Title: Lecture 2: Supersymmetric Field Theory and Topological Twists

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$d=2$ $N=(2,2)$ SUSY σ-model

— top. twist (A/B-model)

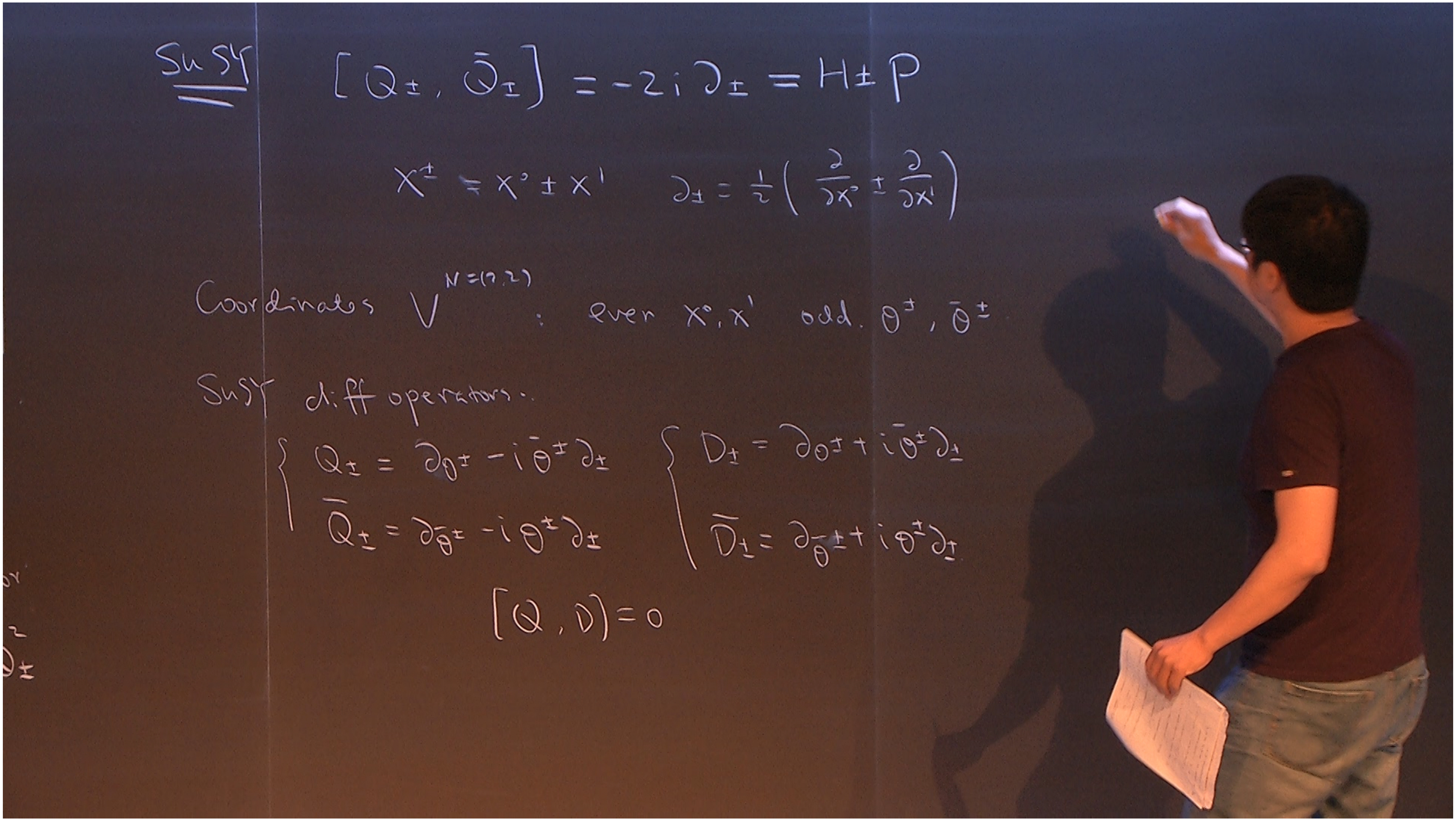
— BV formalism

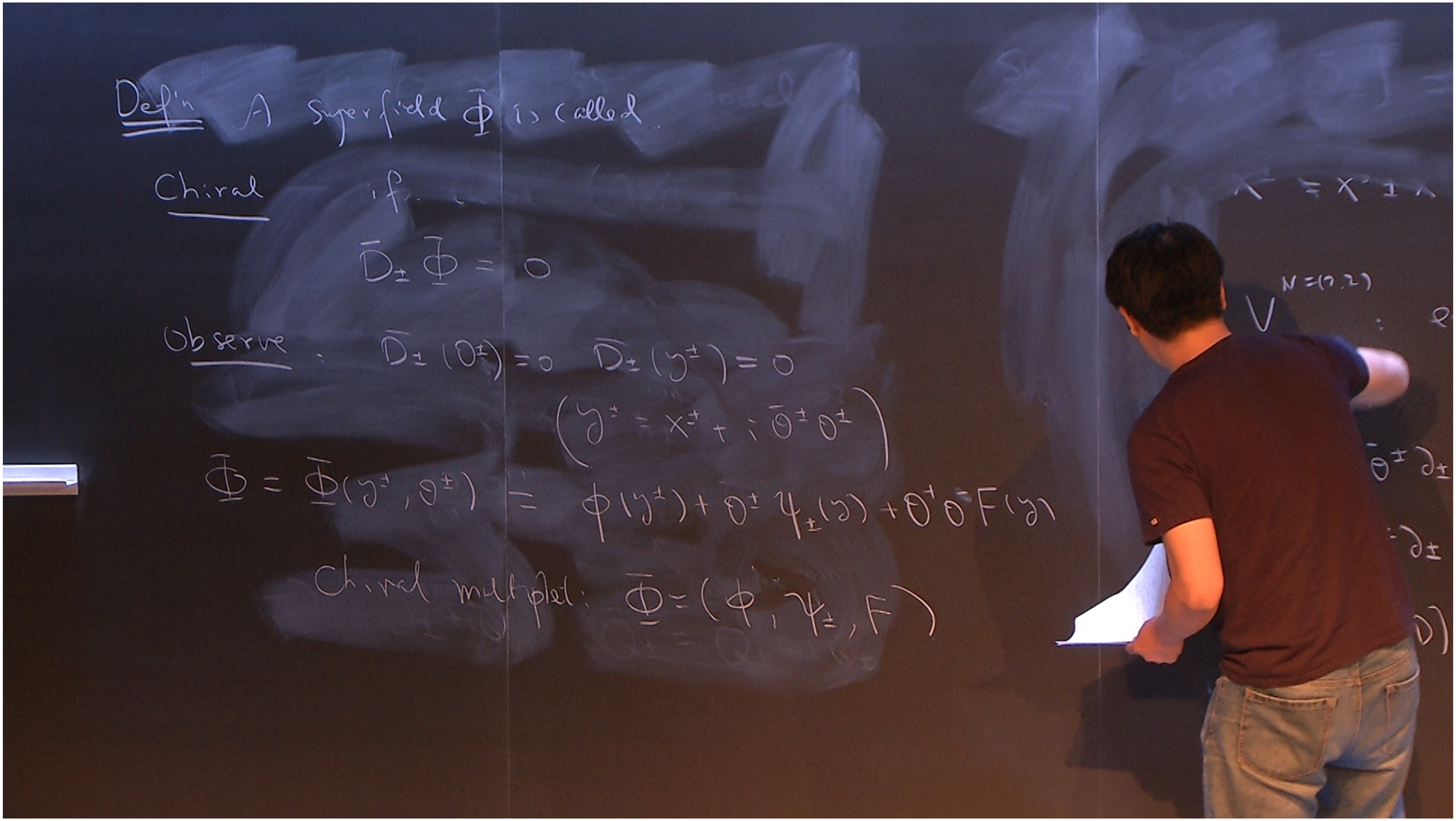
$$V^{N=(2,2)} = \mathbb{R}^{1,1} \times \Pi \mathbb{R}_+^{\oplus 2} \times \Pi \mathbb{R}_-^{\oplus 2}$$

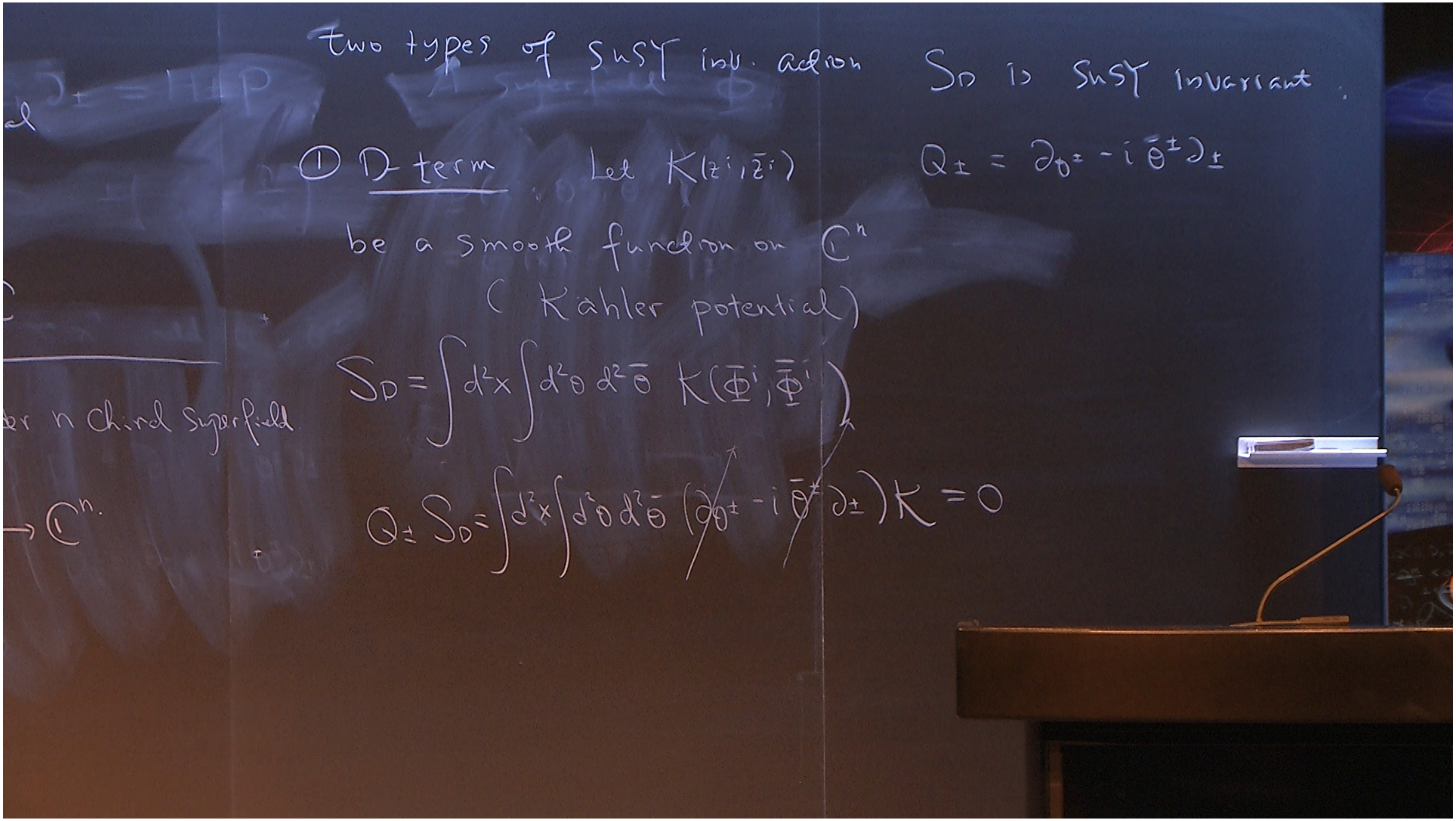
$$= \mathbb{R}^{1,1} \times \Pi S_+ \times \Pi S_-$$

