Title: Lecture 6: Factorization Algebras and the General Structure of QFT

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Collection: QFT for Mathematicians

Date: June 26, 2019 - 11:00 AM

URL: http://pirsa.org/19060020

Noether's Theorem

as formulated by Costello-Guillian

statement of symmetry $\subset \mathcal{E}$

$\bigcup_{\alpha}^{BD} \Sigma \longrightarrow \mathbb{R}^{0|2}$

→ First
classical, 0-dim'l, unshifted
(= Hamiltonian mechanics)

(X, ω) symplectic manifold

(symetry) = $(\mathbb{R}^{0|2} \longrightarrow \mathrm{Symplect}(X))$

(imp'ble observables) = $(\mathbb{R}^{0|2} \longrightarrow \mathcal{O}(X))$

$\Rightarrow (\mathcal{O}(X), \{,\}, \tau)$

$\mathbb{R}^{0|2} \longrightarrow \mathrm{Symplect}(X)$

al, 0-dim'l, unshifted

amiltonian mechanics)

ω) symplectic manifold

(g → Symplect(X))

ariables) = (g → O(X))

→ (O(X), {,}) f

→ Symplect(X)

We have ∧ class in $H^1(X)$

as an obstruction.

$$0 \to \mathfrak{c} \to \mathfrak{g} \to \mathfrak{f} \to \cdots$$

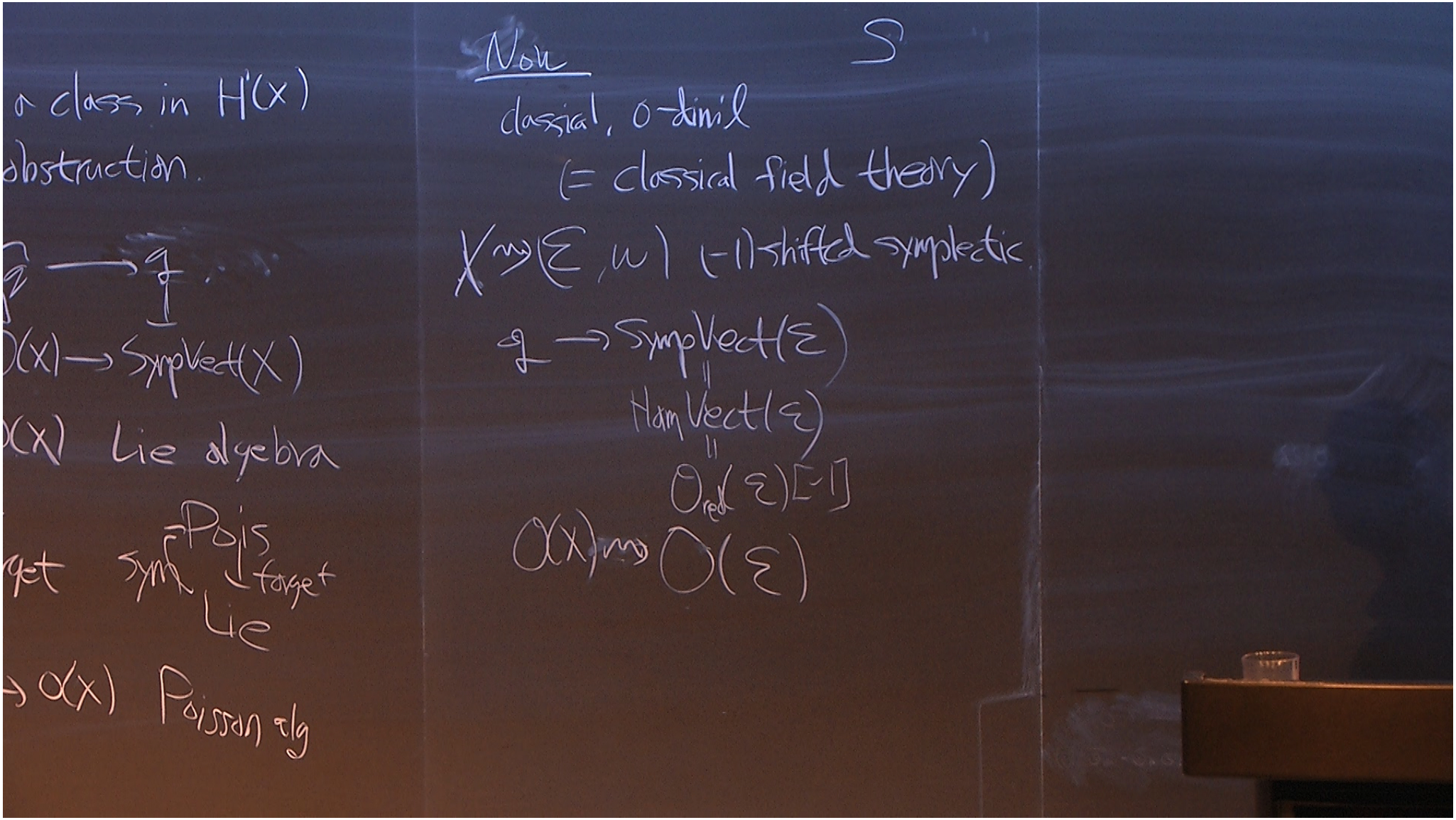
$$0 \to \mathbb{C} \to O(X) \to \mathrm{Symplect}(X)$$

