Title: Lecture 2: Supersymmetric Field Theory and Topological Twists

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Collection: QFT for Mathematicians

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$d=2$ $N=(2,2)$ SUSY σ-model

— top. twist (A/B-model)

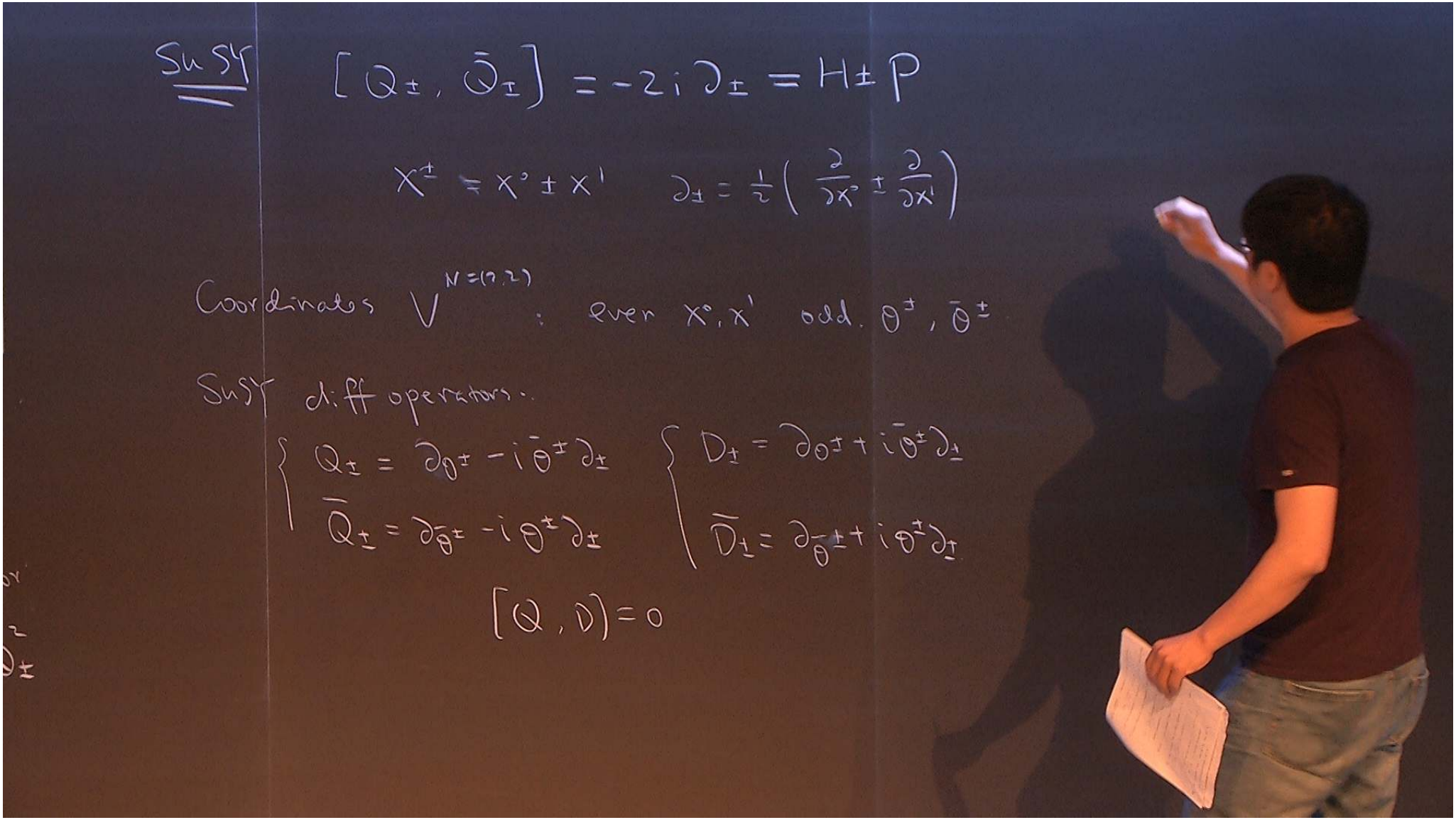
— BV formalism

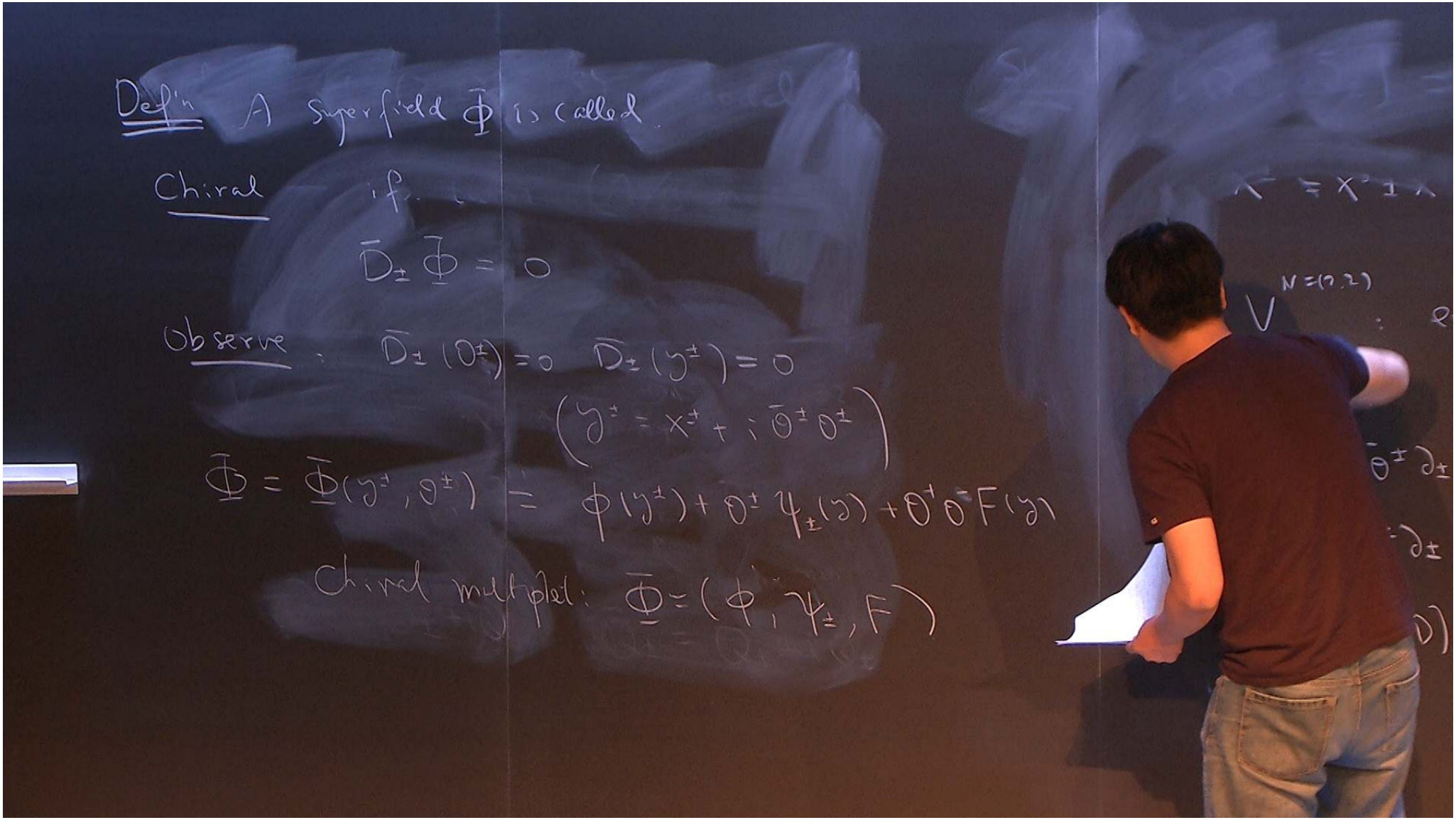
$$V^{N=(2,2)} = \mathbb{R}^{1,1} \times \Pi \mathbb{R}_+^{\oplus 2} \times \Pi \mathbb{R}_-^{\oplus 2}$$

$$= \mathbb{R}^{1,1} \times \Pi S_+ \times \Pi S_-$$

($S_\pm$: complex spinor)

$$Q_\pm = Q_\pm^1 + i Q_\pm^2 \qquad \bar{Q}_\pm = Q_\pm^1 - i Q_\pm^2$$


SuSY
$[Q_\pm, \bar{Q}_\mp] = -2i\partial_\pm = H \pm P$
$x^\pm = x^0 \pm x^1 \quad \partial_\pm = \frac{1}{2}\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right)$
Coordinates $V^{N=(1,1)}$: even x^0, x^1 odd. $\theta^\pm, \bar{\theta}^\pm$
SuSY diff operators.
$\begin{cases} Q_\pm = \partial_{\theta^\pm} - i\bar{\theta}^\pm \partial_\pm \\ \bar{Q}_\pm = \partial_{\bar{\theta}^\pm} - i\theta^\pm \partial_\pm \end{cases} \quad \begin{cases} D_\pm = \partial_{\theta^\pm} + i\bar{\theta}^\pm \partial_\pm \\ \bar{D}_\pm = \partial_{\bar{\theta}^\pm} + i\theta^\pm \partial_\pm \end{cases}$
$[Q, D) = 0$