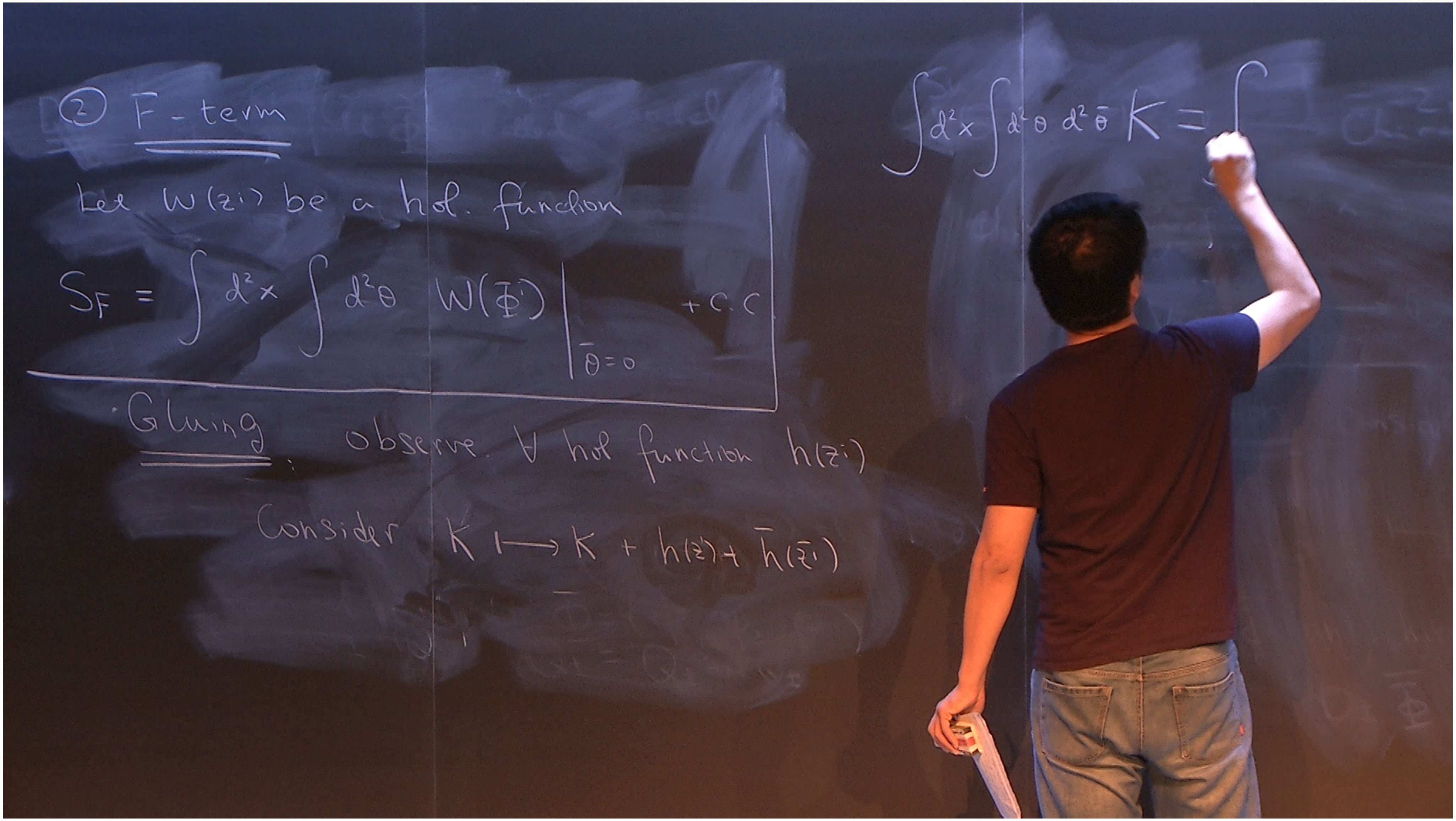
(S_+, S_-: complex spinor)

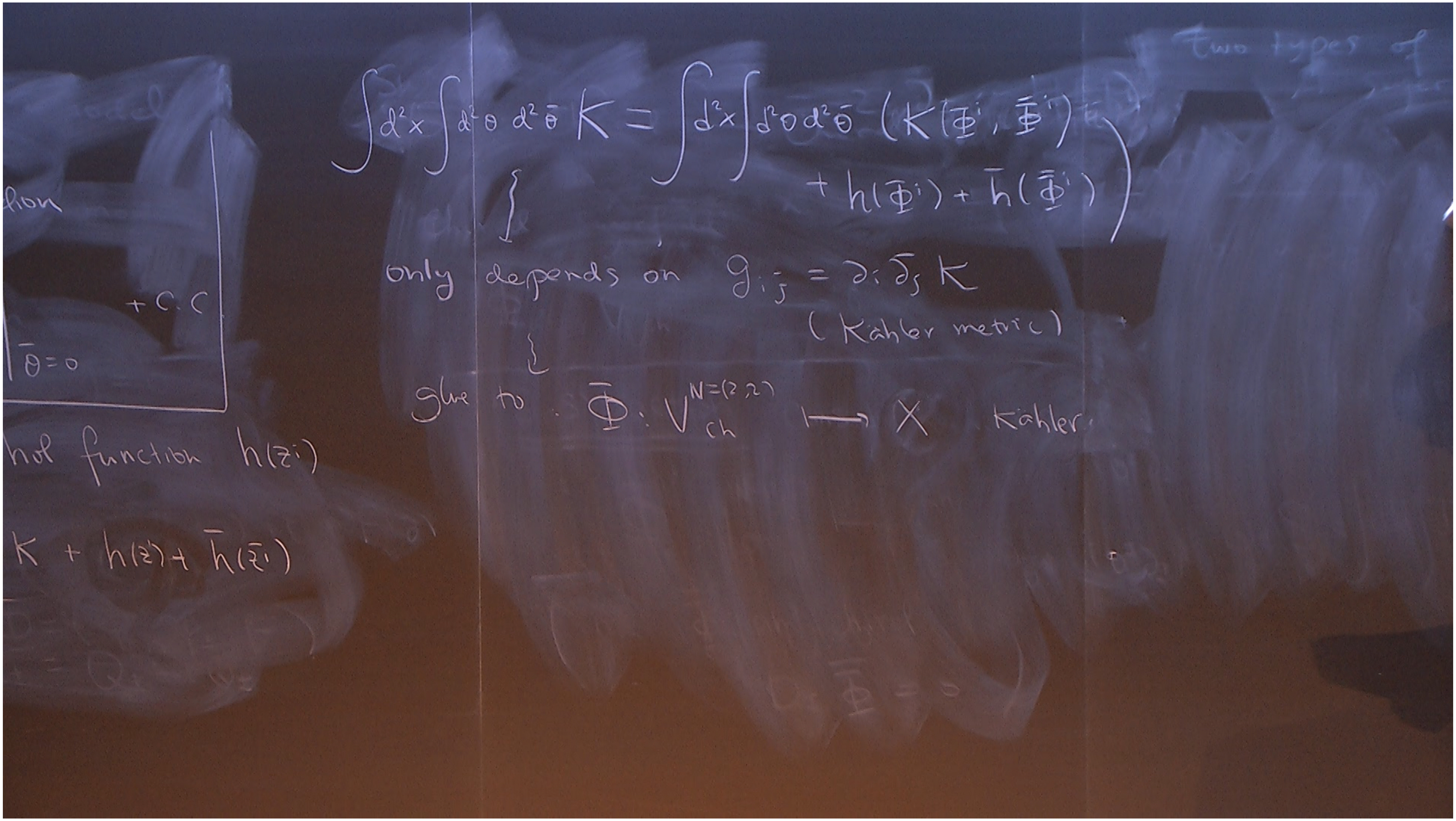
$$Q_\pm = Q^1_\pm + i Q^2_\pm \qquad \bar{Q}_\pm = Q^1_\pm - i Q^2_\pm$$

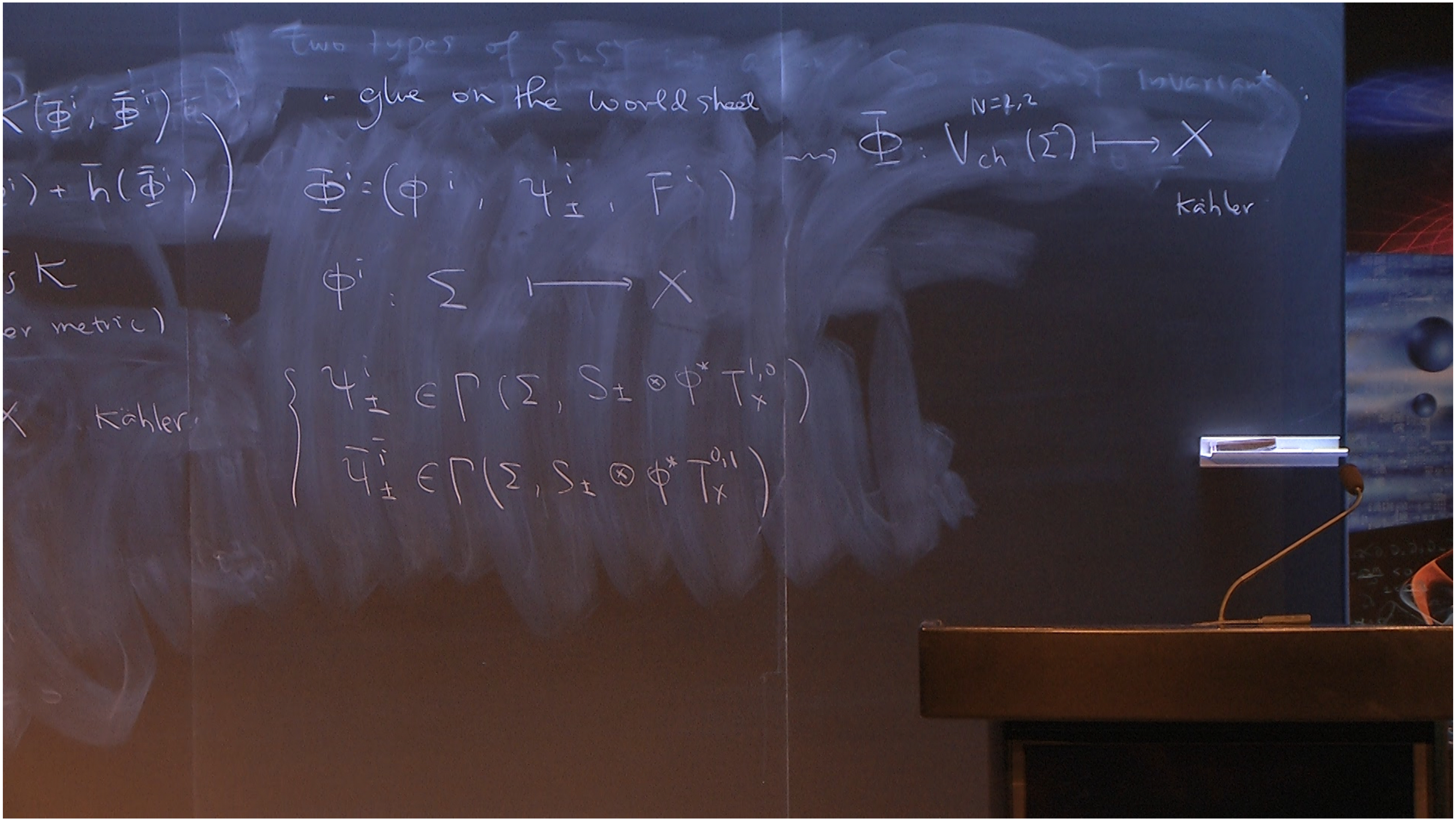

SUSY
$[Q_\pm, \bar{Q}_\pm] = -2i\partial_\pm = H \pm P$
$x^\pm = x^0 \pm x^1$
$\partial_\pm = \frac{1}{2}\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right)$
Coordinates $V^{N=(2,2)}$: even x^0, x^1 odd. $\theta^\pm, \bar{\theta}^\pm$
SUSY diff operators.
$Q_\pm = \partial_{\theta^\pm} - i\bar{\theta}^\pm \partial_\pm$
$\bar{Q}_\pm = \partial_{\bar{\theta}^\pm} - i\theta^\pm \partial_\pm$
$D_\pm = \partial_{\theta^\pm} + i\bar{\theta}^\pm \partial_\pm$
$\bar{D}_\pm = \partial_{\bar{\theta}^\pm} + i\theta^\pm \partial_\pm$
$[Q, D] = 0$