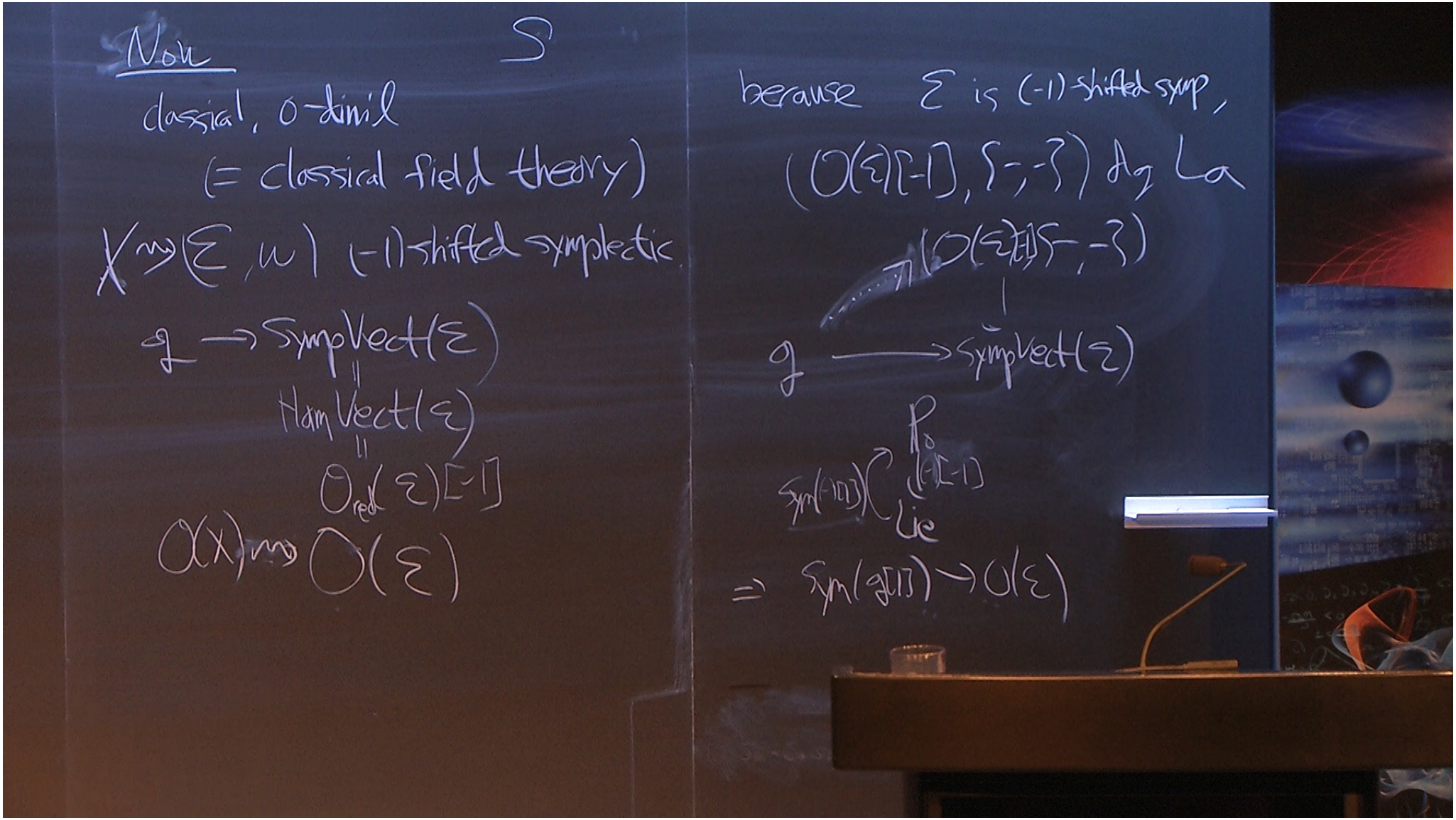
$\mathfrak{g} \to O(X)$ Lie algebra

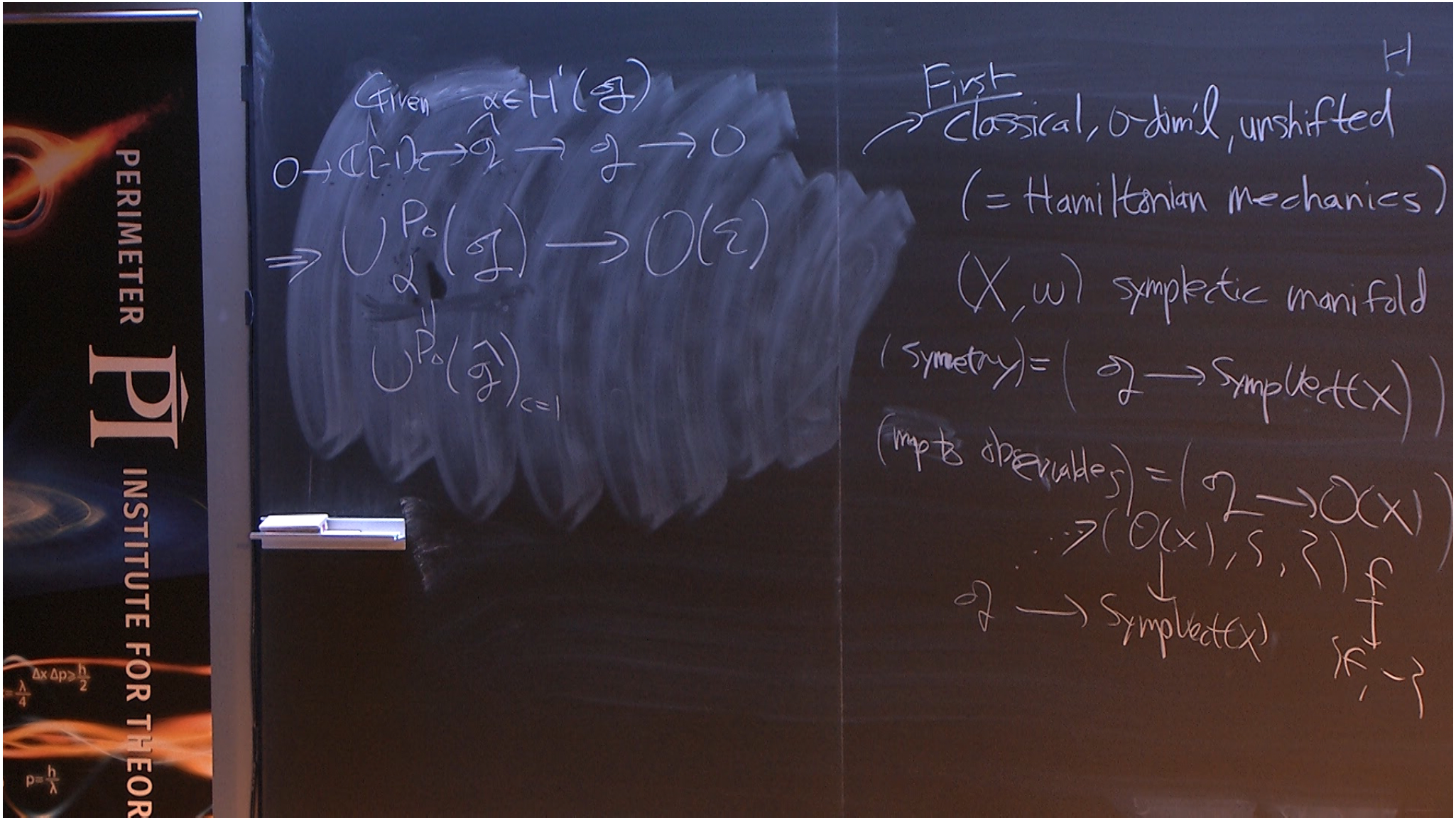
$U(\mathfrak{g})$ —Ass, forget→ Lie

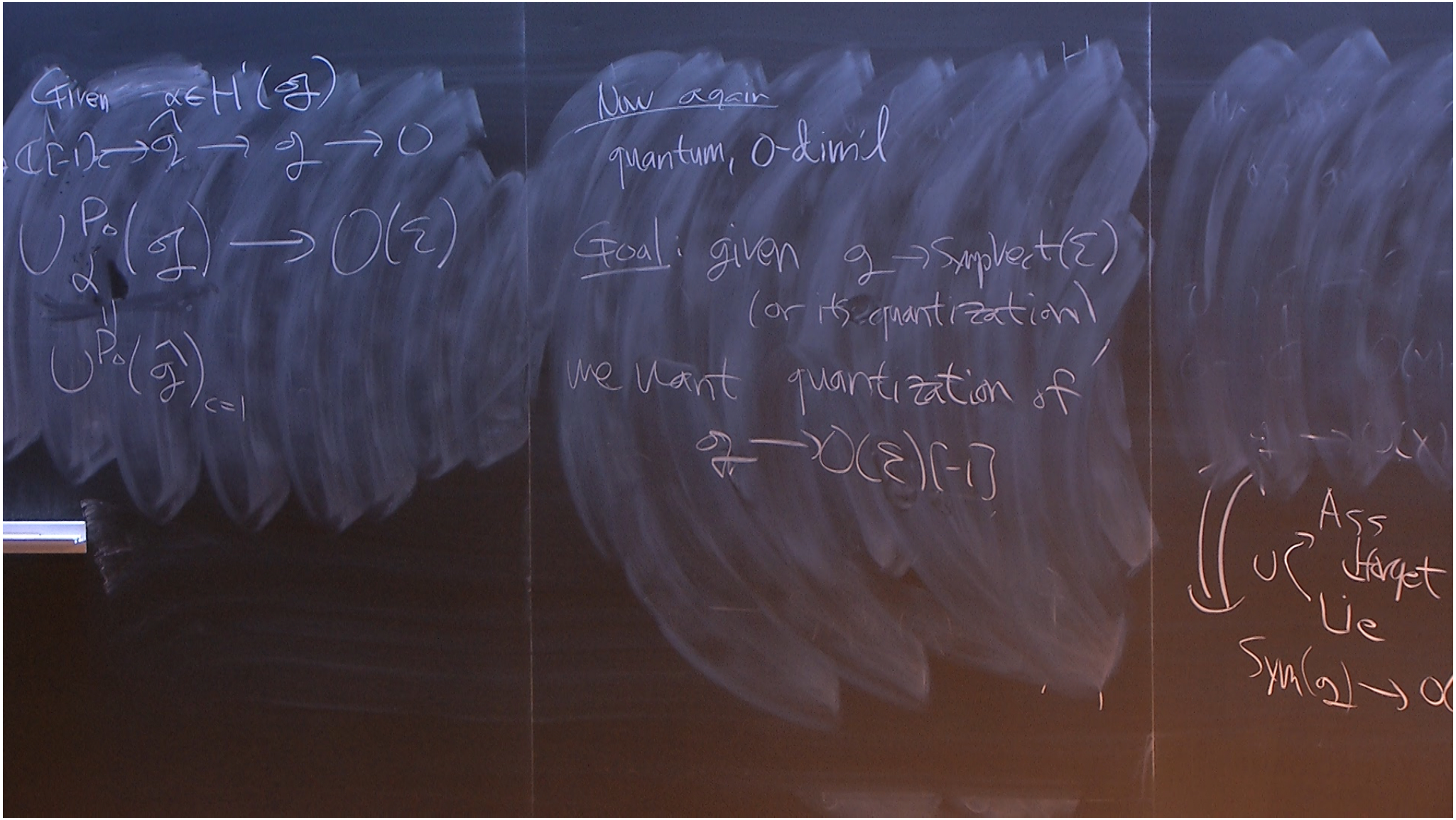
$\mathrm{Sym}(\mathfrak{g})$ —Pois, forget→ Lie

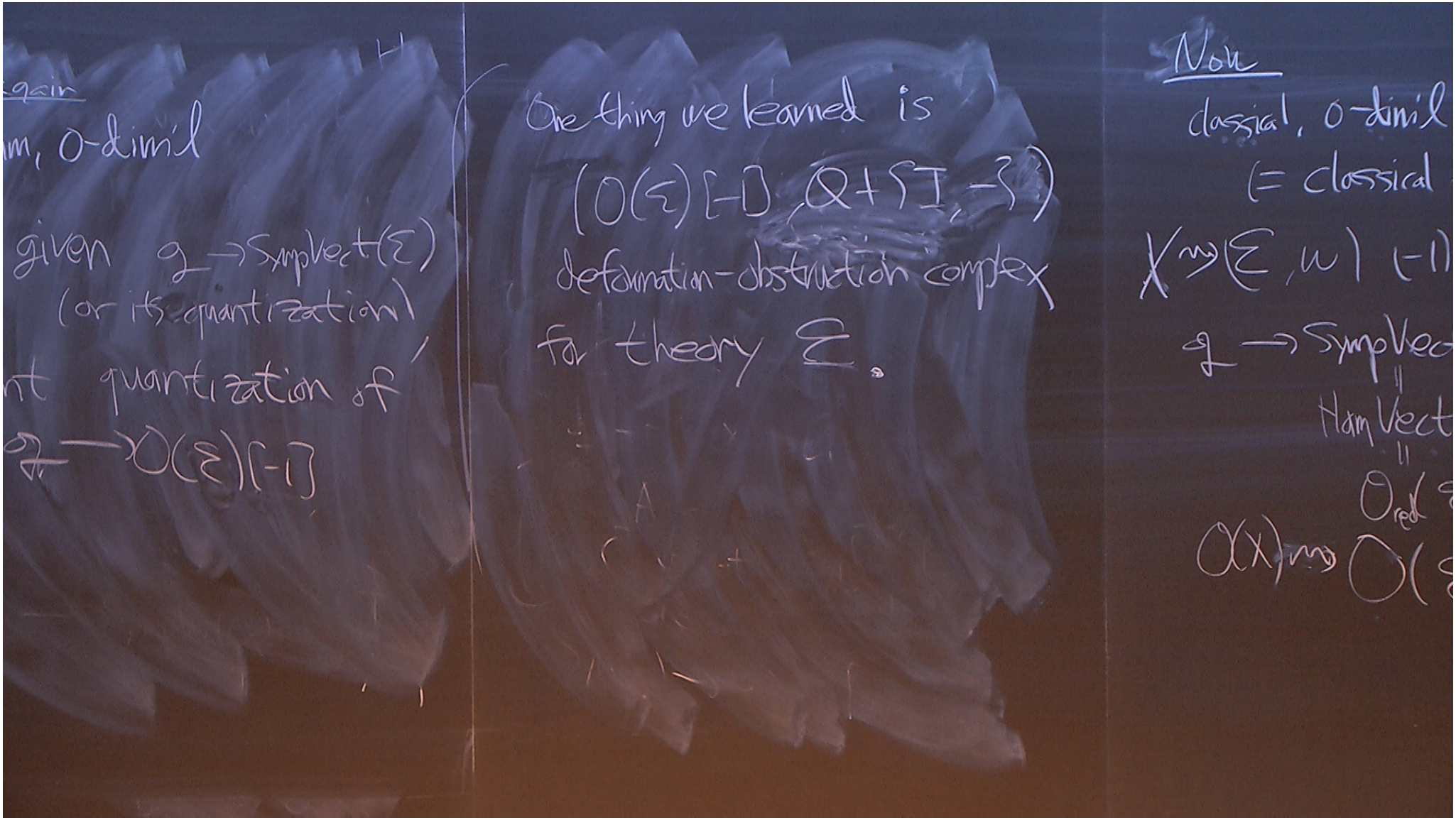
$\mathrm{Sym}(\mathfrak{g}) \to O(X)$ Poisson alg

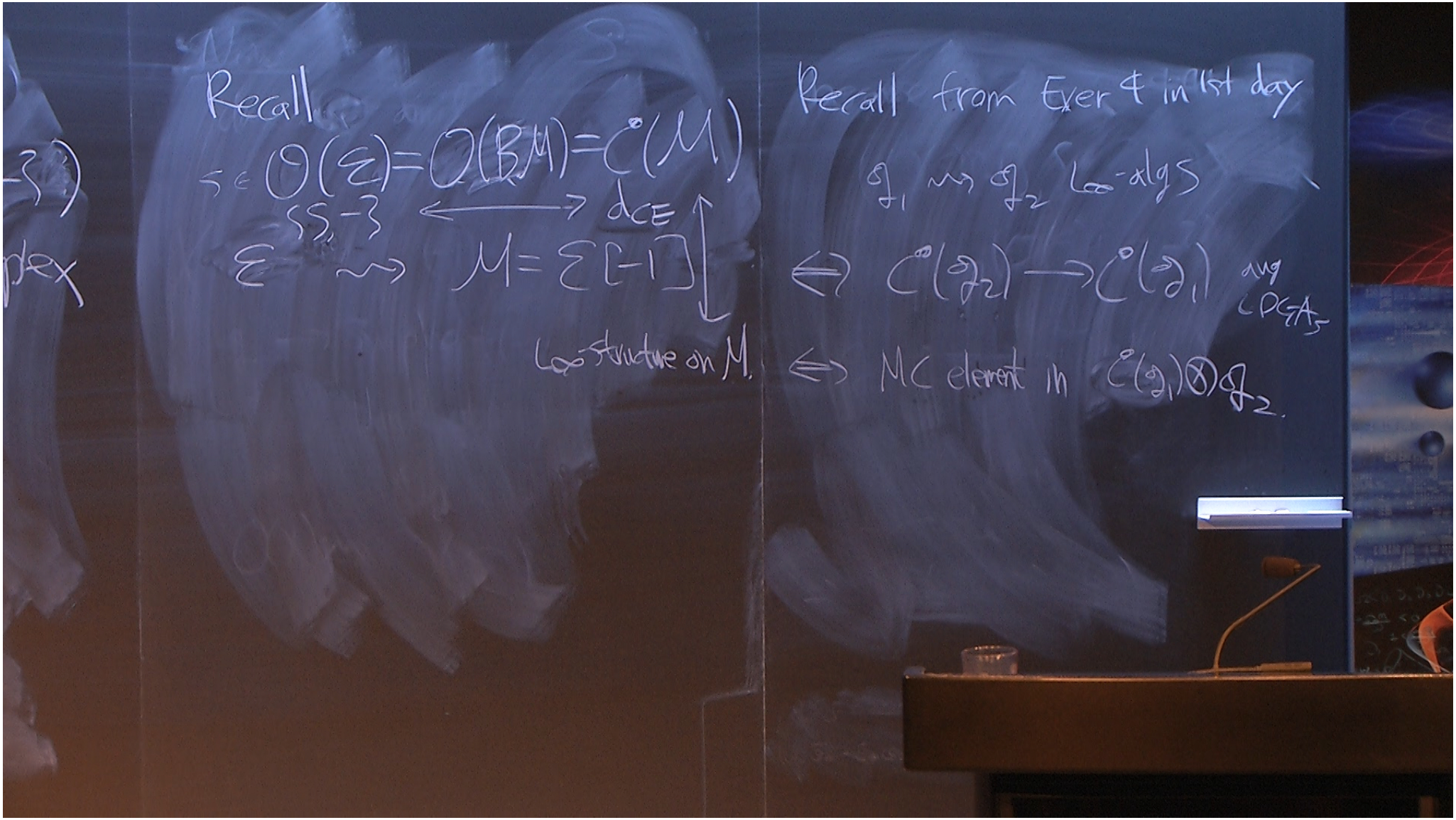

class in H'(X)
obstruction.
)(X) → SympVect(X)
)(X) Lie algebra
Pois
get Sym forget
Lie
→ O(X) Poisson alg
Non
classical, 0-dim'l
(= classical field theory)
X ≃ (Σ, ω) (-1)-shifted symplectic.
g → SympVect(Σ)
HamVect(Σ)
O_red(Σ)[-1]
O(X) ≃ O(Σ)

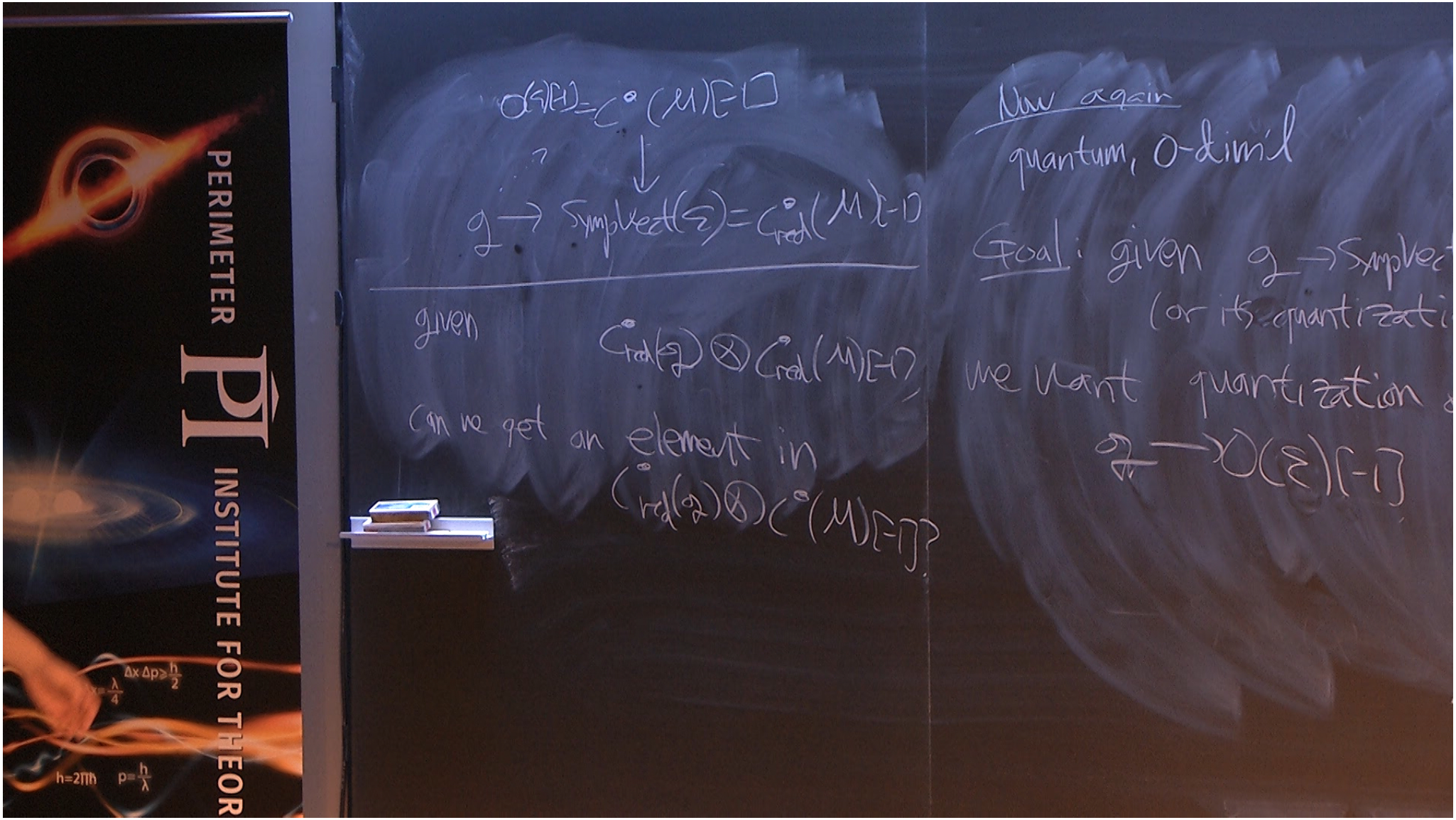

Non
classical, 0-dim'l
(= classical field theory)
$\mathcal{X} \rightsquigarrow (\mathcal{E}, \omega)$ (-1)-shifted symplectic
$\mathfrak{g} \to \mathrm{SymplVect}(\mathcal{E})$
$\mathrm{HamVect}(\mathcal{E})$
$\mathcal{O}_{red}(\mathcal{E})[-1]$
$\mathcal{O}(\mathcal{X}) \rightsquigarrow \mathcal{O}(\mathcal{E})$
because $\mathcal{E}$ is (-1)-shifted symp,
$(\mathcal{O}(\mathcal{E})[-1], \{-,-\})$ A_∞ La
$(\mathcal{O}(\mathcal{E})[1], \{-,-\})$
$\mathfrak{g} \longrightarrow \mathrm{SymplVect}(\mathcal{E})$
$\mathrm{Sym}(\mathfrak{g}[1])$
Lie
$\Rightarrow \mathrm{Sym}(\mathfrak{g}[1]) \to \mathcal{O}(\mathcal{E})$

