Title: Lecture 5: Factorization Algebras and the General Structure of QFT

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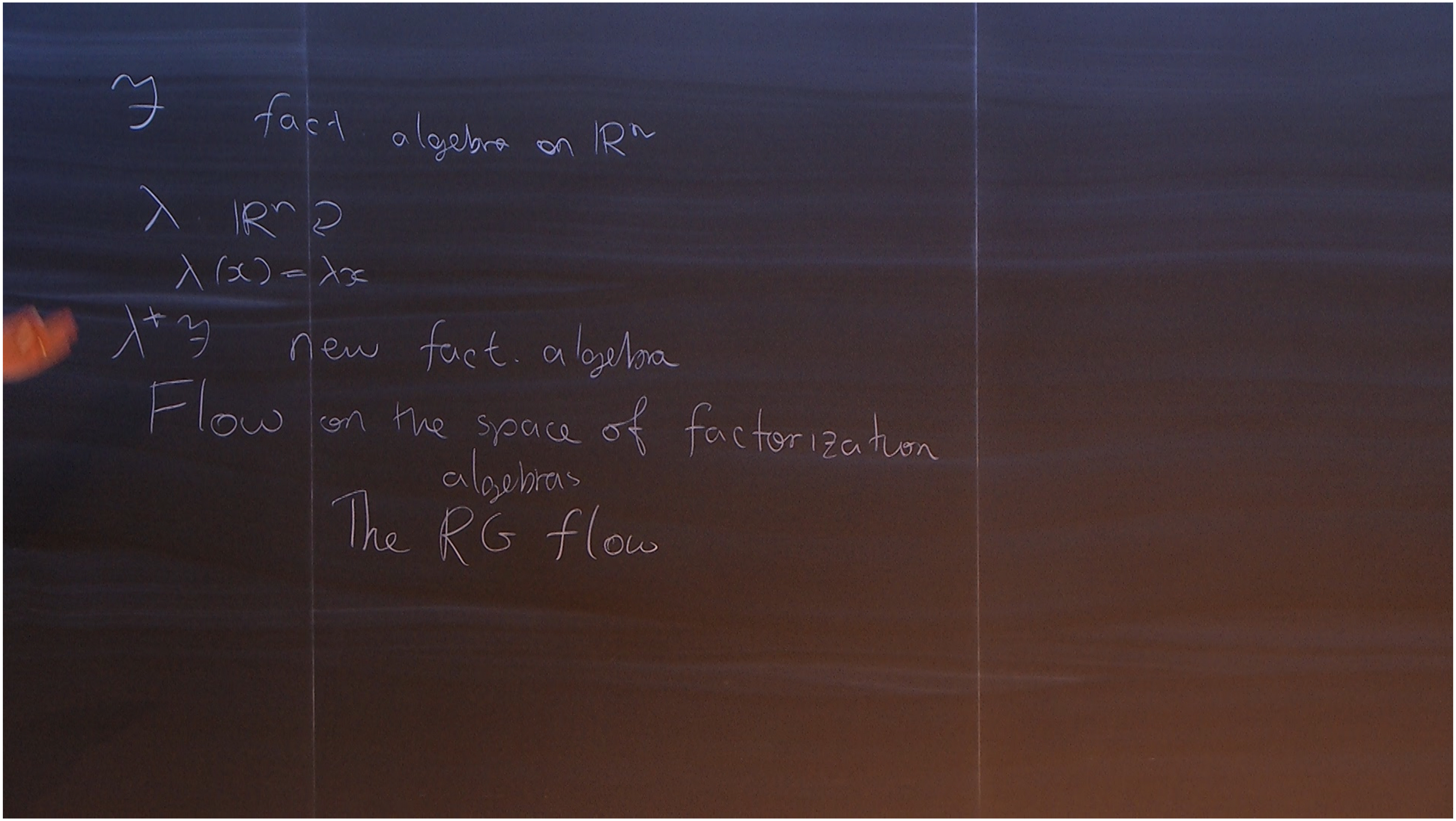
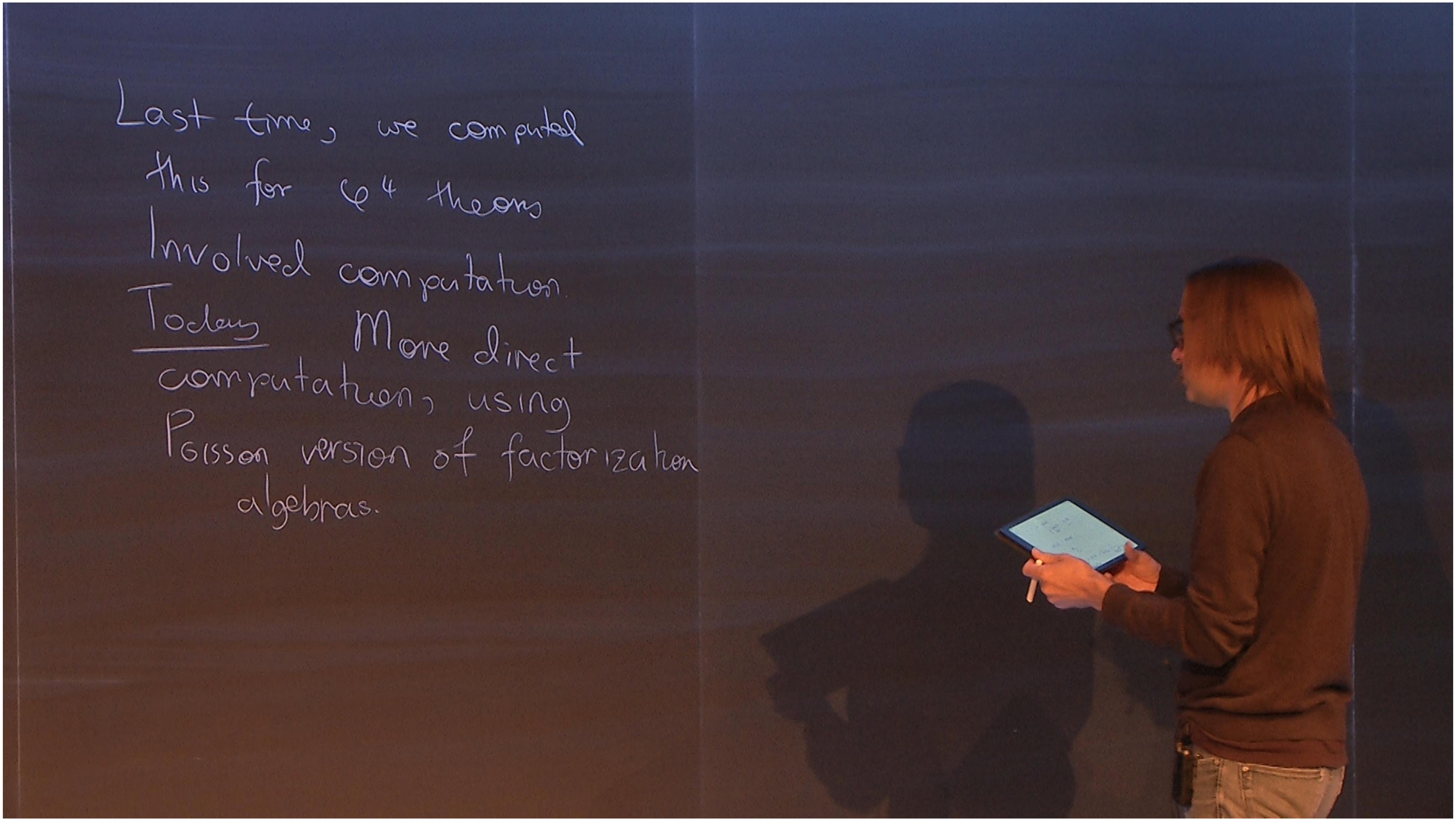
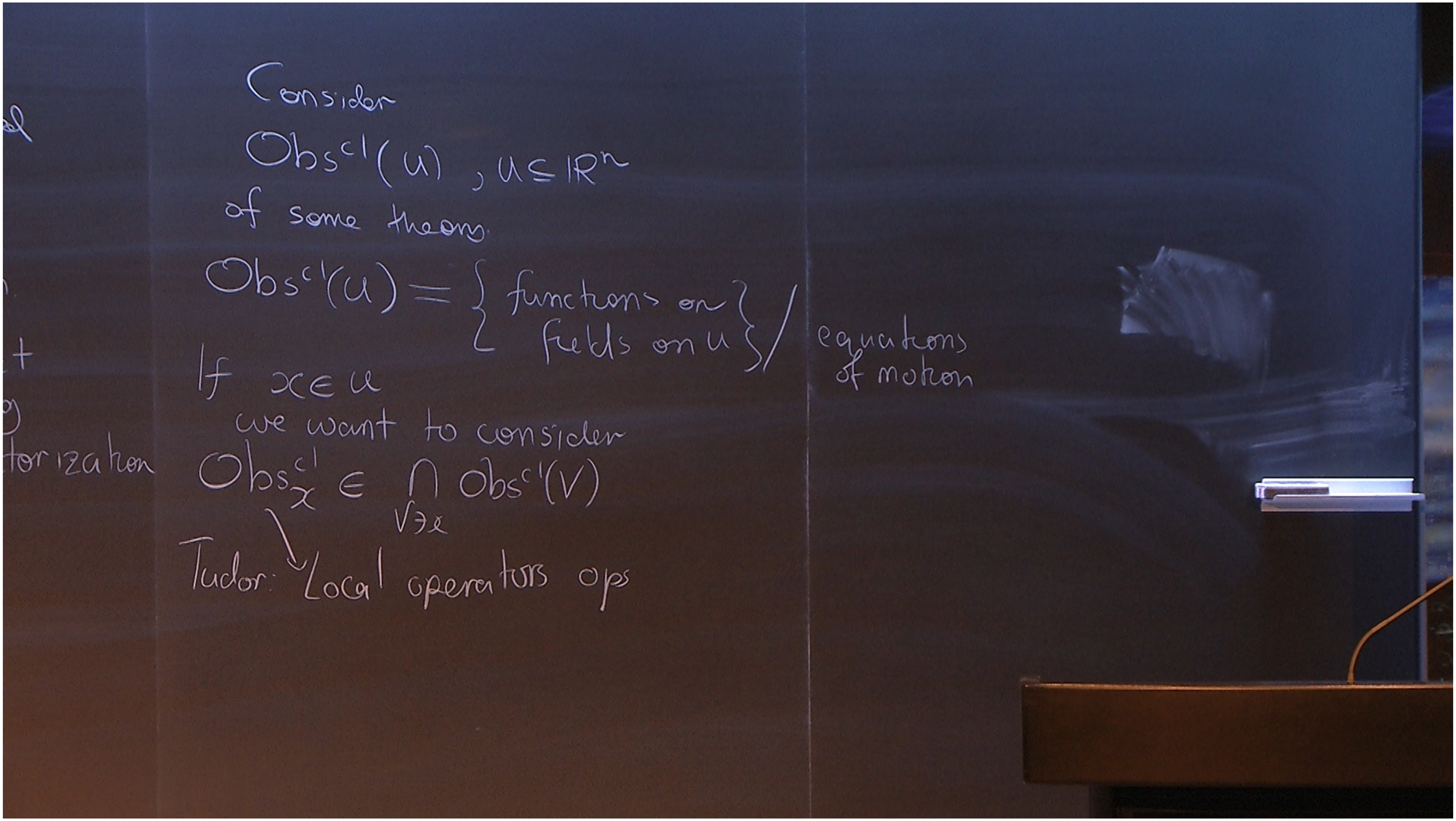
Collection: QFT for Mathematicians