Def'n A superfield $\bar{\Phi}$ is called
Chiral if
$\bar{D}_\pm \bar{\Phi} = 0$
Observe: $\bar{D}_\pm(\theta^\pm) = 0$ $\bar{D}_\pm(y^\pm) = 0$
$(y^\pm = x^\pm + i\bar{\theta}^\pm\theta^\pm)$
$\bar{\Phi} = \bar{\Phi}(y^\pm, \theta^\pm) = \phi(y^\pm) + \theta^\pm\psi_\pm(y) + \theta^1\theta^2 F(y)$
Chiral multiplet: $\bar{\Phi} = (\phi, \psi_\pm, F)$
$N=(2,2)$

two types of SUSY inv. action

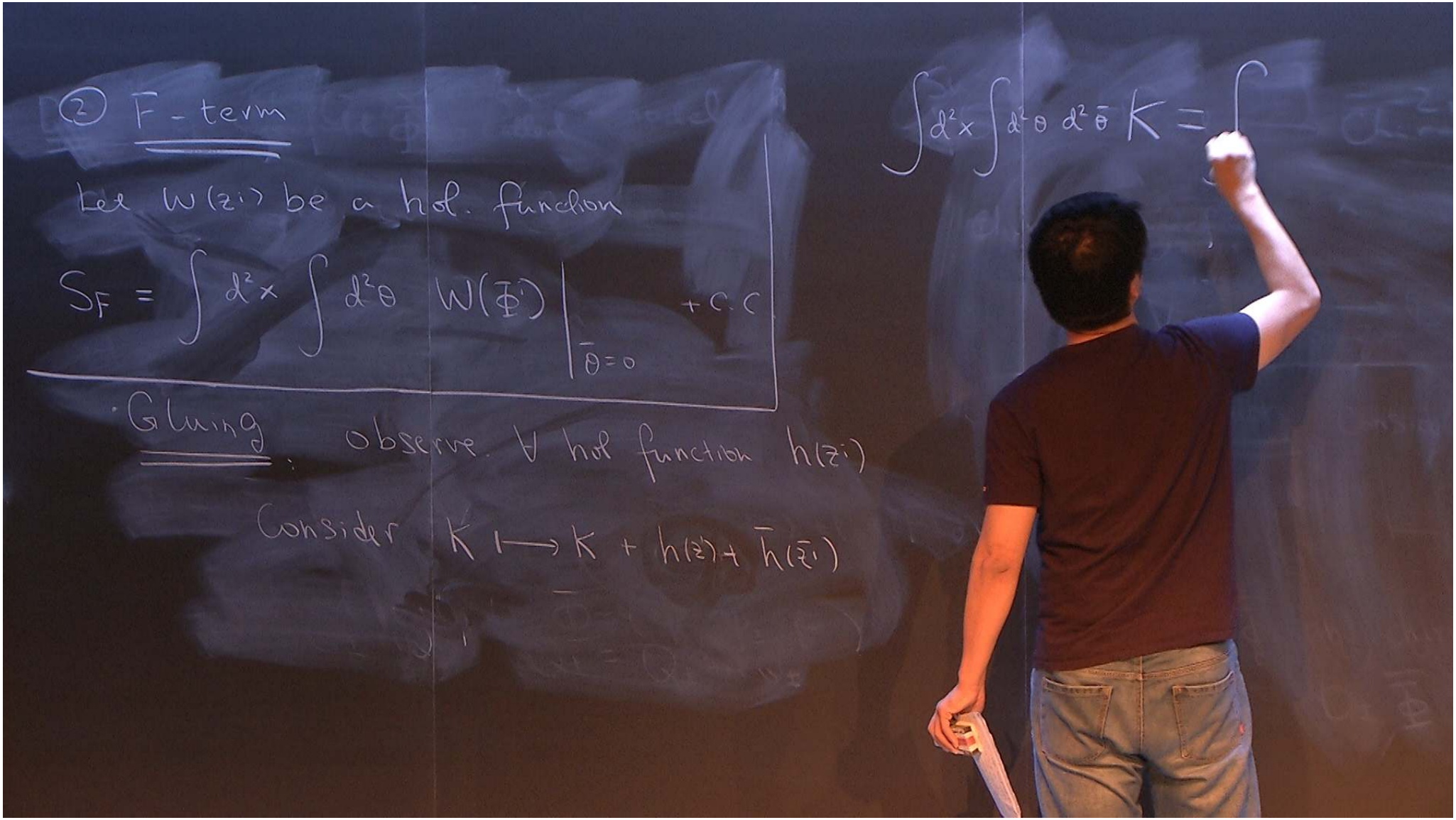
① D-term Let $K(z^i, \bar{z}^i)$ be a smooth function on $\mathbb{C}^n$ (Kähler potential)

$$S_D = \int d^2x \int d^2\theta\, d^2\bar{\theta}\; K(\Phi^i, \bar{\Phi}^i)$$

$$Q_\pm S_D = \int d^2x \int d^2\theta\, d^2\bar{\theta}\; (\partial_{\theta^\pm} - i\bar{\theta}^\pm \partial_\pm) K = 0$$

S_D is SUSY invariant.

$$Q_\pm = \partial_{\theta^\pm} - i\bar{\theta}^\pm \partial_\pm$$


② F-term
Let W(Φ^i) be a hol. function
$S_F = \int d^2x \int d^2\theta \; W(\Phi^i)\Big|_{\bar\theta=0} + c.c$
· Gluing: observe. ∀ hol function h(z^i)
Consider K ⟼ K + h(z^i) + h̄(z̄^i)
$\int d^2x \int d^2\theta\, d^2\bar\theta\, K = \int$

$$\int d^2x \int d^2\theta\, d^2\bar{\theta}\, K = \int d^2x \int d^2\theta\, d^2\bar{\theta}\, \big(K(\Phi^i, \bar{\Phi}^i) + h(\Phi^i) + \bar{h}(\bar{\Phi}^i)\big)$$

only depends on $g_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K$

(Kähler metric)