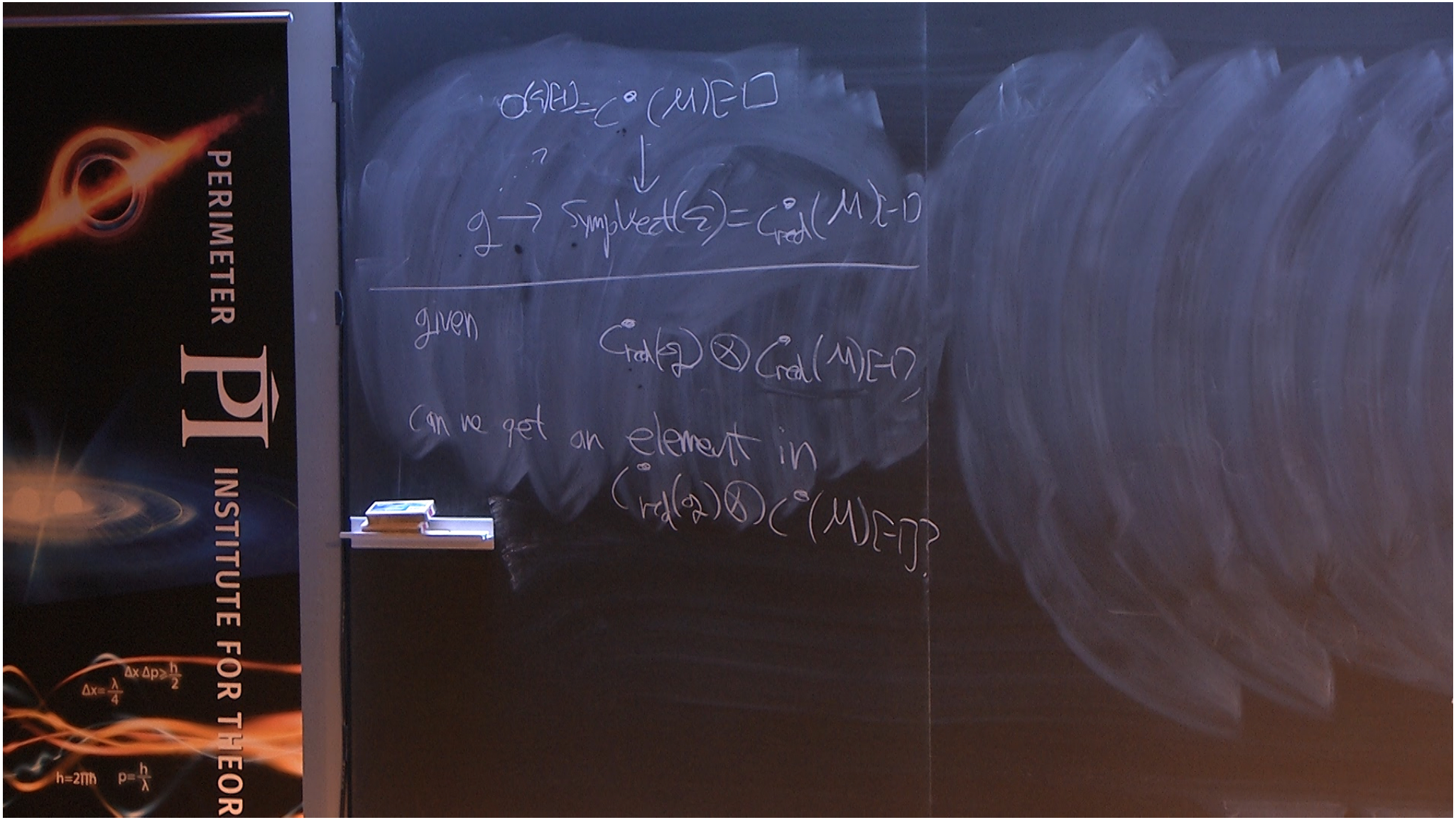

PERIMETER
INSTITUTE FOR THEOR
Given
$\alpha \in H^1(\mathfrak{g})$
$0 \to \mathbb{C}[-1] \to \mathfrak{a} \to \mathfrak{g} \to 0$
$\Rightarrow \bigcup_{\alpha} P_0(\mathfrak{g}) \to O(\mathcal{E})$
$\bigcup_{\alpha} P_0(\mathfrak{g})_{cl}$
First
classical, 0-dim'l, unshifted
H^1
(= Hamiltonian mechanics)
(X, ω) symplectic manifold
(symmetry) = $(\mathfrak{g} \to \mathrm{Symplect}(X))$
(mptd observables) = $(\mathfrak{a} \to O(X))$
$\to (O(X), \{,\}, \cdot)$
$\mathfrak{g} \to \mathrm{Symplect}(X)$

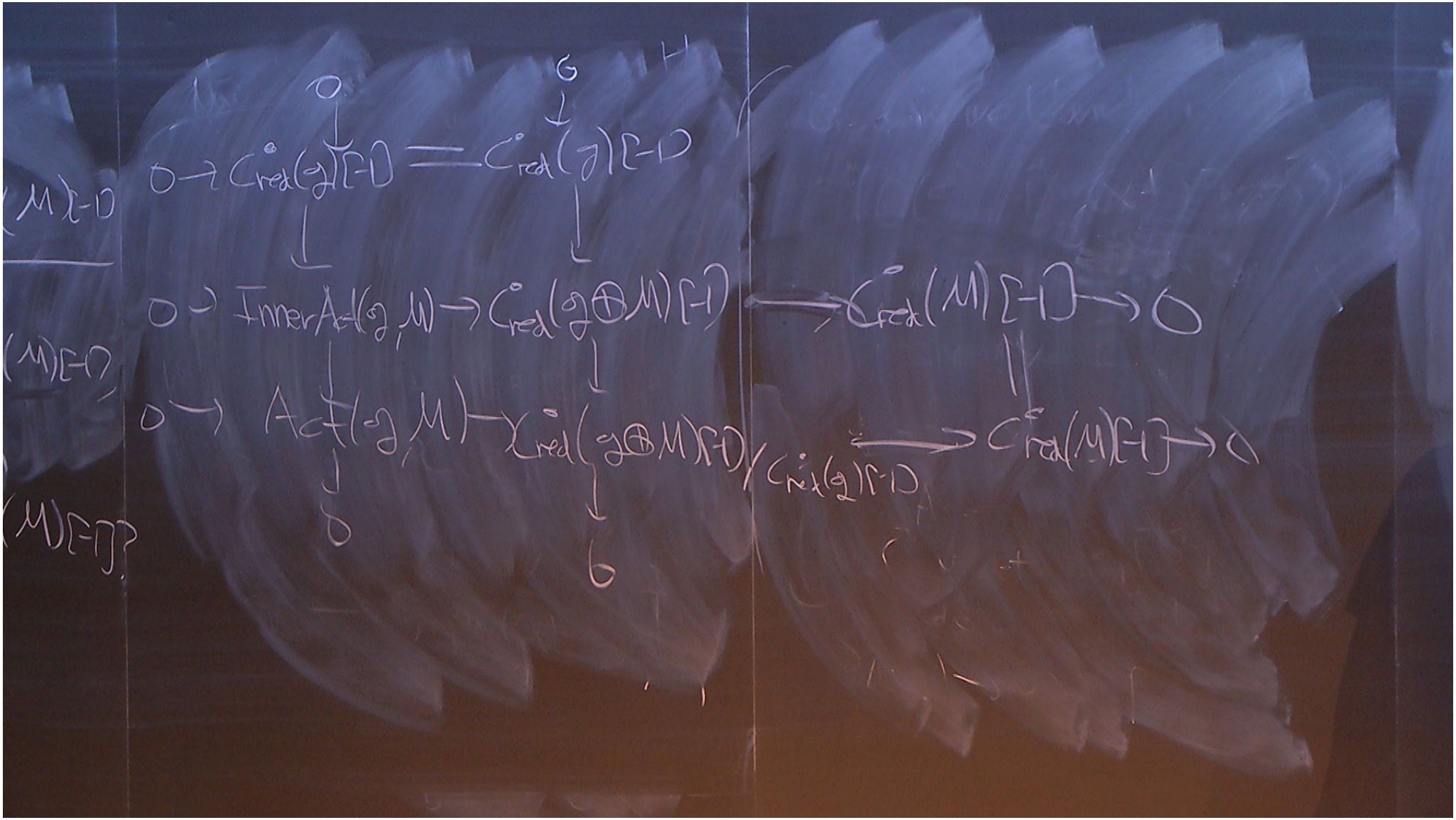

Given $\alpha \in H^1(\mathfrak{g})$
$\mathbb{C}[-1] \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$
$\cup P_0(\mathfrak{g}) \to O(\Sigma)$
α
$\cup P_0(\tilde{\mathfrak{g}})_{c=1}$
New again
quantum, 0-dim'l
Goal: given $\mathfrak{g} \to \mathrm{Symplect}(\Sigma)$
(or its quantization)
we want quantization of
$\mathfrak{g} \to O(\Sigma)[-1]$
Ass
Target
Lie
$\mathrm{Sym}(\mathfrak{g}) \to$

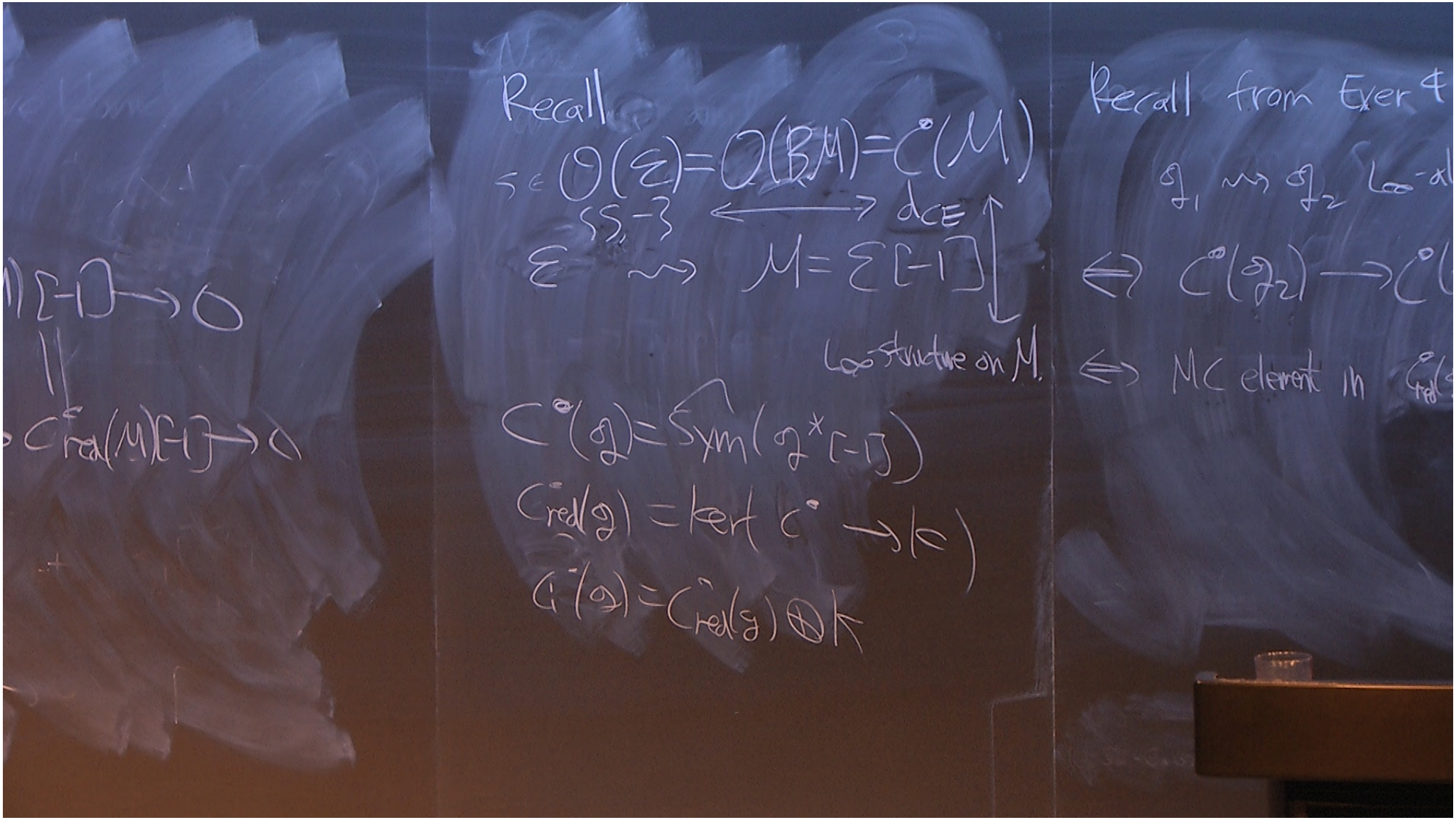

gain
m, 0-dimil
given $\mathfrak{g} \to \mathrm{SymplVect}(\mathcal{E})$
(or its quantization)
nt quantization of
$\mathfrak{g} \to \mathcal{O}(\mathcal{E})[-1]$
One thing we learned is
$(\mathcal{O}(\mathcal{E})[-1], Q + \{I, -\})$
deformation-obstruction complex
for theory $\mathcal{E}$.
Now
classical, 0-dimil
(= classical
$X^{\omega}(\mathcal{E}, \omega)$ (-1)
$\mathfrak{g} \to$ SymplVec
HamVect
$\mathcal{O}_{red}$
$\mathcal{O}(X)^{\omega}$ $\mathcal{O}($