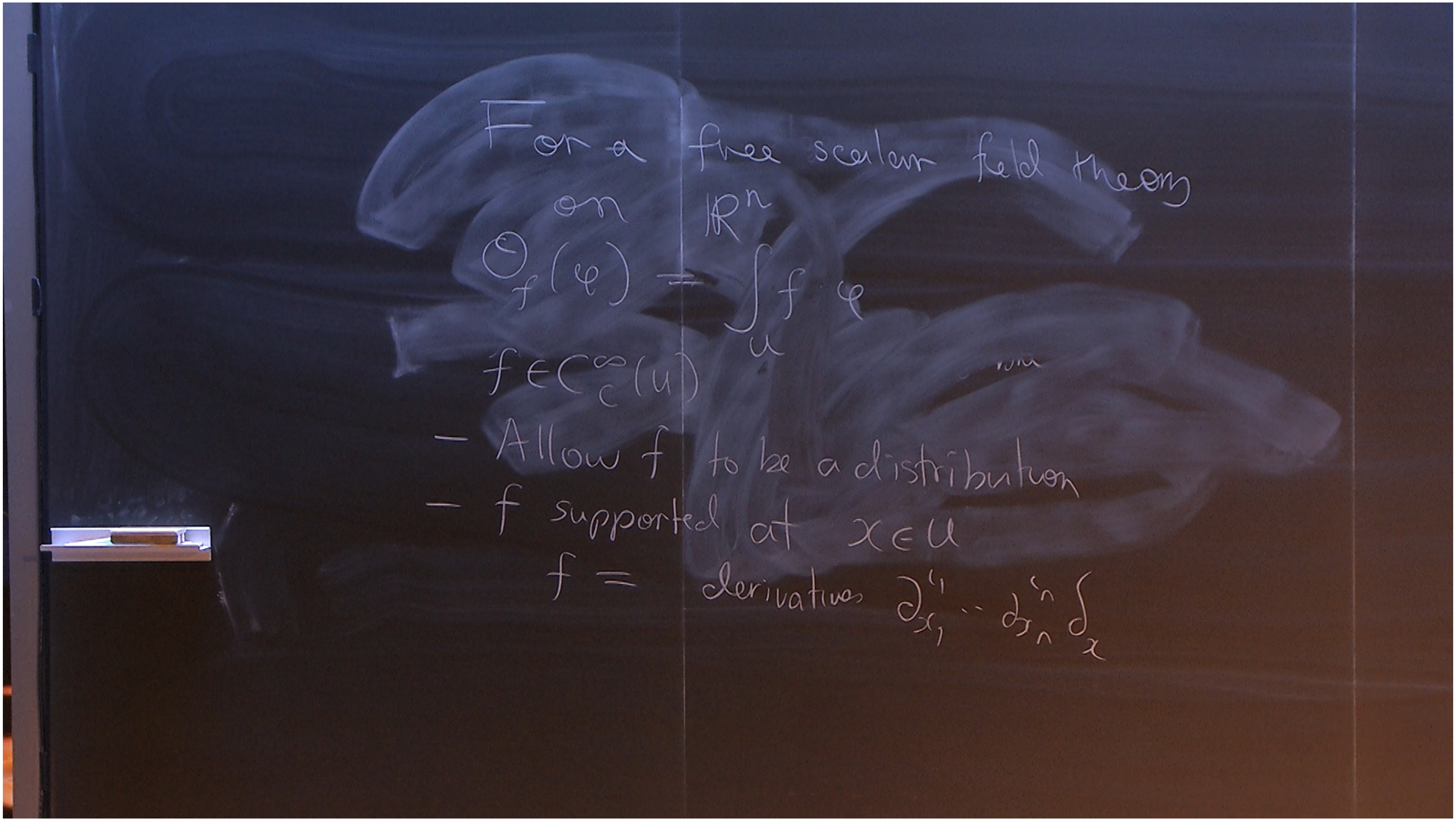
Date: June 24, 2019 - 11:00 AM

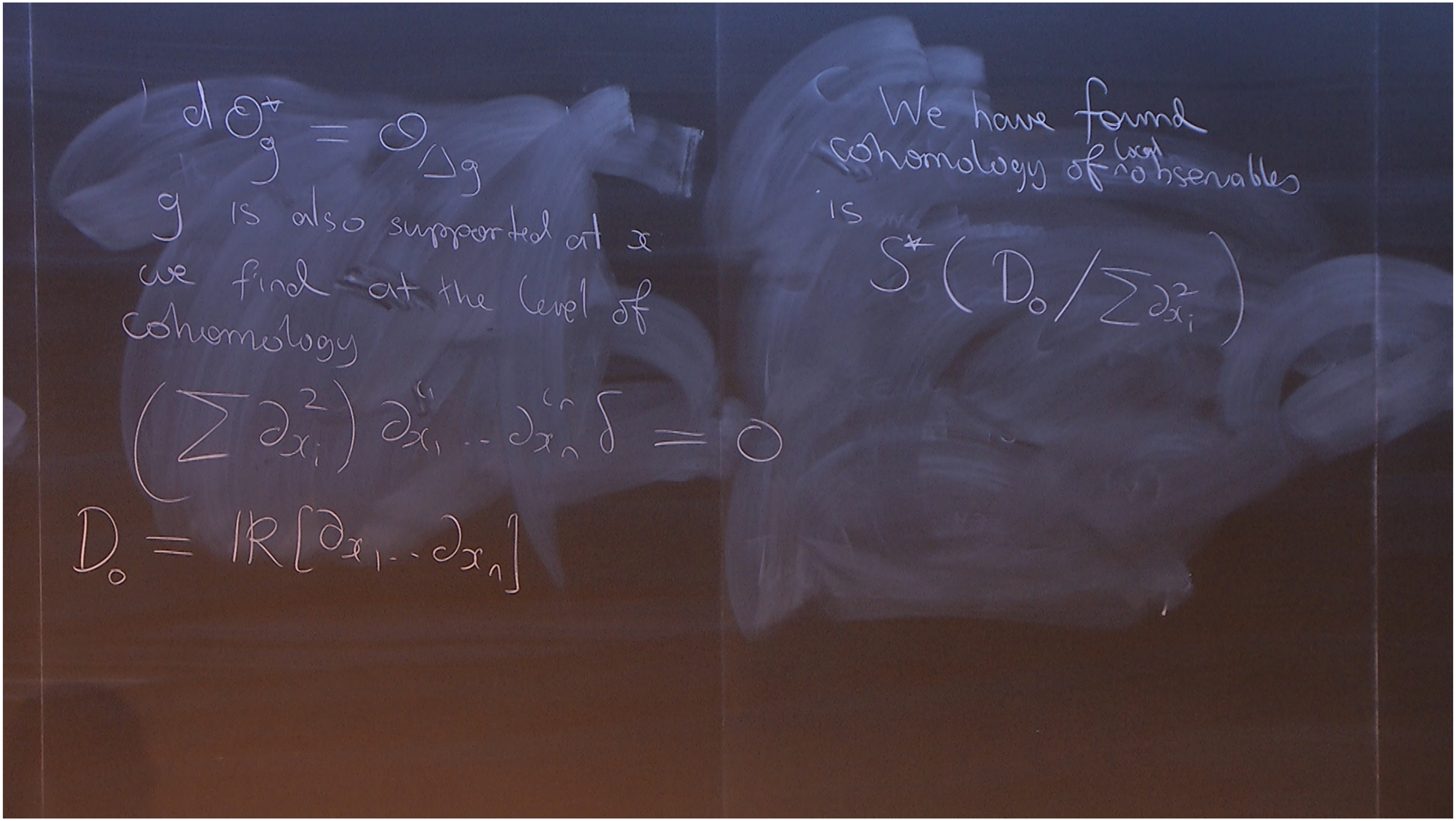
URL: http://pirsa.org/19060016

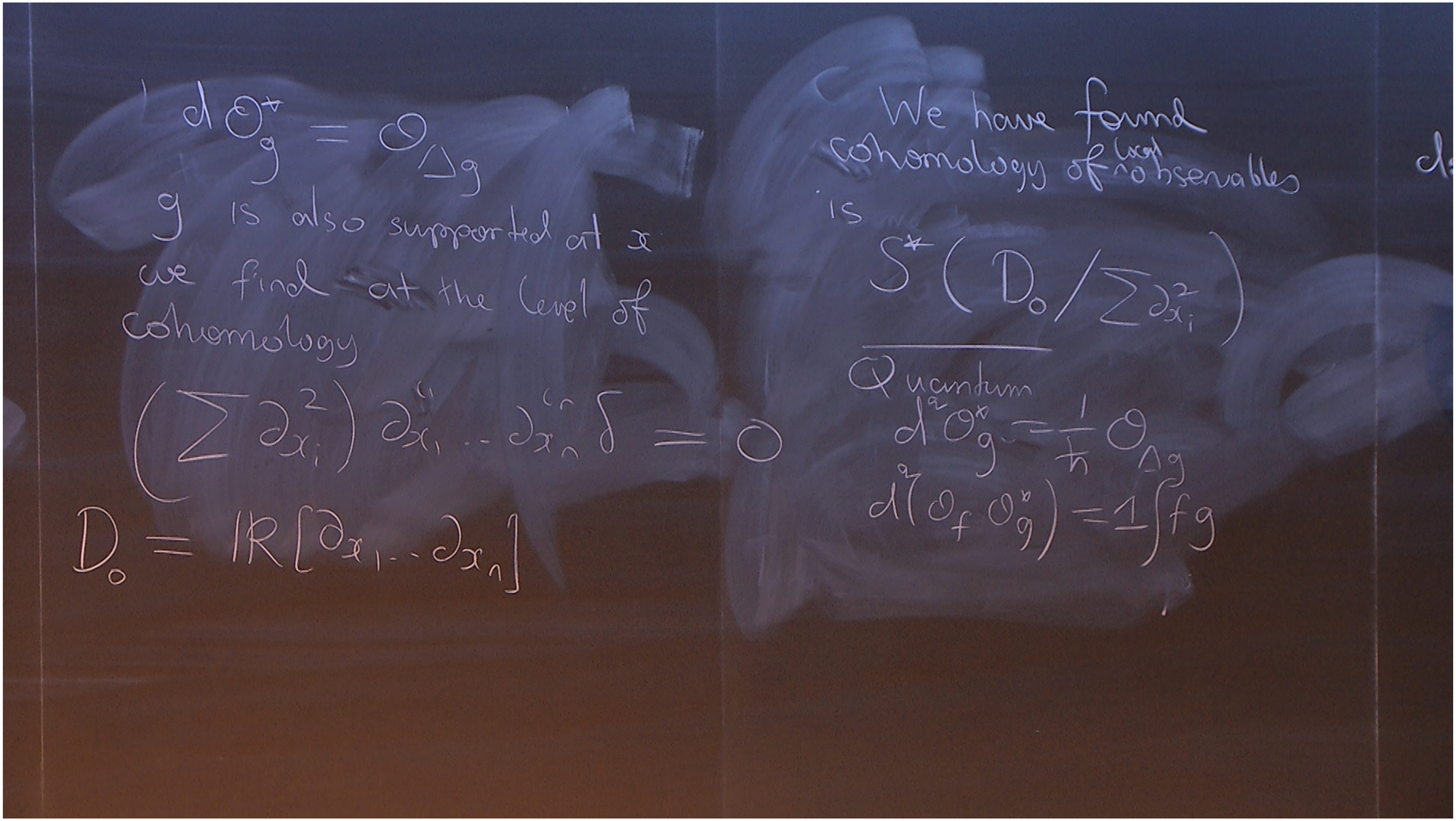

F fact algebra on ℝ^n
λ: ℝ^n ↺
λ(x) = λx
λ*F new fact. algebra
Flow on the space of factorization algebras
The RG flow


Last time, we computed
this for φ^4 theory
Involved computation.
Today More direct
computation, using
Poisson version of factorization
algebras.


Consider
Obs^cl(U), U ⊆ ℝ^n
of some theory
Obs^cl(U) = { functions on fields on U } / equations of motion
If x ∈ U
we want to consider
Obs^cl_x ∈ ∩_{V∋x} Obs^cl(V)
Tudor: Local operators ops
torization


For a free scalar field theory
on $\mathbb{R}^n$
$\mathcal{O}_f(\varphi) = \int_U f \cdot \varphi$
$f \in C_c^\infty(U)$
– Allow f to be a distribution
– f supported at $x \in U$
$f =$ derivatives $\partial_{x_1}^{i_1} \cdots \partial_{x_n}^{i_n} \delta_x$


$d\mathcal{O}^{*}_{g} = \mathcal{O}_{\Delta g}$
g is also supported at x
we find at the level of cohomology
$\left(\sum \partial_{x_i}^2\right) \partial_{x_1}^{i_1} \cdots \partial_{x_n}^{i_n} \delta = 0$
$D_0 = \mathbb{R}[\partial_{x_1} \ldots \partial_{x_n}]$
We have found cohomology of local observables is
$S^{*}\left(D_0 / \sum \partial_{x_i}^2\right)$



$d\,\mathcal{O}_g^* = \mathcal{O}_{\Delta g}$
g is also supported at x
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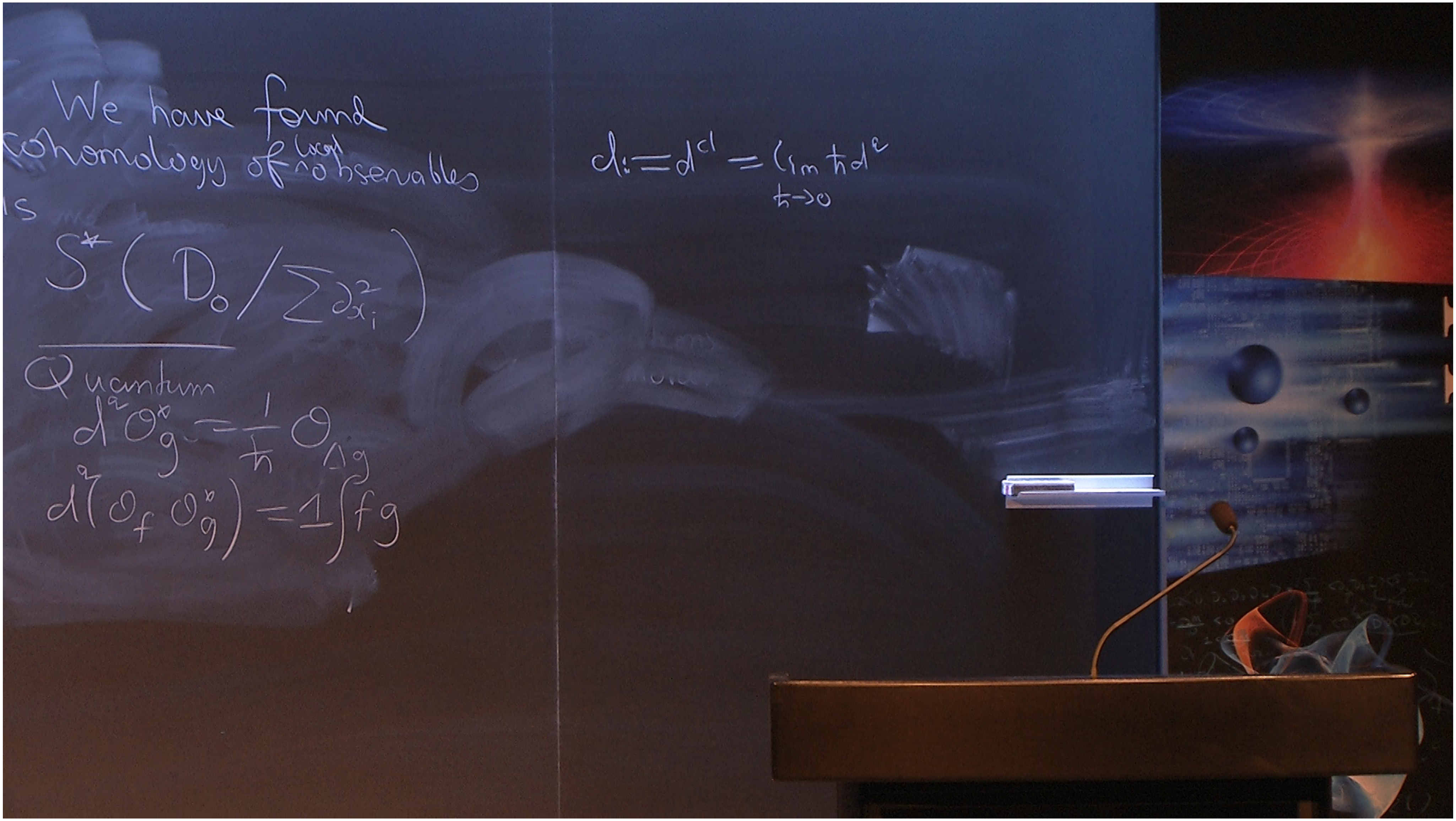
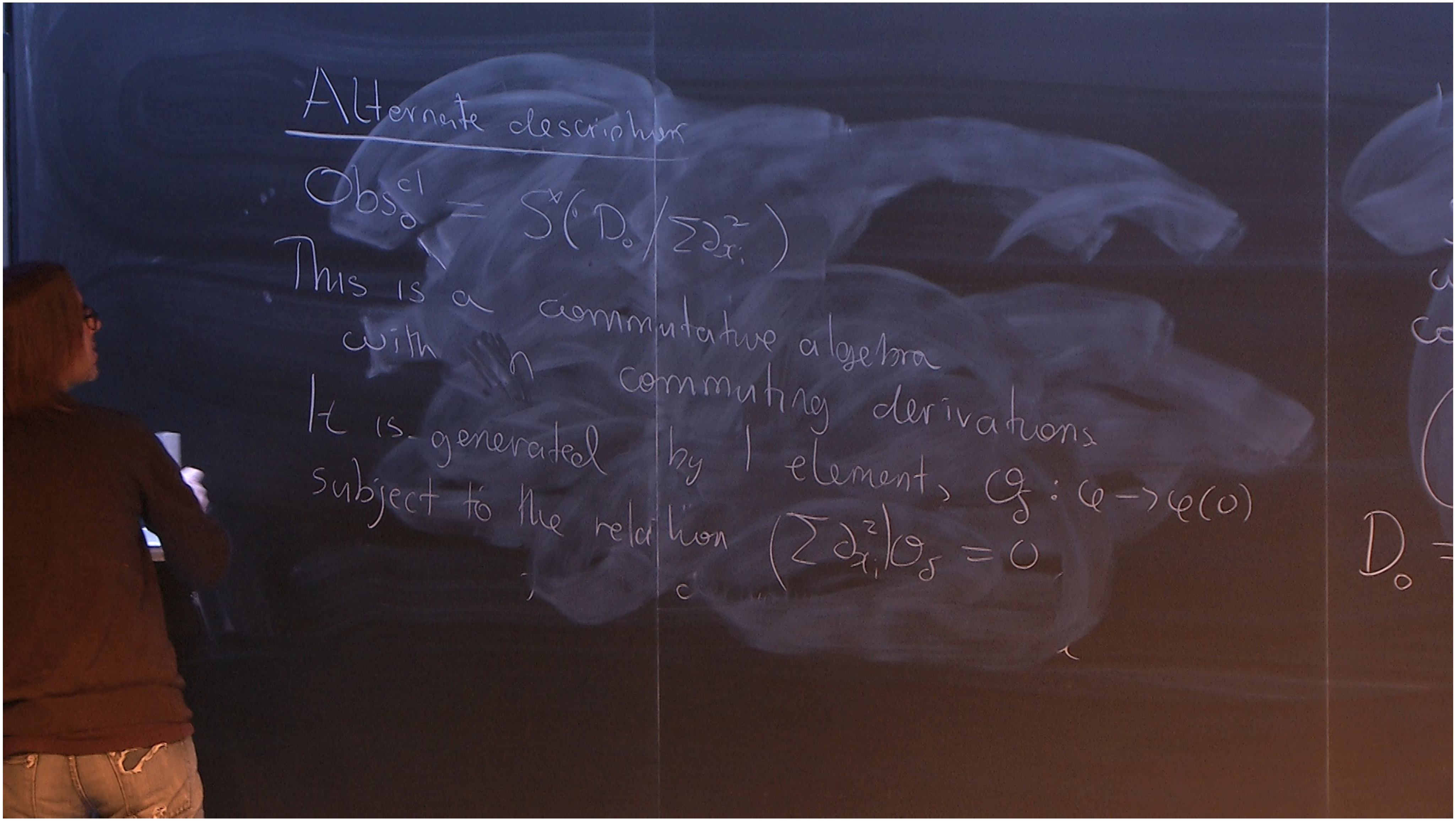
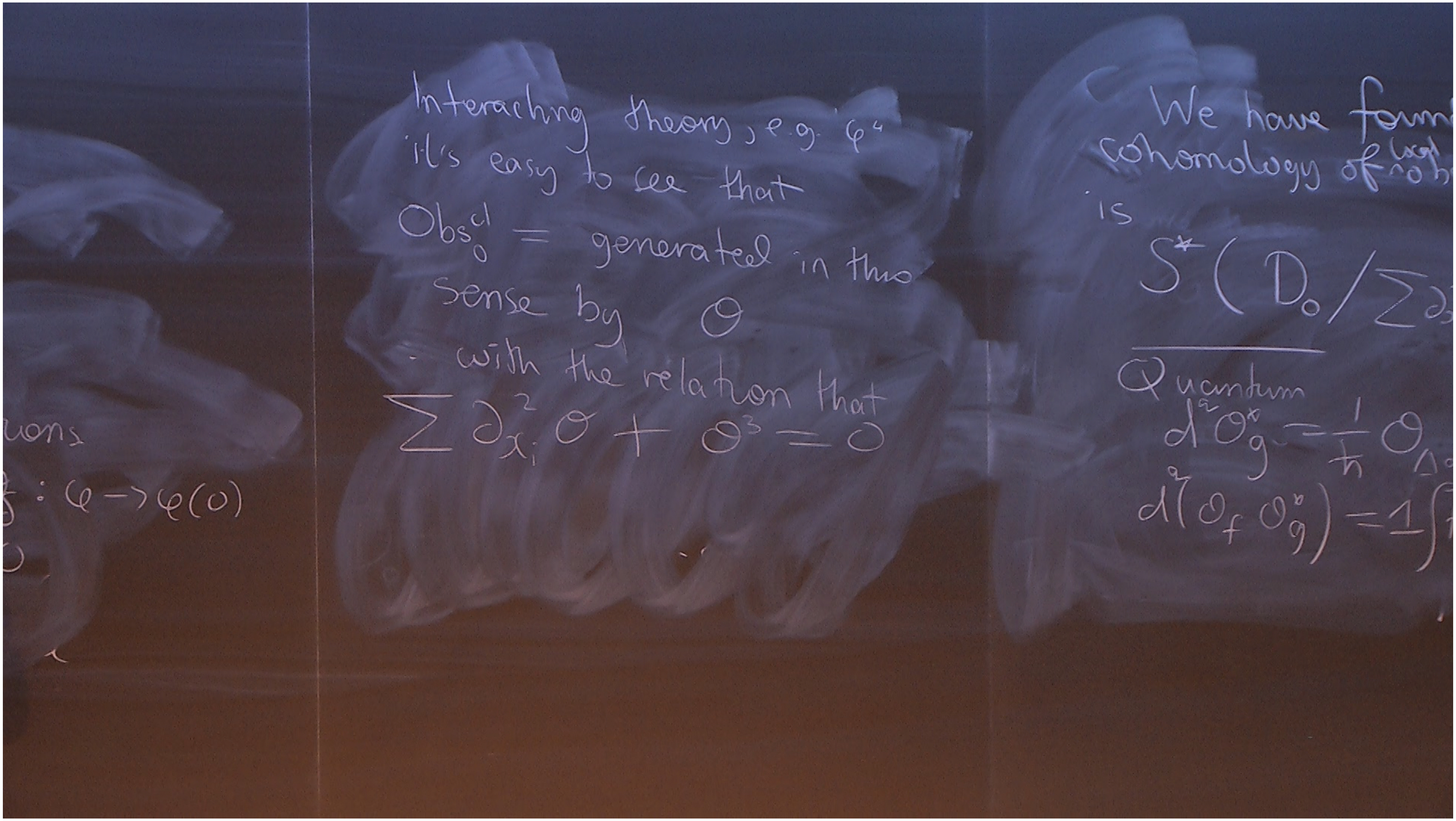
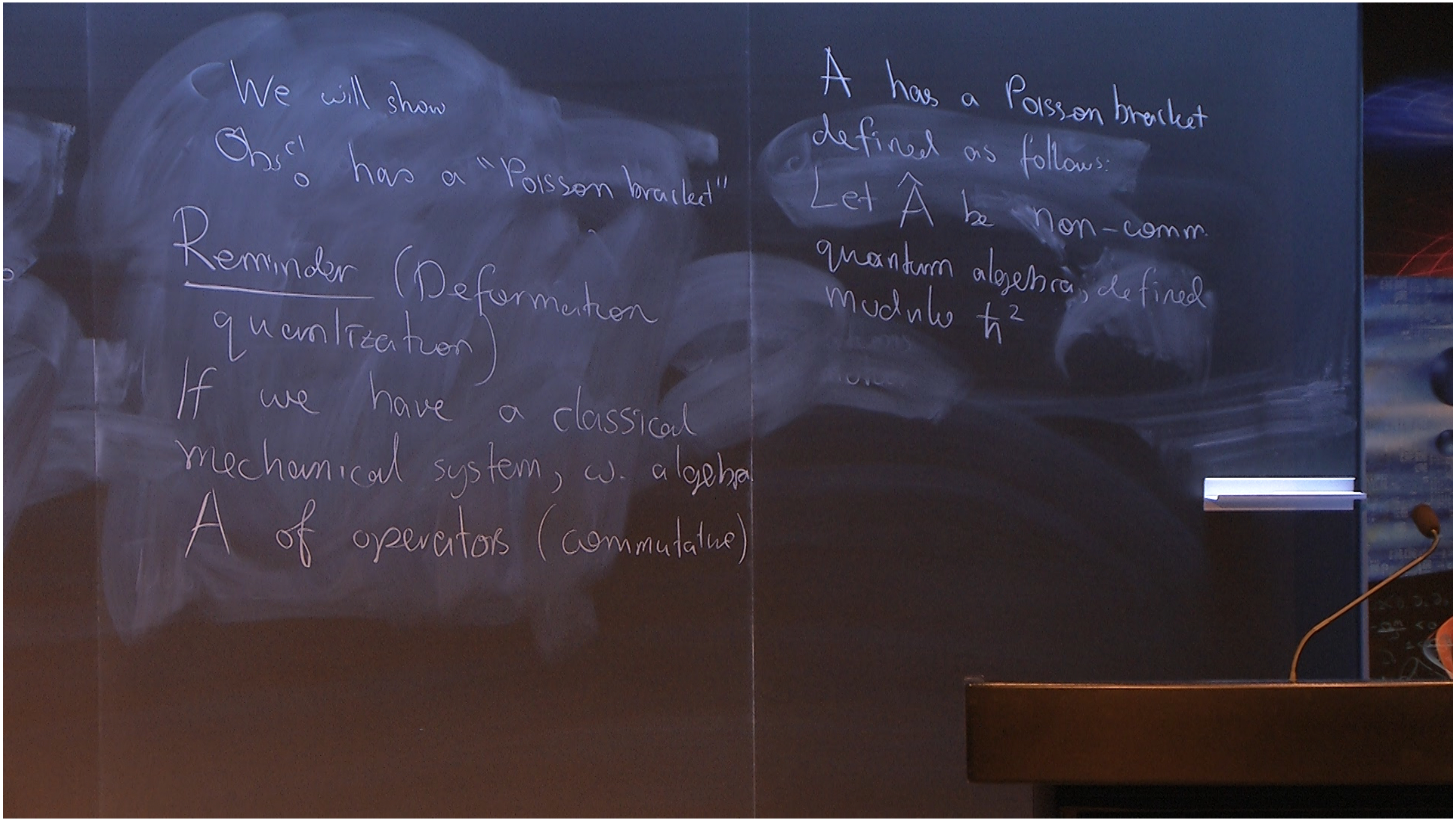
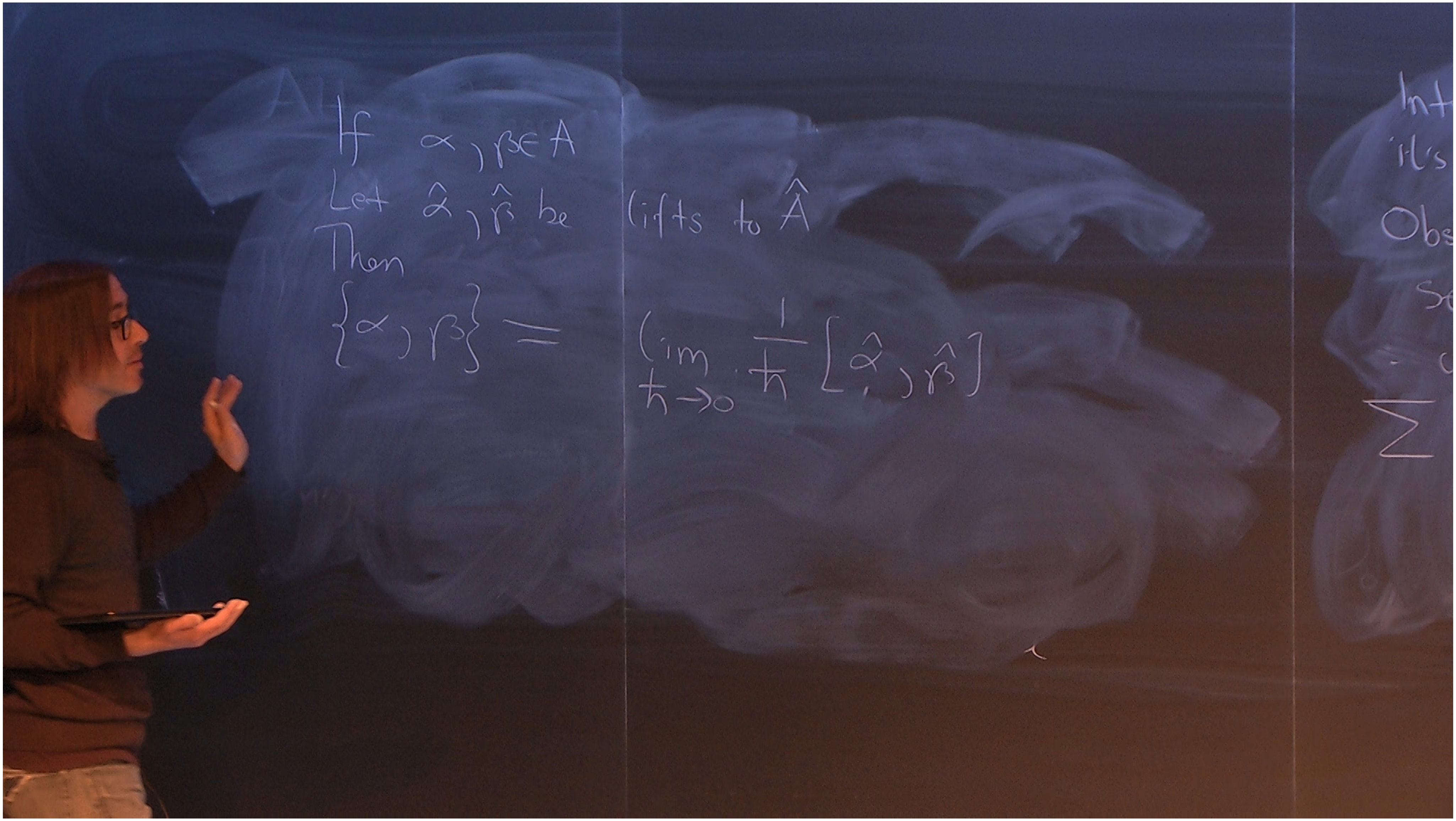
$\left(\sum \partial_{x_i}^2\right) \partial_{x_1}^{i_1} \cdots \partial_{x_n}^{i_n} \delta = 0$
$D_0 = \mathbb{R}[\partial_{x_1} \cdots \partial_{x_n}]$
We have found cohomology of (local) observables is
$S^*\left(D_0 / \sum \partial_{x_i}^2\right)$
Quantum
$d^q \mathcal{O}_g^* = \frac{1}{\hbar} \mathcal{O}_{\Delta g}$
$d^q(\mathcal{O}_f \mathcal{O}_g^*) = 1 \int fg$