Give to $\bar{\Phi} : V_{ch}^{N=(2,2)} \longrightarrow X$ Kähler

two types of

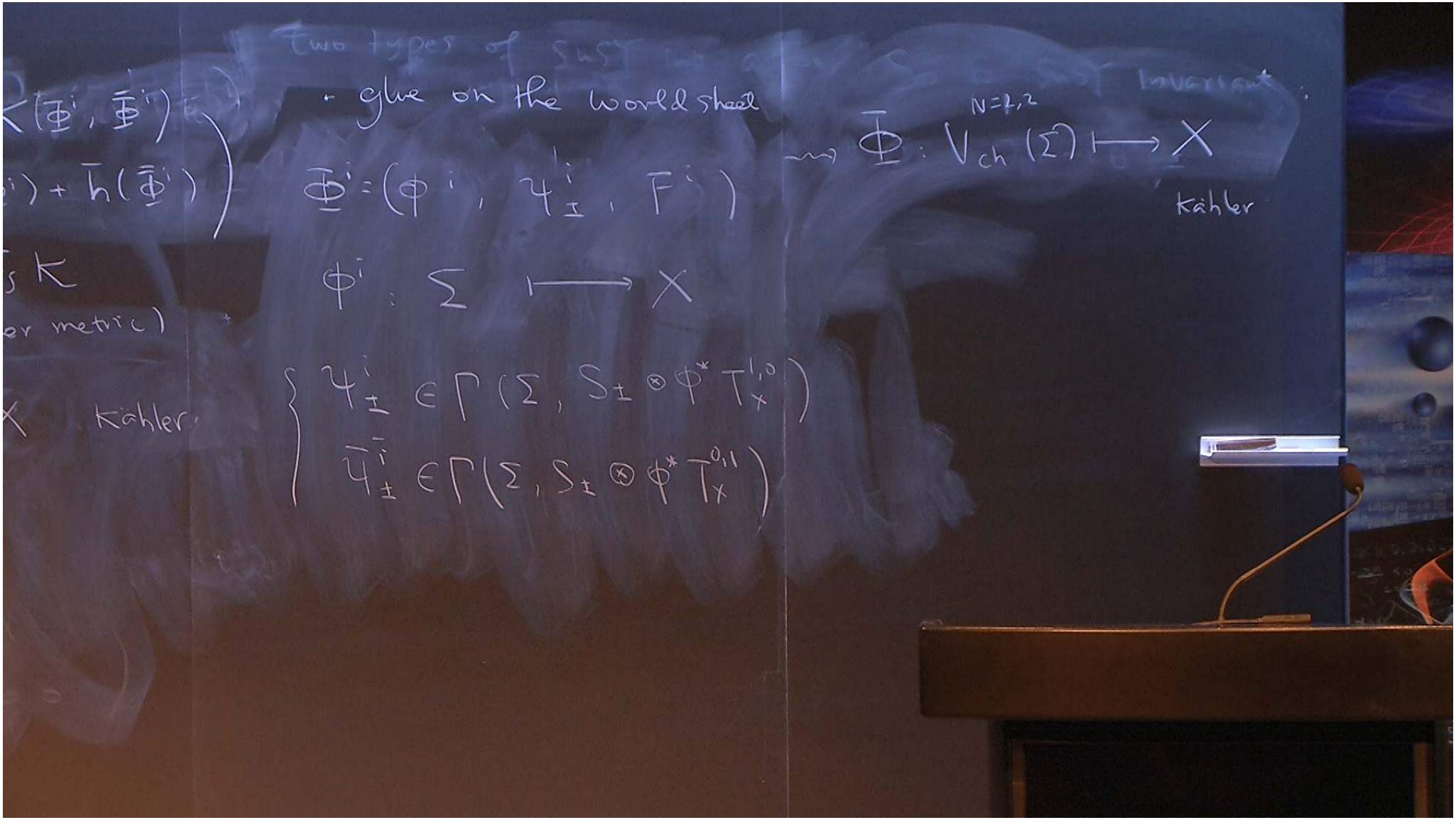
tion

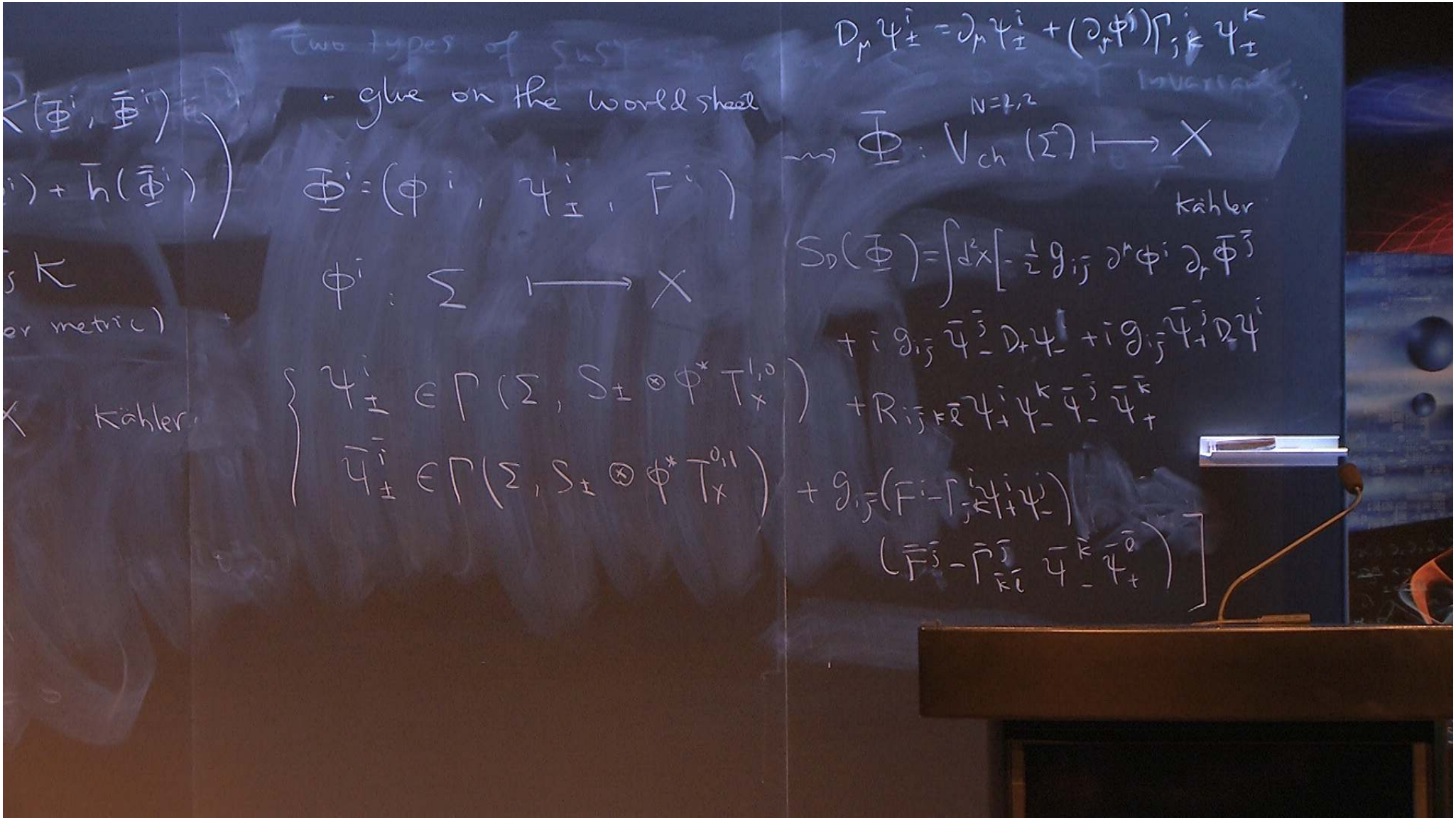
$+ c.c$

$\bar{\theta} = 0$

hol function $h(z^i)$

$K + h(z^i) + \bar{h}(\bar{z}^i)$


Two types of SUSY
glue on the world sheet
$\Phi^i = (\phi^i, \psi^i_\pm, F^i)$
$\phi^i : \Sigma \longmapsto X$
$\psi^i_\pm \in \Gamma(\Sigma, S_\pm \otimes \phi^* T^{1,0}_X)$
$\bar{\psi}^{\bar{i}}_\pm \in \Gamma(\Sigma, S_\pm \otimes \phi^* T^{0,1}_X)$
$\Phi : V_{ch}(\Sigma) \longmapsto X$
N=2,2
Kähler
Kähler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$\psi^i_\pm \in \Gamma(\Sigma, S_\pm \otimes \phi^* T^{1,0}_X)$
$\bar\psi^{\bar i}_\pm \in \Gamma(\Sigma, S_\pm \otimes \phi^* T^{0,1}_X)$
$D_\mu \psi^i_\pm = \partial_\mu \psi^i_\pm + (\partial_\mu \phi^j)\Gamma^i_{jk}\psi^k_\pm$
$N=2,2$
$\Phi : V_{ch}(\Sigma) \longmapsto X$
Kähler
$S_D(\Phi) = \int d^2x \Big[-\frac{1}{2} g_{i\bar j}\, \partial^\mu \phi^i \partial_\mu \bar\phi^{\bar j}$
$+ i g_{i\bar j} \bar\psi^{\bar j}_- D_+ \psi^i_- + i g_{i\bar j} \bar\psi^{\bar j}_+ D_- \psi^i$
$+ R_{i\bar j k \bar l} \psi^i_+ \psi^k_- \bar\psi^{\bar j}_- \bar\psi^{\bar k}_+$
$+ g_{i\bar j}(F^i - \Gamma^i_{jk}\psi^i_+\psi^j_-)$
$(\bar F^{\bar j} - \bar\Gamma^{\bar j}_{\bar k \bar l} \bar\psi^{\bar k}_- \bar\psi^{\bar l}_+) \Big]$
Kähler

Topological Twist

$\Phi : V_{ch}^{N=(2,2)}(\Sigma) \longmapsto X$

under SuSy $\delta = \epsilon^{\pm} Q_{\pm} + \bar{\epsilon}^{\pm} \bar{Q}_{\pm}$

$\epsilon^{\pm}, \bar{\epsilon}^{\pm}$ sections of Spin bundles.

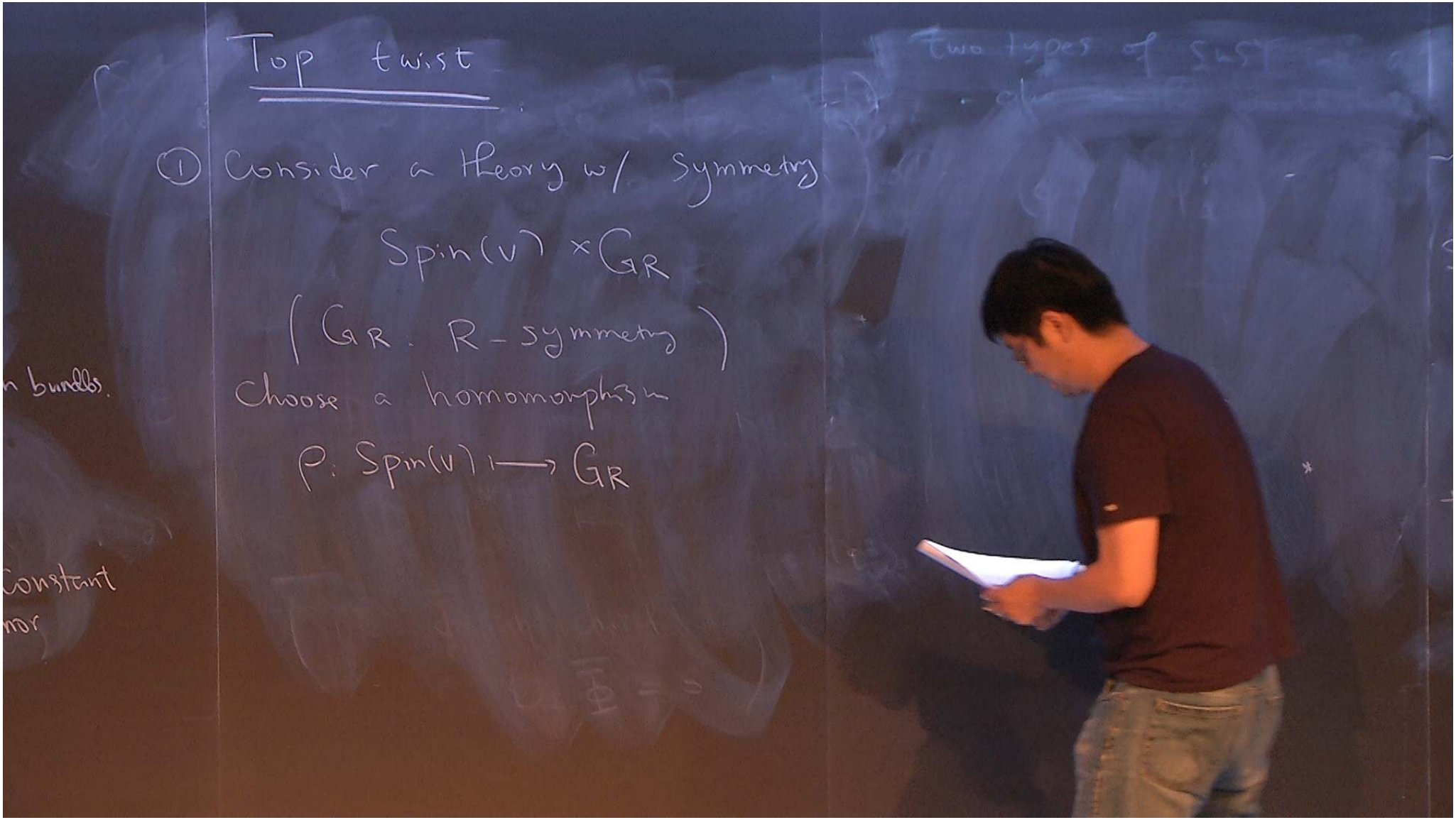
$\delta S = \int_{\Sigma} (\nabla_{\mu} \epsilon^{\pm}) G^{\mu}_{\pm} + (\nabla_{\mu} \bar{\epsilon}^{\pm}) \bar{G}^{\mu}_{\pm}$