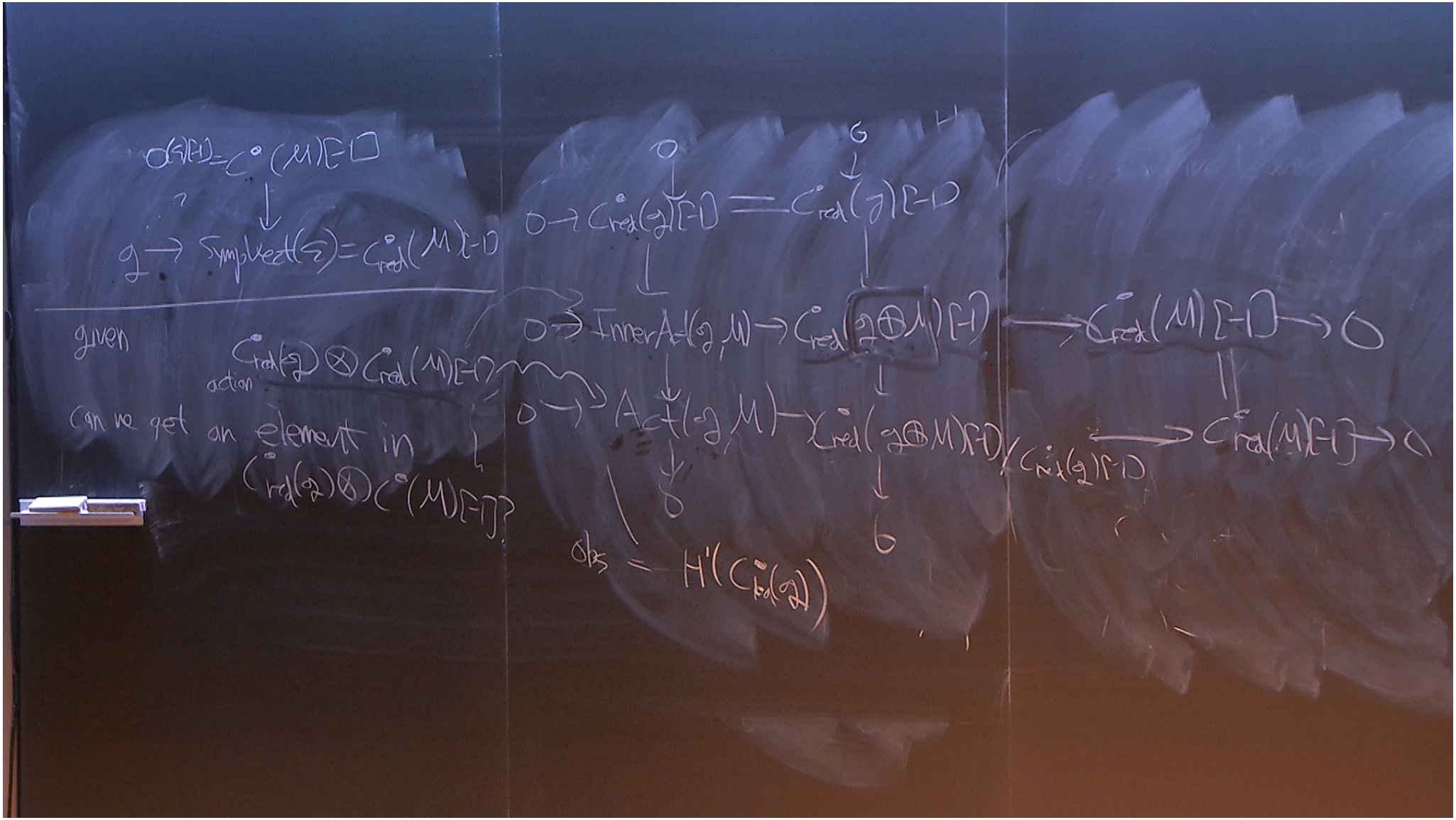

Recall
$\mathcal{O}(\mathcal{E}) = \mathcal{O}(B\mathcal{M}) = C^\infty(\mathcal{M})$
d_{CE}
$\mathcal{M} = \mathcal{E}[-1]$
L_∞ structure on $\mathcal{M}$.
Recall from Exer 4 in 1st day
$\mathfrak{g}_1 \rightsquigarrow \mathfrak{g}_2$ L_∞-algs
$\Leftrightarrow C^\bullet(\mathfrak{g}_2) \longrightarrow C^\bullet(\mathfrak{g}_1)$ alg CDGAs
$\Leftrightarrow$ MC element in $C^\bullet(\mathfrak{g}_1) \otimes \mathfrak{g}_2$.

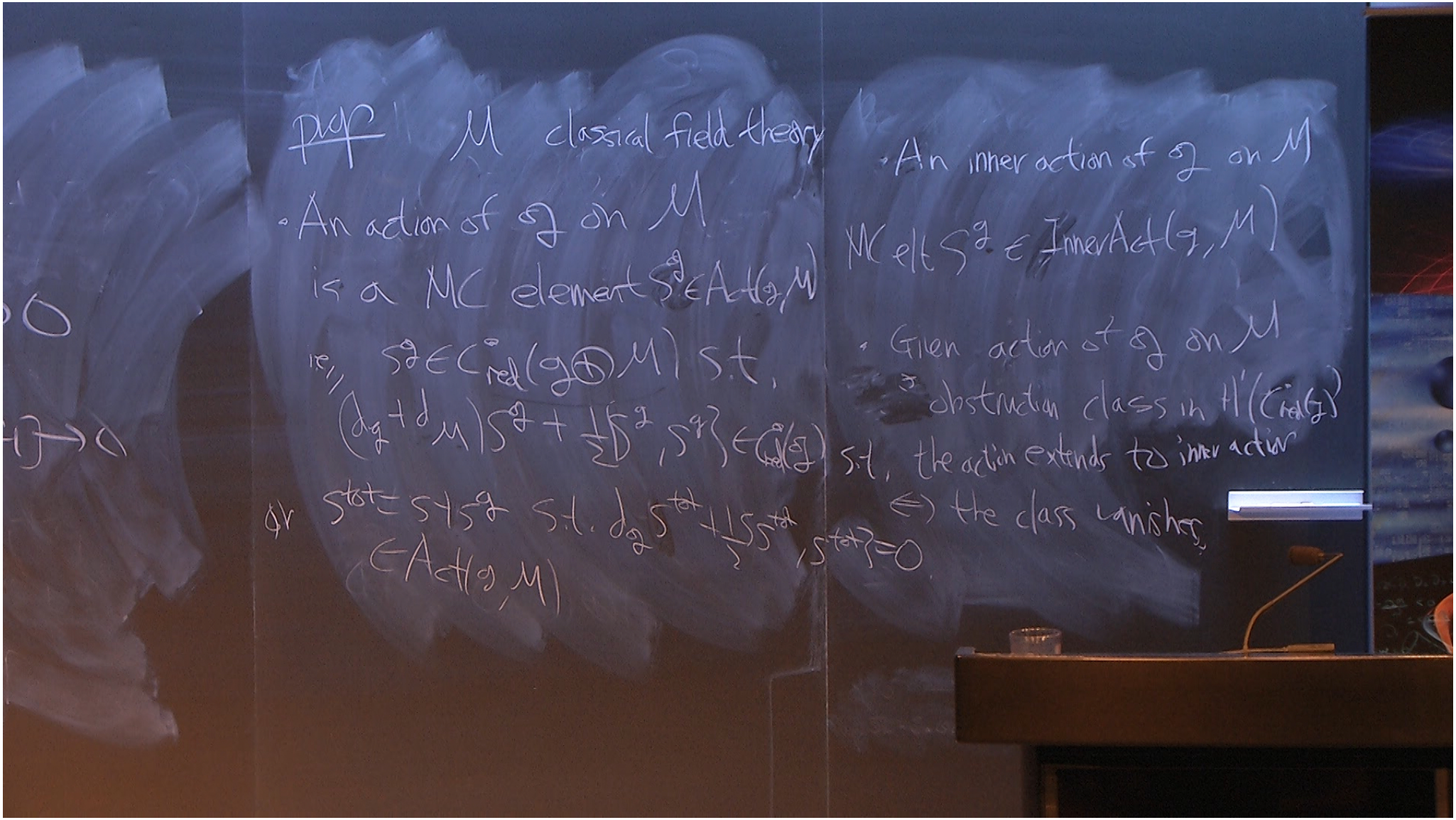

PERIMETER
INSTITUTE FOR THEOR
$\Delta x \Delta p \geq \frac{h}{2}$
$h=2\pi\hbar$
$p=\frac{h}{\lambda}$
$\mathcal{O}(\mathcal{E})[-1] = C^\infty(M)[-1]$
$g \to \mathrm{Symplect}(\mathcal{E}) = C^\infty_{\mathrm{red}}(M)[-1]$
given $C^\infty_{\mathrm{red}}(\mathcal{E}) \otimes C^\infty_{\mathrm{red}}(M)[-1]$
can we get an element in $(C^\infty_{\mathrm{red}}(\mathcal{E}) \otimes C^\infty(M)[-1])$?
Now again
quantum, 0-dim'l
Goal: given $g \to$ Symplect
(or its quantizati
we want quantization
$\mathcal{E} \to \mathcal{O}(\mathcal{E})[-1]$

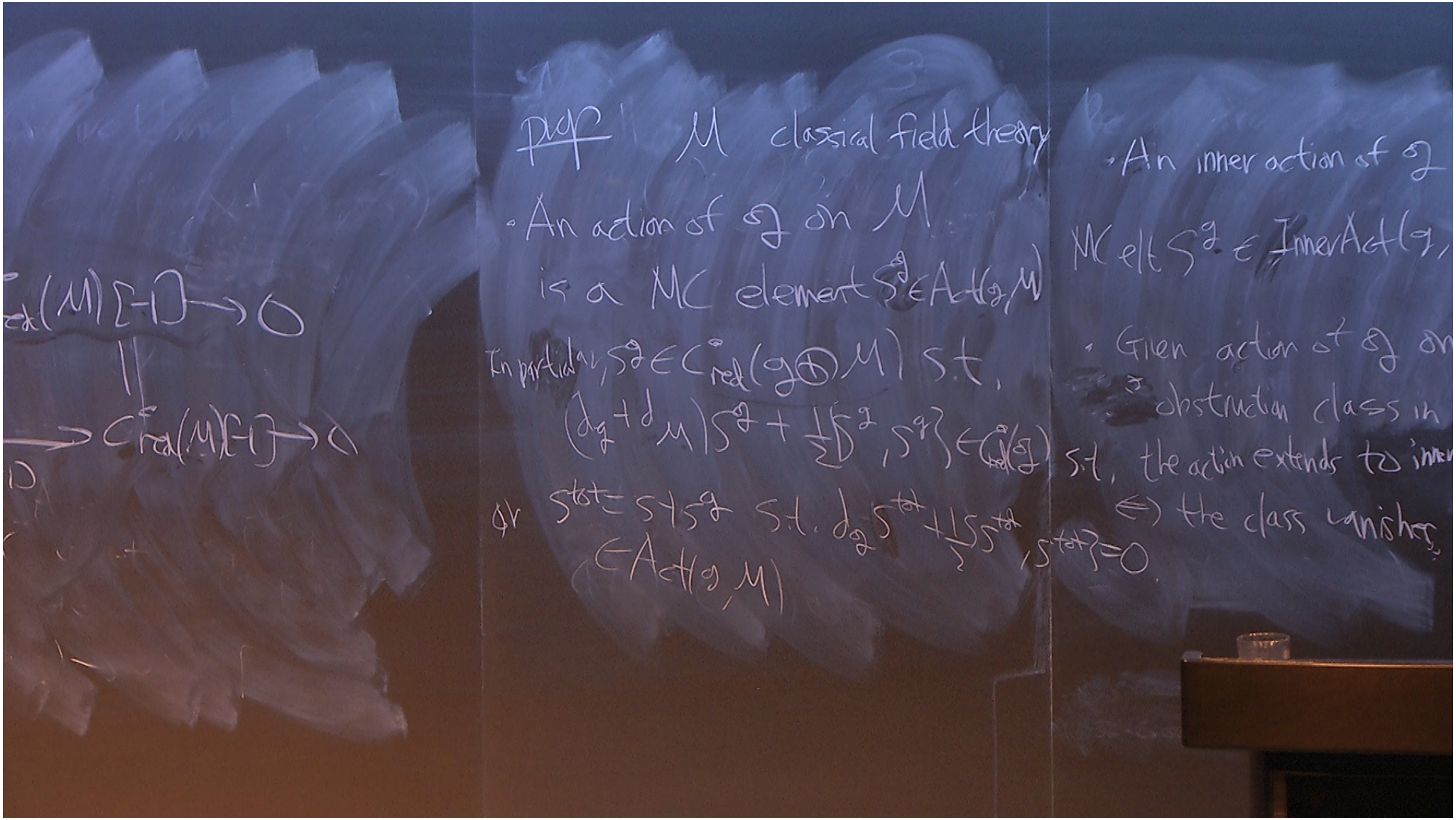

PERIMETER
INSTITUTE FOR THEOR
$\Delta x \Delta p \geq \frac{h}{2}$
$\Delta x = \frac{\lambda}{4}$
$h = 2\pi\hbar$
$p = \frac{h}{\lambda}$
$C^\infty(M)^{\Gamma=0} = \square$
$g \to \mathrm{Symplect}(\Sigma) = C^\infty_{\mathrm{red}}(M)^{\Gamma=0}$
given
$C^\infty_{\mathrm{red}}(\Sigma) \otimes C^\infty_{\mathrm{red}}(M)^{\Gamma=0}$
can we get an element in
$(C^\infty_{\mathrm{red}}(\Sigma) \otimes C^\infty(M))^{\Gamma=0}$?