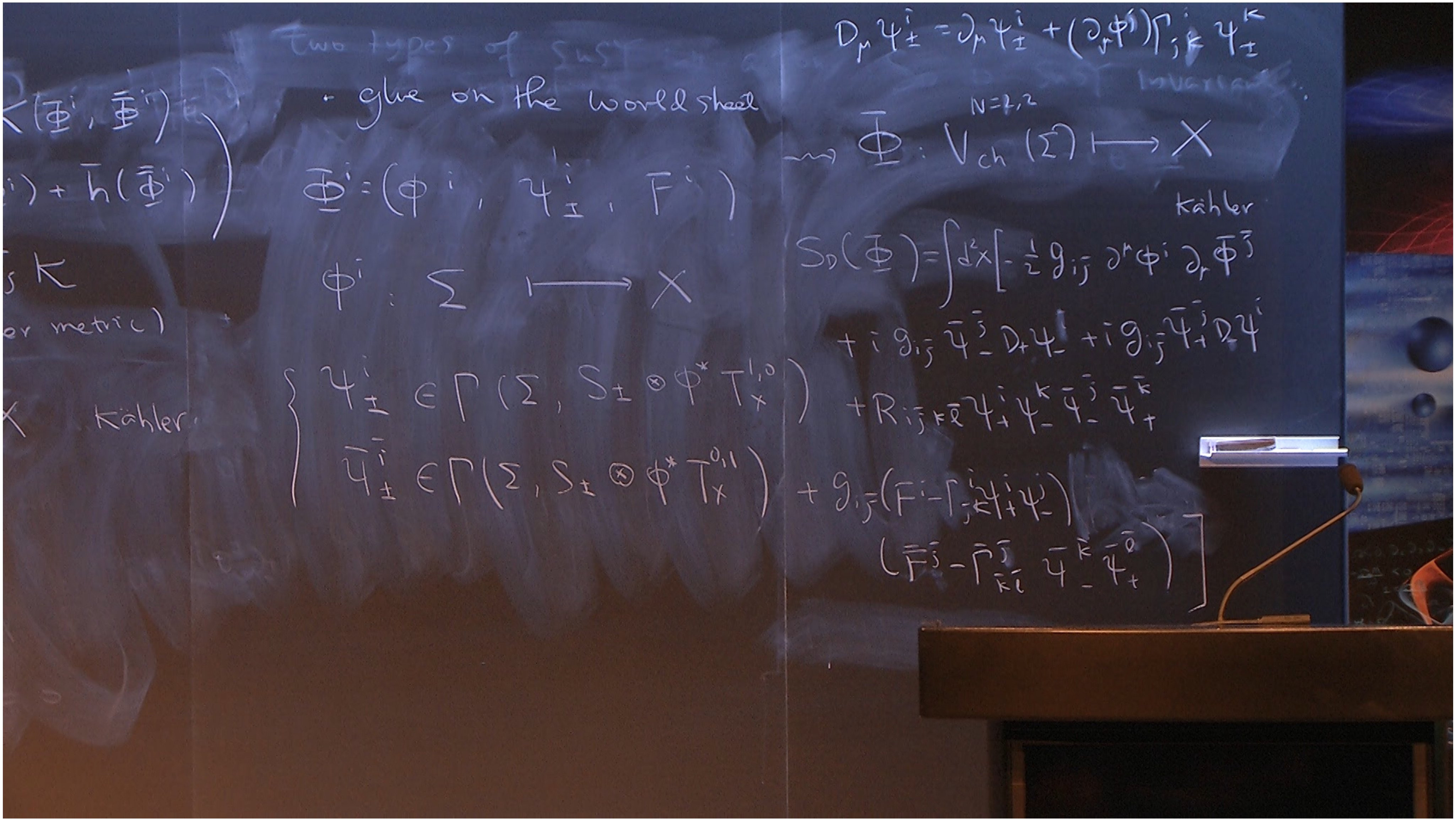

Def'n A superfield Φ is called
Chiral if
$\bar{D}_\pm \Phi = 0$
Observe: $\bar{D}_\pm (\theta^\pm) = 0$ $\bar{D}_\pm (y^\pm) = 0$
$(y^\pm = x^\pm + i \bar{\theta}^\pm \theta^\pm)$
$\Phi = \Phi(y^\pm, \theta^\pm) = \phi(y^\pm) + \theta^\pm \psi_\pm(y) + \theta^+ \theta^- F(y)$
Chiral multiplet: $\Phi = (\phi, \psi_\pm, F)$


two types of SUSY inv. action
① D-term Let K(z^i, z̄^i)
be a smooth function on ℂ^n
(Kähler potential)
S_D = ∫d²x ∫d²θ d²θ̄ K(Φ^i, Φ̄^i)
Q± S_D = ∫d²x ∫d²θ d²θ̄ (∂θ± − i θ̄± ∂±) K = 0
S_D is SUSY invariant.
Q± = ∂θ± − i θ̄± ∂±
n Chiral Superfield
→ ℂ^n.


② F-term
Let W(zi) be a hol. function
$S_F = \int d^2x \int d^2\theta \, W(\Phi^i)\Big|_{\bar\theta=0} + c.c$
· Gluing : observe ∀ hol function h(zi)
Consider $K \longmapsto K + h(z) + \bar h(\bar z^i)$
$\int d^2x \int d^2\theta \, d^2\bar\theta \, K = \int$


two types of
$\int d^2x \int d^2\theta\, d^2\bar\theta\, K = \int d^2x \int d^2\theta\, d^2\bar\theta \left(K(\Phi^i, \bar\Phi^{i}) + h(\Phi^i) + \bar h(\bar\Phi^i) \right)$
only depends on $g_{i\bar j} = \partial_i \bar\partial_j K$
(Kähler metric)
glue to $\Phi : V^{N=(2,2)}_{ch} \longmapsto X$ Kähler
+ c.c
$\bar\theta = 0$
hol function $h(z^i)$
$K + h(z) + \bar h(\bar z^i)$


· glue on the world sheet
Φ : V_ch (Σ) ⟼ X
N=1,2
Kähler
Φ^i = (φ^i , ψ^i_± , F^i)
φ^i : Σ ⟼ X
ψ^i_± ∈ Γ(Σ, S_± ⊗ φ* T^{1,0}_X)
ψ̄^ī_± ∈ Γ(Σ, S_± ⊗ φ* T^{0,1}_X)
Kähler
er metric)


Two types of SUSY
glue on the world sheet
N=1,2
Kähler
metric)
Kähler

Topological Twist

$$\Phi : V^{N=(2,2)}_{ch}(\Sigma) \longrightarrow X$$

under SUSY $\delta = \epsilon^{\pm} Q_{\pm} + \bar{\epsilon}^{\pm} \bar{Q}_{\pm}$

$\epsilon^{\pm}, \bar{\epsilon}^{\pm}$ sections of Spin bundles.

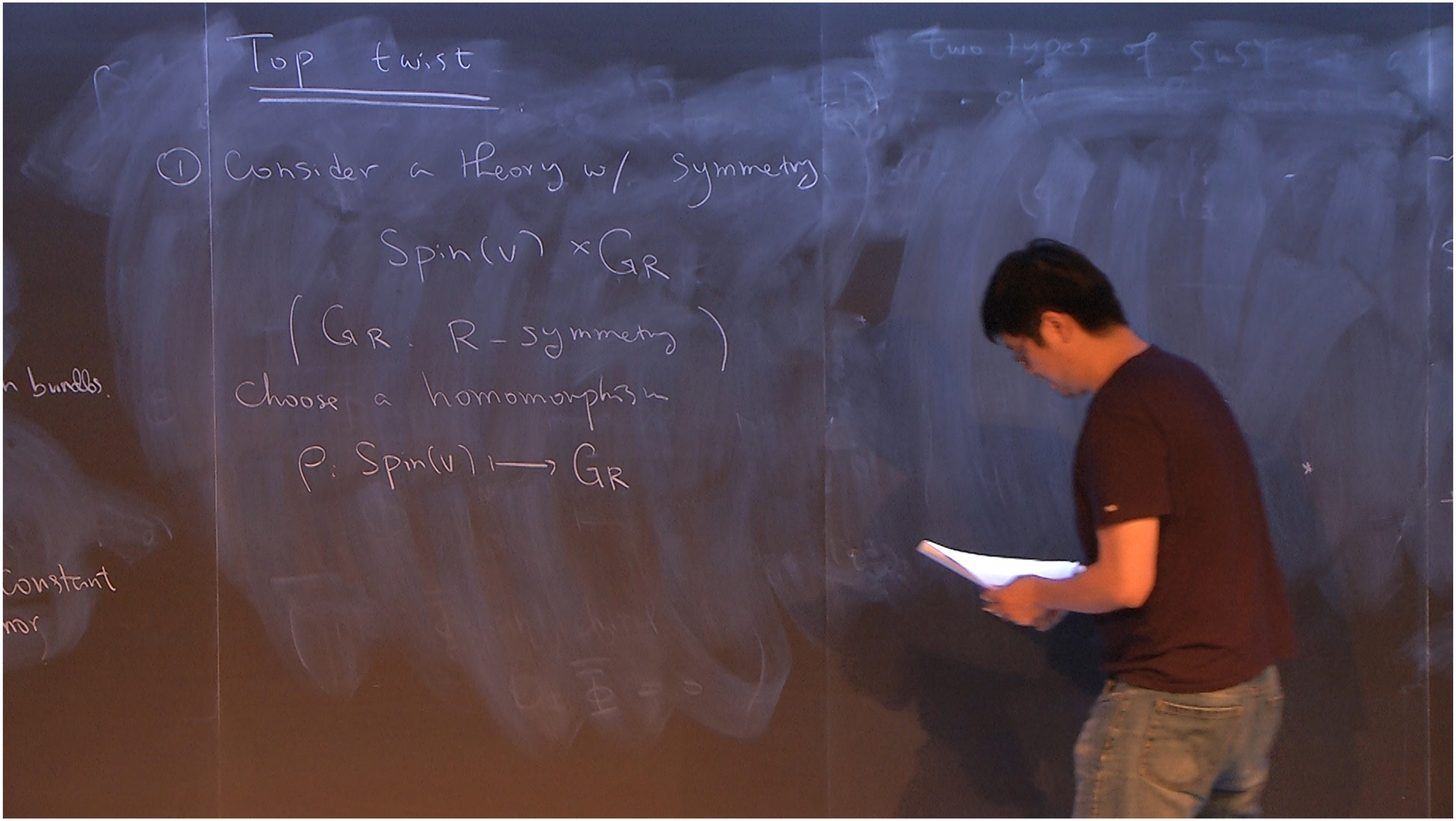
$$\delta S = \int_{\Sigma} (\nabla_{\mu} \epsilon^{\pm}) G^{\mu}_{\pm} + (\nabla_{\mu} \bar{\epsilon}^{\pm}) \bar{G}^{\mu}_{\pm}$$

Global SUSY $\rightsquigarrow$ $\nabla \epsilon^{\pm} = 0$ covariant constant spinor

$$\int d^2x \int d^2\theta \, d^2\bar{\theta} \, K = \int d^2x \int d$$

only depends on g

glue to $\Phi : V^{N=(2,2)}_{ch}$


Top twist
① Consider a theory w/ symmetry
Spin(V) × G_R
(G_R : R-symmetry)
Choose a homomorphism
ρ: Spin(V) ⟶ G_R