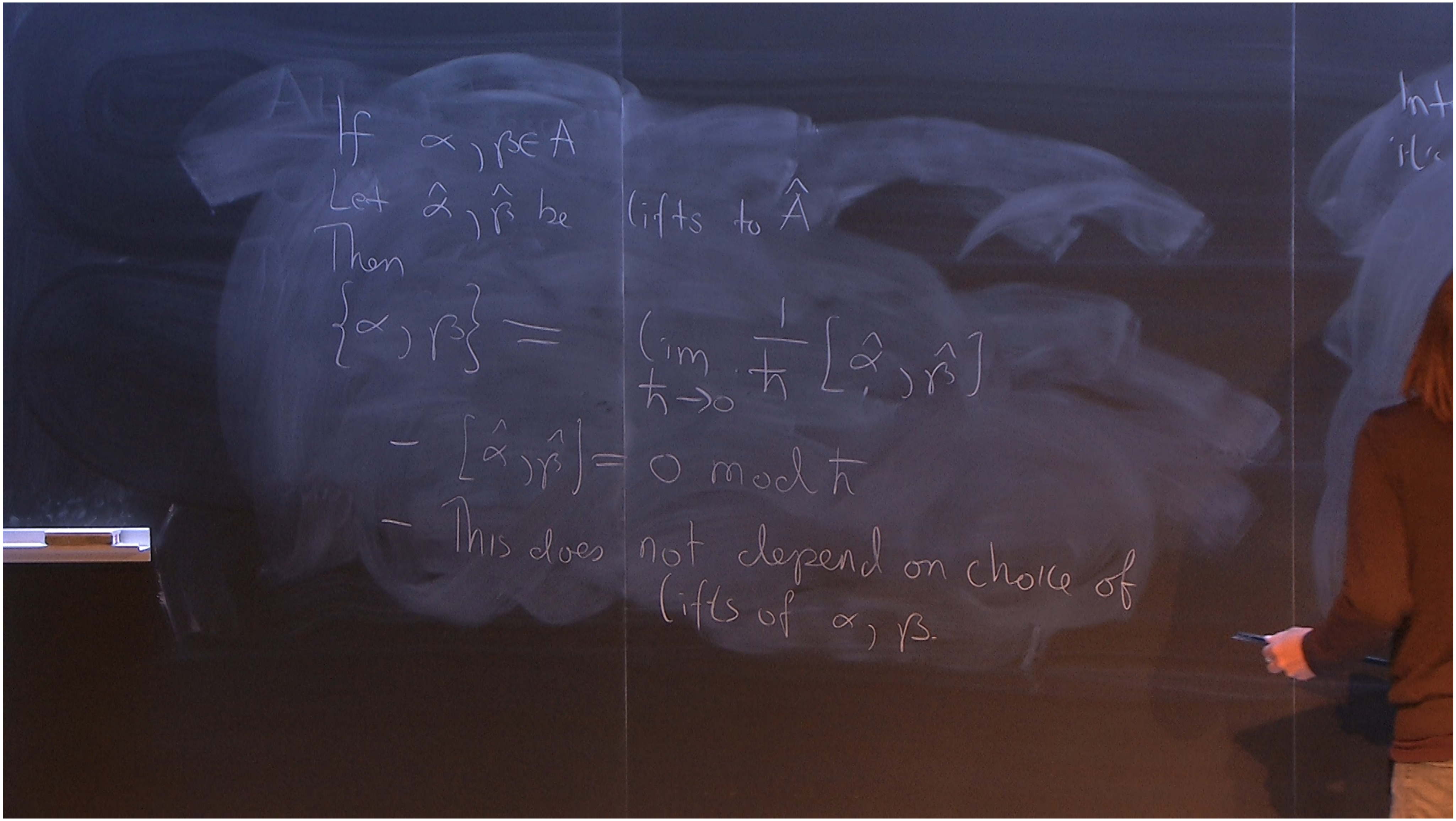

We have found
cohomology of (local) observables
is
S*(D0/Σ∂²xi)
Quantum
dq O*g = 1/ħ OΔg
dq(Of O*g) = 1 ∫fg
d := dcl = lim ħ→0 ħ dq

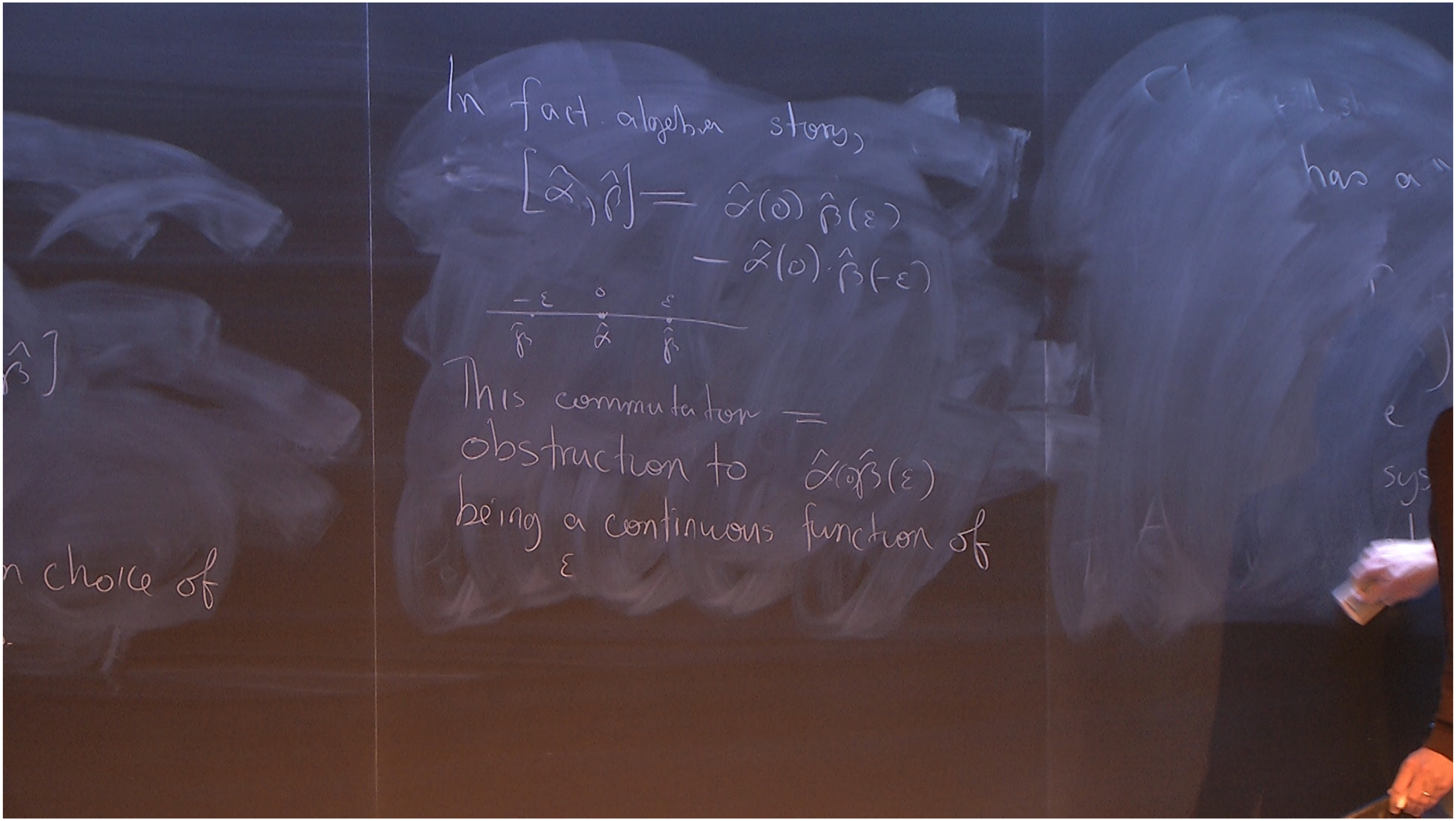

Alternate description
Obs0cl = S*(D0 / Σ∂xi2)
This is a commutative algebra
with n commuting derivations
It is generated by 1 element, Oδ : φ → φ(0)
subject to the relation (Σ∂xi2) Oδ = 0