Global SuSy $\rightsquigarrow \nabla \epsilon^{\pm} = 0$ Covariant Constant Spinor

$\int d^2x \int d^2\theta \, d^2\bar{\theta} \, K = \int d^2x \int$

only depends on $g_{\cdot}$

Glue to $\bar{\Phi} : V_{ch}^{N=(2,2)}$


Top twist
① Consider a theory w/ symmetry
Spin(v) × G_R
(G_R : R-symmetry)
Choose a homomorphism
ρ : Spin(v) ⟶ G_R
two types of
bundles.
constant

Top twist

① Consider a theory w/ symmetry

$$\mathrm{Spin}(V) \times G_R$$

$(G_R$: R-symmetry$)$

Choose a homomorphism

$$\rho : \mathrm{Spin}(V) \longmapsto G_R$$

② Consider the new Poincare symmetry

$$\mathrm{Spin}(V) \xrightarrow{1 \times \rho} \mathrm{Spin}(V) \times G_R \curvearrowright S$$

③ Find $Q \in S$ such that

$Q \in$ trivial rep of $\mathrm{Spin}(V)$ inside S

and $[Q, Q] = 0$

$Q \rightsquigarrow$ global SUSY

$D_\mu \psi^i_\pm =$

$\rightsquigarrow \Phi :$

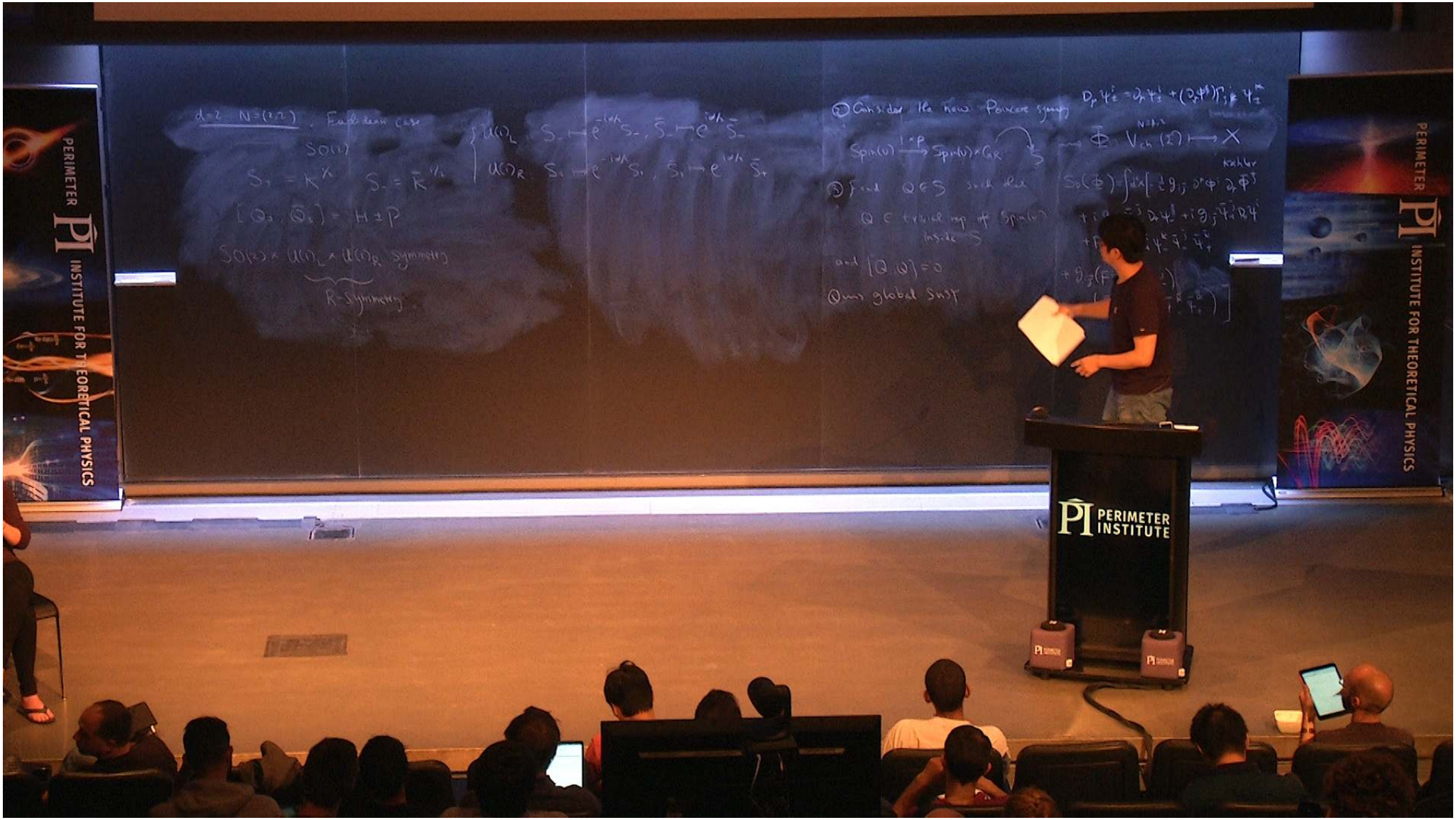
$S_D(\Phi) = \int$

$+ i g_{i\bar{j}}$

$+ R_{i\bar{j}k\bar{l}}$

$+ g_{i\bar{j}}(F^i -$

$(\bar{F}^{\bar{j}} -$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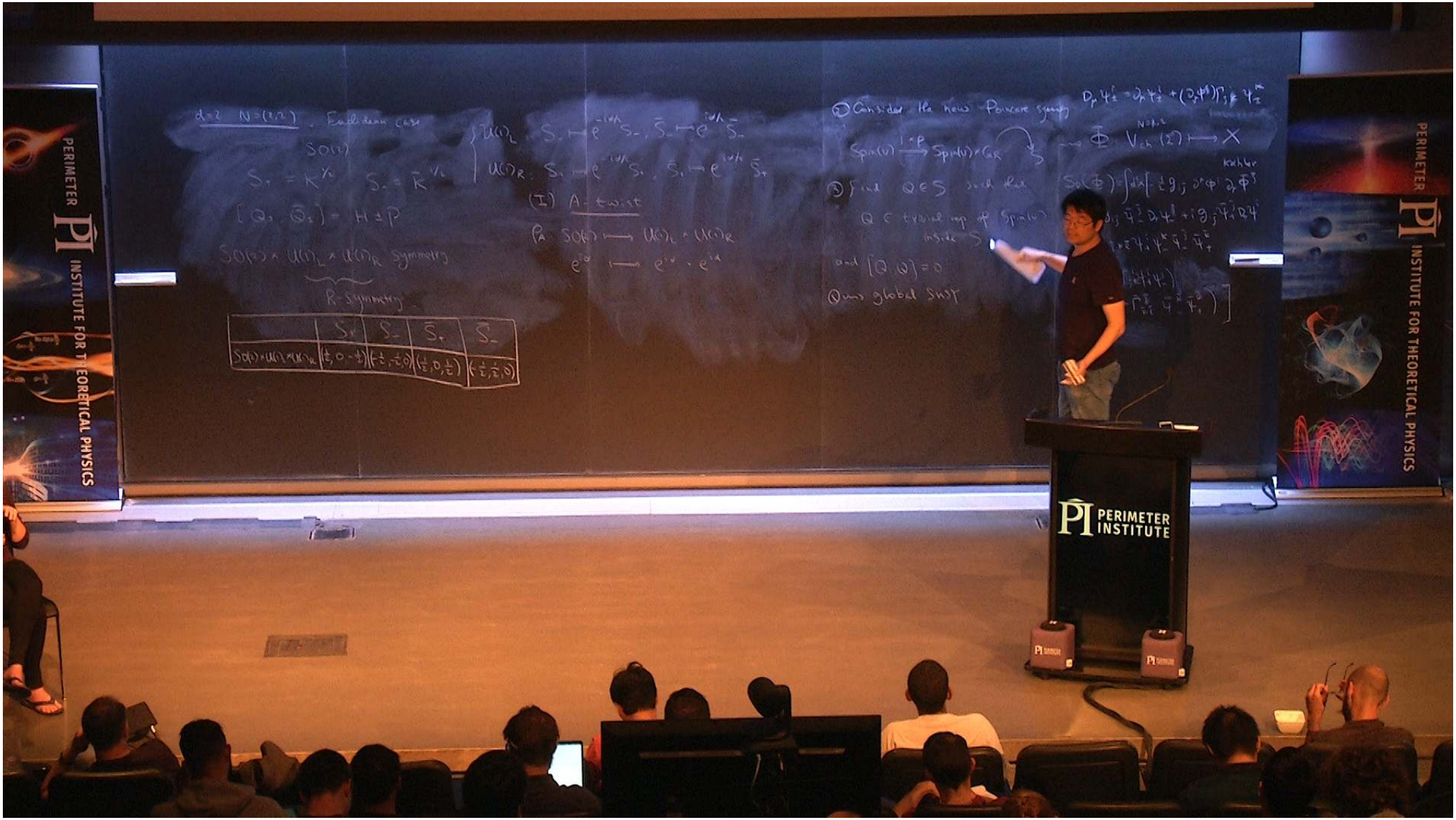
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$\longrightarrow e^{-i\alpha/2} S_-,\ \bar{S}_- \longmapsto e^{i\alpha/2} \bar{S}_-$