Top twist

① Consider a theory w/ symmetry

$$\mathrm{Spin}(V) \times G_R$$

(G_R : R-symmetry)

Choose a homomorphism

$$\rho : \mathrm{Spin}(V) \longmapsto G_R$$

② Consider the new Poincare symmetry

$$\mathrm{Spin}(V) \xrightarrow{1 \times \rho} \mathrm{Spin}(V) \times G_R \curvearrowright S$$

③ Find $Q \in S$ such that

$Q \in$ trivial rep of $\mathrm{Spin}(V)$ inside S

and $[Q, Q] = 0$

④ $\rightsquigarrow$ global SUSY

$D_\mu \psi^i_\pm =$

$\rightsquigarrow \Phi :$

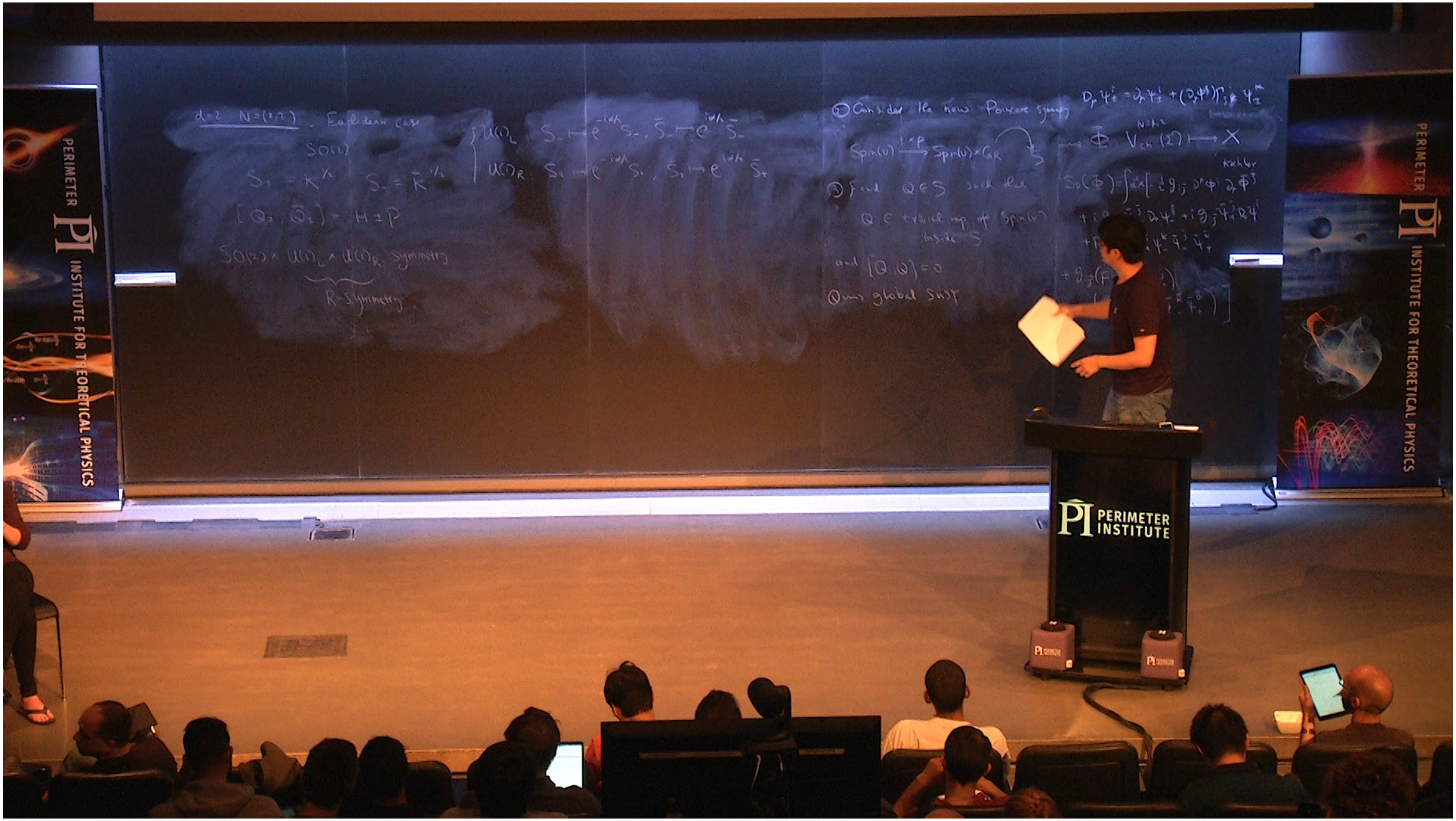
$S_D(\Phi) = \int$

$+ i g_{i\bar{j}}$

$+ R_{i\bar{j}k\bar{l}}$

$+ g_{i\bar{j}}(F^i -$

$(\bar{F}^{\bar{j}} -$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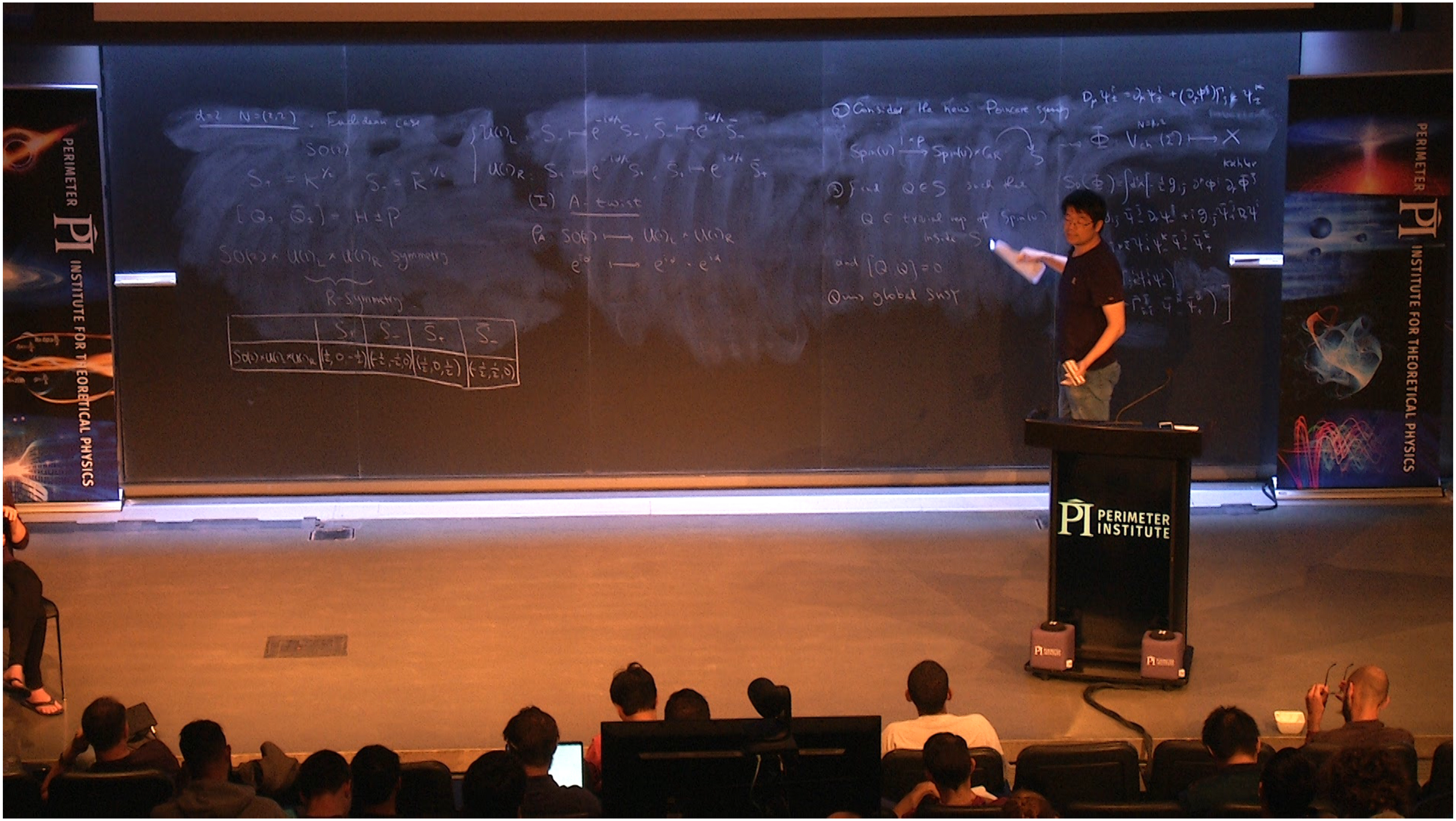
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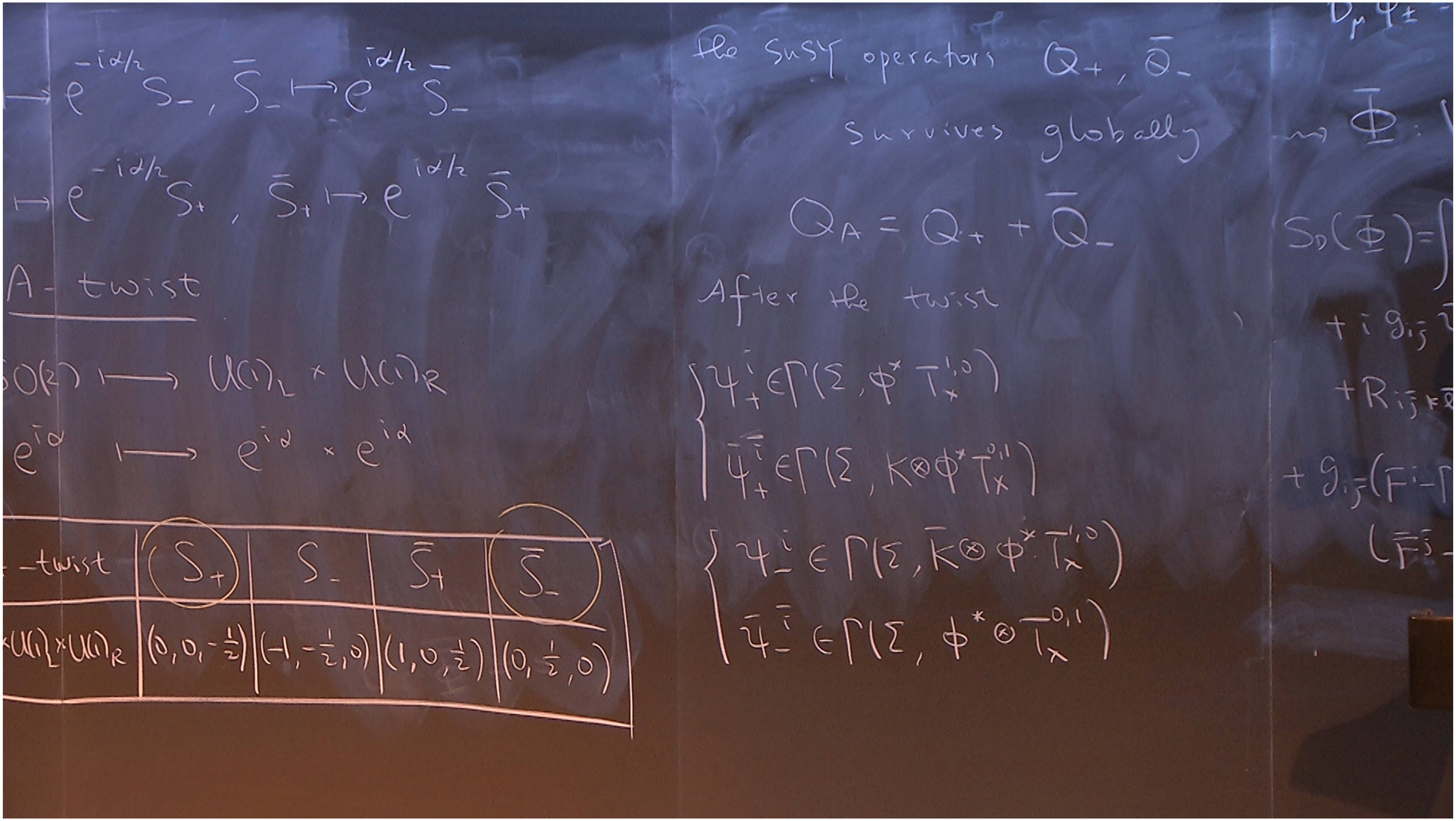
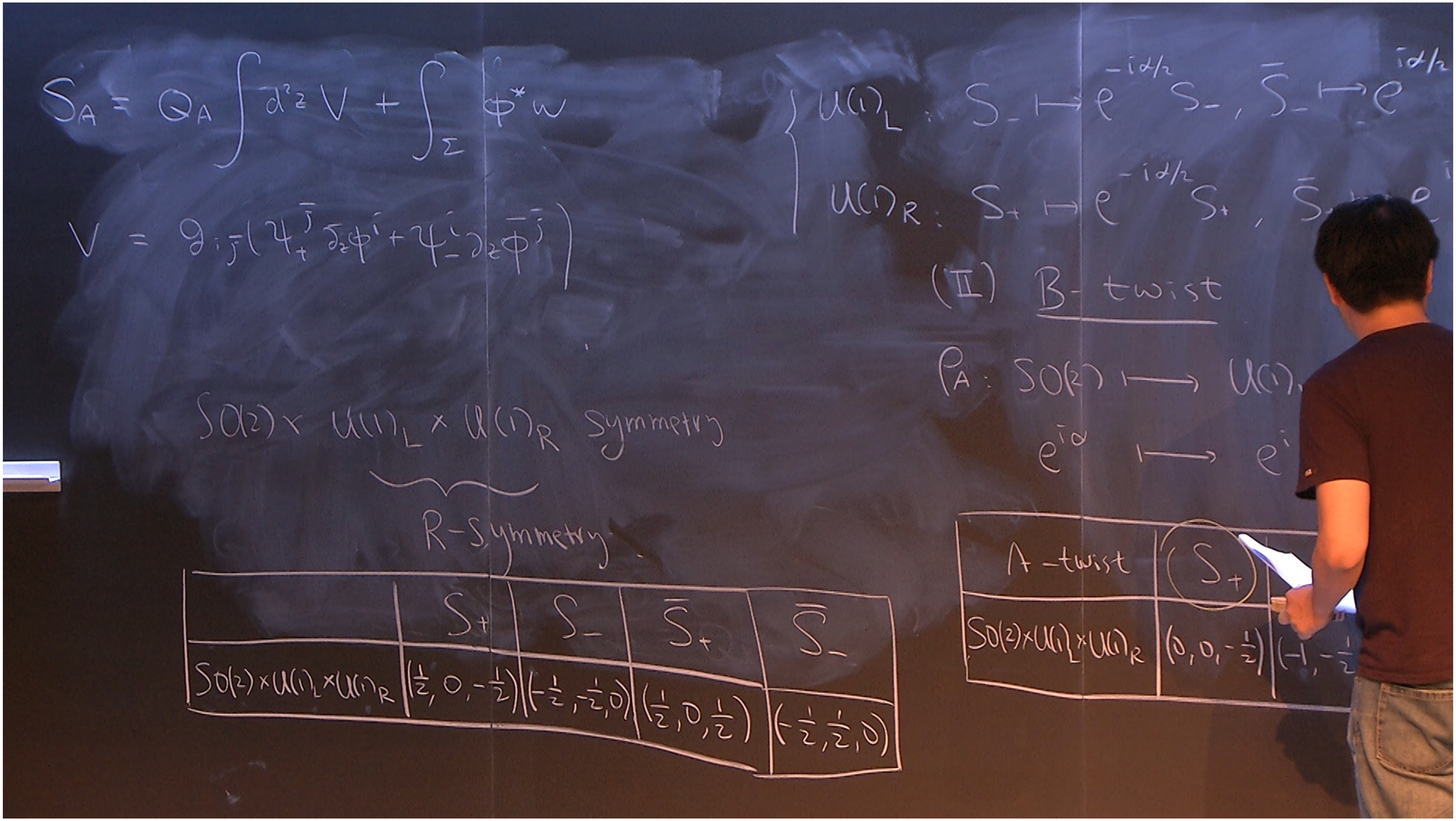
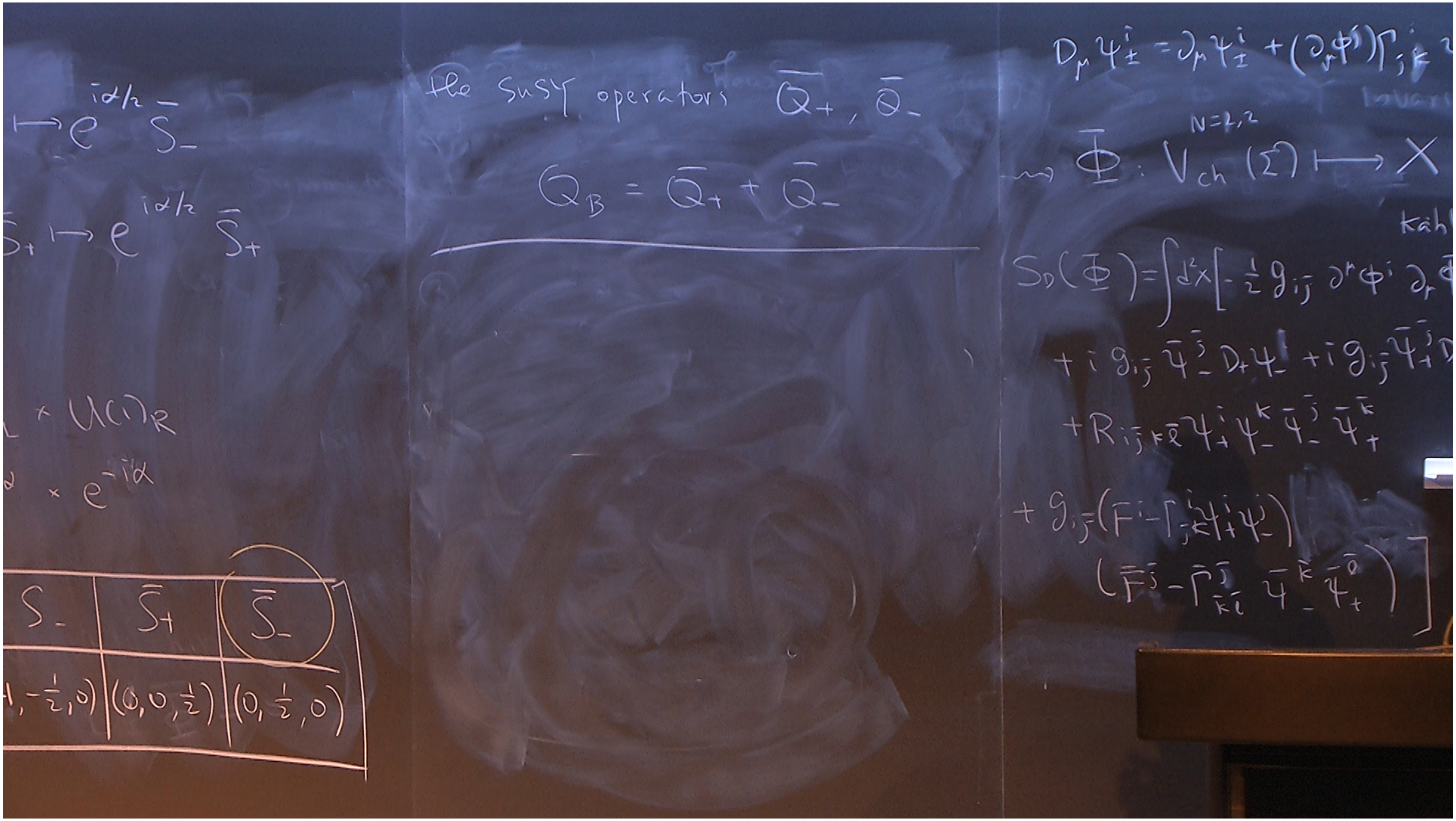

$\mapsto e^{-i\alpha/2} S_-$, $\bar{S}_- \mapsto e^{i\alpha/2} \bar{S}_-$
$\mapsto e^{-i\alpha/2} S_+$, $\bar{S}_+ \mapsto e^{i\alpha/2} \bar{S}_+$
A - twist
$SO(2) \longmapsto U(1)_L \times U(1)_R$
$e^{i\alpha} \longmapsto e^{i\alpha} \times e^{i\alpha}$
-twist
S_+
S_-
$\bar{S}_+$
$\bar{S}_-$
$\times U(1)_L \times U(1)_R$
$(0, 0, -\frac{1}{2})$
$(-1, -\frac{1}{2}, 0)$
$(1, 0, \frac{1}{2})$
$(0, \frac{1}{2}, 0)$
the SUSY operators Q_+, $\bar{Q}_-$
survives globally
$Q_A = Q_+ + \bar{Q}_-$
After the twist
$\psi_+^i \in \Gamma(\Sigma, \phi^* T_X^{1,0})$
$\bar{\psi}_+^{\bar{i}} \in \Gamma(\Sigma, K \otimes \phi^* T_X^{0,1})$