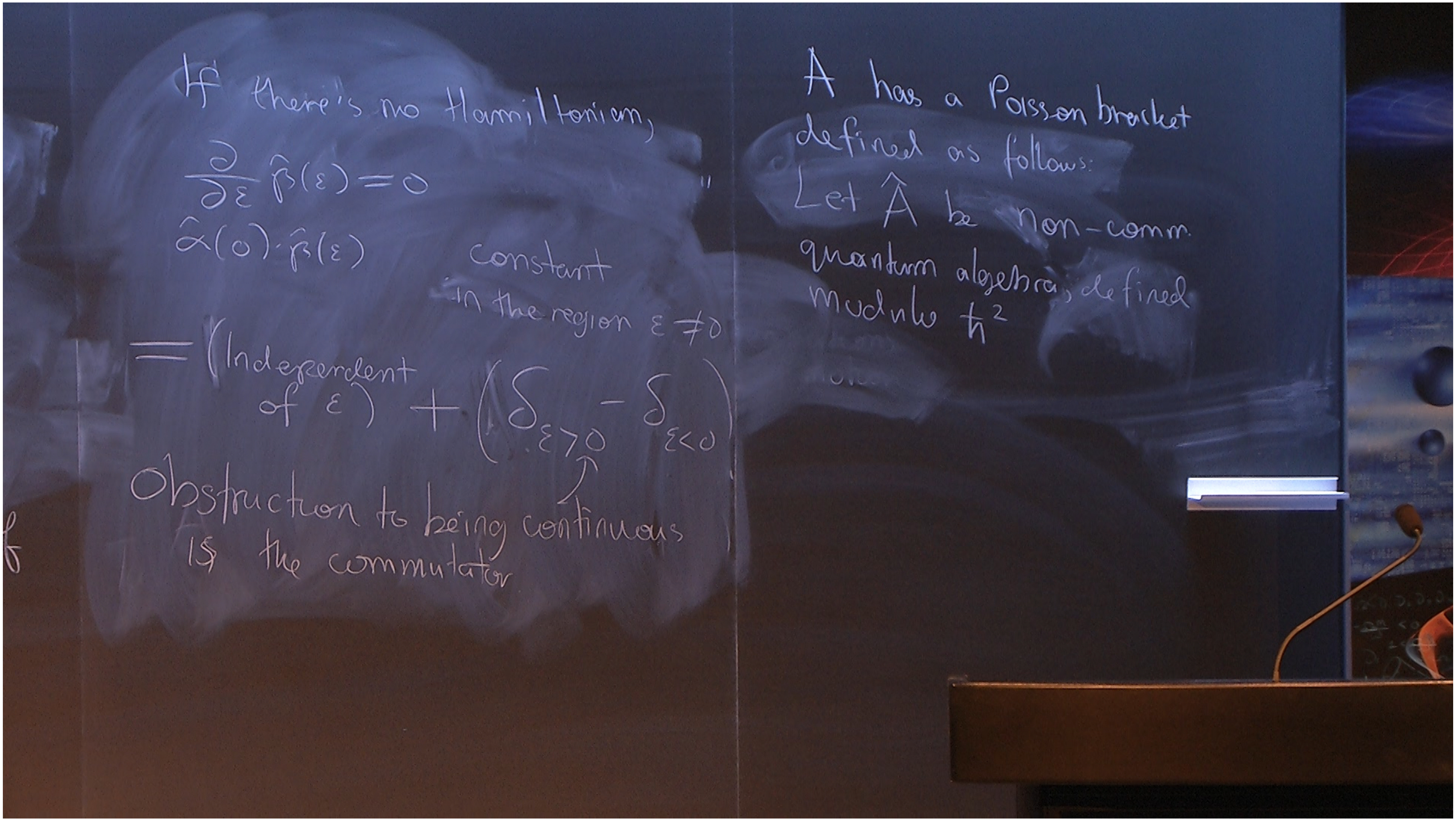

Interacting theory, e.g. φ^4
it's easy to see that
Obs_0^{cl} = generated in this sense by $\mathcal{O}$
with the relation that
$\sum \partial_{x_i}^2 \mathcal{O} + \mathcal{O}^3 = 0$
$: \varphi \to \varphi(0)$
We have found
cohomology of
is
$S^*(D_0 / \sum \partial$
Quantum
$d^q \mathcal{O}_g^* = \frac{1}{\hbar} \mathcal{O}$
$d^q(\mathcal{O}_f \mathcal{O}_g^*) = \frac{1}{\hbar}\int$

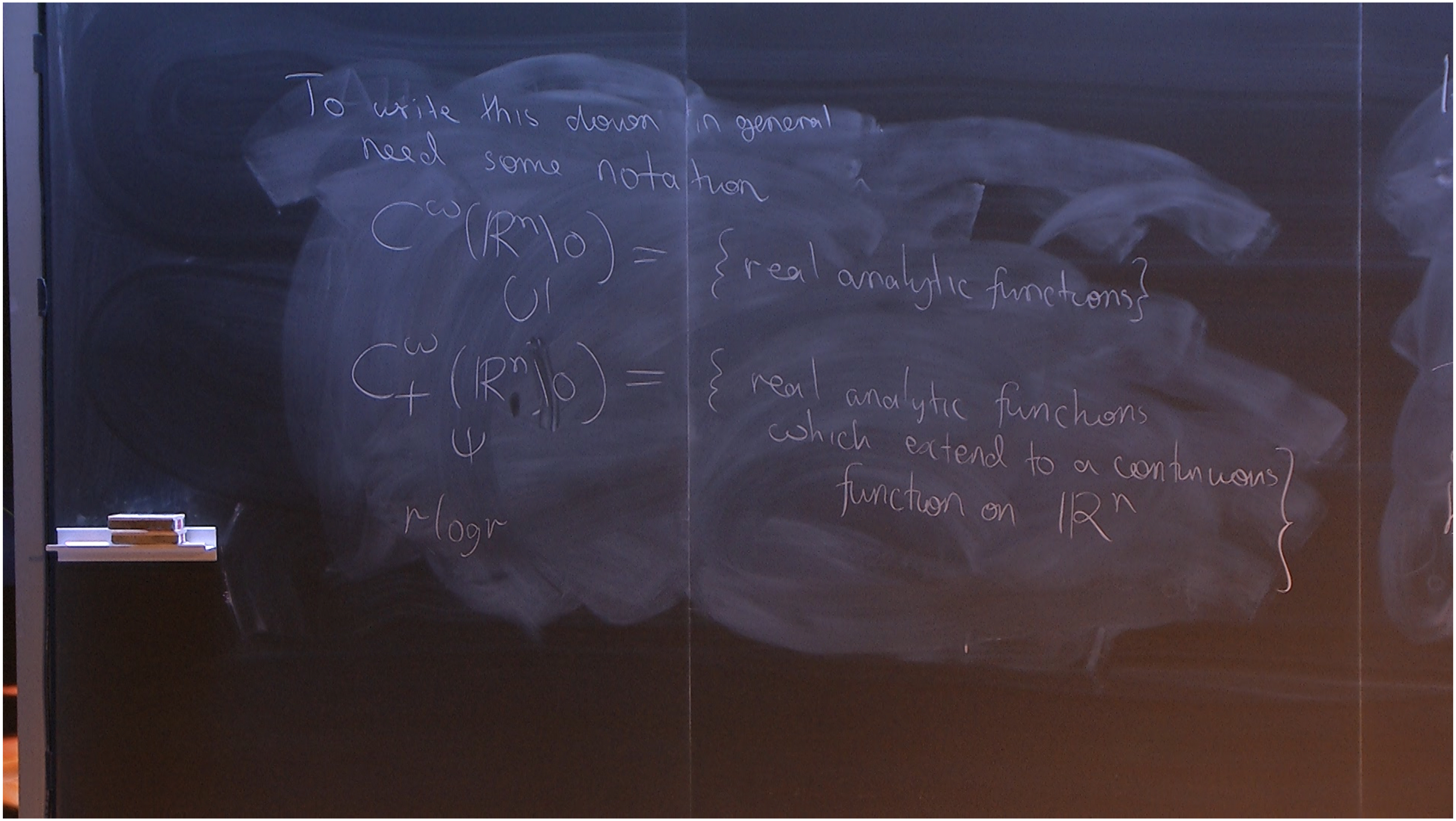

We will show
$O_{hs,0}^{cl}$ has a "Poisson bracket"
Reminder (Deformation quantization)
If we have a classical mechanical system, w. algebra A of operators (commutative)
A has a Poisson bracket defined as follows:
Let $\hat{A}$ be non-comm. quantum algebra, defined modulo $\hbar^2$


If $\alpha, \beta \in A$
Let $\hat{\alpha}, \hat{\beta}$ be lifts to $\hat{A}$
Then
$\{\alpha, \beta\} = \lim_{\hbar \to 0} \frac{1}{\hbar} [\hat{\alpha}, \hat{\beta}]$


If α, β ∈ A
Let α̂, β̂ be lifts to Â
Then
{α, β} = lim 1/ħ [α̂, β̂]
ħ→0
- [α̂, β̂] = 0 mod ħ
- This does not depend on choice of
lifts of α, β.


In fact. algebra story,
$[\hat{\alpha}, \hat{\beta}] = \hat{\alpha}(0) \cdot \hat{\beta}(\varepsilon) - \hat{\alpha}(0) \cdot \hat{\beta}(-\varepsilon)$
This commutator =
obstruction to $\hat{\alpha}(0)\hat{\beta}(\varepsilon)$
being a continuous function of ε
choice of
has a


If there's no Hamiltonian,
$\frac{\partial}{\partial \varepsilon} \hat{\beta}(\varepsilon) = 0$
$\hat{\alpha}(0) \cdot \hat{\beta}(\varepsilon)$
constant
in the region $\varepsilon \neq 0$
$= \left(\text{Independent of } \varepsilon \right) + \left(\delta_{\varepsilon > 0} - \delta_{\varepsilon < 0} \right)$
Obstruction to being continuous
is the commutator
A has a Poisson bracket
defined as follows:
Let $\hat{A}$ be non-comm.
quantum algebra, defined
modulo $\hbar^2$


To write this down in general
need some notation
C^ω(ℝ^n \ 0) = {real analytic functions}
C^ω_+(ℝ^n \ 0) = {real analytic functions which extend to a continuous function on ℝ^n}
r log r

Theorem

Consider any classical field theory

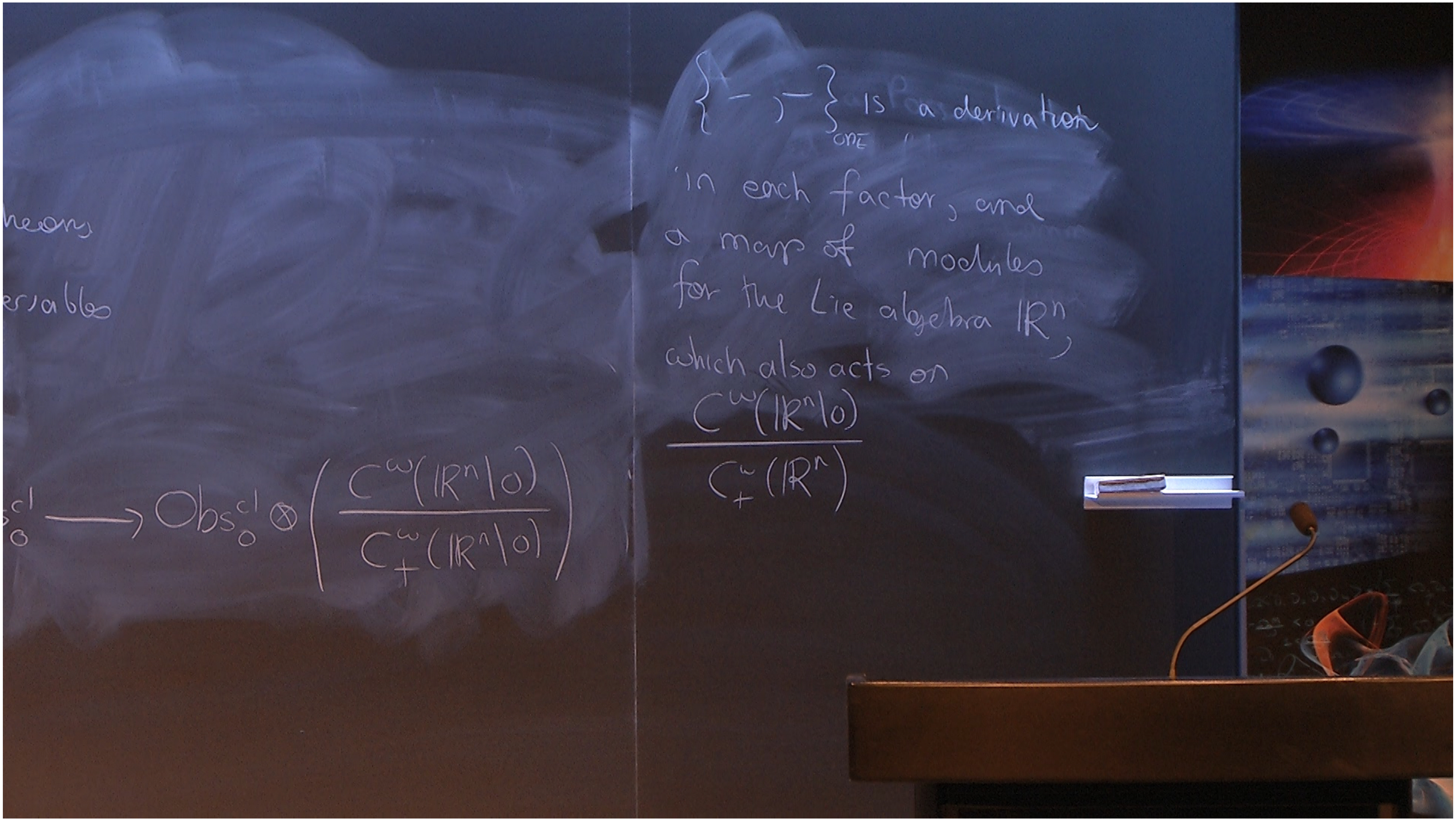
Let Obs_0^{cl} = point observables (taking cohomology)

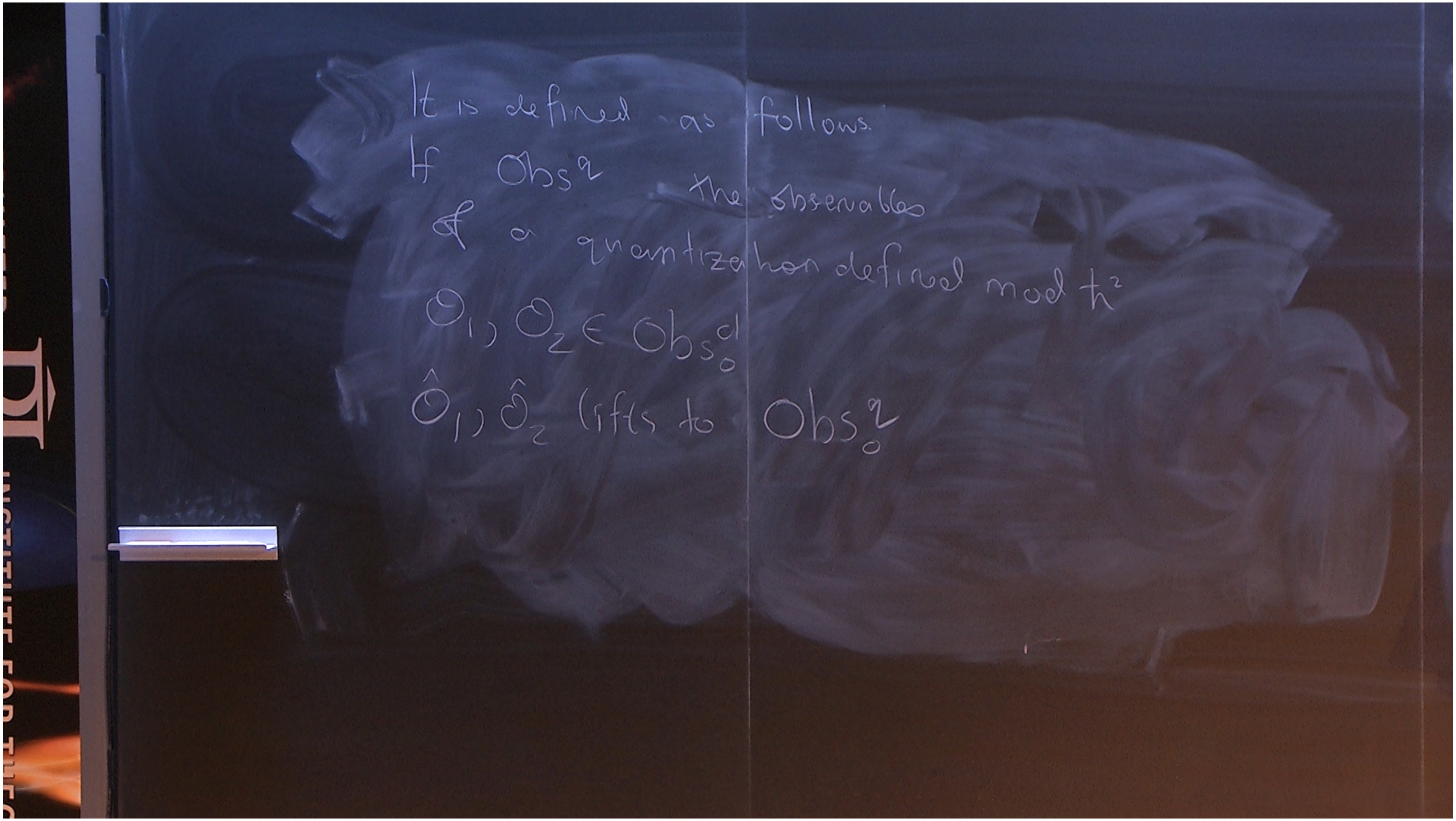
Then, $\exists$ a map

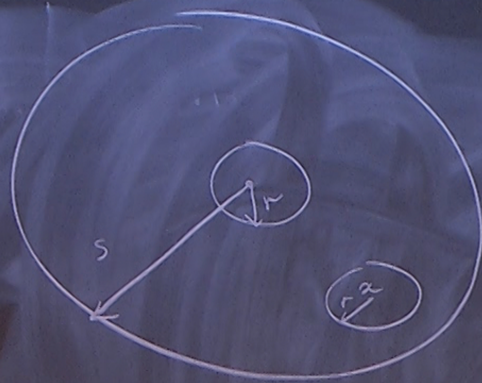
$$\{-,-\}_{OPE} : \mathrm{Obs}_0^{cl} \otimes \mathrm{Obs}_0^{cl} \longrightarrow \mathrm{Obs}_0^{cl} \otimes \left(\frac{C^\omega(\mathbb{R}^n \setminus 0)}{C_+^\omega(\mathbb{R}^n \setminus 0)} \right)$$

$\{-,-$

in eac


{ - , - } is a derivation
in each factor, and
a map of modules
for the Lie algebra IR^n,
which also acts on
C^ω(IR^n|0) / C^ω_+(IR^n)
Obs^cl_0 ⊗ (C^ω(IR^n|0) / C^ω_+(IR^n|0))


It is defined as follows.
If Obs^q the observables
of a quantization defined mod ħ^2
O_1, O_2 ∈ Obs^cl_0
Ô_1, Ô_2 lifts to Obs^q_0