$\longmapsto e^{-i\alpha/2} S_+,\ \bar{S}_+ \longmapsto e^{i\alpha/2} \bar{S}_+$

A - twist

$SO(2) \longrightarrow U(1)_L \times U(1)_R$

$e^{i\alpha} \longmapsto e^{i\alpha} \times e^{i\alpha}$

-twist	S_+	S_-	$\bar{S}_+$	$\bar{S}_-$
$U(1)_L \times U(1)_R$	$(0, 0, -\frac{1}{2})$	$(-1, -\frac{1}{2}, 0)$	$(1, 0, \frac{1}{2})$	$(0, \frac{1}{2}, 0)$

the susy operators $Q_+, \bar{Q}_-$ survives globally

$$Q_A = Q_+ + \bar{Q}_-$$

After the twist

$$\begin{cases} \eta_+^i \in \Gamma(\Sigma, \Phi^* T_X^{1,0}) \\ \bar{\psi}_+^{\bar{i}} \in \Gamma(\Sigma, K \otimes \Phi^* T_X^{0,1}) \end{cases}$$

$$\begin{cases} \eta_-^i \in \Gamma(\Sigma, \bar{K} \otimes \Phi^* T_X^{1,0}) \\ \bar{\eta}_-^{\bar{i}} \in \Gamma(\Sigma, \Phi^* \otimes T_X^{0,1}) \end{cases}$$

$D_\mu \varphi_\pm$ -

$\rightsquigarrow \bar{\Phi}$:

$S_D(\bar{\Phi}) = \int$

$+ i g_{i\bar{j}}$

$+ R_{i\bar{j}}$

$+ g_{i\bar{j}}(F^i -$

$(\bar{F}^{\bar{j}} -$

$$S_A = Q_A \int d^2z\, V + \int_\Sigma \phi^* w$$

$$V = g_{i\bar{j}} \left(\eta_+^{\bar{j}} \bar{\partial}_z \phi^i + \eta_-^i \partial_z \bar{\phi}^{\bar{j}} \right)$$

$$\begin{cases} U(1)_L : S_- \mapsto e^{-id/2} S_-, \ \bar{S}_- \mapsto e^{id/2} \\ U(1)_R : S_+ \mapsto e^{-id/2} S_+, \ \bar{S}_+ \mapsto e^{id/2} \end{cases}$$

(II) B-twist

$\rho_A : SO(2) \longrightarrow U(1)_V$

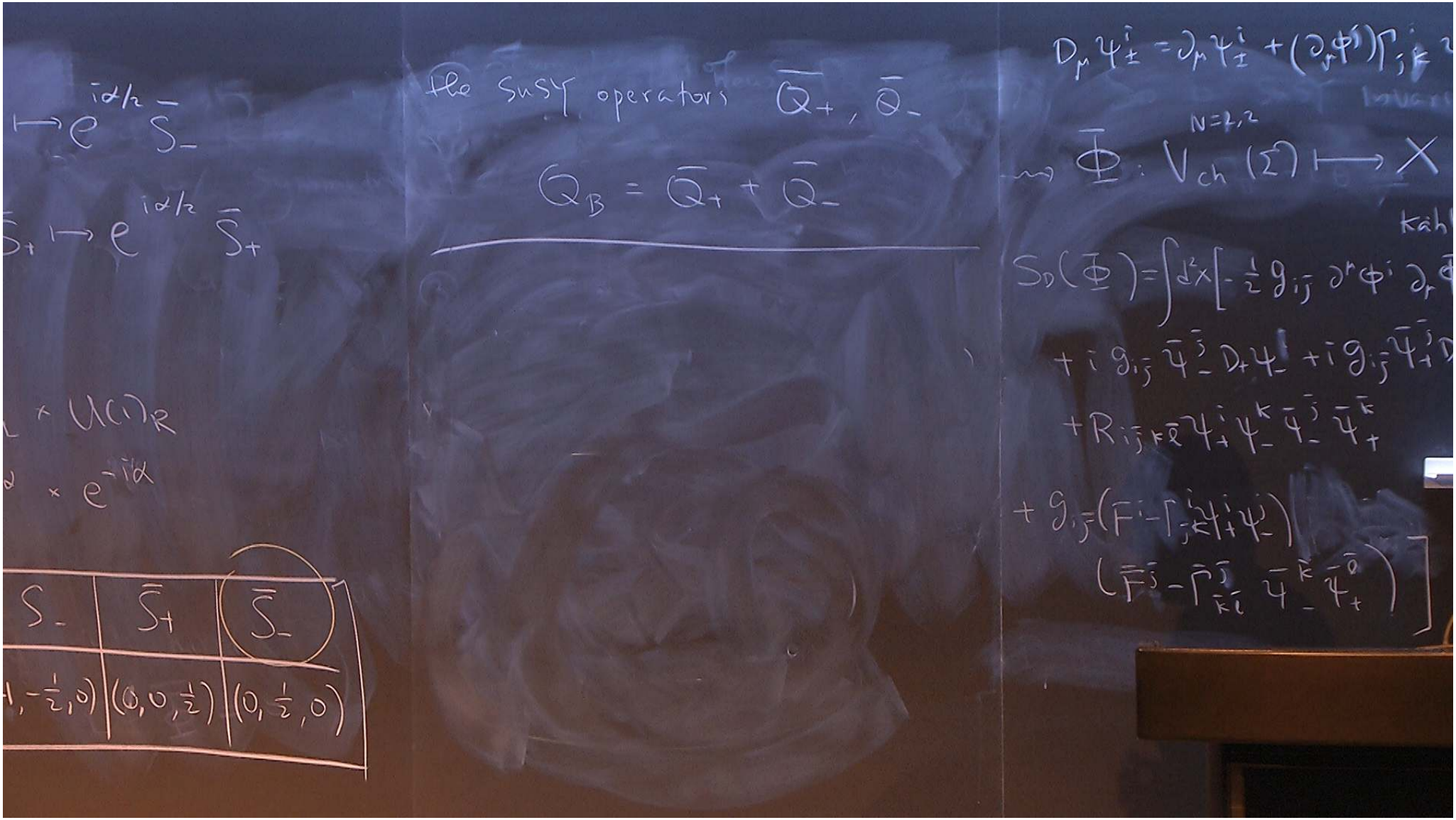
$e^{i\alpha} \longmapsto e^{i\alpha}$

$SO(2) \times U(1)_L \times U(1)_R$ symmetry

$\underbrace{\qquad\qquad}$ R-Symmetry

	S_+	S_-	$\bar{S}_+$	$\bar{S}_-$
$SO(2) \times U(1)_L \times U(1)_R$	$(\frac{1}{2}, 0, -\frac{1}{2})$	$(-\frac{1}{2}, -\frac{1}{2}, 0)$	$(\frac{1}{2}, 0, \frac{1}{2})$	$(-\frac{1}{2}, \frac{1}{2}, 0)$

A-twist	S_+	
$SO(2) \times U(1)_L \times U(1)_R$	$(0, 0, -\frac{1}{2})$	$(-1, -\frac{1}{2}$


$\mapsto e^{i\alpha/2} S_-$
$\bar{S}_+ \mapsto e^{i\alpha/2} \bar{S}_+$
$\times U(1)_R$
$\times e^{-i\alpha}$
S_- | $\bar{S}_+$ | $\bar{S}_-$
$(1, -\frac{1}{2}, 0)$ | $(0, 0, \frac{1}{2})$ | $(0, \frac{1}{2}, 0)$
the susy operators $\bar{Q}_+$, $\bar{Q}_-$
$Q_B = \bar{Q}_+ + \bar{Q}_-$
$D_\mu \psi^i_\pm = \partial_\mu \psi^i_\pm + (\partial_\mu \phi^j) \Gamma^i_{jk}$
$N=1,2$
$\Phi : V_{ch}(\Sigma) \longmapsto X$
Kah
$S_D(\Phi) = \int d^2x \left[-\frac{1}{2} g_{ij} \partial^\mu \phi^i \partial_\mu$
$+ i g_{ij} \bar{\psi}^j_- D_+ \psi^i_- + i g_{ij} \bar{\psi}^j_+$
$+ R_{ij\bar{k}\ell} \psi^i_+ \psi^k_- \bar{\psi}^j_- \bar{\psi}^\ell_+$
$+ g_{ij} (F^i - \Gamma^i_{jk} \psi^j_+ \psi^k_-)$
$(\bar{F}^j - \bar{\Gamma}^j_{k\ell} \bar{\psi}^k_- \bar{\psi}^\ell_+) \Big]$

Let (X, d_X), (Y, d_Y) be two dg space, $(\mathcal{O}_X, d_X)$, $(\mathcal{O}_Y, d_Y)$ dga

$\begin{cases} X \text{ is equipped w/ } \int : \mathcal{O}_X \longrightarrow \mathbb{C} \text{ of deg} = -k \\ Y \text{ is equipped w/ a symplectic form of deg} = k-1 \end{cases}$

$\Longrightarrow$ map(X, Y) is (-1)-shifted symplectic

$d_X, d_Y \rightsquigarrow \{S_X, -\}, \{S_Y, -\}$

$S = S_X + S_Y$ satisfies $\{S, S\} = 0$ (CME)

$\begin{cases} U(1)_L : S_- \mapsto e^{-i\alpha/2} S_-, \bar{S}_- \mapsto \\ U(1)_R : S_+ \mapsto e^{-i\alpha/2} S_+, \bar{S}_+ \mapsto \end{cases}$

(II) B-twist

$\rho_A : SO(2) \longrightarrow U(1)_L \times$

$e^{i\alpha} \longmapsto e^{i\alpha} \times$

B-twist	S_+	S_-
$SO(2) \times U(1)_L \times U(1)_R$	$(1, 0, -\frac{1}{2})$	$(-1, -\frac{1}{2},$

T) be two

$(\mathcal{O}_X, d_X)$, $(\mathcal{O}_Y, d_Y)$ dga

$\int : \mathcal{O}_X \longrightarrow \mathbb{C}$ of deg $= -k$

a symplectic form of deg $= k-1$

$P(X, Y)$ is (-1)-Shifted Symplectic

$\rightsquigarrow \{S_X, -\}$, $\{S_Y, -\}$

$S = S_X + S_Y$ satisfies $\{S, S\} = 0$

(CME)

Example : Poisson σ-model

① $T[1]\Sigma = (\Sigma, \Omega^\cdot(\Sigma))$ $\left(= \Sigma_{dR}\right)$

$\mathcal{O}(T[1]\Sigma) = \Omega^\cdot(\Sigma) \xrightarrow{\int} \mathbb{C}$ deg $= -2$

