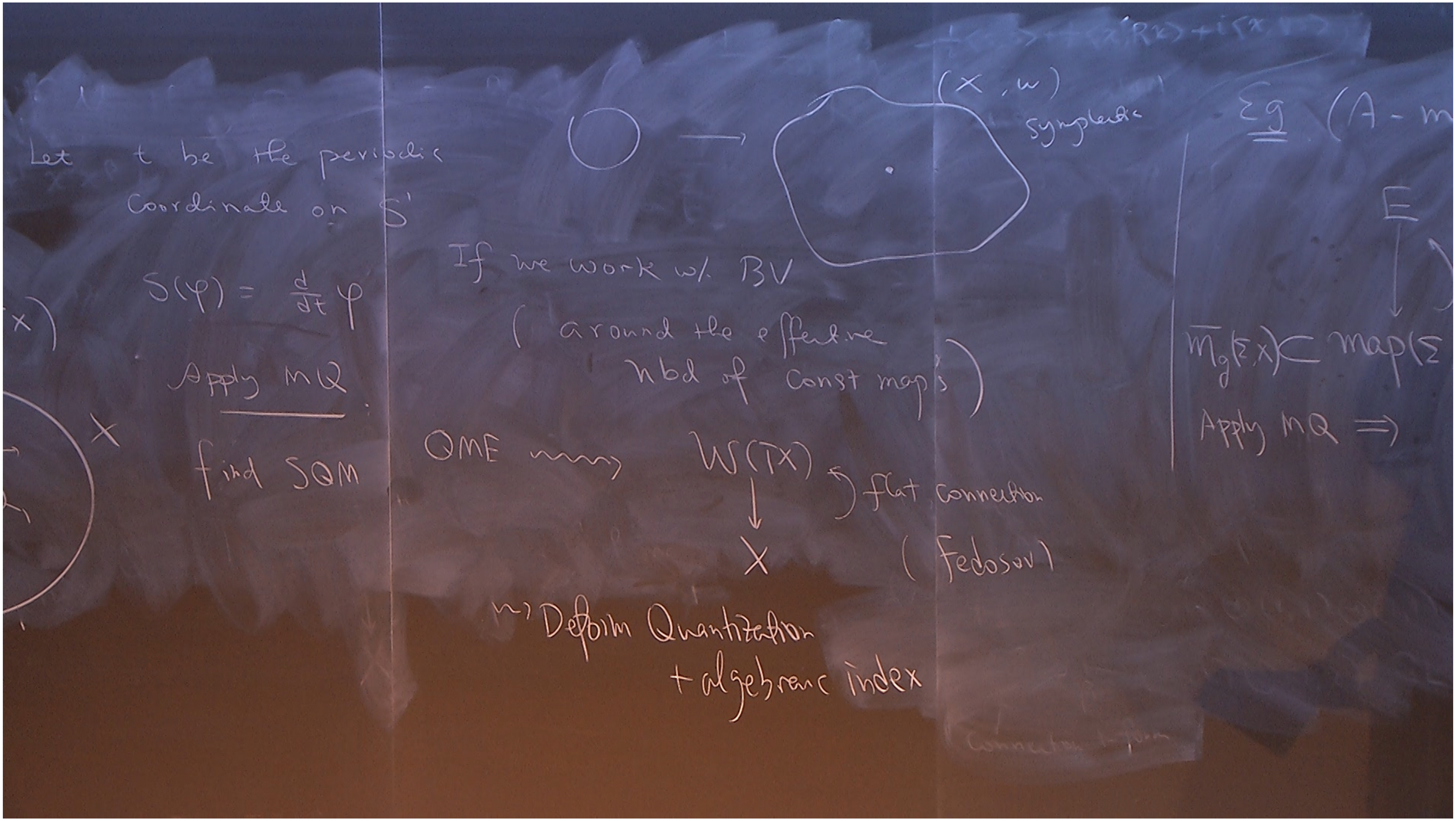
$$\downarrow \big) s = \bar\partial \qquad s(\varphi) = \bar\partial \varphi$$

$$\overline{m}_g(\Sigma, X) \subset map(\Sigma, X)$$

Apply MQ $\Rightarrow$ $\Rightarrow$ A-model action

$$+ \ [\overline{m}_g(\Sigma, X)]^{vir}.$$


Eg SUSY QM
S' ↦ X
TLX = E
S = ∂t
E|φ = Γ(S', φ*TX)
X = M ⊂ LX
Let t be the periodic coordinate on S'
S(φ) = d/dt φ
Apply MQ
find SQM


Let t be the periodic
Coordinate on S'
S(φ) = d/dt φ
Apply MQ
find SQM
(X, ω)
symplectic
If we work w/ BV
(around the effective
nbd of const maps)
QMF ⇝ W(TX)
flat connection
X
(Fedosov)
⇝ Deform Quantization
+ algebraic index
Σg (A - m
E
M̄g(Σ,X) ⊂ map(Σ
Apply MQ ⟹

(X, ω) symplectic

Σ_g (A-model) $\Sigma \xrightarrow{\varphi} X$

f we work w/ BV

(around the effective nbd of const maps)

$E \rightsquigarrow W(TX) \to X$ flat connection (Fedosov)

$\rightsquigarrow$ Deform Quantization + algebraic index

$Q + [I, -]_*$

$E \xrightarrow{S = \bar\partial} \overline{M}_g(\Sigma, X) \subset \mathrm{map}(\Sigma, X)$

$E|_\varphi = \Gamma(\Sigma, Hom(T^{0,1}_\Sigma, \varphi^*$

$S(\varphi) = \bar\partial\varphi$

Apply HQ $\Rightarrow$ $\Rightarrow$ A-model action + top. term

$[\overline{M}_g(\Sigma, X)]^{vir}$

$QI + \hbar\Delta I + \frac{1}{2}\{I, I\} = 0$

renormalize

$QI + \frac{1}{2}[I, I]_* = 0$