$\hat{\mathcal{O}}_1(0) \in \mathrm{Obs}^q(D(0,r)) \quad \forall r$

$\hat{\mathcal{O}}_2(x) \in \mathrm{Obs}^q(D(x,r)) \quad \forall r$

$\hat{\mathcal{O}}_1(0)\,\hat{\mathcal{O}}_2(x) \in \mathrm{Obs}^q(D(0,s))$

$\forall x$, not too close to 0, $|x| < s-r$

Better

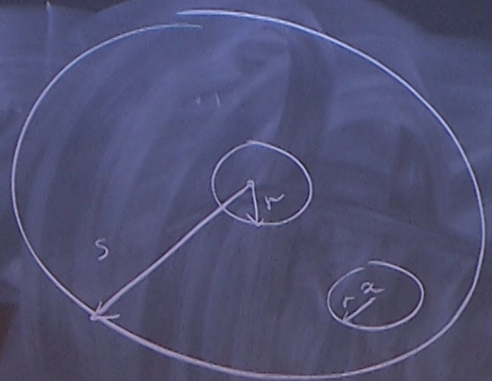
$\forall x \neq 0$

$|x| < s$

$\hat{\mathcal{O}}_1(0) \cdot \hat{\mathcal{O}}_2(x)$

extends across $x = 0$

mod $\hbar$



$\hat{O}_1(0) \in Obs^q(D(0,r)) \quad \forall r$

$\hat{O}_2(x) \in Obs^q(D(x,r)) \quad \forall r$

$\hat{O}_1(0)\,\hat{O}_2(x) \in Obs^q(D(0,s))$

$\forall x$, not too close to 0, $|x| < s-r$

Better

$\forall x \neq 0$

$|x| < s$

$\hat{O}_1(0) \cdot \hat{O}_2(x)$

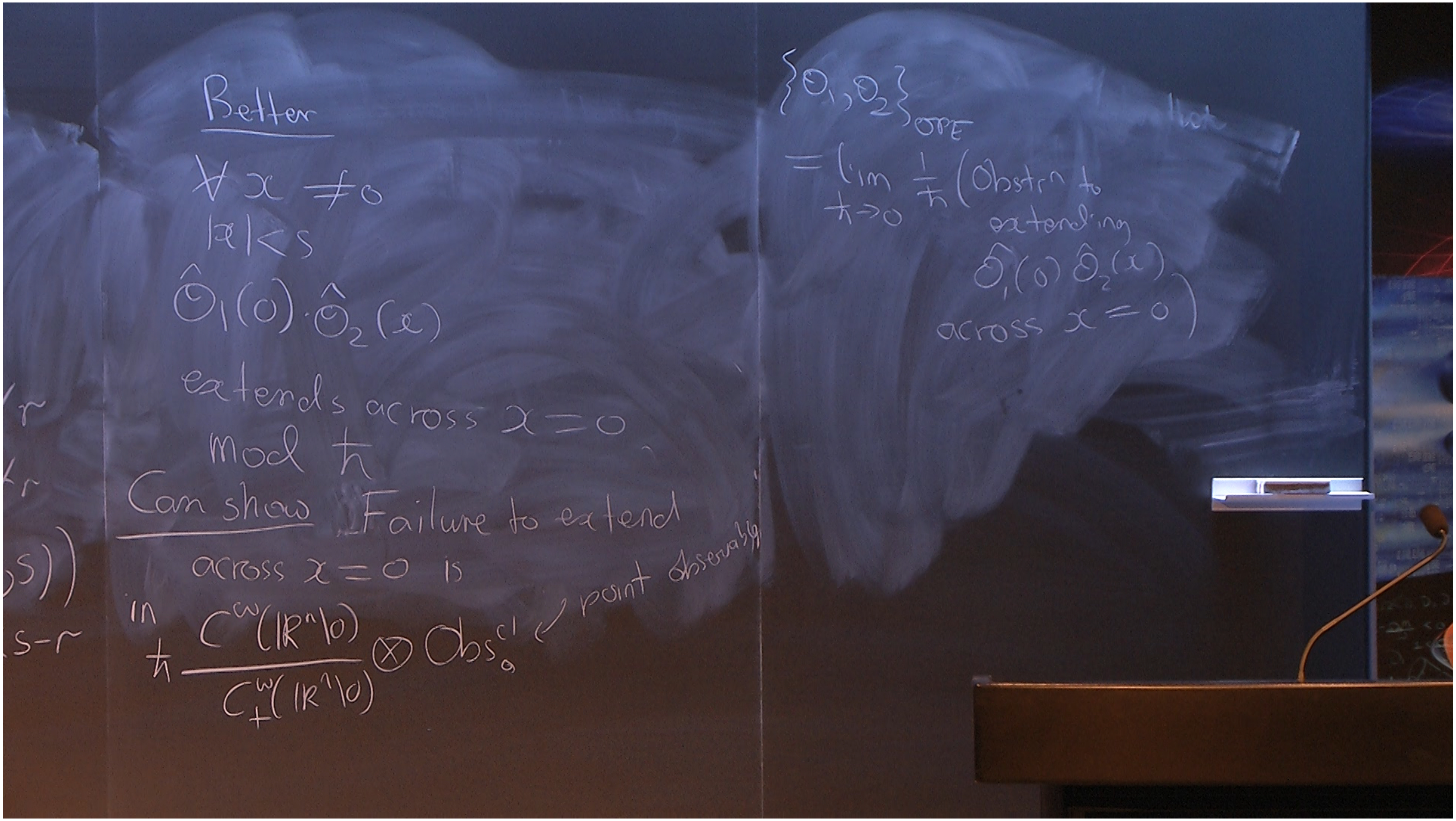
extends across $x = 0$

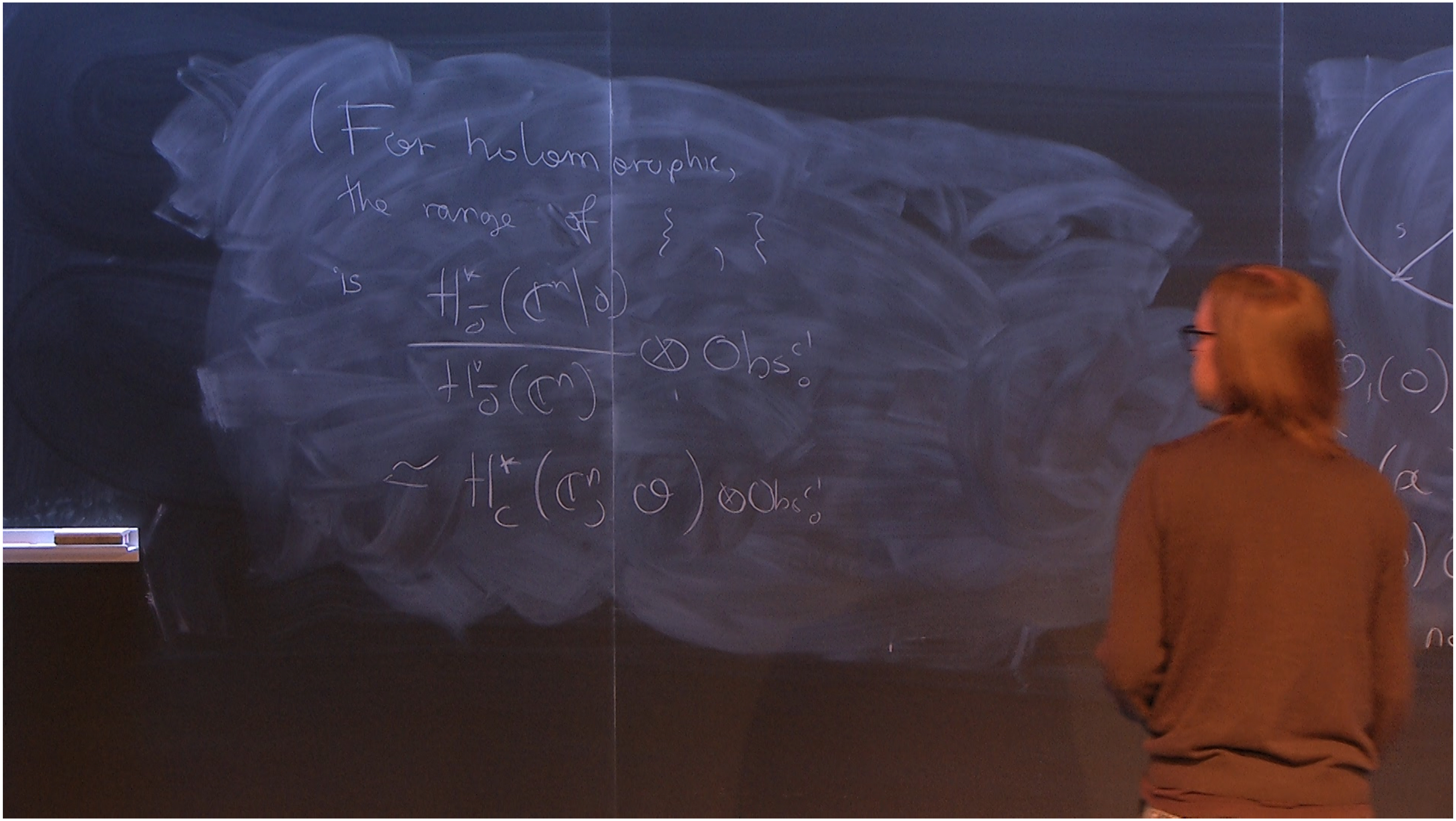
mod $\hbar$

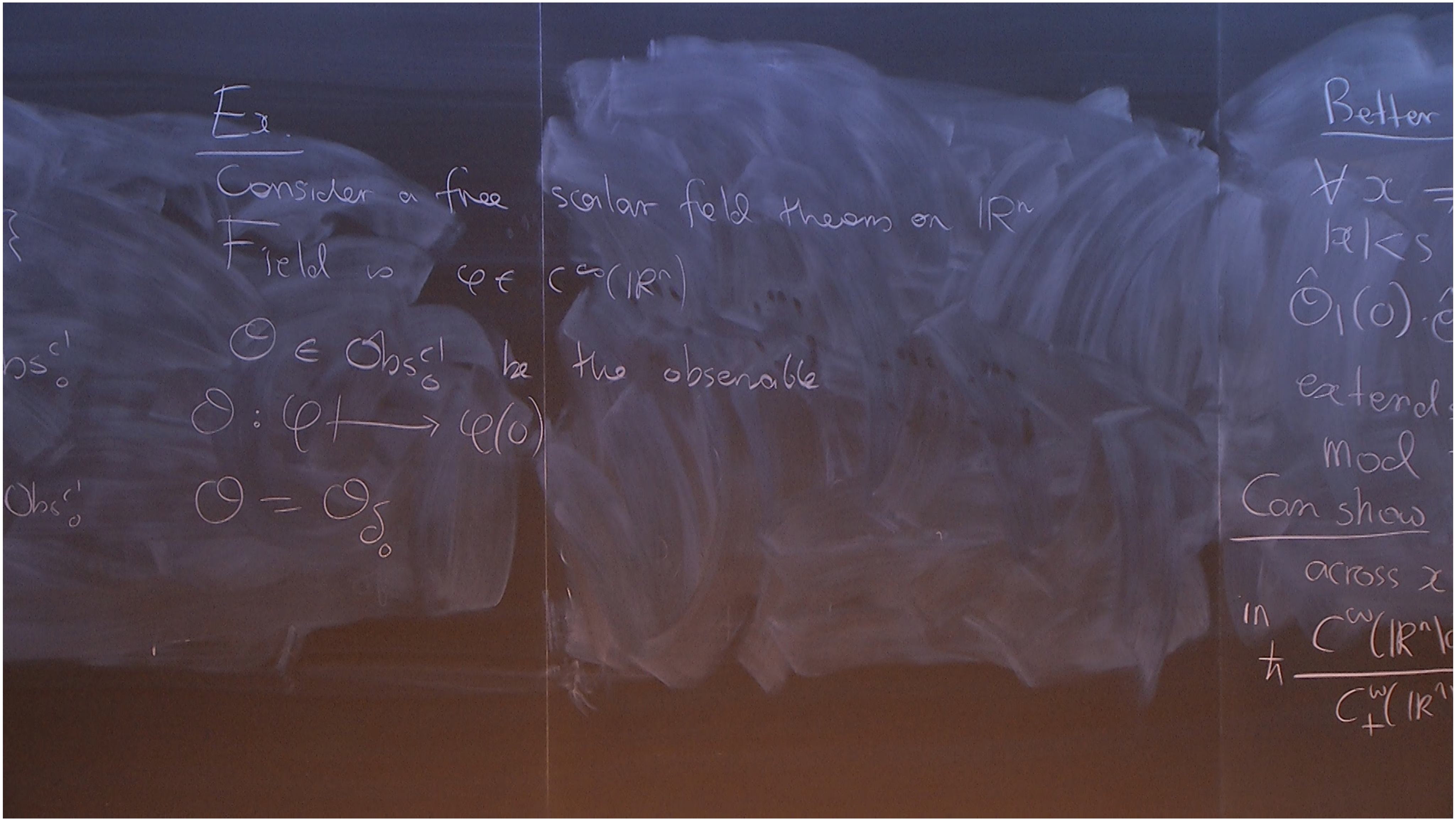
Can show: Failure to extend across $x = 0$ is in

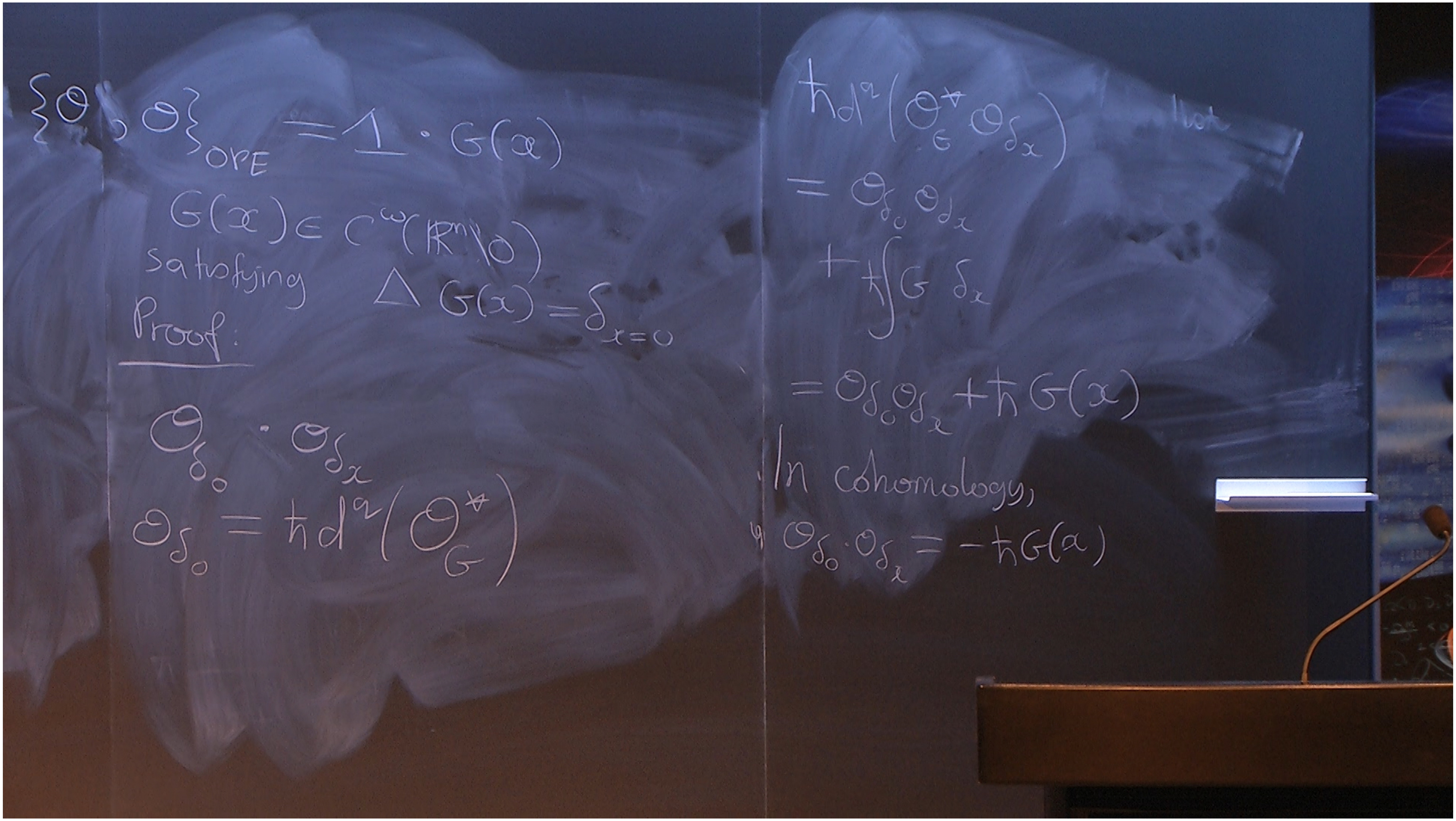
$$\hbar \, \frac{C^\omega(\mathbb{R}^n \setminus 0)}{C^\omega_+(\mathbb{R}^n \setminus 0)} \otimes Obs^{cl}_0$$