$\psi_-^i \in \Gamma(\Sigma, \bar{K} \otimes \phi^* T_X^{1,0})$
$\bar{\psi}_-^{\bar{i}} \in \Gamma(\Sigma, \phi^* \otimes T_X^{0,1})$
$D_\mu \psi_\pm$
$\bar{\Phi}$
$S_D(\bar{\Phi}) = \int$
$+ i g_{i\bar{j}}$
$+ R_{i\bar{j}k\bar{l}}$
$+ g_{i\bar{j}}(F^i -$


S_A = Q_A ∫ d²z V + ∫_Σ φ*w
V = g_{ij̄}(ψ_+^j̄ ∂_z̄ φ^i + ψ_-^i ∂_z φ̄^j̄)
U(1)_L : S_- ↦ e^{-iα/2} S_- , S̄_- ↦ e^{iα/2}
U(1)_R : S_+ ↦ e^{-iα/2} S_+ ,
(II) B-twist
ρ_A : SO(2) ⟶ U(1)
e^{iα} ⟶
SO(2) × U(1)_L × U(1)_R symmetry
R-symmetry
S_+
S_-
S̄_+
S̄_-
SO(2)×U(1)_L×U(1)_R
(1/2, 0, -1/2)
(-1/2, -1/2, 0)
(1/2, 0, 1/2)
(-1/2, 1/2, 0)
A-twist
S_+
SO(2)×U(1)_L×U(1)_R
(0, 0, -1/2)


the SUSY operators $\bar{Q}_+$, $\bar{Q}_-$
$Q_B = \bar{Q}_+ + \bar{Q}_-$
$\mapsto e^{i\alpha/2} \bar{S}_-$
$\bar{S}_+ \mapsto e^{i\alpha/2} \bar{S}_+$
$\times U(1)_R$
$\times e^{-i\alpha}$
S_- | $\bar{S}_+$ | $\bar{S}_-$
$(1, -\frac{1}{2}, 0)$ | $(0, 0, \frac{1}{2})$ | $(0, \frac{1}{2}, 0)$
$D_\mu \psi^i_\pm = \partial_\mu \psi^i_\pm + (\partial_\mu \phi^j)\Gamma^i_{jk}$
$\Phi : V_{ch}(\Sigma) \longmapsto X$
Kähl
$S_D(\Phi) = \int d^2x [-\frac{1}{2} g_{i\bar{j}} \partial^\mu \phi^i \partial_\mu \bar{\phi}$
$+ i g_{i\bar{j}} \bar{\psi}^{\bar{j}}_- D_+ \psi^i_- + i g_{i\bar{j}} \bar{\psi}^{\bar{j}}_+ D$
$+ R_{i\bar{j}k\bar{l}} \psi^i_+ \psi^k_- \bar{\psi}^{\bar{j}}_- \bar{\psi}^{\bar{l}}_+$
$+ g_{i\bar{j}} (F^i - \Gamma^i_{jk} \psi^j_+ \psi^k_-)$
$(\bar{F}^{\bar{j}} - \Gamma^{\bar{j}}_{\bar{k}\bar{l}} \bar{\psi}^{\bar{k}}_- \bar{\psi}^{\bar{l}}_+)]$