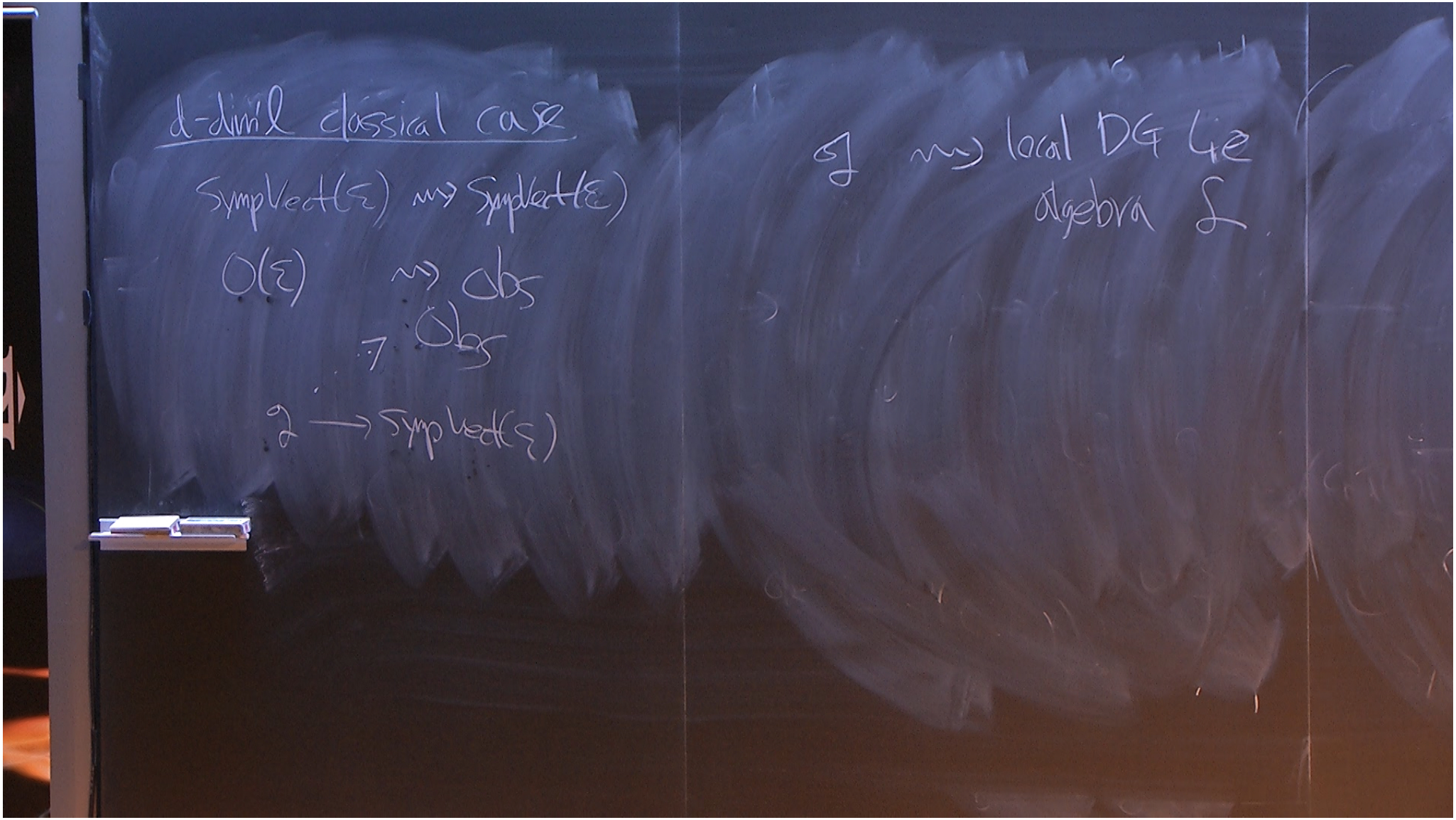



Recall
$\mathfrak{g}$ $\Theta(\mathfrak{g}) = \Omega(B\mathfrak{g}) = C^\bullet(\mathfrak{g})$
$\mathfrak{g}[-1]$ $\longleftrightarrow$ d_{CE}
$\mathcal{E}$ $\rightsquigarrow$ $\mathcal{M} = \mathcal{E}[-1]$
L_∞ structure on $\mathcal{M}$.
$C^\bullet(\mathfrak{g}) = \mathrm{Sym}(\mathfrak{g}^*[-1])$
$C^\bullet_{red}(\mathfrak{g}) = \ker(C^\bullet \to \mathbb{K})$
$C^\bullet(\mathfrak{g}) = C^\bullet_{red}(\mathfrak{g}) \oplus \mathbb{K}$
Recall from Exer 4
$\mathfrak{g}_1 \rightsquigarrow \mathfrak{g}_2$ L_∞-al
$\Leftrightarrow$ $C^\bullet(\mathfrak{g}_2) \to C^\bullet($
$\Leftrightarrow$ MC element in $C^\bullet_{red}($


given
Can we get an element in
$(C^\bullet_{red}(\mathfrak{g}) \otimes C^\bullet(M))[-1]$?
$C^\bullet_{red}(\mathfrak{g}) \otimes C^\bullet_{red}(M)[-1]$
action
$\mathfrak{g} \to$ Symplect$(\Sigma) = C^\bullet_{red}(M)[-1]$
$C^\bullet(\Sigma)[-1]$
$0 \to C^\bullet_{red}(\mathfrak{g})[-1] = C^\bullet_{red}(\mathfrak{g})[-1]$
$0 \to$ InnerAct$(\mathfrak{g}, M) \to C^\bullet_{red}(\mathfrak{g} \oplus M)[-1] \to C^\bullet_{red}(M)[-1] \to 0$
$0 \to$ Act$(\mathfrak{g}, M) \to C^\bullet_{red}(\mathfrak{g} \oplus M)[-1] / C^\bullet_{red}(\mathfrak{g})[-1] \to C^\bullet_{red}(M)[-1] \to 0$
$Obs = H^1(C^\bullet_{red}(\mathfrak{g}))$


pf: M classical field theory
• An action of g on M
is a MC element $S^g \in Act(g, M)$
i.e. $S^g \in C^\bullet_{red}(g \oplus M)$ s.t.
$(d_g + d_M) S^g + \frac{1}{2}\{S^g, S^g\} \in C^\bullet_{red}(g)$
& $S^{tot} = S + S^g$ s.t. $d_g S^{tot} + \frac{1}{2}\{S^{tot}, S^{tot}\} = 0$
$\in Act(g, M)$
• An interaction of g on M
MC elt $S^g \in InnerAct(g, M)$
• Given action of g on M
∃ obstruction class in $H^1(C^\bullet_{red} g)$
s.t. the action extends to interaction
⟺ the class vanishes.


$C_{red}(M)[-1] \to 0$
$\to C_{red}(M)[-1] \to 0$
pf
M classical field theory
• An action of $\mathfrak{g}$ on M
is a MC element $S^{\mathfrak{g}} \in Act(\mathfrak{g}, M)$
In particular, $S^{\mathfrak{g}} \in C^{\bullet}_{red}(\mathfrak{g} \oplus M)$ s.t.
$(d_{\mathfrak{g}} + d_M) S^{\mathfrak{g}} + \frac{1}{2}\{S^{\mathfrak{g}}, S^{\mathfrak{g}}\} \in C^{\bullet}_{red}(\mathfrak{g})$
or $S^{tot} = S + S^{\mathfrak{g}}$ s.t. $d_{\mathfrak{g}} S^{tot} + \frac{1}{2}\{S^{tot}, S^{tot}\} = 0$
$\in Act(\mathfrak{g}, M)$
• An inner action of $\mathfrak{g}$
MC elt $S^{\mathfrak{g}} \in InnerAct(\mathfrak{g},$
• Given action of $\mathfrak{g}$ on
obstruction class in
s.t. the action extends to inner
$\Leftrightarrow$ the class vanishes.


d-dim'l classical case
SymVect(ξ) ⇝ SymVect(ξ)
O(ξ) ⇝ Obs
Obs
2 ⟶ SymVect(ξ)
g ⇝ local DG Lie algebra g

$\mathcal{L}$ ~ local DG Lie algebra $\mathfrak{L}$

$\mathfrak{L}$ sheaf of sections of bundle together with d, l_2 ↑ → diff. operators;

Ex

- (G-bundle) Σ Riemann surface, $\mathfrak{g}$ Lie alg

$\mathfrak{L}_\Sigma : U \mapsto (\Omega^{0,*}(U) \otimes \mathfrak{g}, \bar\partial, [\cdot,\cdot])$