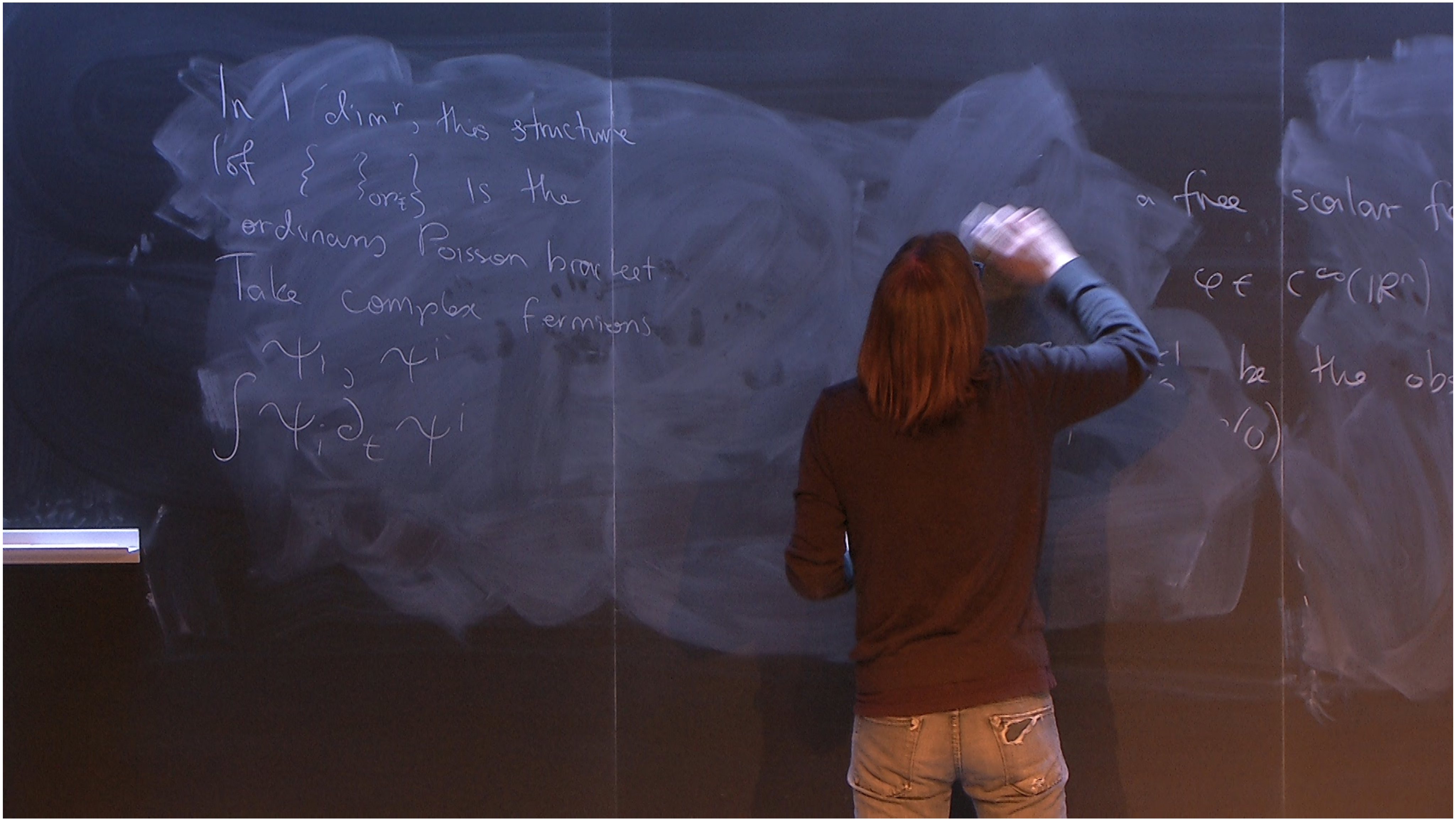
← point observable

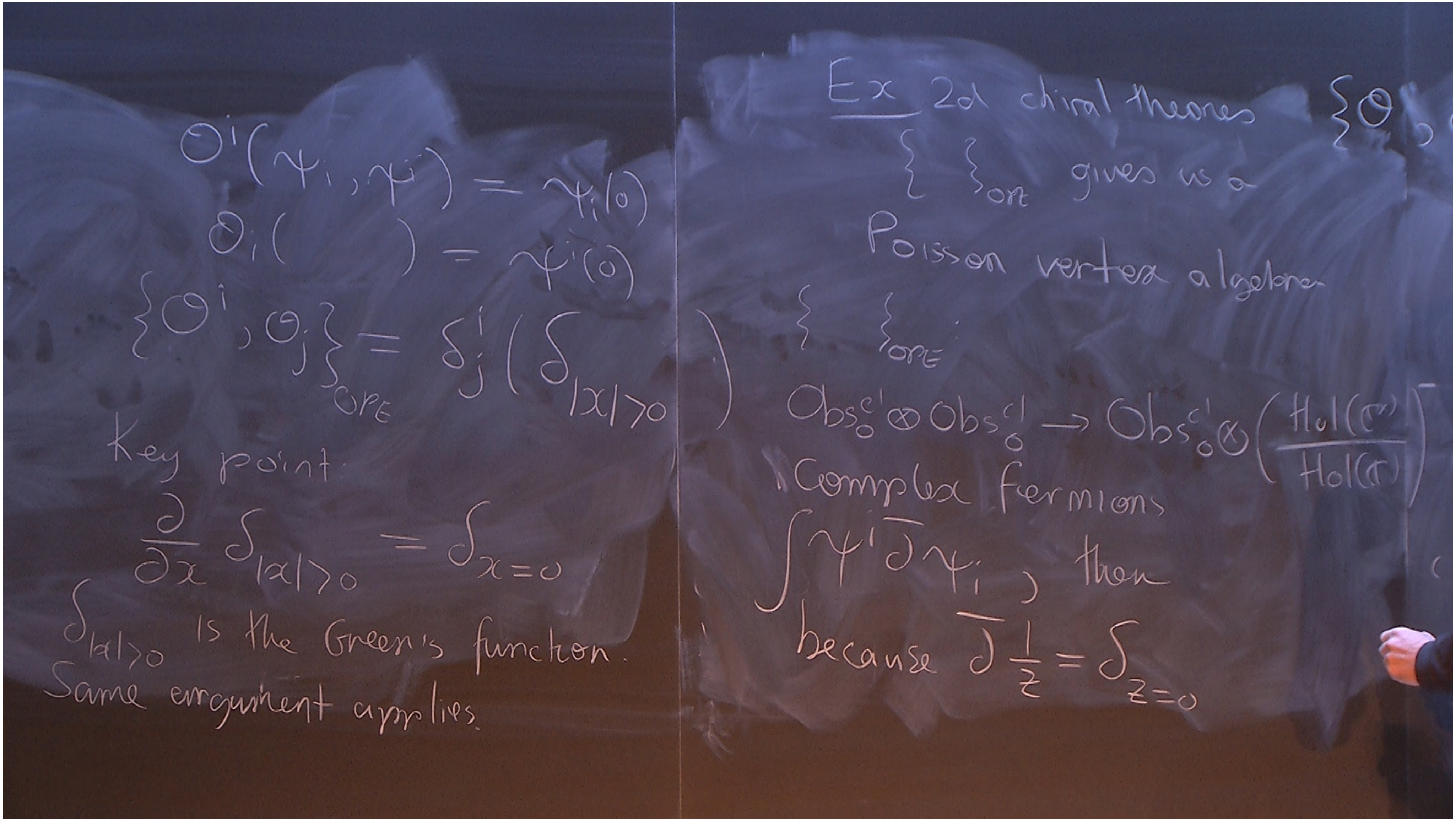

Better
∀ x ≠ 0
|x| < s
Ô1(0) · Ô2(x)
extends across x = 0
mod ħ
Can show Failure to extend
across x = 0 is
in
ħ C^ω(ℝ^n\0) / C+^ω(ℝ^n\0) ⊗ Obs0^cl ← point observables
{O1, O2}OPE
= lim ħ→0 1/ħ (Obstrn to
extending
Ô1(0) Ô2(x)
across x = 0)

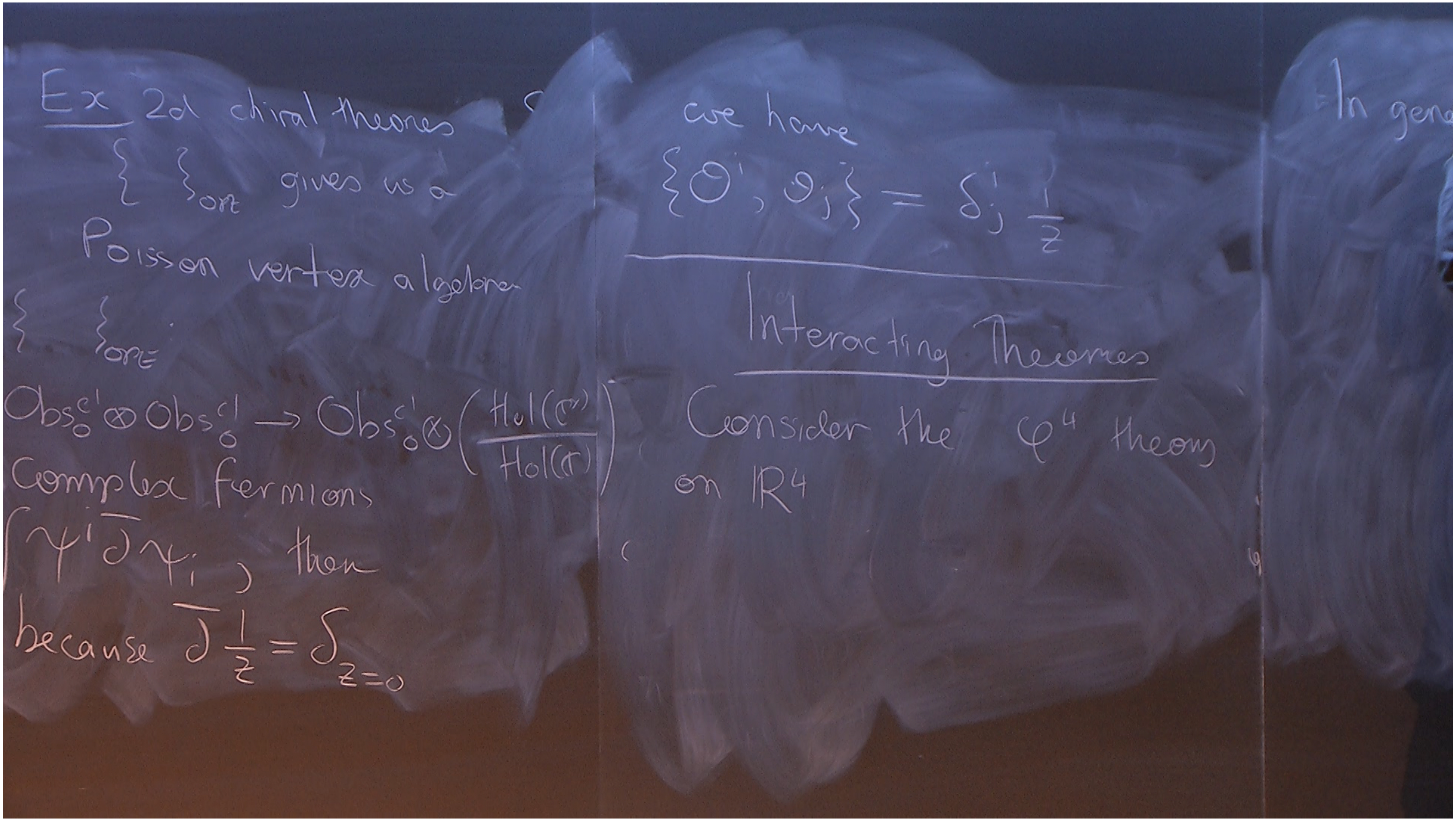

(For holomorphic,
the range of { , }
is

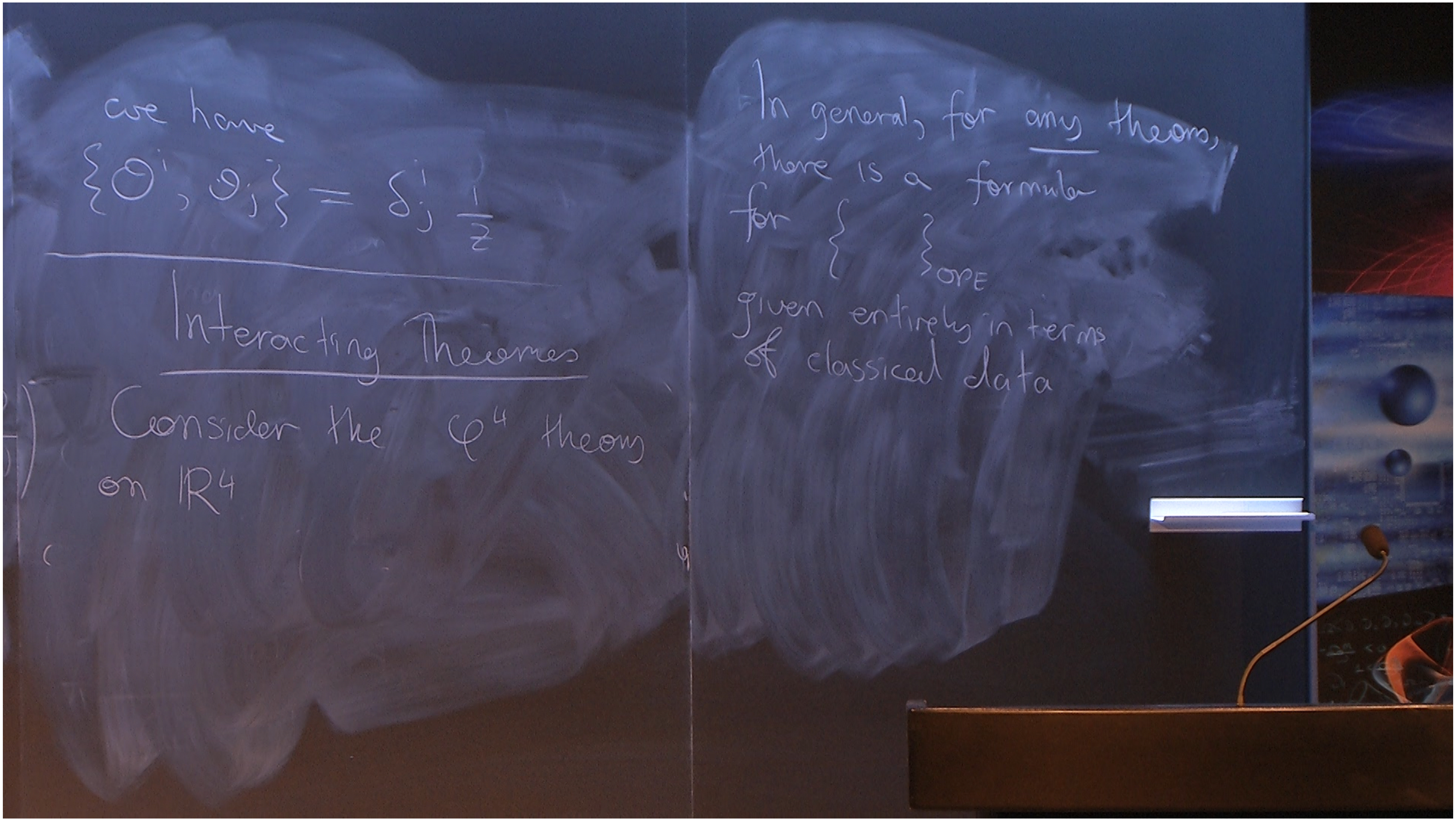

Ex.
Consider a free scalar field theory on $\mathbb{R}^n$
Field is $\varphi \in C^\infty(\mathbb{R}^n)$
$\mathcal{O} \in Obs^{cl}_0$ be the observable
$\mathcal{O} : \varphi \longmapsto \varphi(0)$
$\mathcal{O} = \mathcal{O}_{\delta_0}$
Better
extend
mod
Can show
across

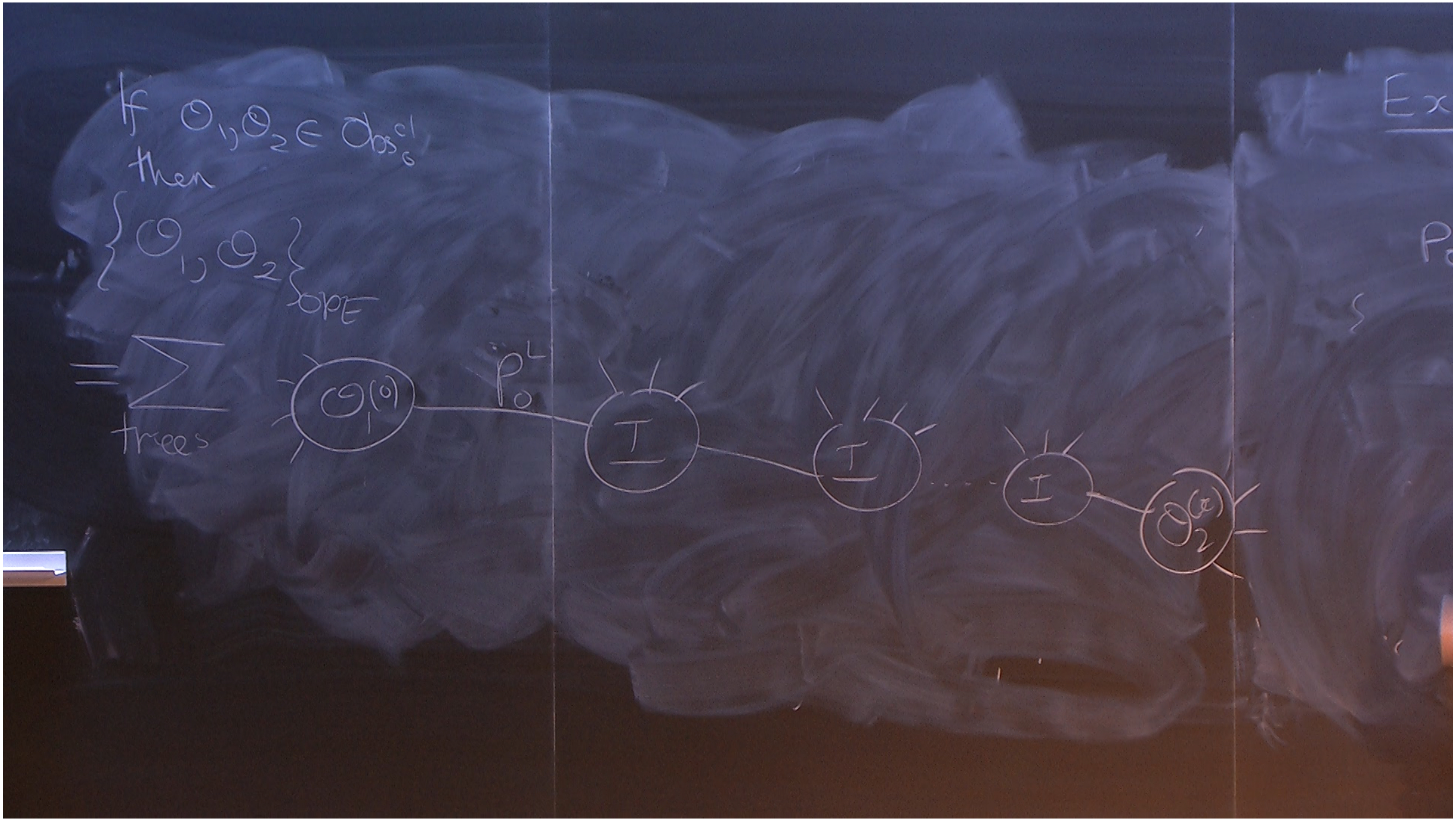

{O, O}_OPE = 1 · G(x)
G(x) ∈ C^ω(R^n \ 0)
satisfying Δ G(x) = δ_{x=0}
Proof:
O_δ0 · O_δx
O_δ0 = ħ d^q (O*_G)
ħ d^q (O*_G O_δx)
= O_δ0 O_δx
+ ħ ∫ G δ_x
= O_δ0 O_δx + ħ G(x)
In cohomology,
O_δ0 · O_δx = −ħ G(x)