Let (X, d_X), (Y, d_Y) be two dg space, $(\mathcal{O}_X, d_X)$, $(\mathcal{O}_Y, d_Y)$ dga

$\begin{cases} X \text{ is equipped w/ } \int \mathcal{O}_X \longmapsto \mathbb{C} \text{ of deg} = -k \\ Y \text{ is equipped w/ a symplectic form of deg} = k-1 \end{cases}$

$\Longrightarrow$ map(X, Y) is (-1)-shifted symplectic

$d_X, d_Y \rightsquigarrow \{S_X, -\}, \{S_Y, -\}$

$S = S_X + S_Y$ satisfies $\{S, S\} = 0$

(CME)

$\begin{cases} U(1)_L : S_- \longmapsto e^{-i\alpha/2} S_-, \ \bar{S}_- \longmapsto \\ U(1)_R : S_+ \longmapsto e^{-i\alpha/2} S_+, \ \bar{S}_+ \longmapsto \end{cases}$

(II) B-twist

$\rho_A : SO(2) \longmapsto U(1)_L \times$

$e^{i\alpha} \longmapsto e^{i\alpha} \times$

B-twist	S_+	S_-
$SO(2) \times U(1)_L \times U(1)_R$	$(1, 0, -\frac{1}{2})$	$(-1, -\frac{1}{2},$

$\ldots Y)$ be two

$(\ldots, d_X)$, $(\mathcal{O}_Y, d_Y)$ dga

$\int \mathcal{O}_X \longmapsto \mathbb{C}$ of $\deg = -k$

a symplectic form of $\deg = k-1$

$\ldots(X, Y)$ is (-1)-shifted symplectic

$\ldots \rightsquigarrow \{S_X, -\}$, $\{S_Y, -\}$

$\ldots = S_X + S_Y$ satisfies $\{S, S\} = 0$

(CME)

Example Poisson σ-model

① $T[1]\Sigma = (\Sigma, \Omega^{\cdot}(\Sigma))$ $\left(= \Sigma_{dR}\right)$

$\mathcal{O}(T[1]\Sigma) = \Omega^{\cdot}(\Sigma) \xrightarrow{\int} \mathbb{C}$ $\deg = -2$

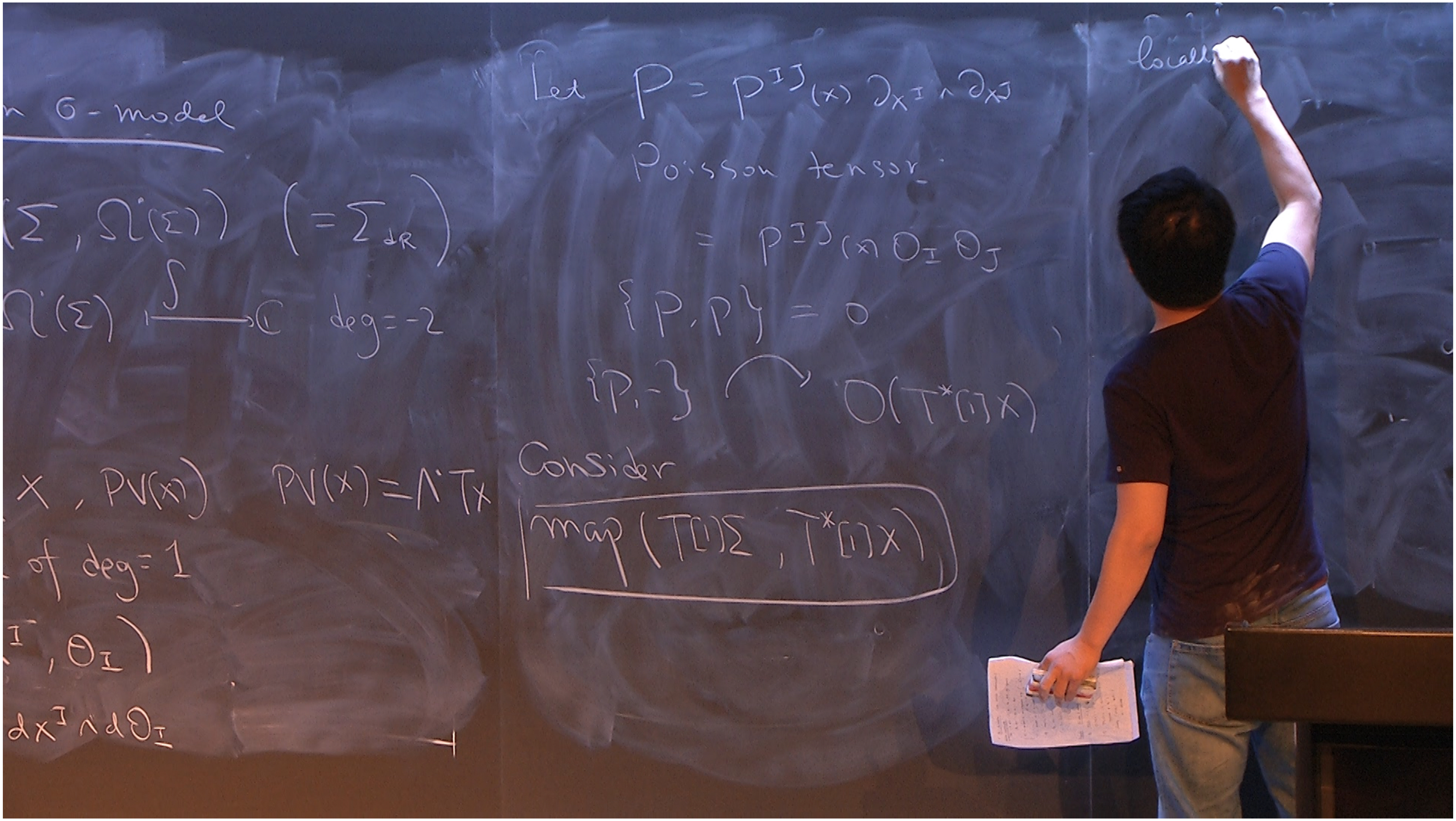
(Σ: surface)

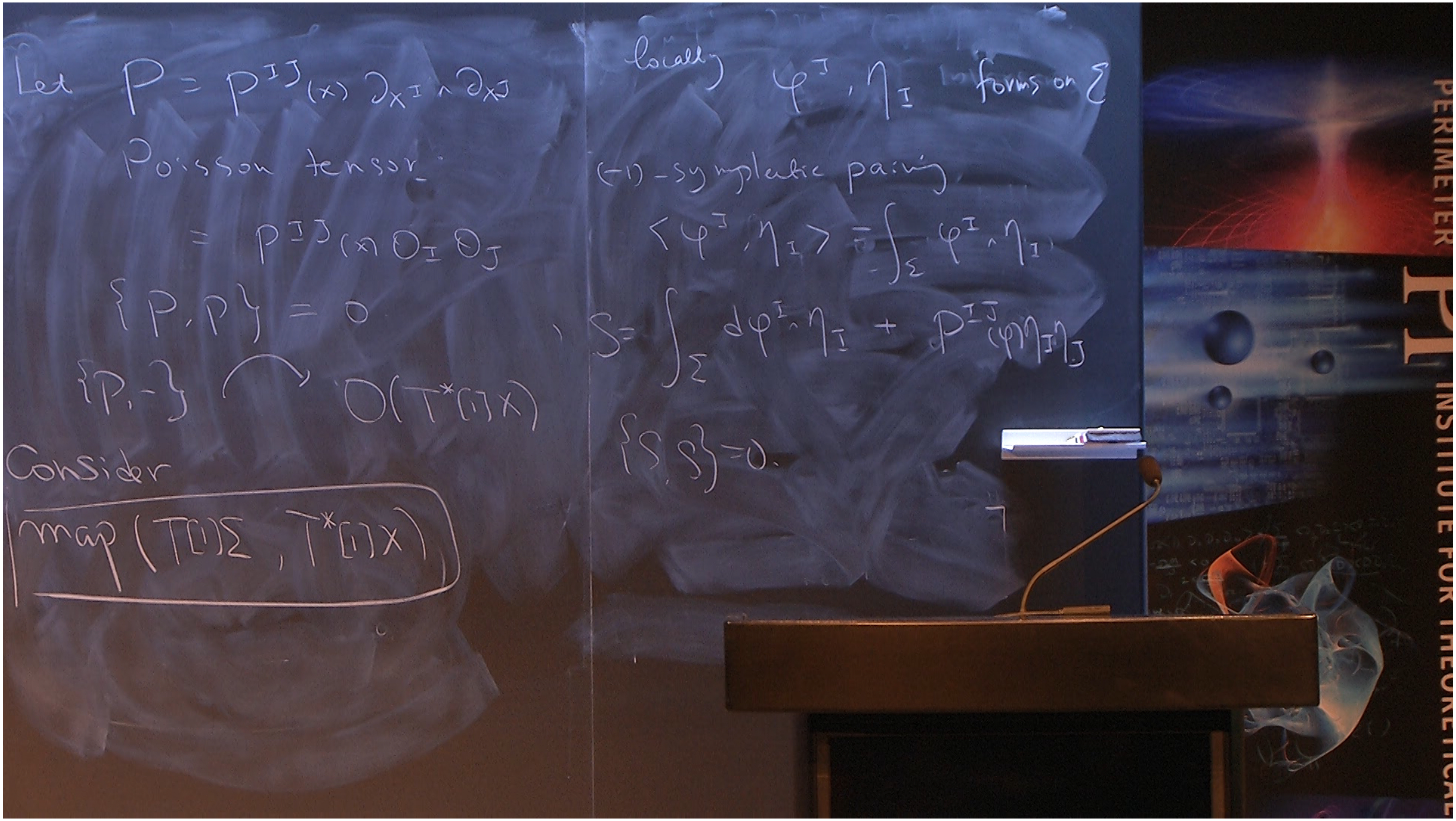
② $T^*[1]X = (X, PV(X))$ $PV(X) = \Lambda^{\cdot} T_X$

symplectic of $\deg = 1$

locally: (X^I, θ_I)

$\omega = dx^I \wedge d\theta_I$


σ-model
Let P = P^IJ(x) ∂x^I ∧ ∂x^J
Poisson tensor
= P^IJ(x) θ_I θ_J
{P, P} = 0
{P, -}
O(T*[1]X)
Consider
map(T[1]Σ, T*[1]X)
deg = -2
PV(X) = ∧·T_X
of deg = 1
locally