(complex structure) X cpx mfd

$\mathfrak{L}^{mf}(X) = (\Omega^{0,*}(X, T_X), \bar\partial, [\cdot,\cdot])$

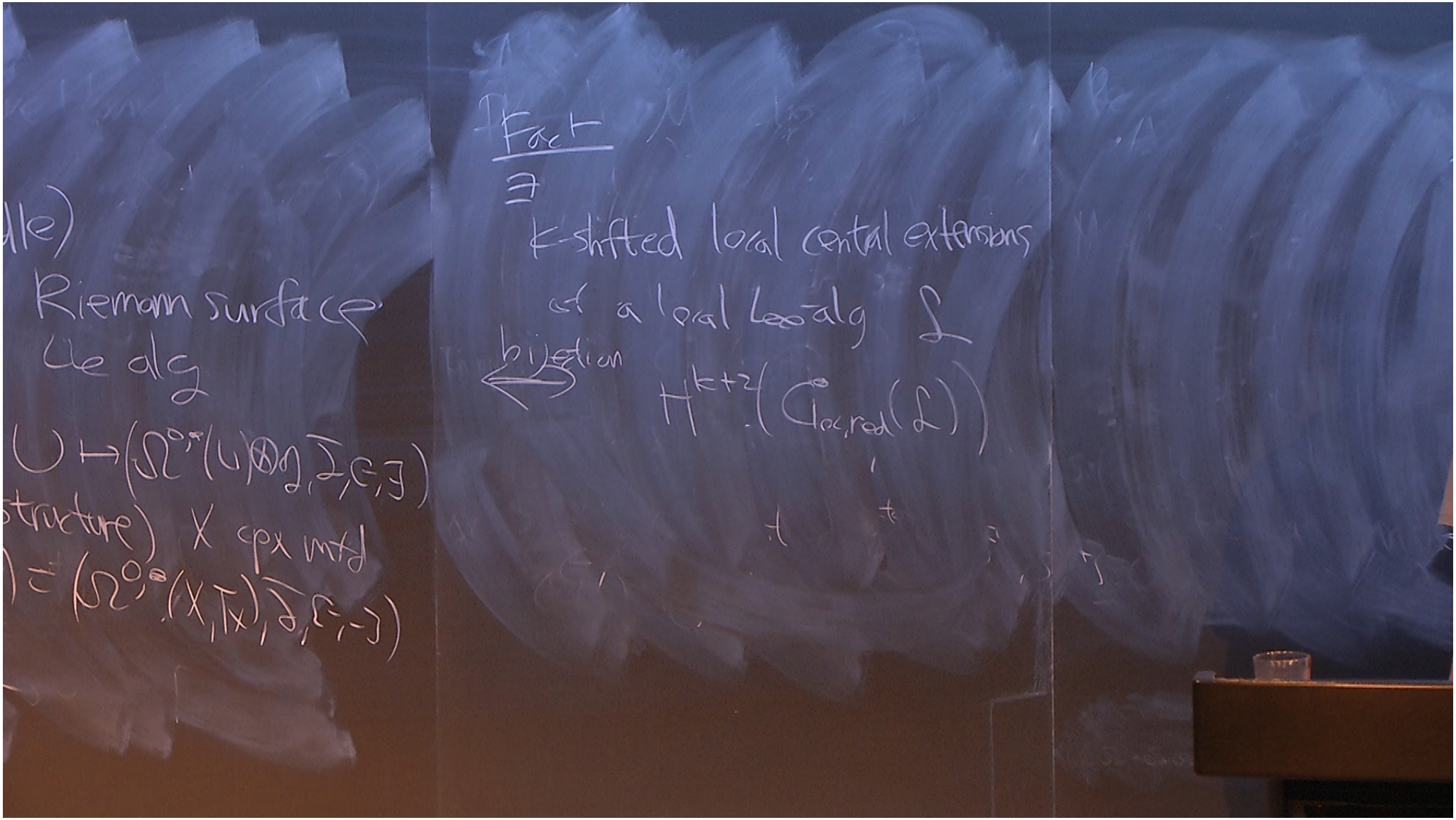
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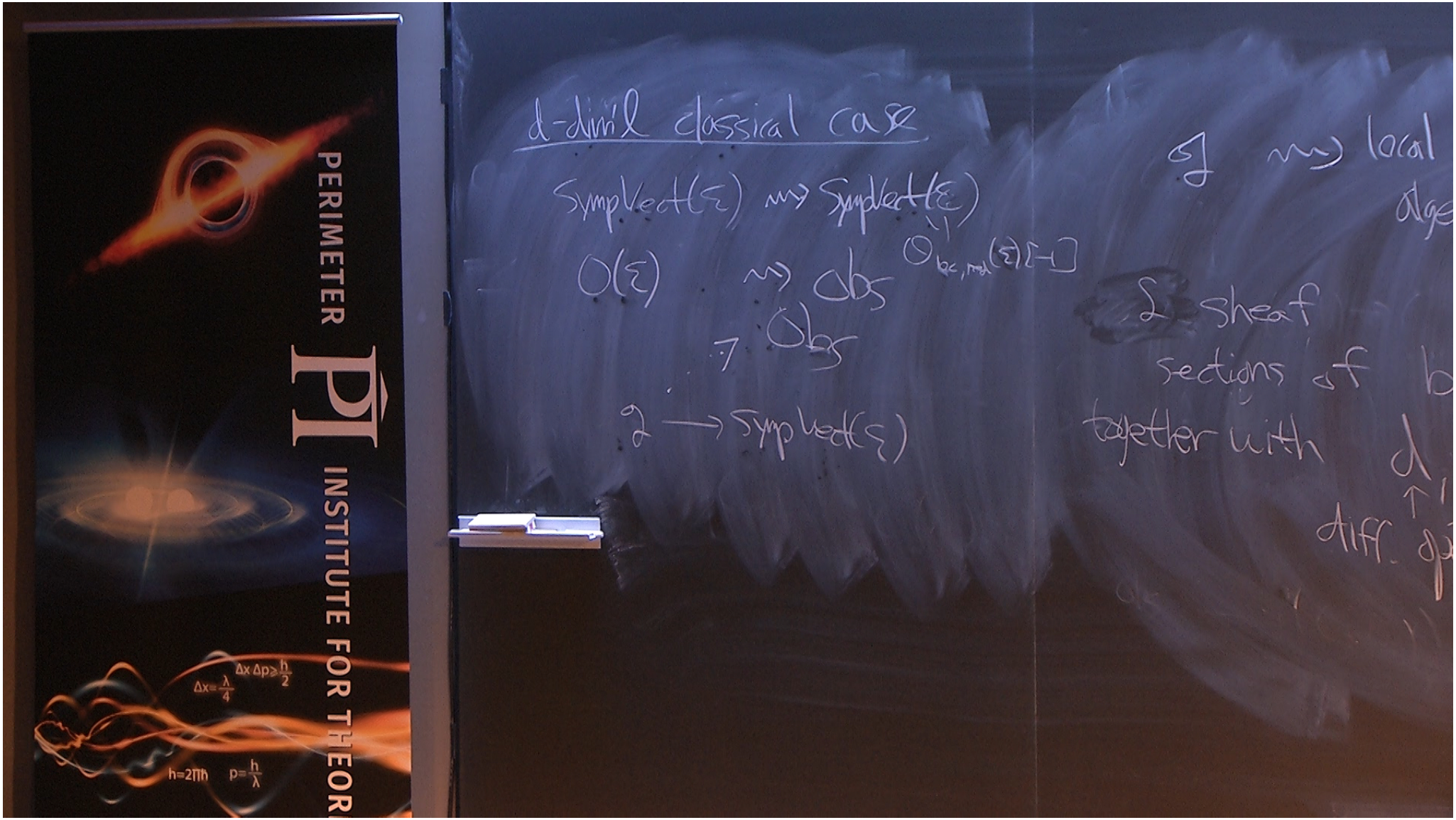
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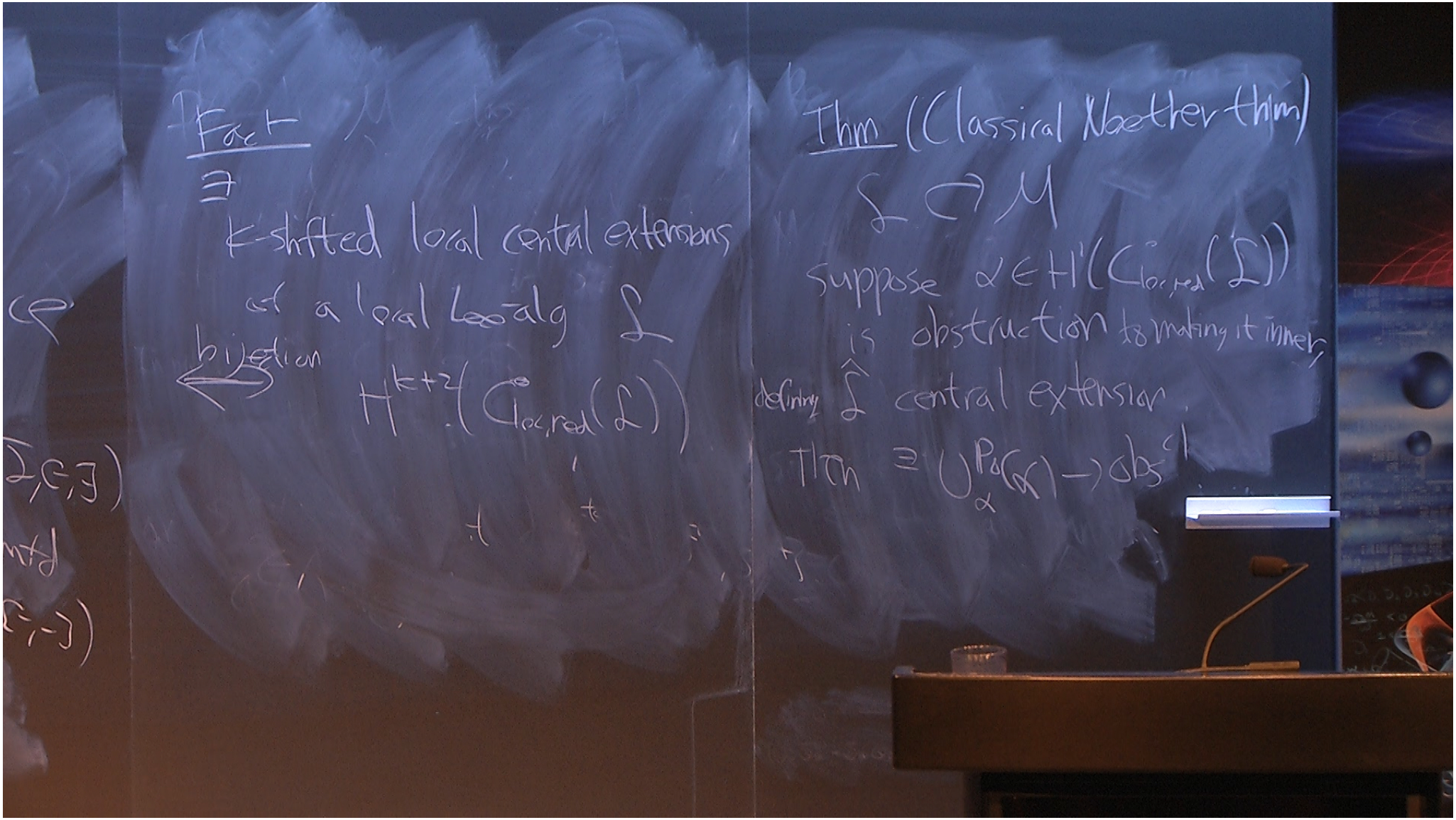
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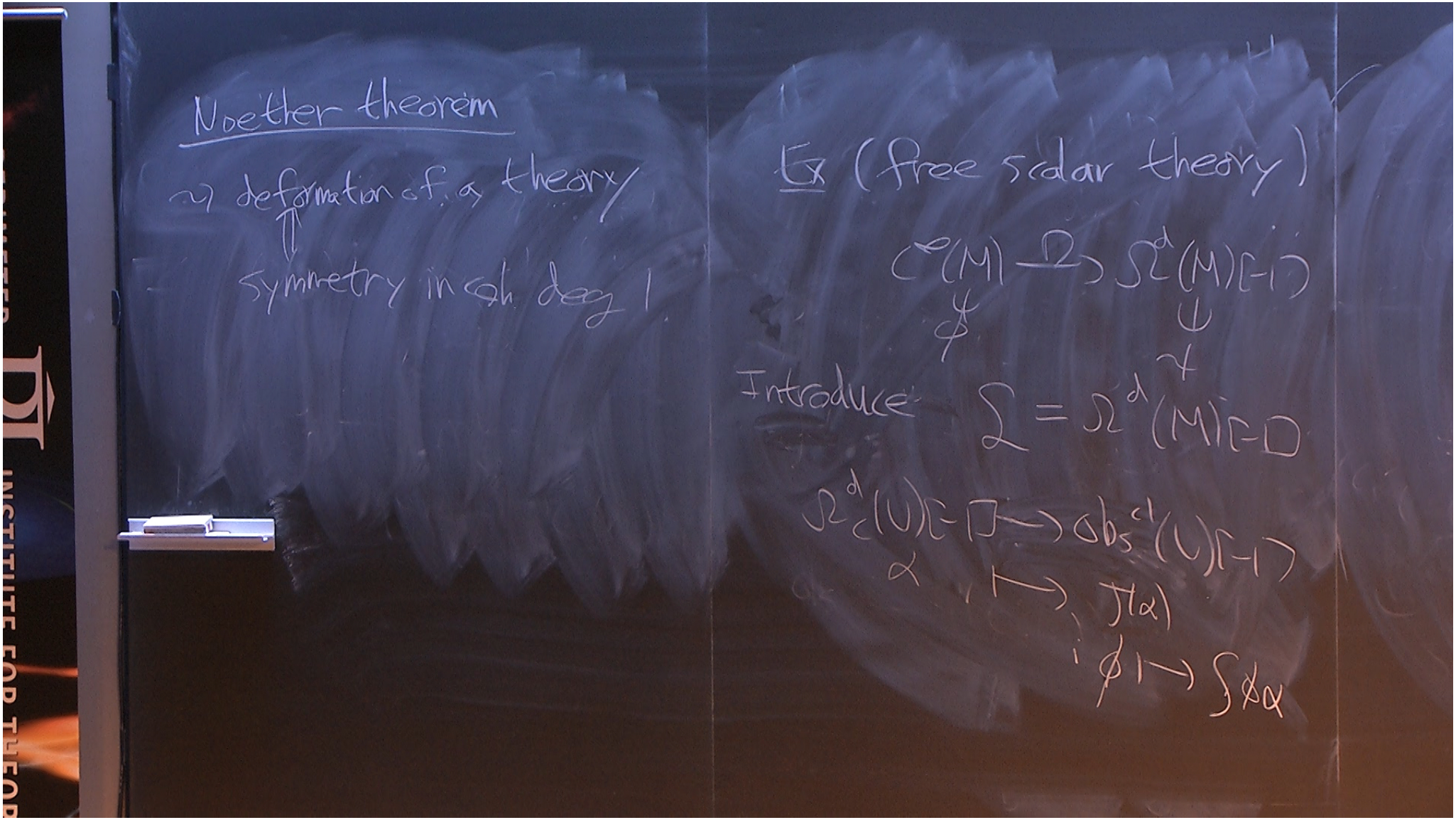
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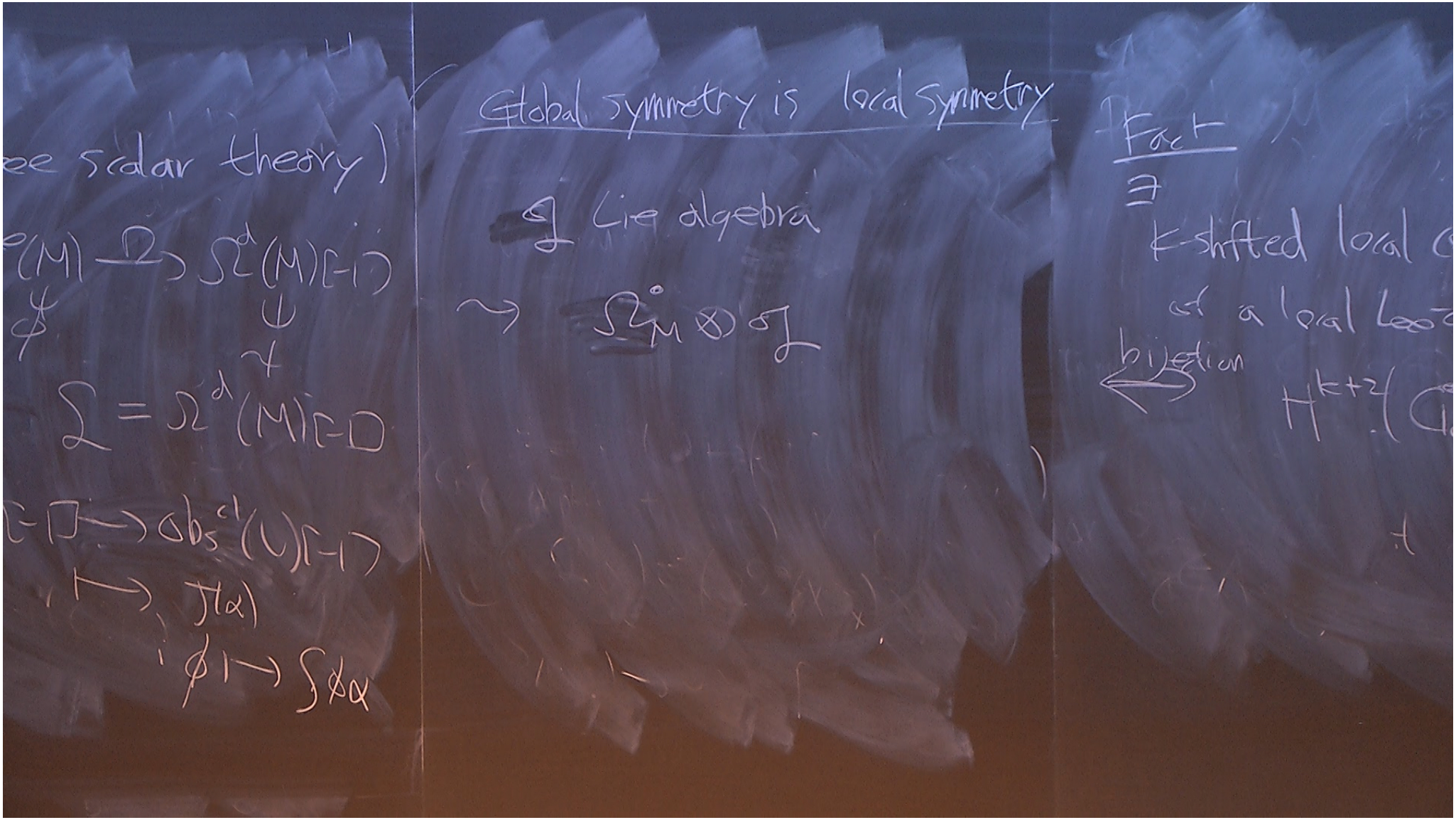
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Fact
∃
k-shifted local central extensions
of a local Lie alg L
bijection
$H^{k+2}(C^\bullet_{loc,red}(L))$


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d-dim'l classical case
SympVect(ξ) ⇝ Sympl(ξ)
O(ξ) ⇝ Obs
Obs
Sympl(ξ)
sheaf
sections of
together with
local
diff.