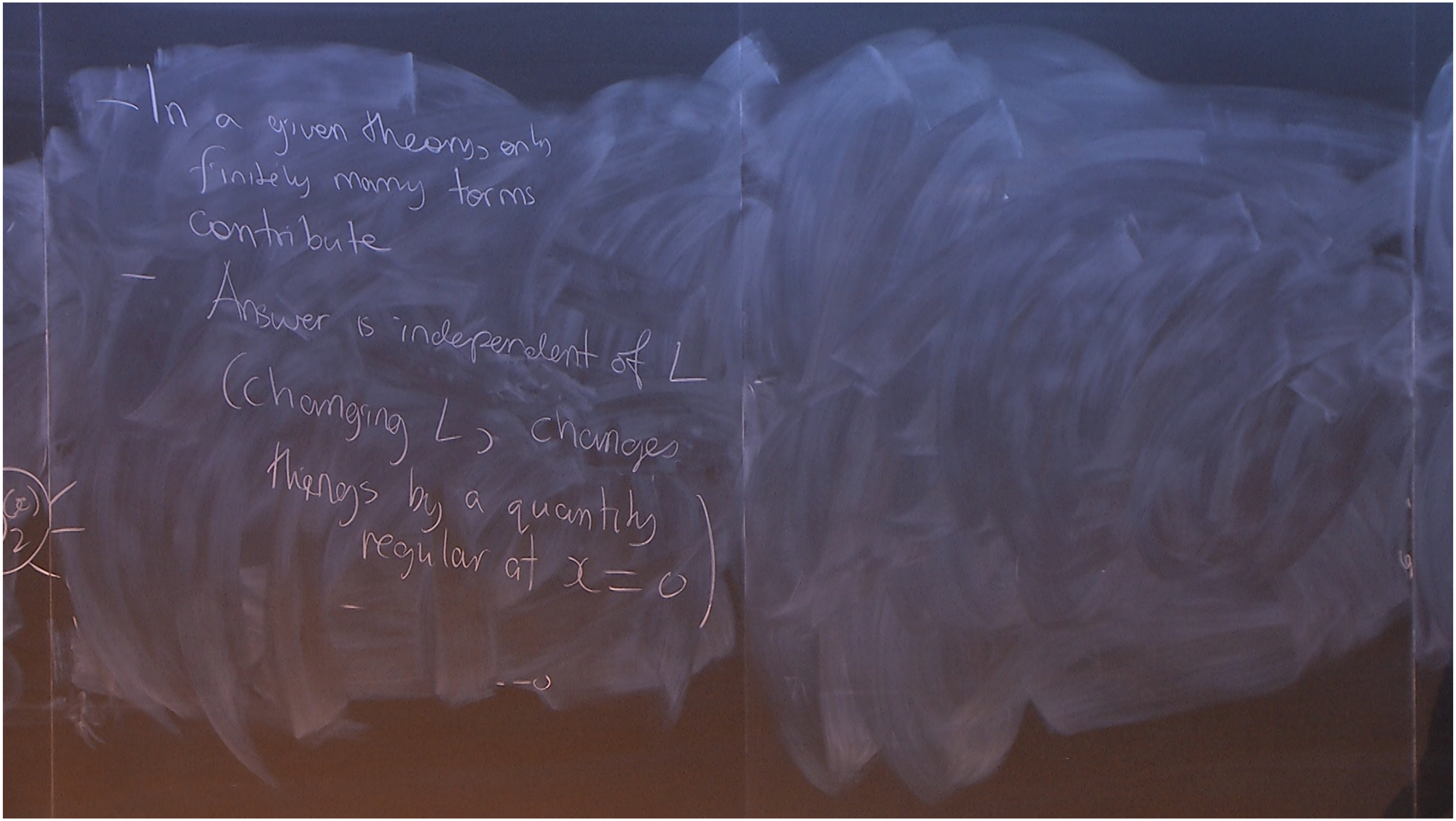

In 1 dim'n, this structure
is the
ordinary Poisson bracket
Take complex fermions
a free scalar
be the

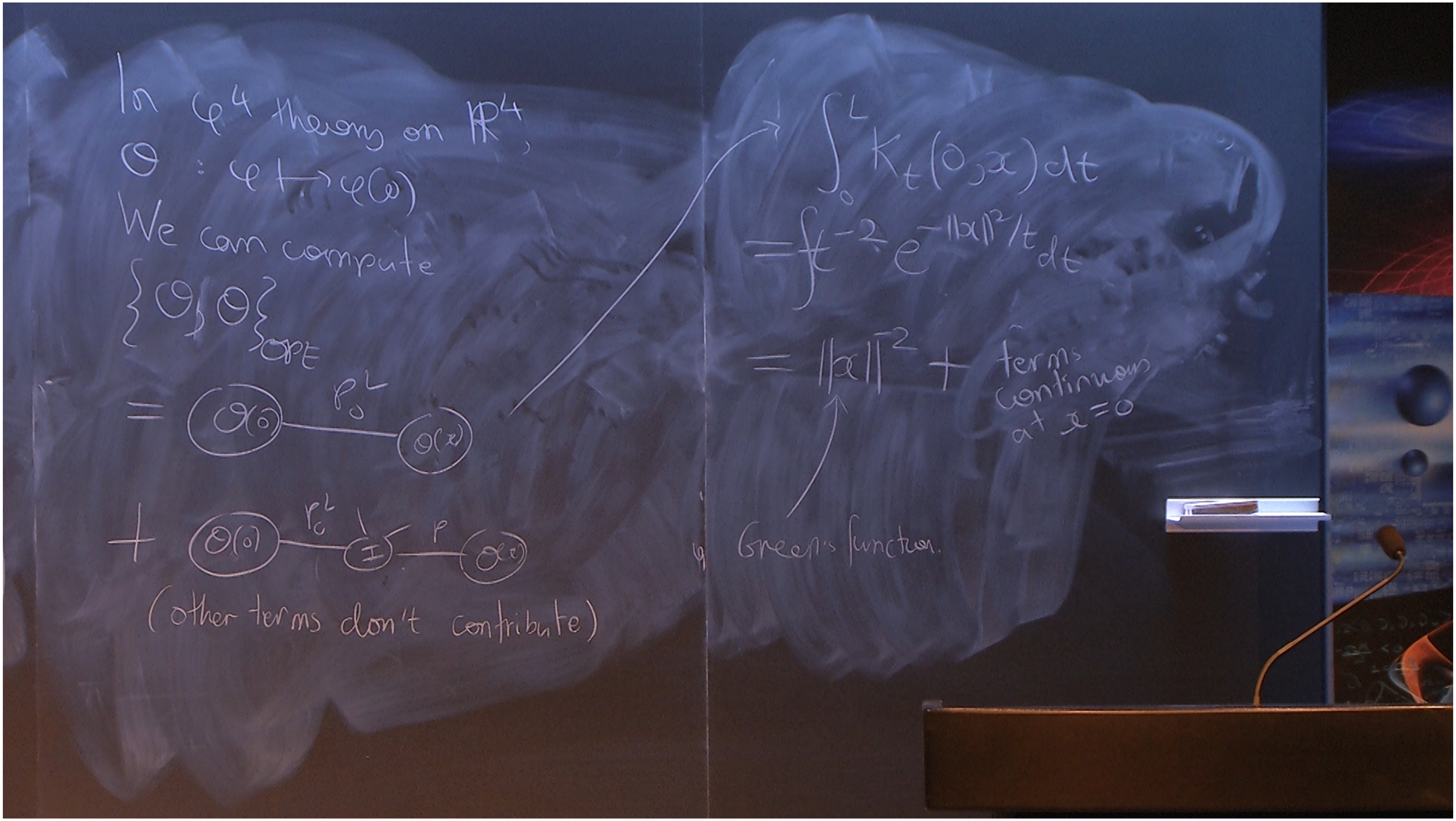

O^i(ψ_i, ψ^i) = ψ_i(0)
O_i( ) = ψ^i(0)
{O^i, O_j}_OPE = δ^i_j(δ_|x|>0)
Key point:
∂/∂x δ_|x|>0 = δ_x=0
δ_|x|>0 is the Green's function.
Same argument applies.
Ex 2d chiral theories {O
{ }_OPE gives us a
Poisson vertex algebra
{ }_OPE
Obs^cl_0 ⊗ Obs^cl_0 → Obs^cl_0 ⊗ (Hol(C^×) / Hol(C))
Complex fermions
∫ψ^i ∂̄ ψ_i, then
because ∂̄ 1/z = δ_z=0

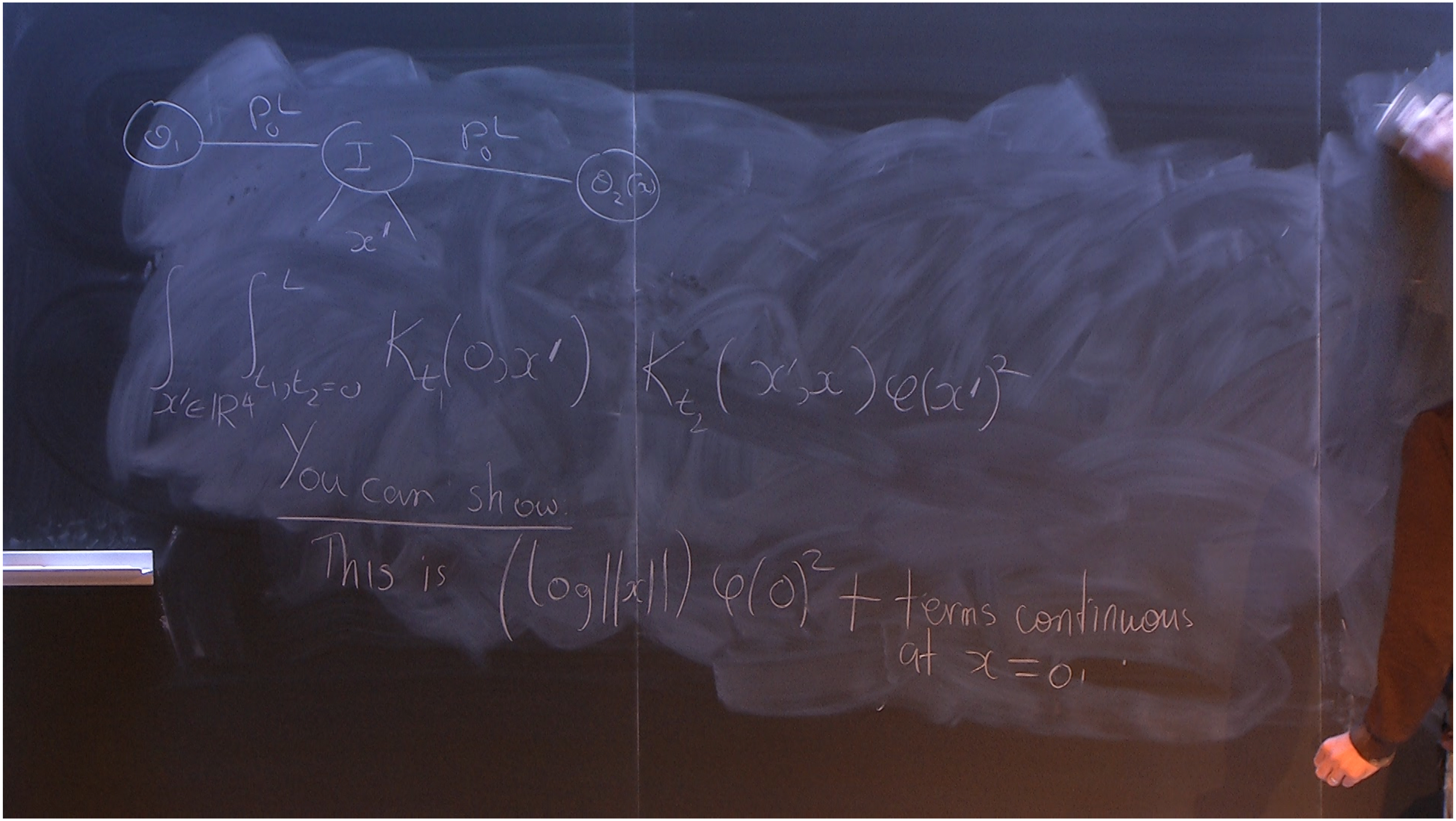

Ex 2d chiral theories
{ }_{orz} gives us a
Poisson vertex algebra
{ }_{orz}:
Obs^{cl}_0 ⊗ Obs^{cl}_0 → Obs^{cl}_0 ⊗ (Hol(r*)/Hol(r))
Complex fermions
∫ψ^i ∂̄ ψ_i , then
because ∂̄ 1/z = δ_{z=0}
we have
{θ^i, θ_j} = δ^i_j 1/z
Interacting Theories
Consider the φ^4 theory
on ℝ^4
In gen

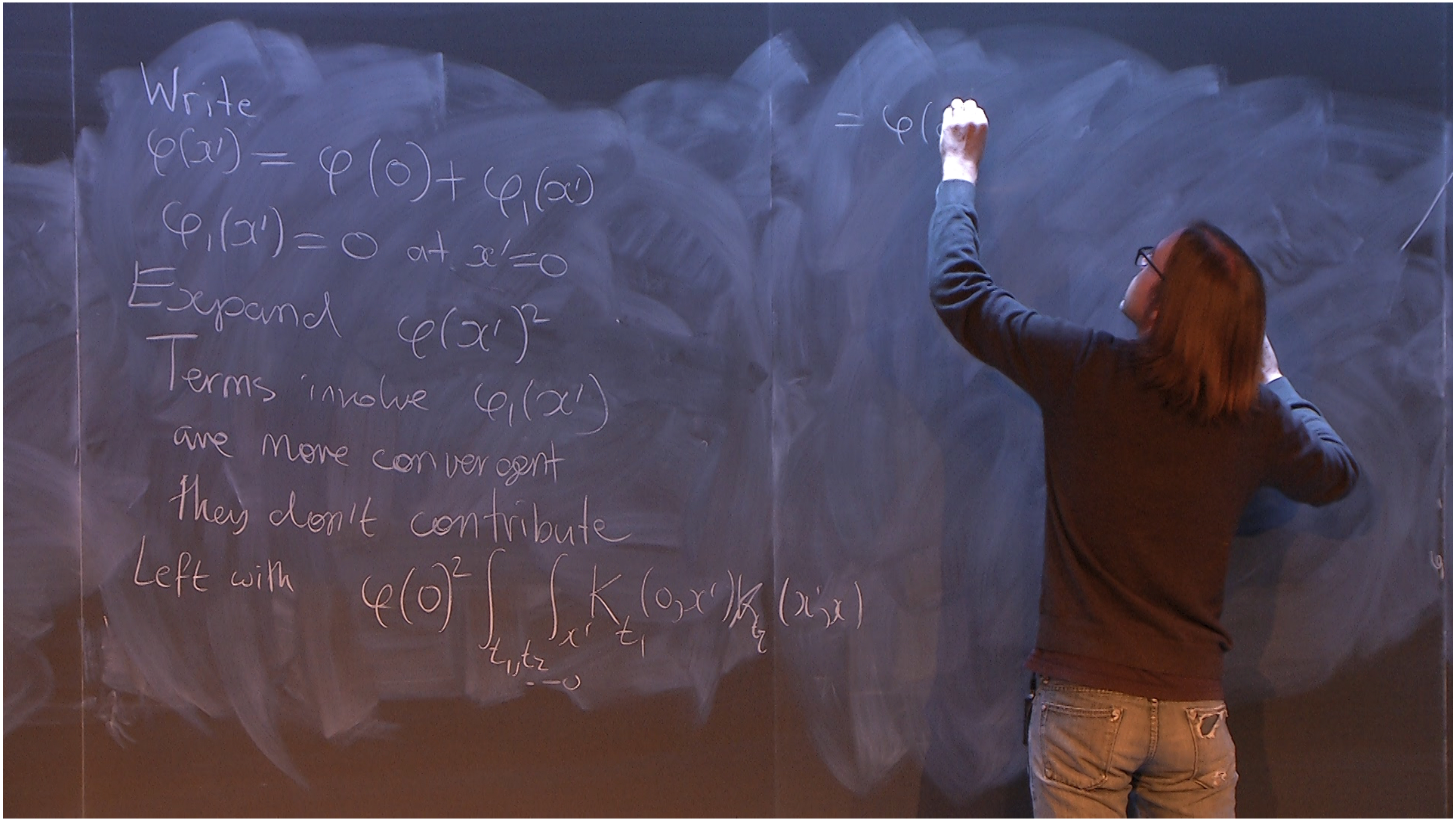

we have
$\{\mathcal{O}^i, \vartheta_j\} = \delta^i_j \frac{1}{z}$
Interacting Theories
Consider the φ^4 theory on $\mathbb{R}^4$
In general, for any theory, there is a formula for $\{\ \}_{OPE}$ given entirely in terms of classical data