Let P = p^{IJ}(x) ∂_{x^I} ∧ ∂_{x^J}
Poisson tensor
= p^{IJ}(x) θ_I θ_J
{P, P} = 0
{P, -} O(T*[1]X)
Consider
map(T[1]Σ, T*[1]X)
locally φ^I, η_I forms on Σ
(-1)-symplectic pairing
⟨φ^I, η_I⟩ = ∫_Σ φ^I ∧ η_I
S = ∫_Σ dφ^I ∧ η_I + P^{IJ}(φ) η_I η_J
{S, S} = 0.
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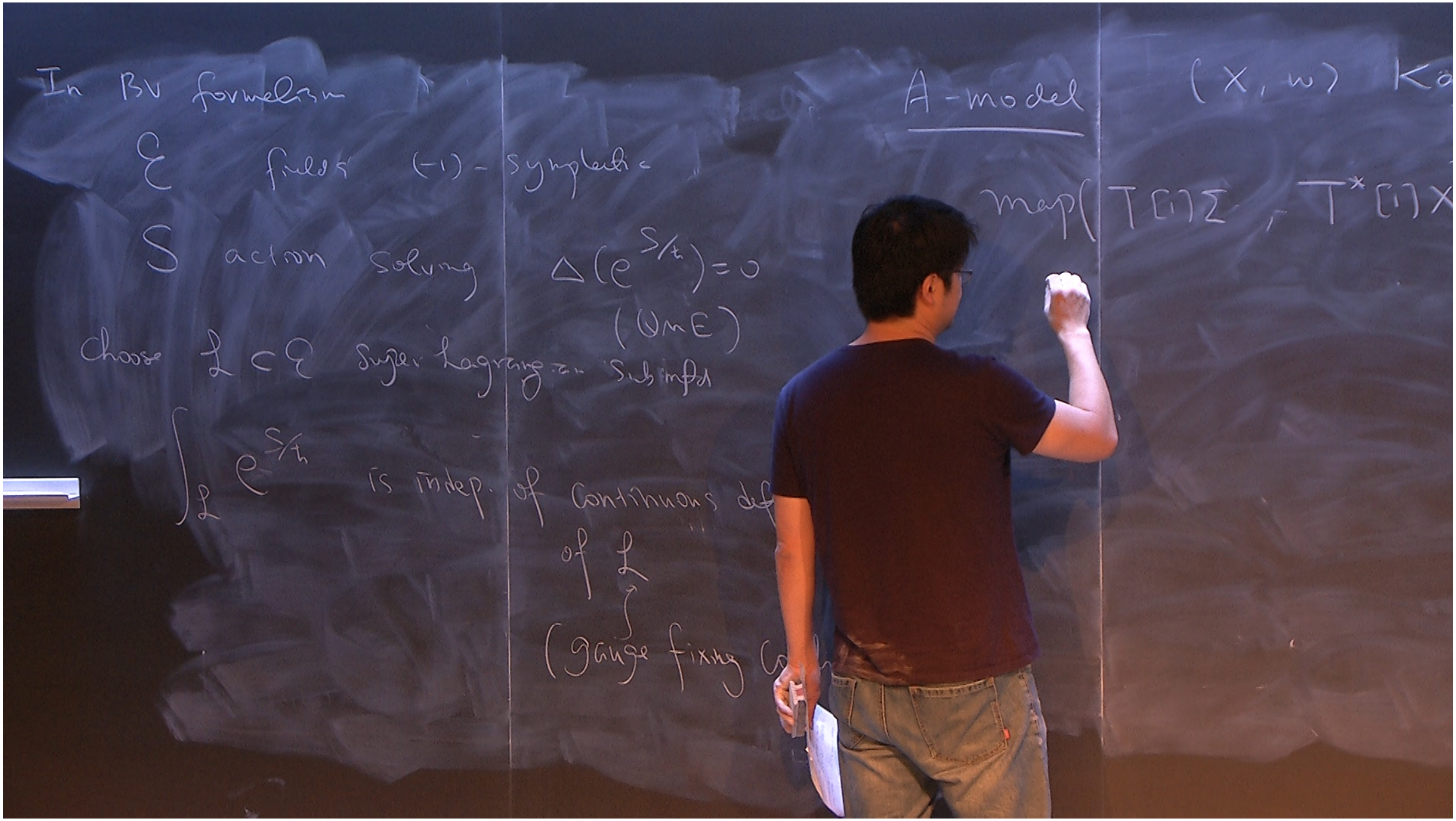
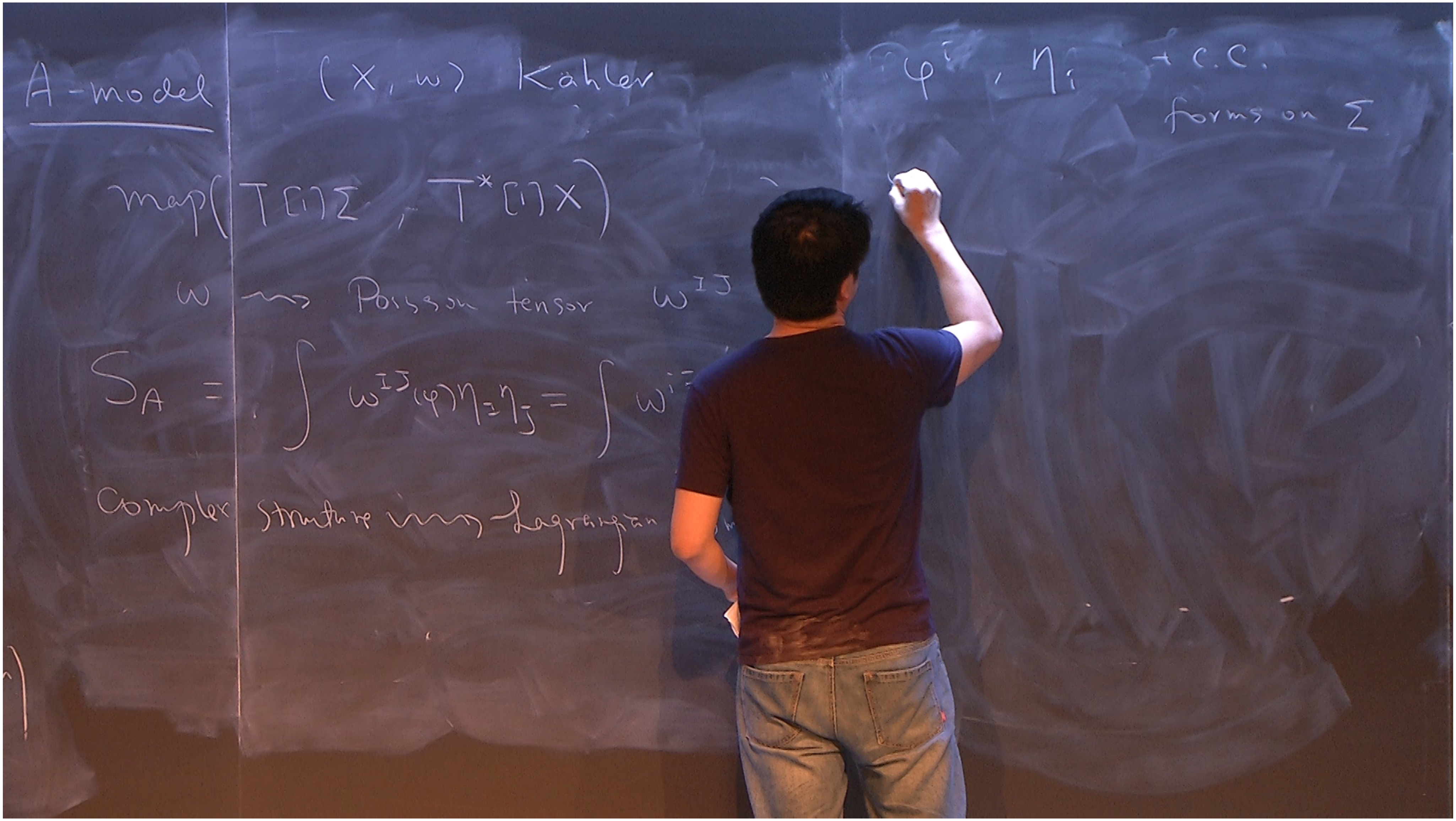
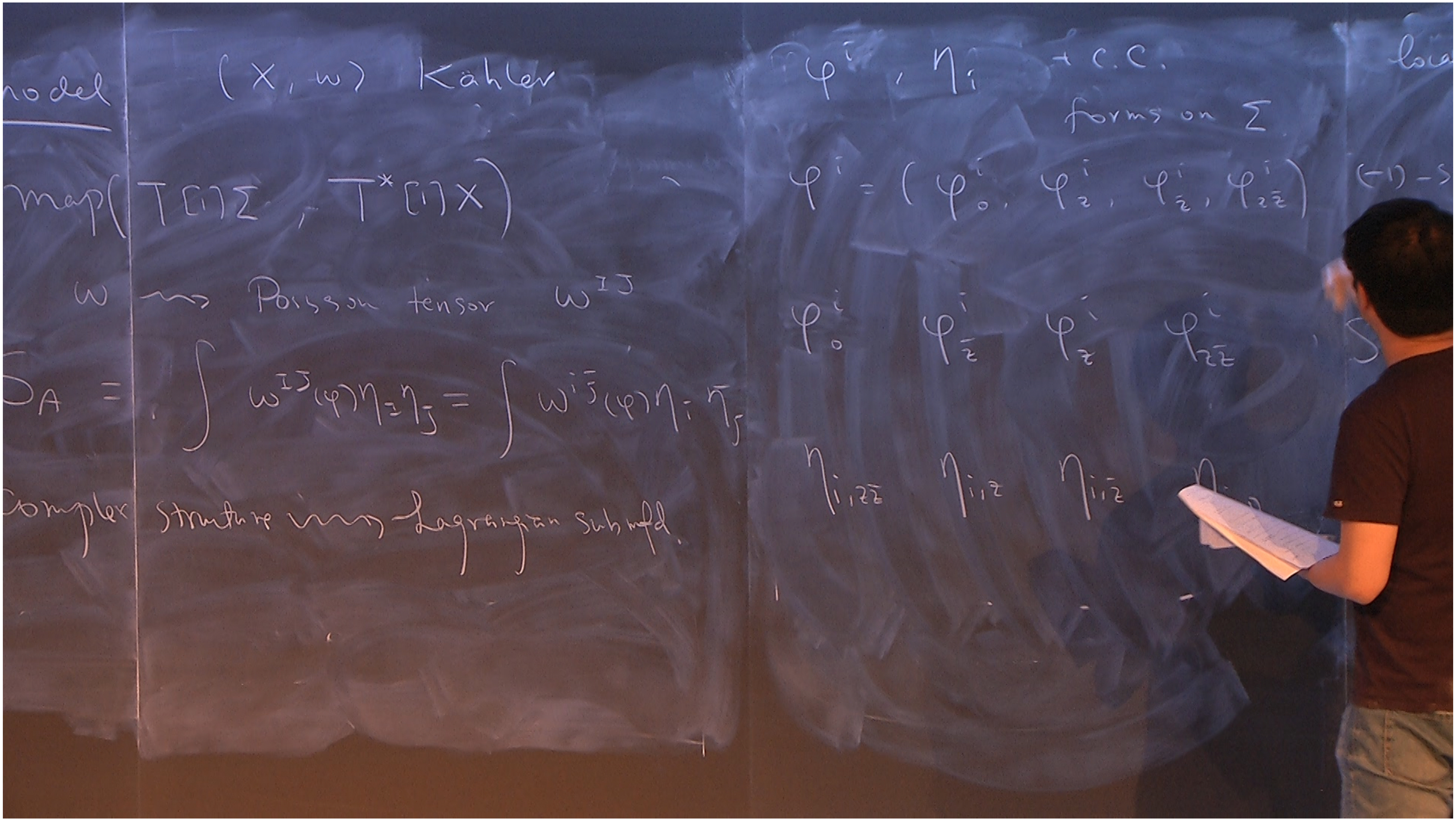
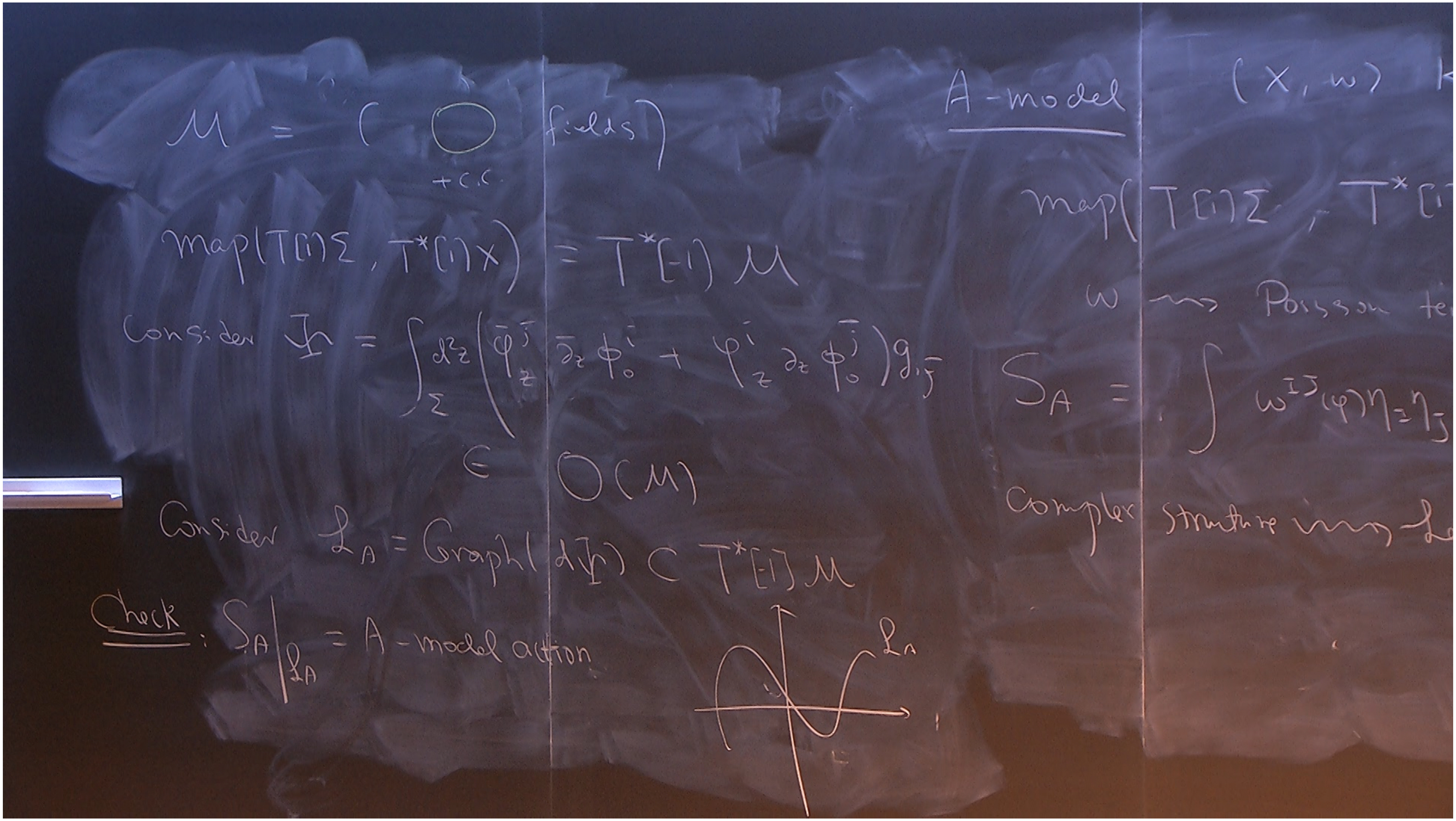
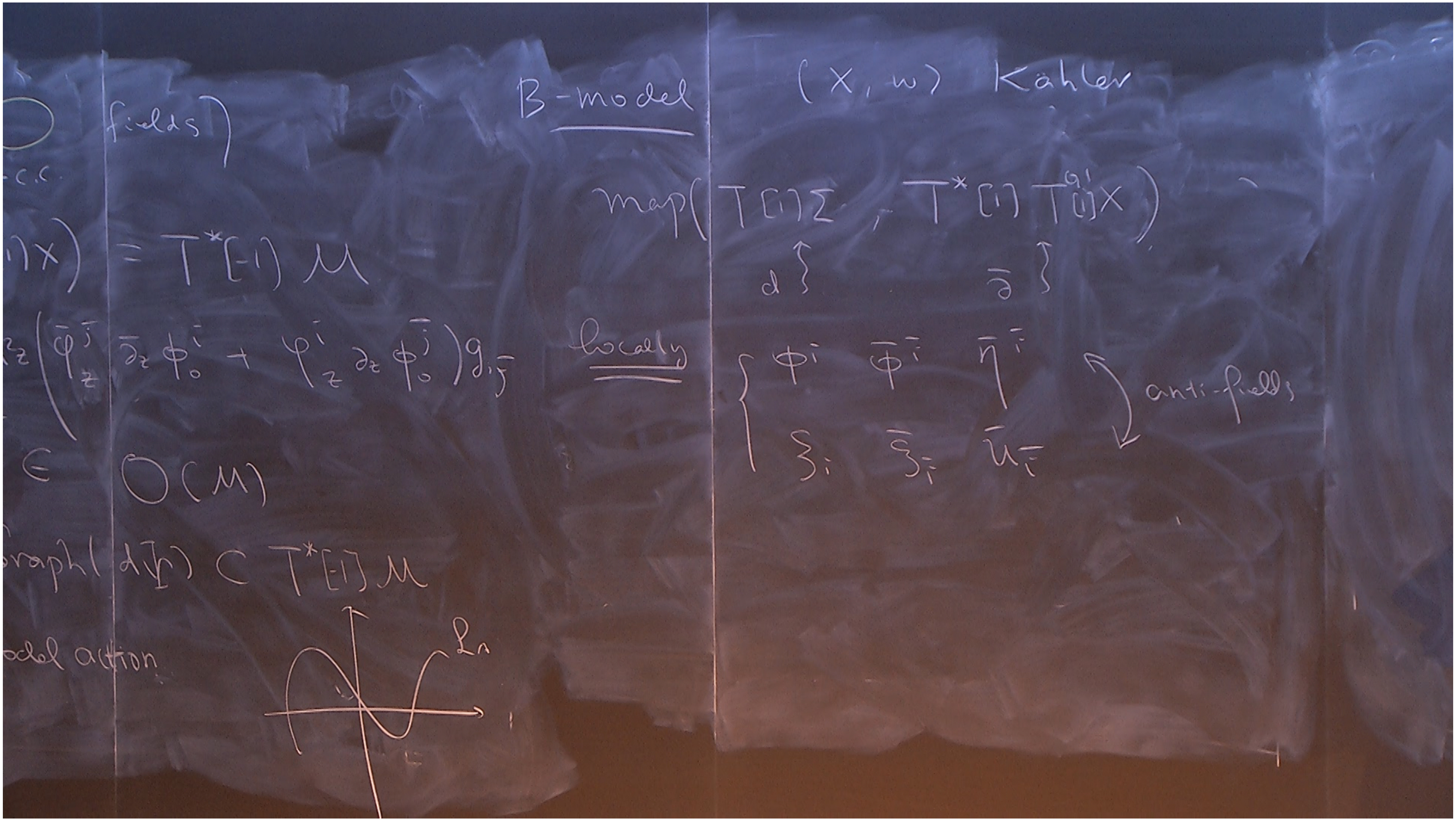
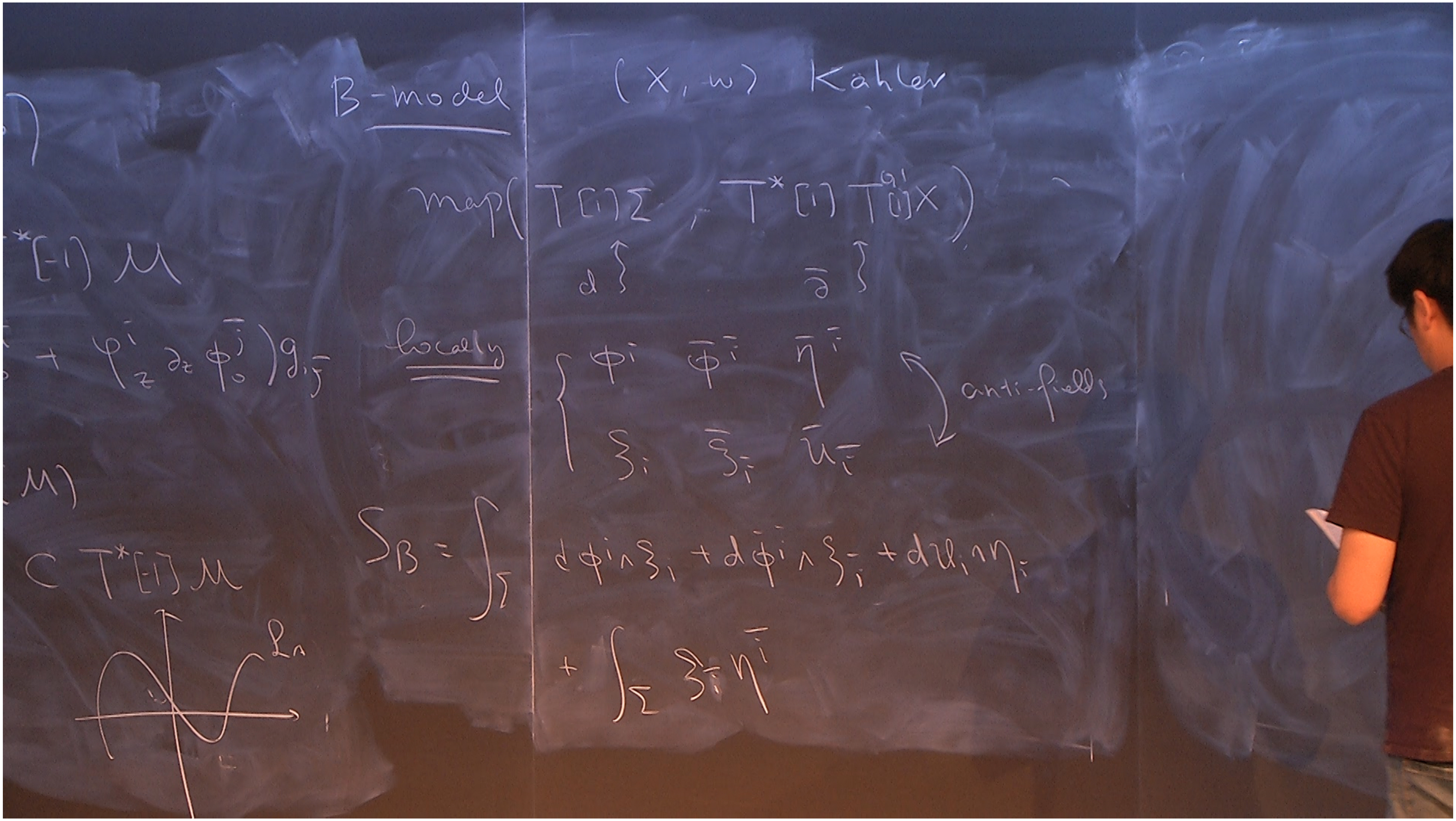