Fact
∃
k-shifted local central extensions
of a local Lie alg $\mathcal{L}$
bijection
$H^{k+2}_{c,red}(\mathcal{L})$
Thm (Classical Noether thm)
$\mathcal{L} \curvearrowright M$
suppose $\alpha \in H^1(G_{loc,red}(\mathcal{L}))$
is obstruction to making it inner,
defining $\hat{\mathcal{L}}$ central extension.
Thm $= \bigcup_\alpha$ Ker(α) → obs


Noether theorem
⇝ deformation of a theory
⇑
symmetry in sh. deg 1
Ex (free scalar theory)
$C^\infty(M) \xrightarrow{\Omega} \Omega^d(M)[-1]$
ϕ ↓ ↓ $\sim$
Introduce $\Sigma = \Omega^d(M)[-1]$
$\Omega^d_c(U)[-1] \longrightarrow Obs^{-1}(U)[-1]$
$\alpha \longmapsto J(\alpha)$
$\phi \longmapsto \int \phi\alpha$


ee scalar theory)
Global symmetry is local symmetry
Lie algebra
Fact
biection

Global symmetry is local symmetry

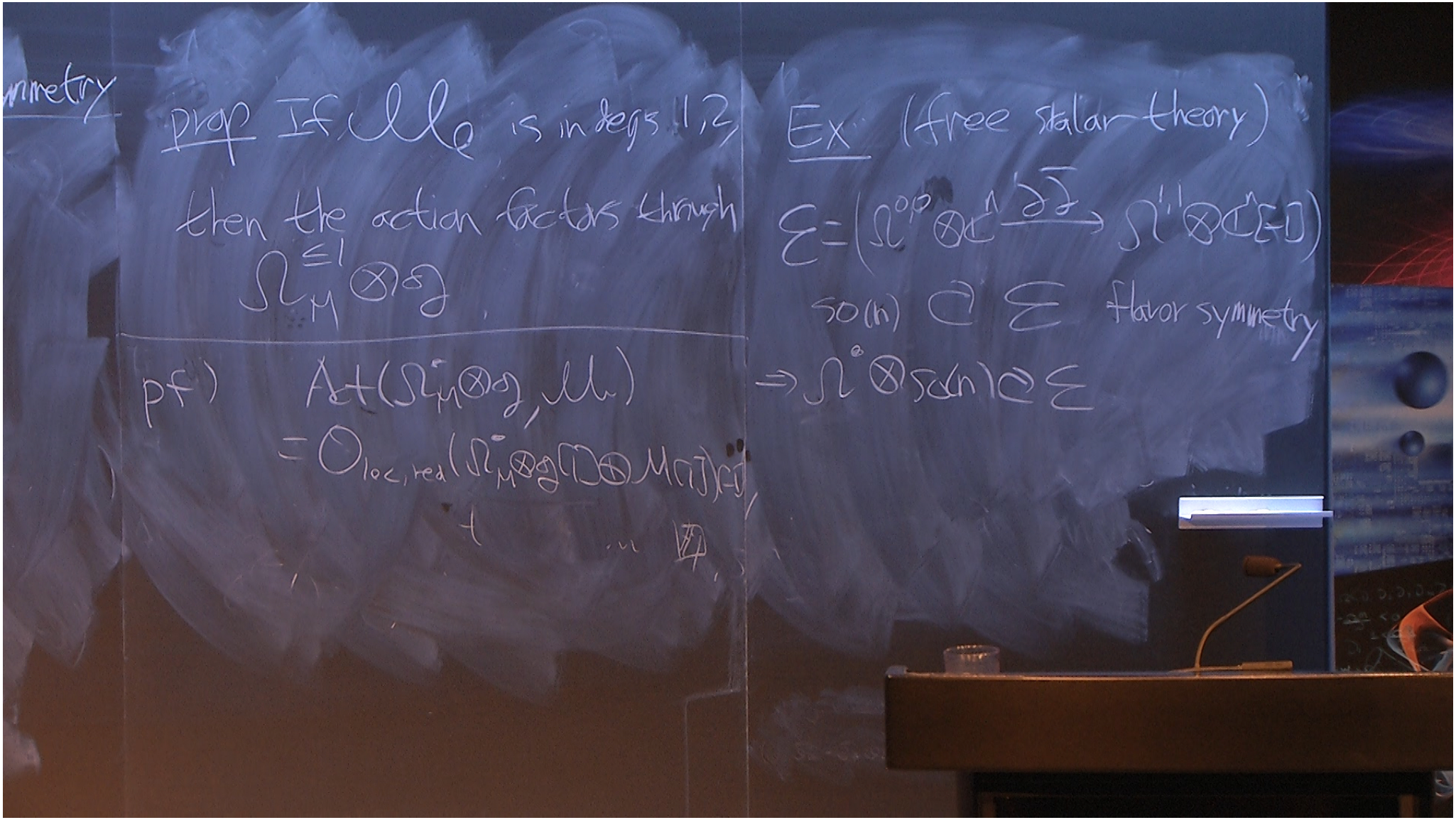
$\mathfrak{g}$ Lie algebra

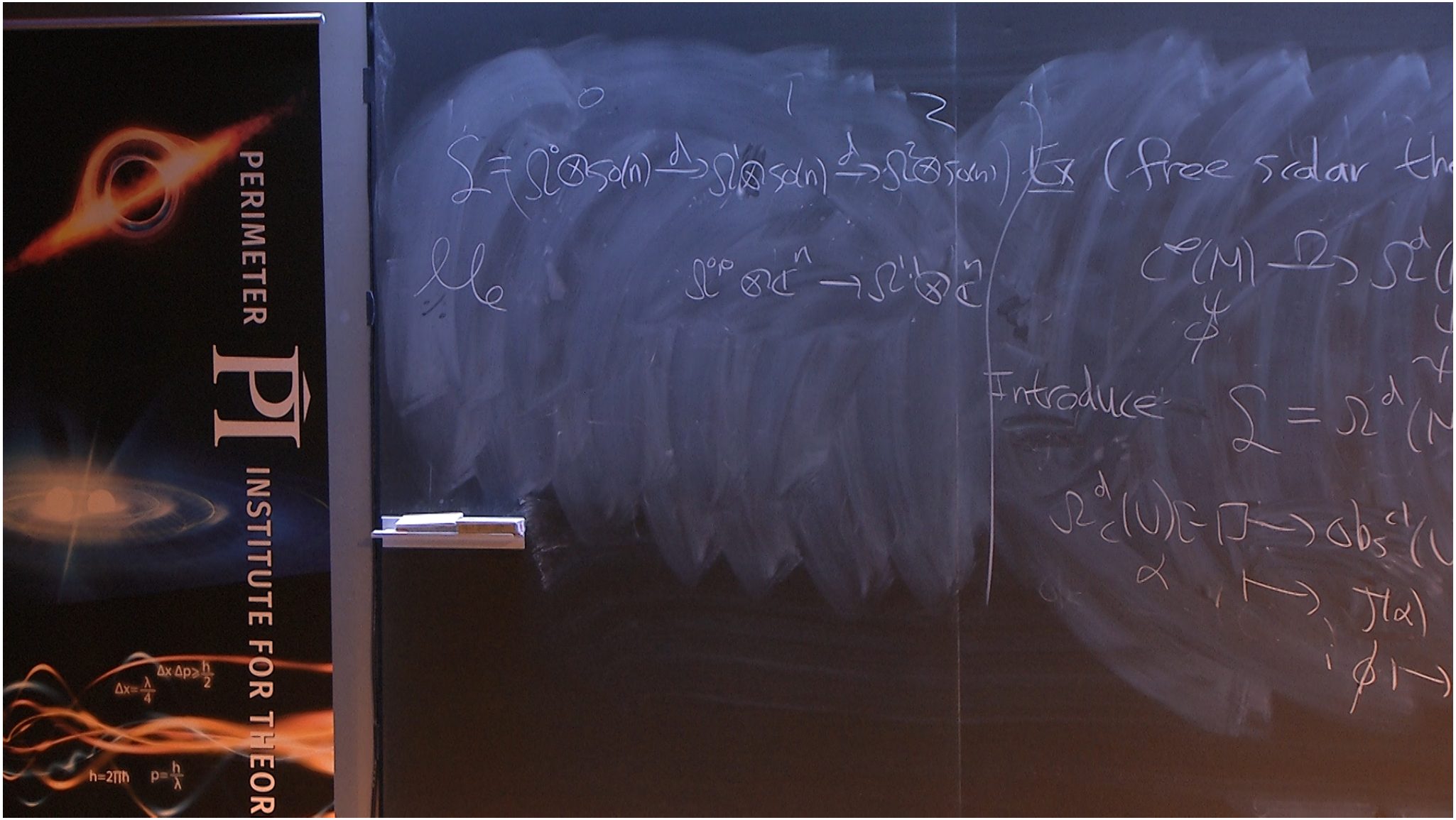
$\rightsquigarrow$ $\Omega^{\cdot}_M \otimes \mathfrak{g}$

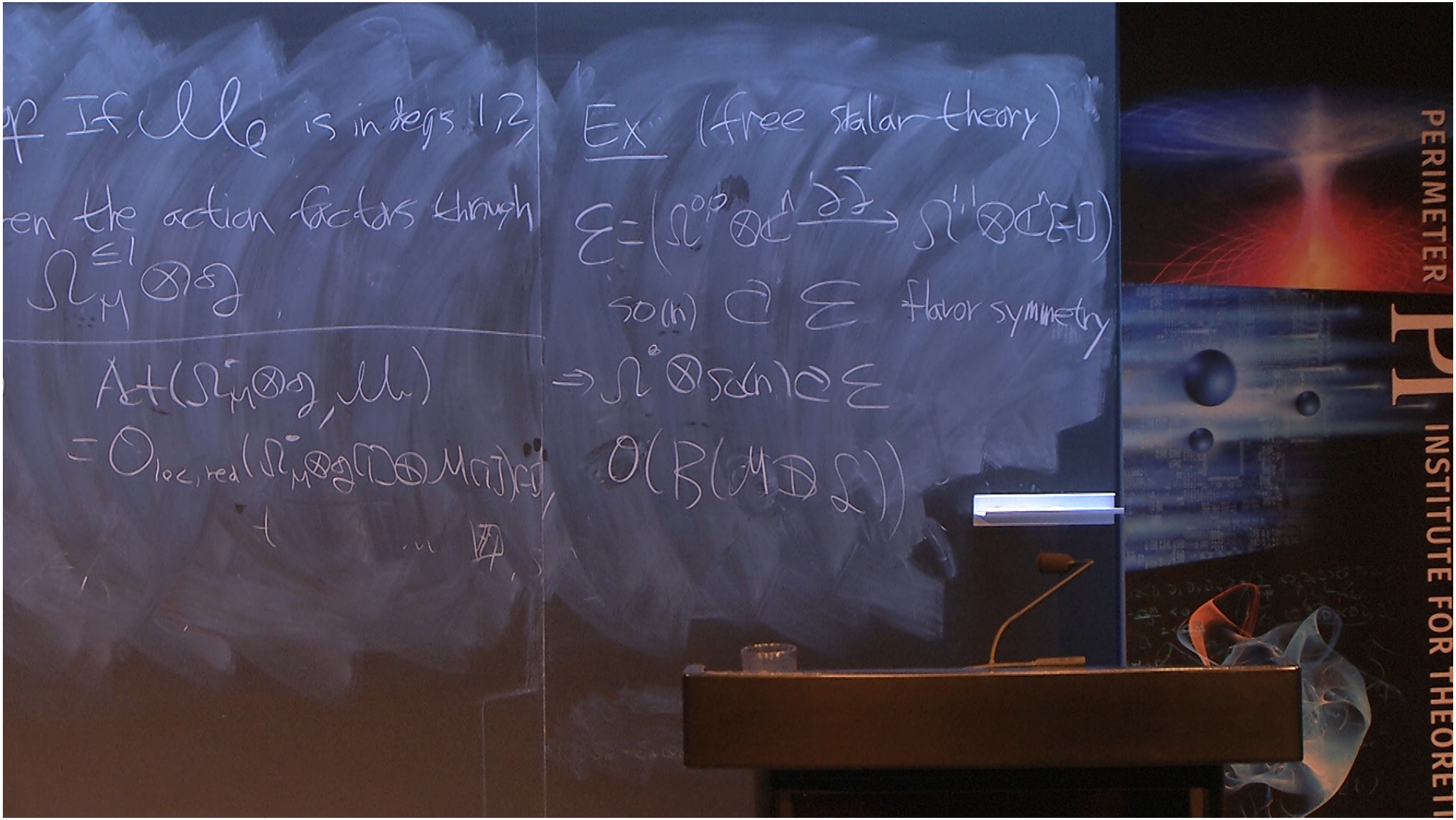
Prop If $\mathcal{U}_\bullet$ is in degs $1, 2$, then the action factors through $\Omega^{\leq 1}_M \otimes \mathfrak{g}$.