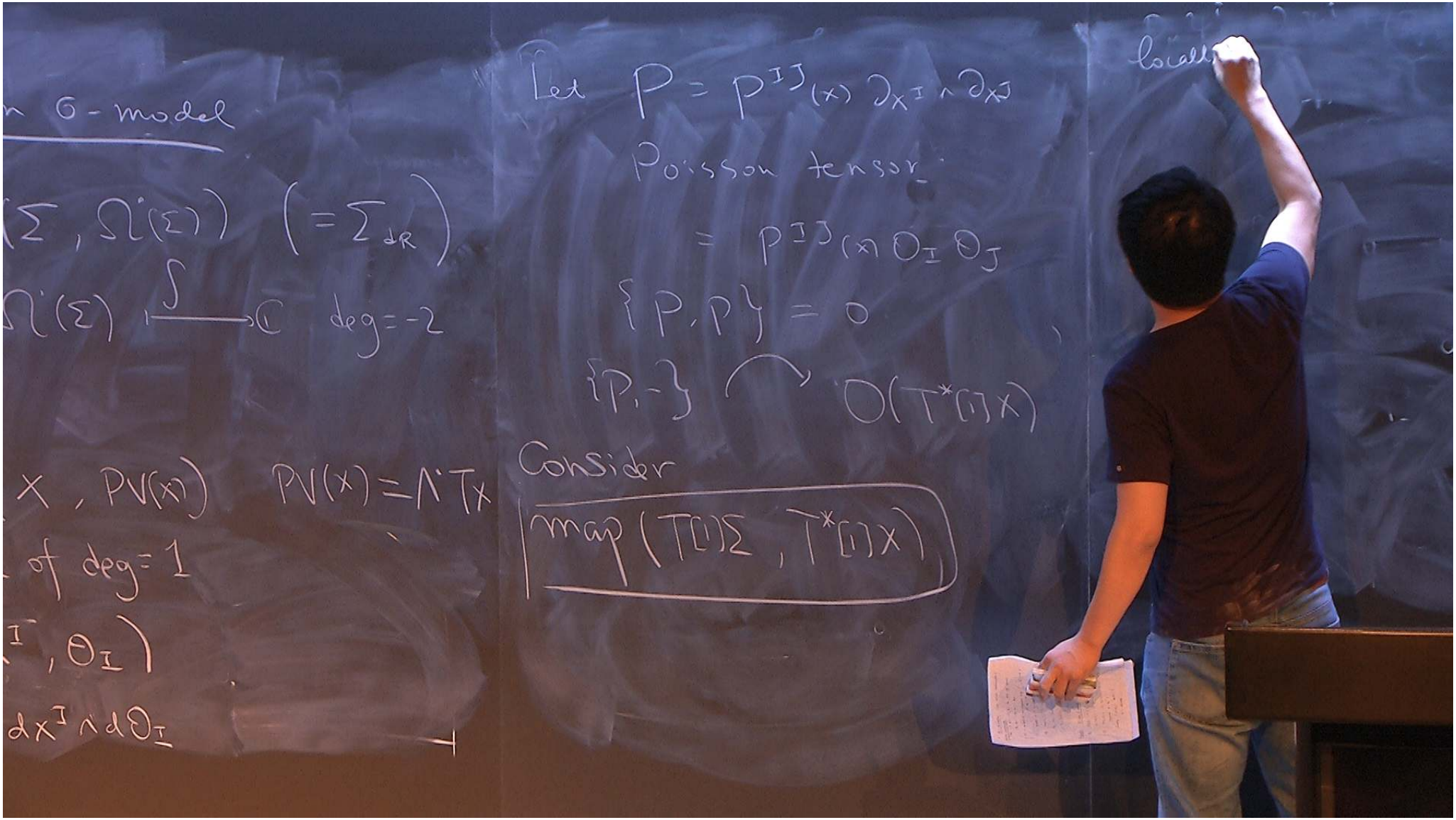
(Σ : Surface)

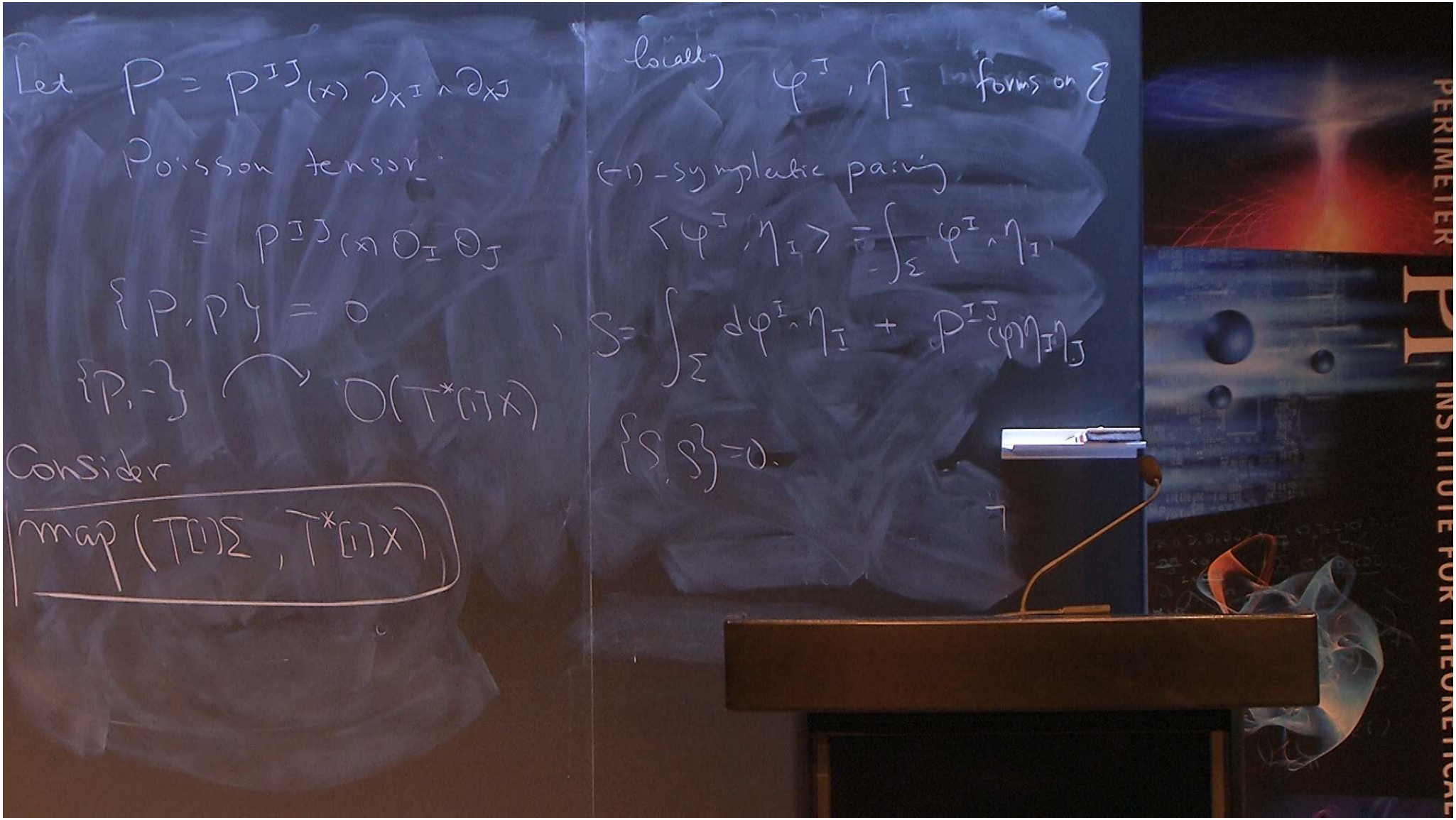
② $T^*[1]X = (X, PV(X))$ $PV(X) = \Lambda^\cdot TX$

Symplectic of deg $= 1$

locally : (X^I, θ_I)

$\omega = dX^I \wedge d\theta_I$


σ-model
$(\Sigma, \Omega^{\cdot}(\Sigma))$ $(= \Sigma_{dR})$
$\Omega^{\cdot}(\Sigma) \xrightarrow{\int} \mathbb{C}$ deg = -2
X, $PV(x)$ $PV(x) = \Lambda^{\cdot} TX$
of deg = 1
$\theta_I)$
$dx^I \wedge d\theta_I$
Let $P = P^{IJ}(x)\, \partial_{x^I} \wedge \partial_{x^J}$
Poisson tensor
$= P^{IJ}(x)\, \theta_I \theta_J$
$\{P, P\} = 0$
$\{P, -\}$ $\mathcal{O}(T^*[1]X)$
Consider
$\mathrm{map}(T[1]\Sigma, T^*[1]X)$