In BV formalism
fields (-1)-symplectic
S action solving
(QME)
Choose
super Lagrangian submfd
is indep. of continuous def
of
A-model
map(


A-model
(X, ω) Kähler
φ^i , η_i + c.c.
forms on Σ
map(T[1]Σ , T*[1]X)
ω ⇝ Poisson tensor ω^IJ
S_A = ∫ ω^IJ(φ)η_I η_J = ∫ ω
Complex structure ⇝ Lagrangian


model
(X, ω) Kähler
map(T[1]Σ, T*[1]X)
ω ⇝ Poisson tensor ω^IJ
S_A = ∫ ω^IJ(φ)η_I η_J = ∫ ω^{i j̄}(φ)η_i η_j̄
Complex structure ⇝ Lagrangian submfd.
φ^i, η_i + c.c.
forms on Σ
φ^i = (φ^i_0, φ^i_z, φ^i_z̄, φ^i_{zz̄})
φ^i_0 φ^i_z̄ φ^i_z φ^i_{z̄z}
η_{i,zz̄} η_{i,z} η_{i,z̄}


A-model
fields
+ c.c.
Consider
Consider
Graph
Check
A-model action
Poisson
Complex structure


fields
B-model
(X, ω) Kähler
locally
anti-fields


B-model
(X, ω) Kähler
locally
anti-fields