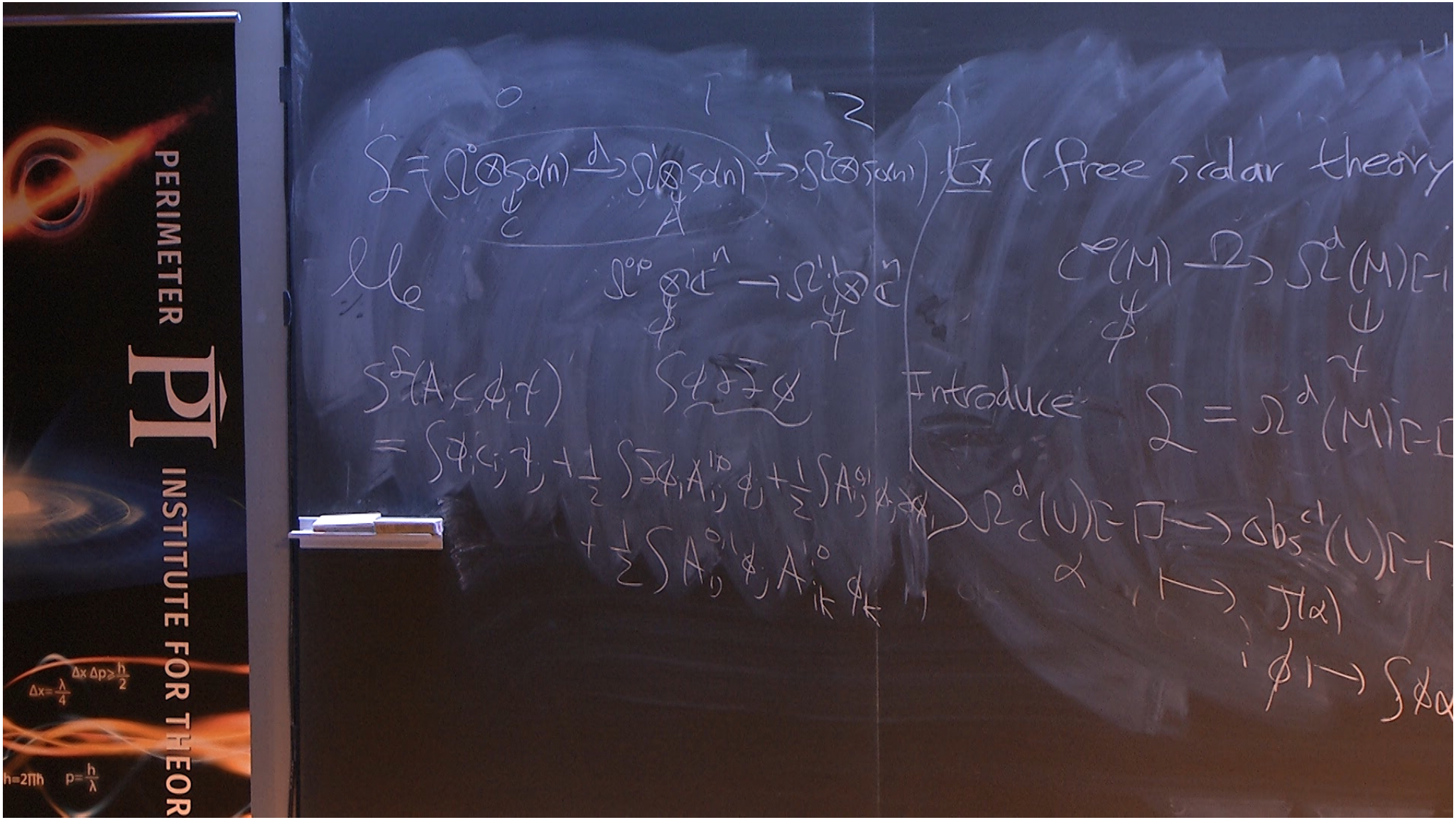
pf) $\mathrm{Act}(\Omega^{\cdot}_M \otimes \mathfrak{g}, \mathcal{U}_\bullet)$

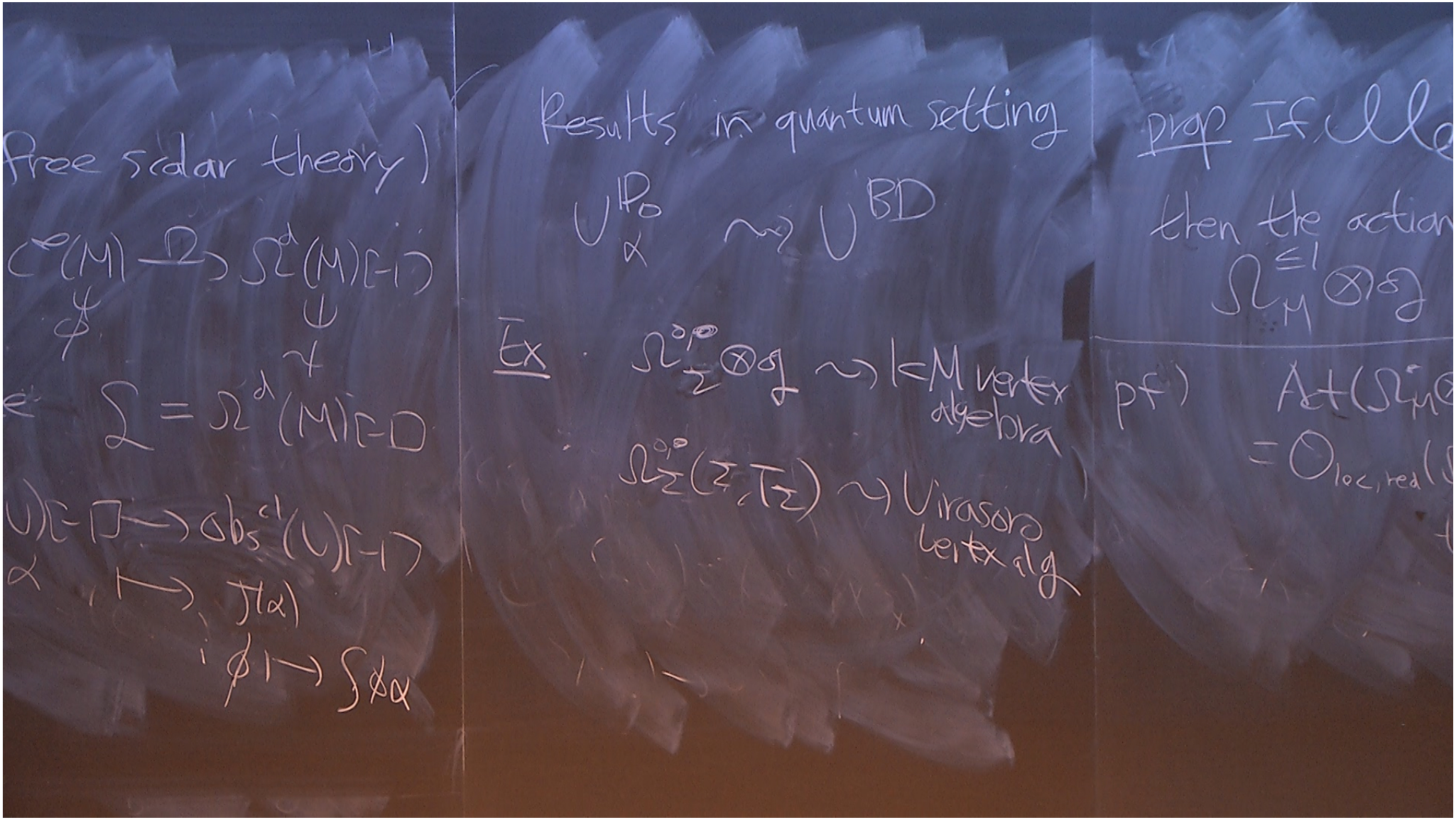
$= \mathcal{O}_{\mathrm{loc,red}}(\Omega^{\cdot}_M \otimes \mathfrak{g}[1] \otimes M(\mathcal{U})_{\mathrm{fl}})$

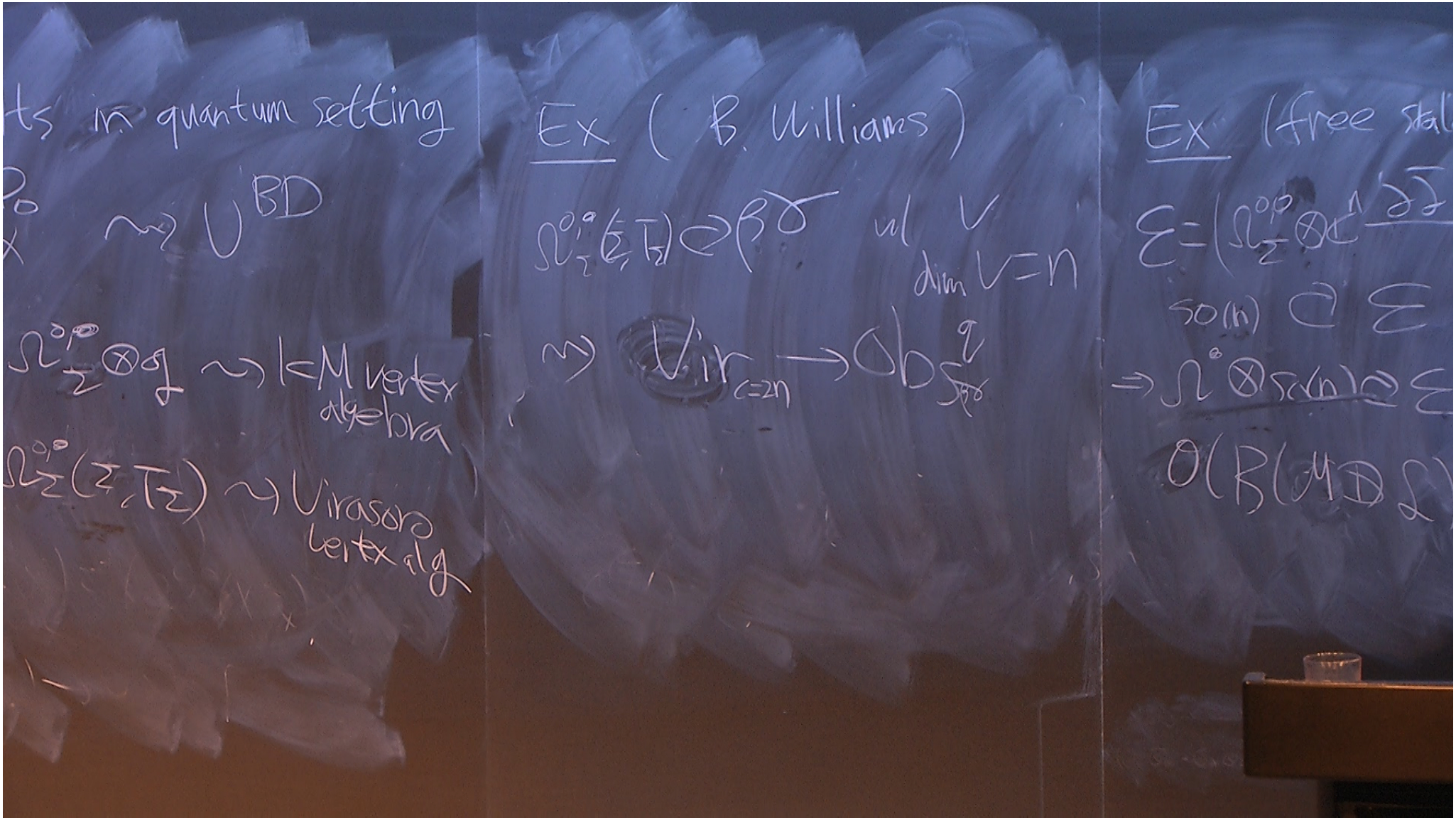

symmetry
Prop If $\mathcal{U}_\bullet$ is in degs 1,2
then the action factors through
$\Omega^{\leq 1}_M \otimes_{\mathcal{O}_M} \mathfrak{g}$
pf) $Act(\Omega^\bullet_M \otimes \mathfrak{g}, \mathcal{U}_\bullet)$
$= \mathcal{O}_{loc,red}(\Omega^\bullet_M \otimes \mathfrak{g}[1] \otimes M(n))^{S_n}$
Ex (free scalar theory)
$\mathcal{E} = (\Omega^{0,0} \otimes \mathbb{C}^n \xrightarrow{\partial\bar{\partial}} \Omega^{1,1} \otimes \mathbb{C}^n[-1])$
$so(n) \subset \mathcal{E}$ flavor symmetry
$\Rightarrow \Omega^\bullet \otimes so(n) \subset \mathcal{E}$


PERIMETER
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Introduce


If $\mathcal{U}_Q$ is in degs 1,2
Ex (free scalar theory)
$\mathcal{E} = (\Omega^{0,0} \otimes \mathbb{C}^\wedge \xrightarrow{\partial\bar{\partial}} \Omega^{1,1} \otimes \mathbb{C}^\wedge[-1])$
$so(n) \subset \mathcal{E}$ flavor symmetry
$\Rightarrow \Omega^\bullet \otimes so(n) \subset \mathcal{E}$
$\mathcal{O}(B(\mathcal{M} \oslash \mathcal{L}))$
PERIMETER
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PERIMETER
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Introduce


free scalar theory)
Results in quantum setting
Prop If
then the action
Ex
pf)


ts in quantum setting
$\rightsquigarrow U^{BD}$
$\Omega^{0,\bullet}_{\Sigma}\otimes\mathfrak{g}$ ~ loop vertex algebra
$\Omega^{0,\bullet}_{\Sigma}(\Sigma,T_{\Sigma})$ ~ Virasoro vertex alg
Ex (B. Williams)
$\Omega^{0,\bullet}_{\Sigma}(\Sigma,T_{\Sigma})\otimes\beta\gamma$ w/ V
dim V = n
$\rightsquigarrow Vir_{c=2n} \to Obs^{q}_{\beta\gamma}$
Ex (free stab
$\mathcal{E}=(\Omega^{0,\bullet}_{\Sigma}\otimes\mathbb{C}^{1}\xrightarrow{\partial}$
$so(n)\subset\mathcal{E}$
$\Rightarrow \Omega^{\bullet}\otimes so(n)\subset\mathcal{E}$
$\mathcal{O}(B(M\oplus$