Let $P = P^{IJ}(x)\,\partial_{x^I} \wedge \partial_{x^J}$
Poisson tensor
$= P^{IJ}(x)\,\theta_I\,\theta_J$
$\{P, P\} = 0$
$\{P, -\} \curvearrowright \mathcal{O}(T^*[1]X)$
Consider
$\mathrm{map}(T[1]\Sigma, T^*[1]X)$
locally φ^I, η_I forms on Σ
(-1)-symplectic pairing
$\langle \varphi^I, \eta_I \rangle = \int_\Sigma \varphi^I \wedge \eta_I$
$S = \int_\Sigma d\varphi^I \wedge \eta_I + P^{IJ}(\varphi)\eta_I\eta_J$
$\{S, S\} = 0.$
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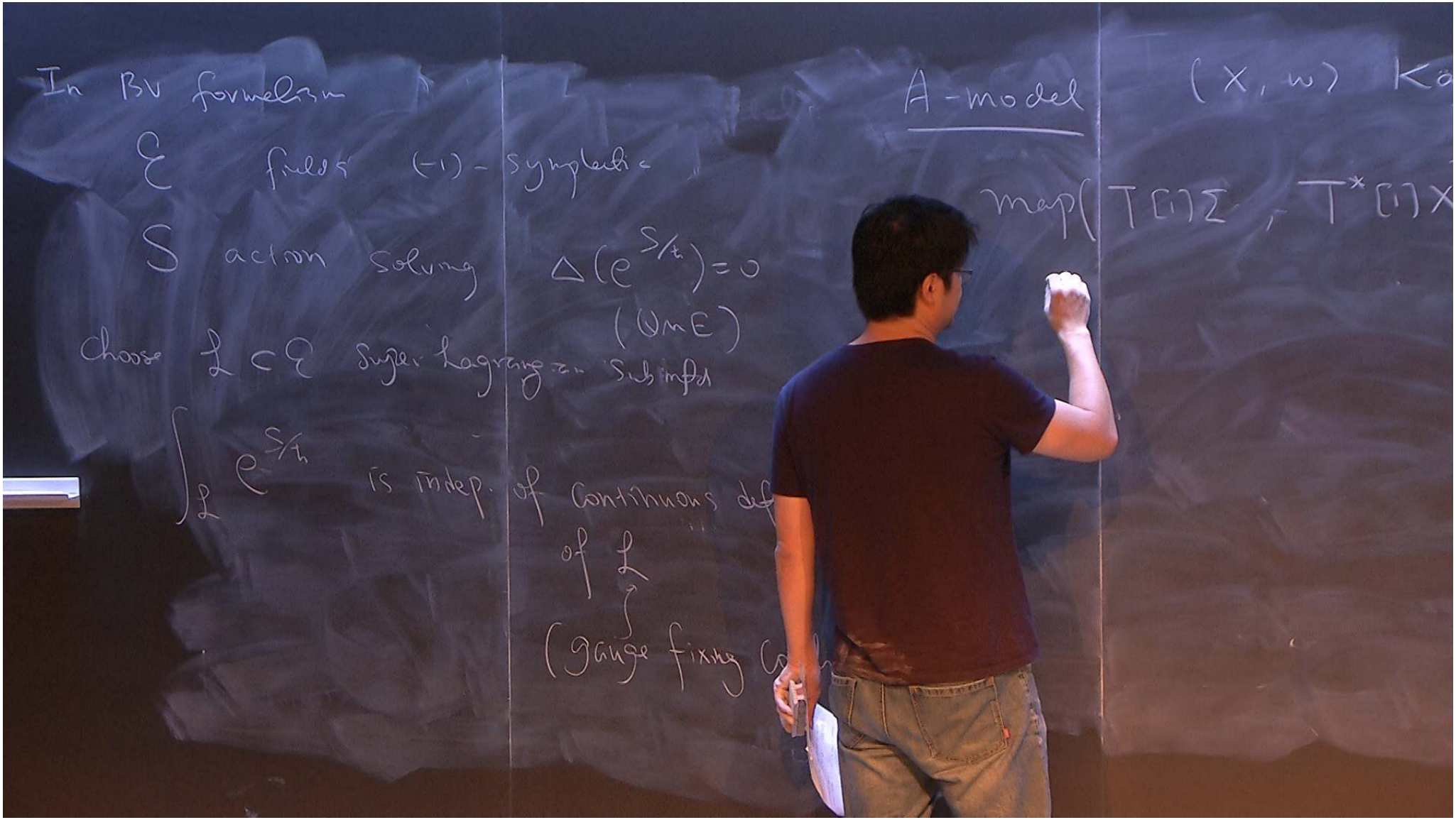
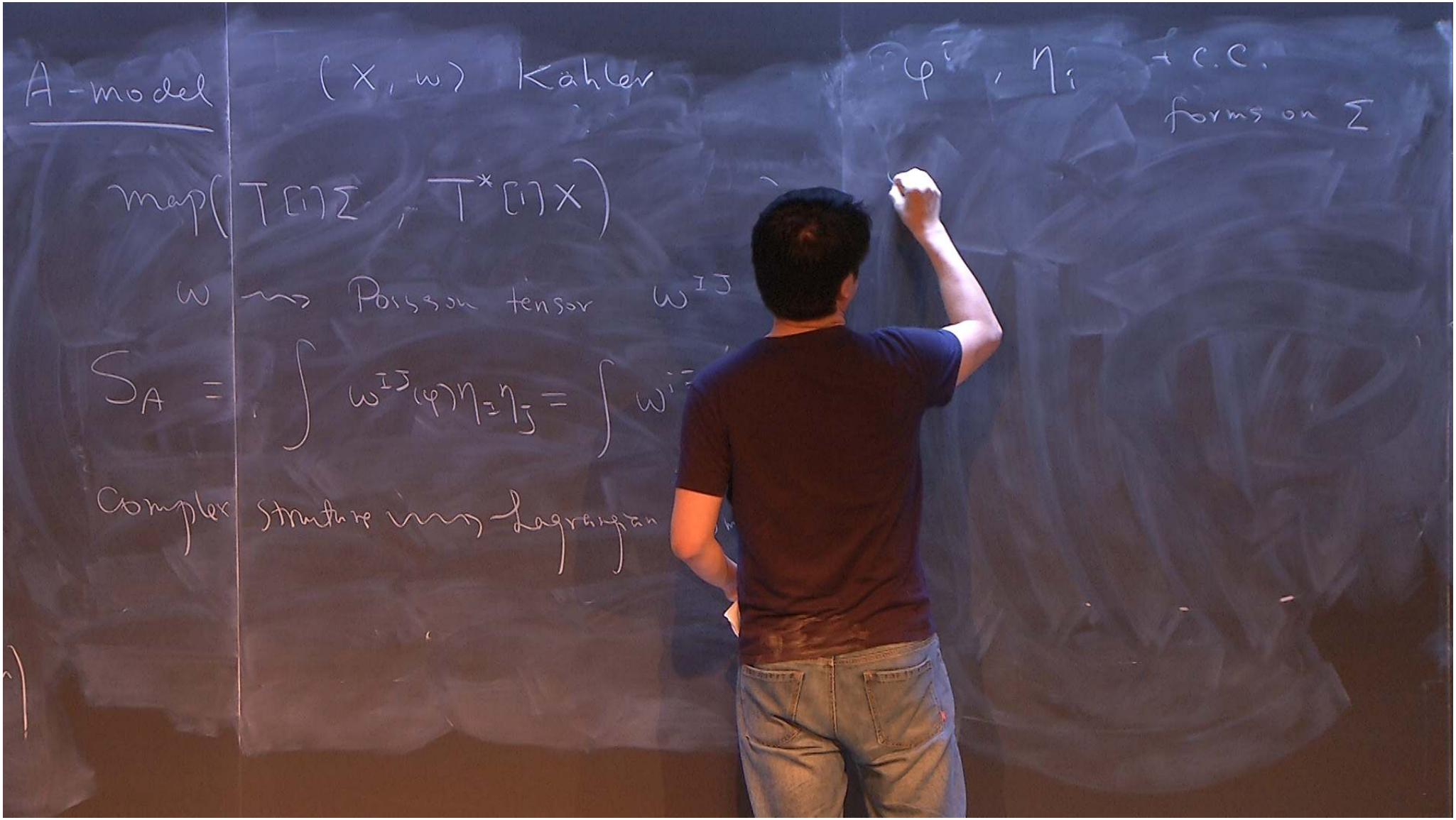
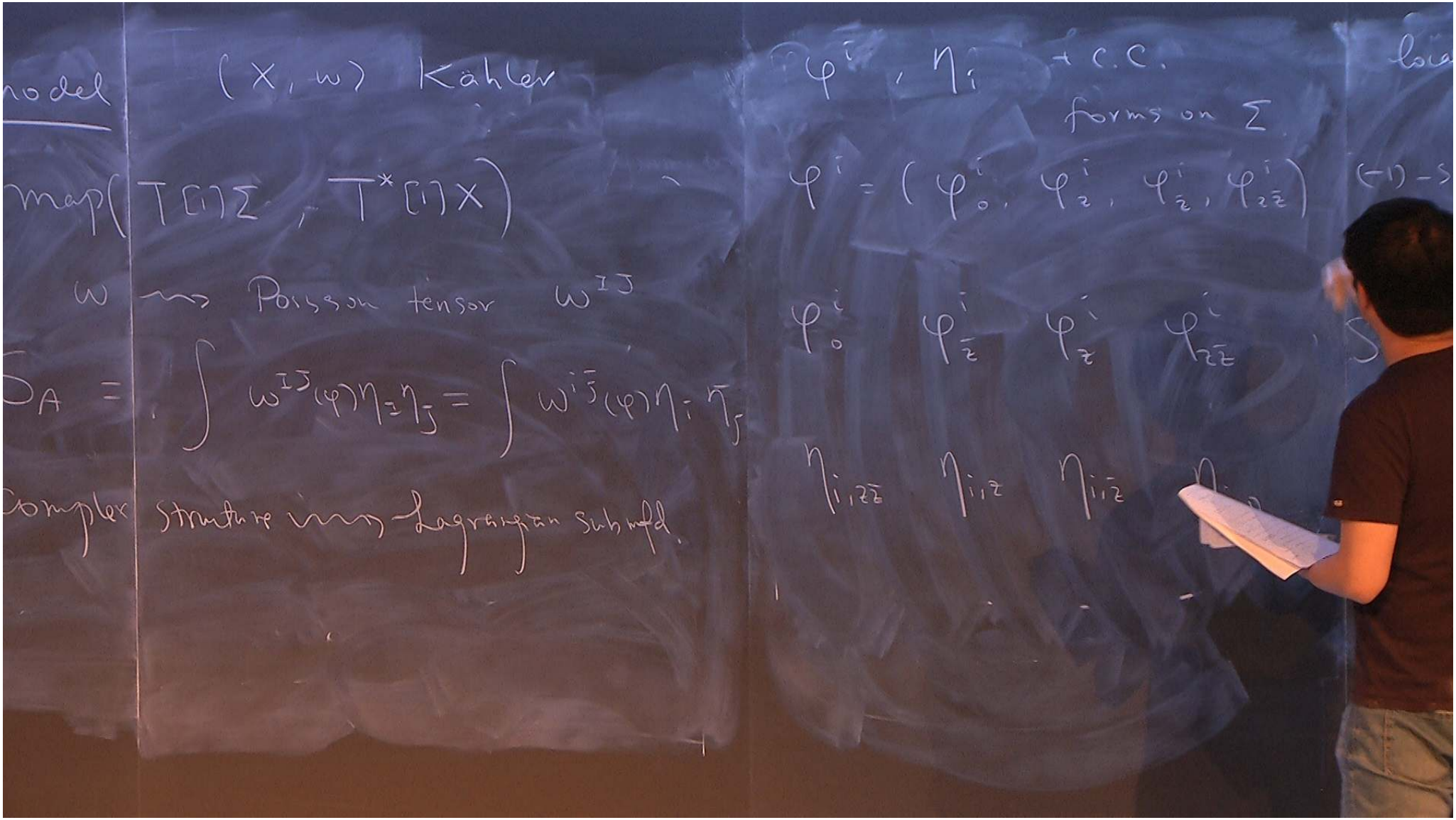