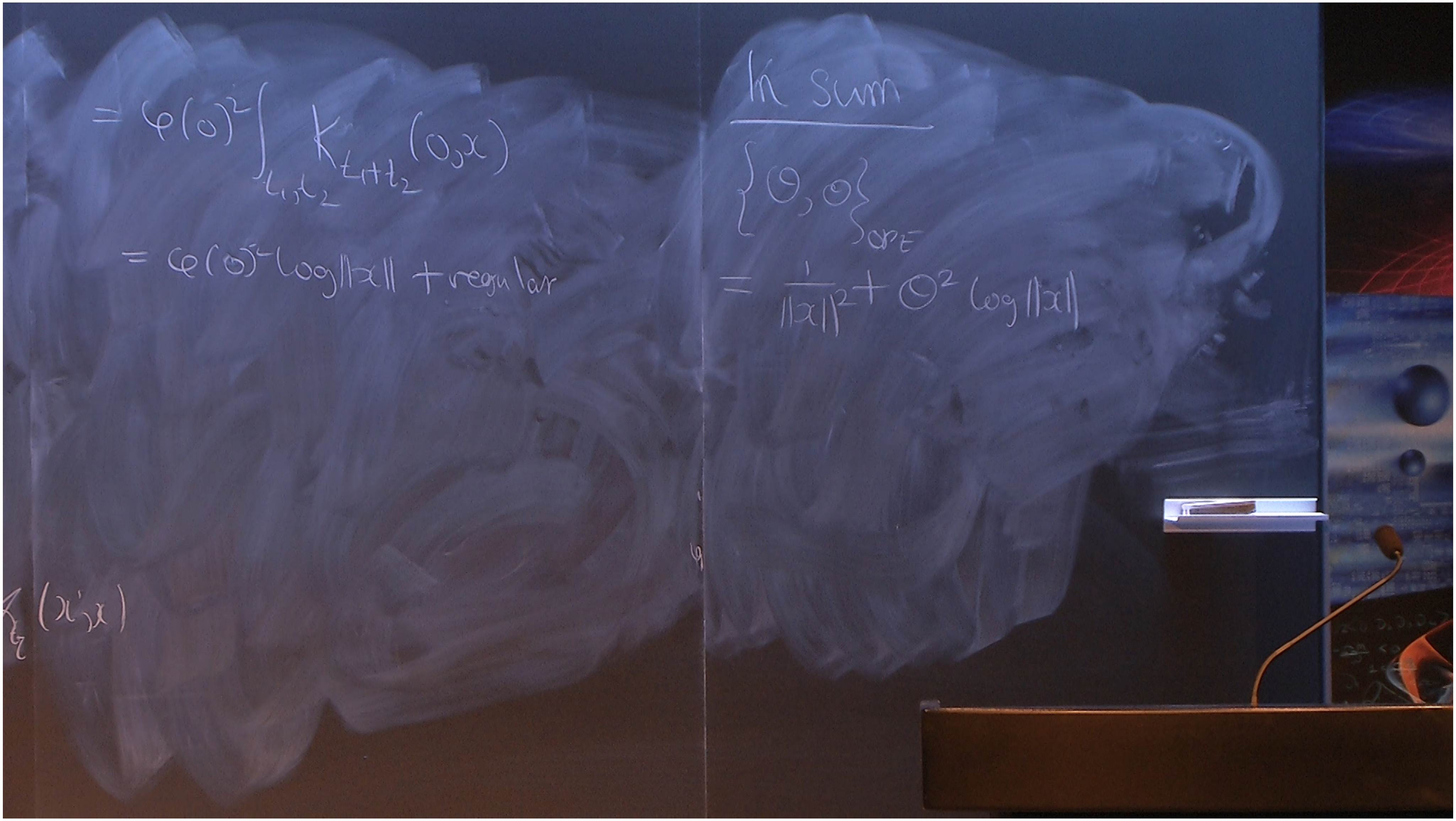

If O1, O2 ∈ Obos
then
{O1, O2}OPE
= Σ trees
I
I
I
Ex

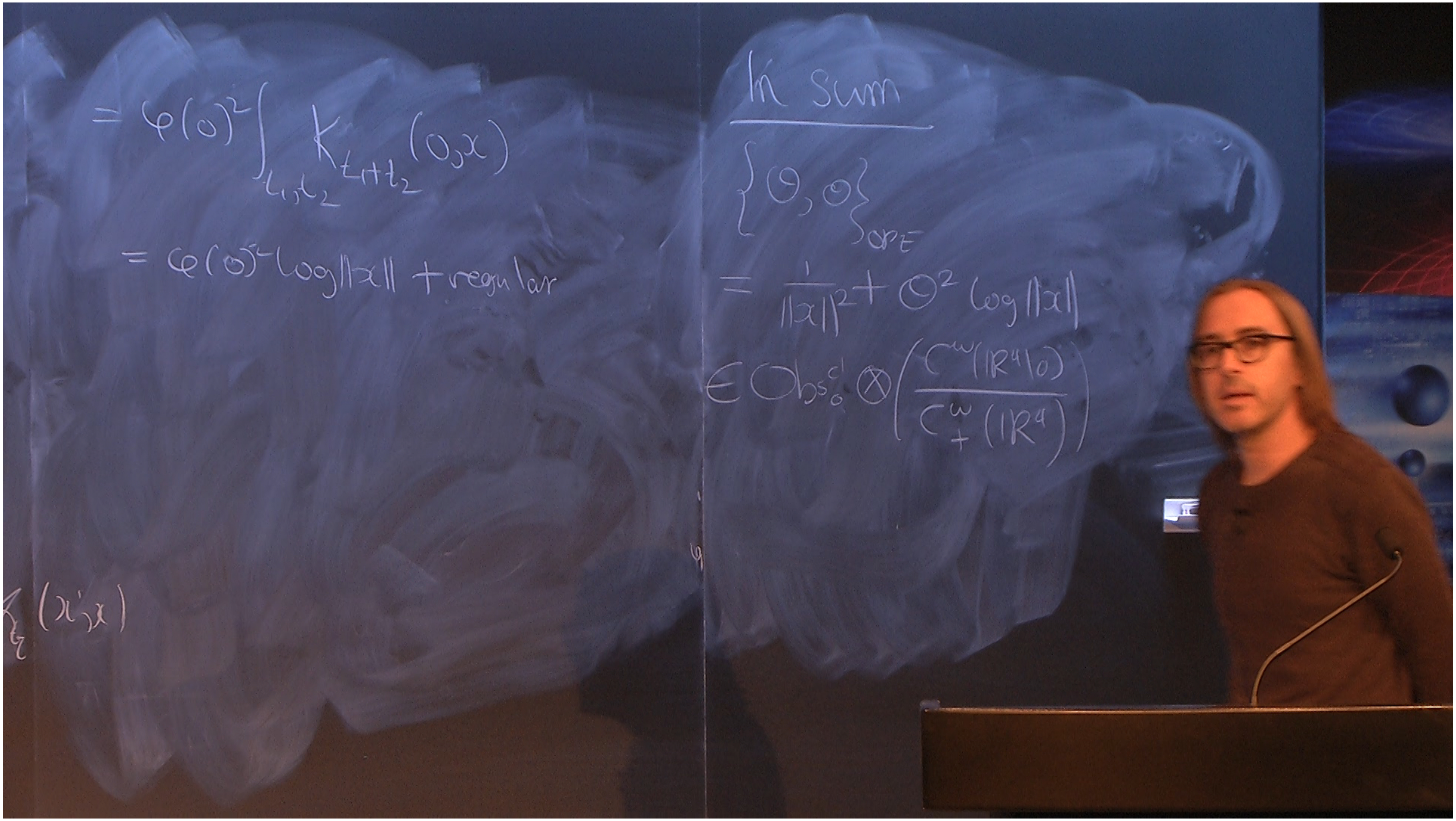

- In a given theory, only finitely many terms contribute
- Answer is independent of L
(changing L, changes things by a quantity regular at x = 0)


In φ^4 theory on ℝ^4,
We can compute
(other terms don't contribute)
Green's function.
terms continuous at x=0


$\mathcal{O}_1 \; P_0^L \; I \; P_0^L \; \mathcal{O}_2(x)$
x'
$\int_{x' \in \mathbb{R}^4} \int_{t_1, t_2 = 0}^{L} K_{t_1}(0, x') \, K_{t_2}(x', x) \, \varphi(x')^2$
You can show:
This is $(\log \|x\|)\, \varphi(0)^2$ + terms continuous at $x = 0$.


Write
φ(x') = φ(0) + φ1(x')
φ1(x') = 0 at x' = 0
Expand φ(x')2
Terms involve φ1(x')
are more convergent
they don't contribute
Left with φ(0)2 ∫t1,t2→0 ∫x' Kt1(0,x') Kt2(x',x)
= φ(


= φ(0)² ∫ K_{t₁+t₂}(0,x)
= φ(0)² log||x|| + regular
In sum
{O, O}_OPE
= 1/||x||² + O² log||x||
(x',x)


$= \varphi(0)^2 \int_{t_1,t_2} K_{t_1+t_2}(0,x)$
$= \varphi(0)^2 \log\|x\| + \text{regular}$
In sum
$\{\mathcal{O},\mathcal{O}\}_{OPE}$
$= \frac{1}{\|x\|^2} + \mathcal{O}^2 \log\|x\|$
$\in \mathrm{Obs}^{cl}_0 \otimes \left(\frac{C^\omega(\mathbb{R}^4 \setminus 0)}{C^\omega_+(\mathbb{R}^4)} \right)$

Chiral complex fermions

ψ_i, ψ^i

Then we have:

$$\{O_i, O^j\}_{OPE} \sim \delta_i^j \frac{1}{z}$$

If we have anti-chiral fermions $\bar{O}_i, \bar{O}^j$

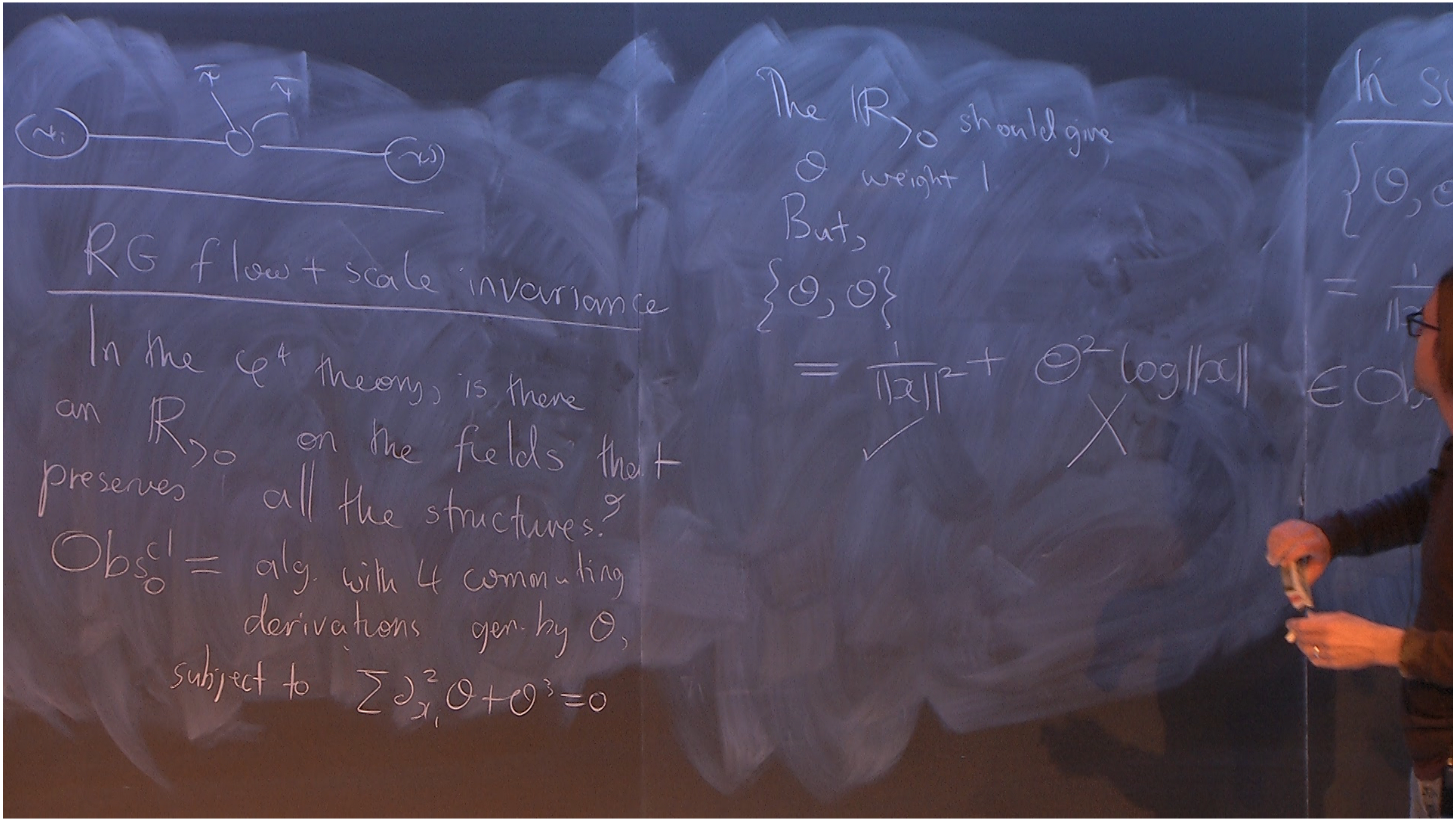
$$\{\bar{O}_i, \bar{O}^j\} = \delta_i^j \frac{1}{\bar{z}}$$

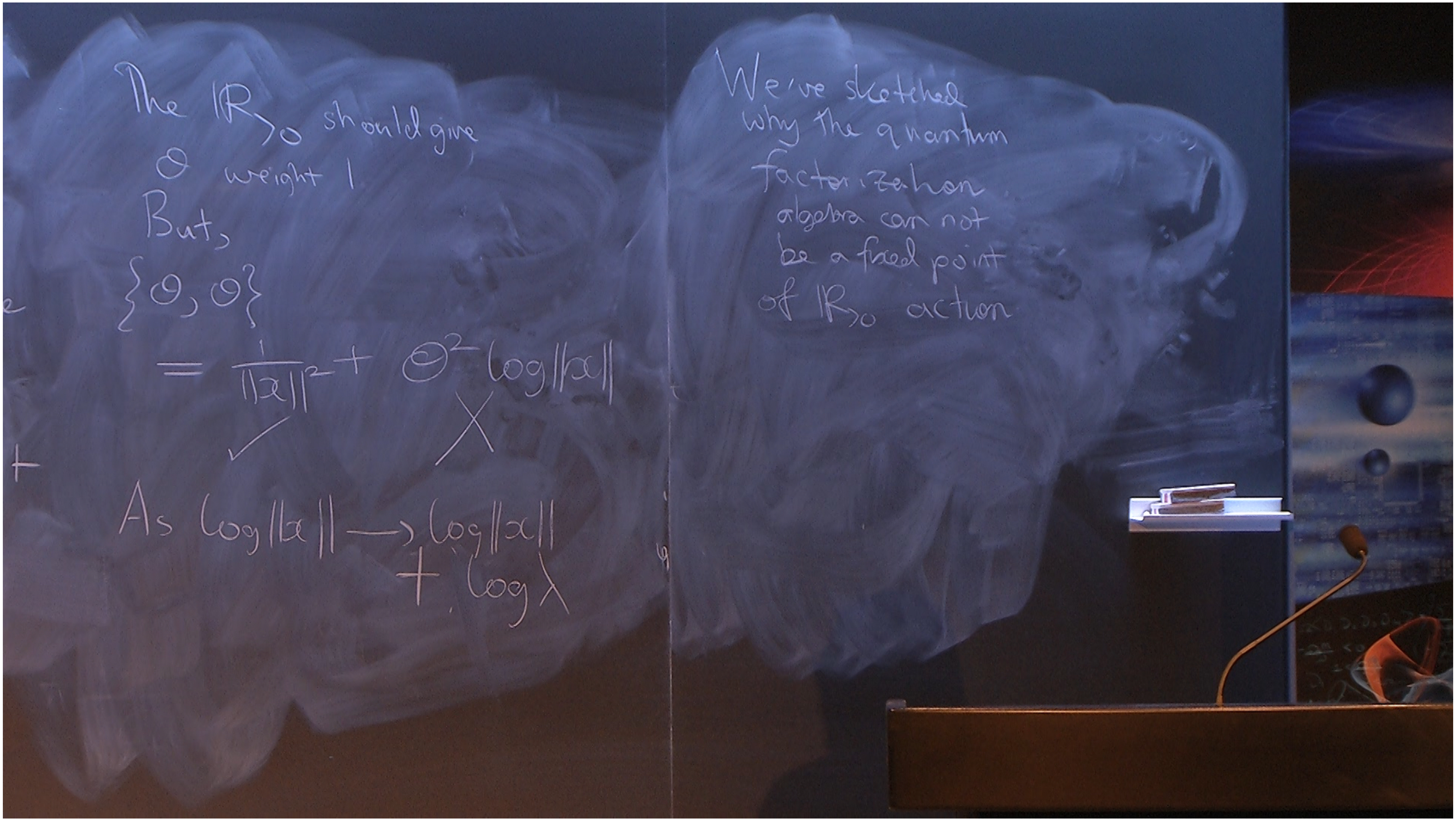
If chiral + anti-chiral are deformed by an interaction

$$\int \psi_i \psi^j \bar{\psi}_k \bar{\psi}^l m^{jl}_{ik}$$

Then, the one loop OPE is deformed by

$$\{\psi_i, \psi^j\} = m^{jl}_{ik} \bar{\psi}_k \bar{\psi}^l \frac{\bar{z}}{z} \log|z|$$


RG flow + scale invariance
In the φ⁴ theory, is there an ℝ>0 on the fields that preserves all the structures?
Obs₀ᶜˡ = alg. with 4 commuting derivations gen. by θ, subject to Σ ∂²xᵢ θ + θ³ = 0
The ℝ>0 should give θ weight 1
But,
{θ, θ} = 1/‖x‖² + θ² log‖x‖


The $\mathbb{R}_{>0}$ should give
$\mathcal{O}$ weight 1
But,
$\{\mathcal{O}, \mathcal{O}\}$
$= \frac{1}{\|x\|^2} + \mathcal{O}^2 \log\|x\|$
As $\log\|x\| \to \log\|x\| + \log\lambda$
We've sketched
why the quantum
factorization
algebra can not
be a fixed point
of $\mathbb{R}_{>0}$ action