In BV formalism
$\mathcal{E}$ fields (-1)-symplectic
S action solving $\Delta(e^{S/\hbar})=0$
(QME)
Choose $\mathcal{L} \subset \mathcal{E}$ super Lagrangian submfd
$\int_{\mathcal{L}} e^{S/\hbar}$ is indep. of continuous def
of $\mathcal{L}$
(gauge fixing
A-model
(X, ω)
map($T[1]\Sigma$, $T^*[1]X$


A-model
(X, ω) Kähler
φ^i , η_i + c.c.
forms on Σ
map(TΣ , T*X)
ω ⇝ Poisson tensor ω^IJ
S_A = ∫ ω^IJ(φ) η_I η_J = ∫ ω^ij
Complex structure ⇝ Lagrangian


model
(X, ω) Kähler
map(T[1]Σ , T*[1]X)
ω ⇝ Poisson tensor ω^{IJ}
S_A = ∫ ω^{IJ}(φ) η_I η_J = ∫ ω^{ij̄}(φ) η_i η̄_j
Complex structure ⇒ Lagrangian submfd.
φ^i , η_i + c.c.
forms on Σ
φ^i = (φ^i_0 , φ^i_z , φ^i_z̄ , φ^i_zz̄)
φ^i_0 φ^i_z̄ φ^i_z φ^i_zz̄
η_{i,zz̄} η_{i,z} η_{i,z̄}

$\mathcal{M} = ($ $\bigcirc$ fields $)$
$+ c.c.$

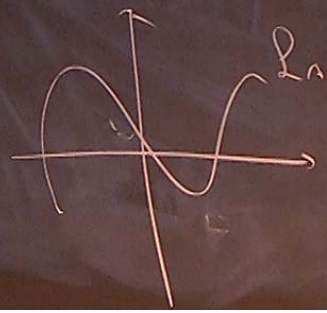
$\mathrm{map}(T[1]\Sigma, T^*[1]X) = T^*[-1]\mathcal{M}$

Consider $\Psi = \int_\Sigma d^2z \left(\bar{\varphi}^{\bar{j}}_{\bar{z}} \partial_z \phi_0^i + \varphi^i_z \partial_{\bar{z}} \phi_0^{\bar{j}} \right) g_{i\bar{j}}$

$\in \mathcal{O}(\mathcal{M})$

Consider $\mathcal{L}_A = \mathrm{Graph}(d\Psi) \subset T^*[-1]\mathcal{M}$

Check: $S_A\big|_{\mathcal{L}_A}$ = A-model action



A-model (X, ω)

$\mathrm{map}(T[1]\Sigma, T^*[1]$

$\omega \rightsquigarrow$ Poisson te

$S_A = \int \omega^{IJ}(\varphi)\eta_I \eta_J$

Complex structure

$\mathcal{O}$ fields)

c.c.

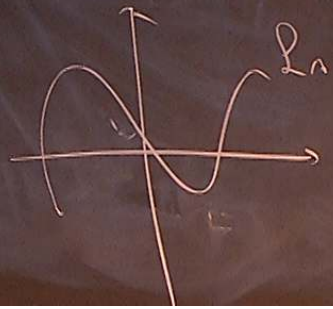
$)X) = T^*[-1]\mathcal{M}$

$\ldots z\left(\bar{\varphi}^{\bar{j}}_{z}\,\bar{\partial}_z \phi^i_0 + \varphi^i_z\,\partial_z \phi^{\bar{j}}_0\right) g_{i\bar{j}}$

$\in \mathcal{O}(\mathcal{M})$

$\mathrm{graph}(d\Psi) \subset T^*[-1]\mathcal{M}$

odel action



B-model (X, ω) Kähler

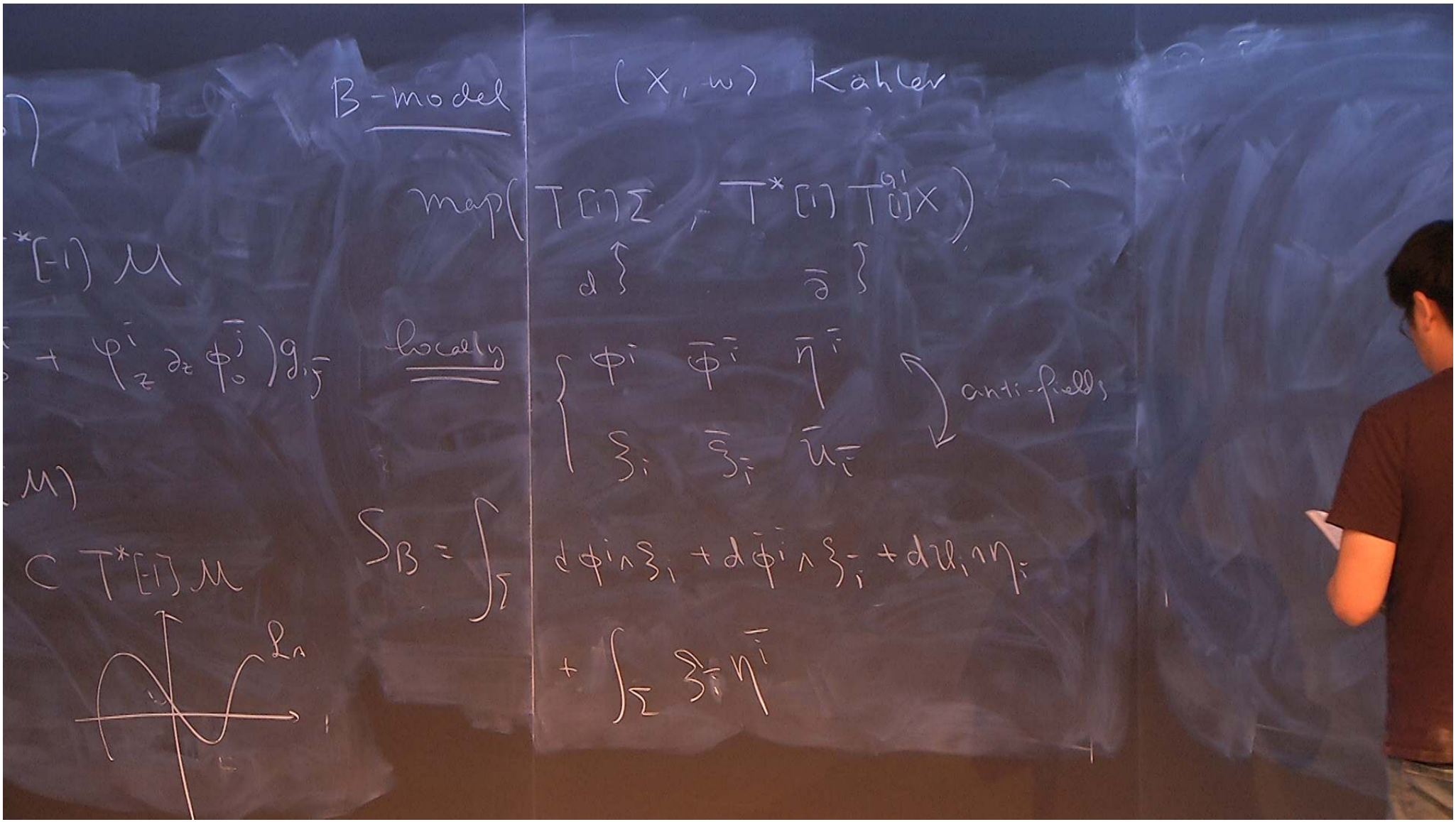
$$\mathrm{map}\left(T[1]\Sigma,\ T^*[1]\,T^{0,1}[1]X\right)$$

$d \uparrow$ $\qquad$ $\bar{\partial} \uparrow$

locally

$$\begin{cases} \phi^i & \bar{\phi}^{\bar{i}} & \bar{\eta}^{\bar{i}} \\ \xi_i & \bar{\xi}_{\bar{i}} & \bar{u}_{\bar{i}} \end{cases}$$

anti-fields


B-model
(X, ω) Kähler
map(T[1]Σ , T*[1] T^{0,1}[1]X)
d
∂̄
locally
φ^i φ̄^ī η̄^ī
ξ_i ξ̄_ī ū_ī
anti-fields
S_B = ∫_Σ dφ^i ∧ ξ_i + dφ̄^ī ∧ ξ_ī + du_i ∧ η_ī
+ ∫_Σ ξ_ī η^ī
T*[-1] M
+ φ^i_z ∂_z φ^j_0) g_{i,j̄}
M)
⊂ T*[-1] M
L_A