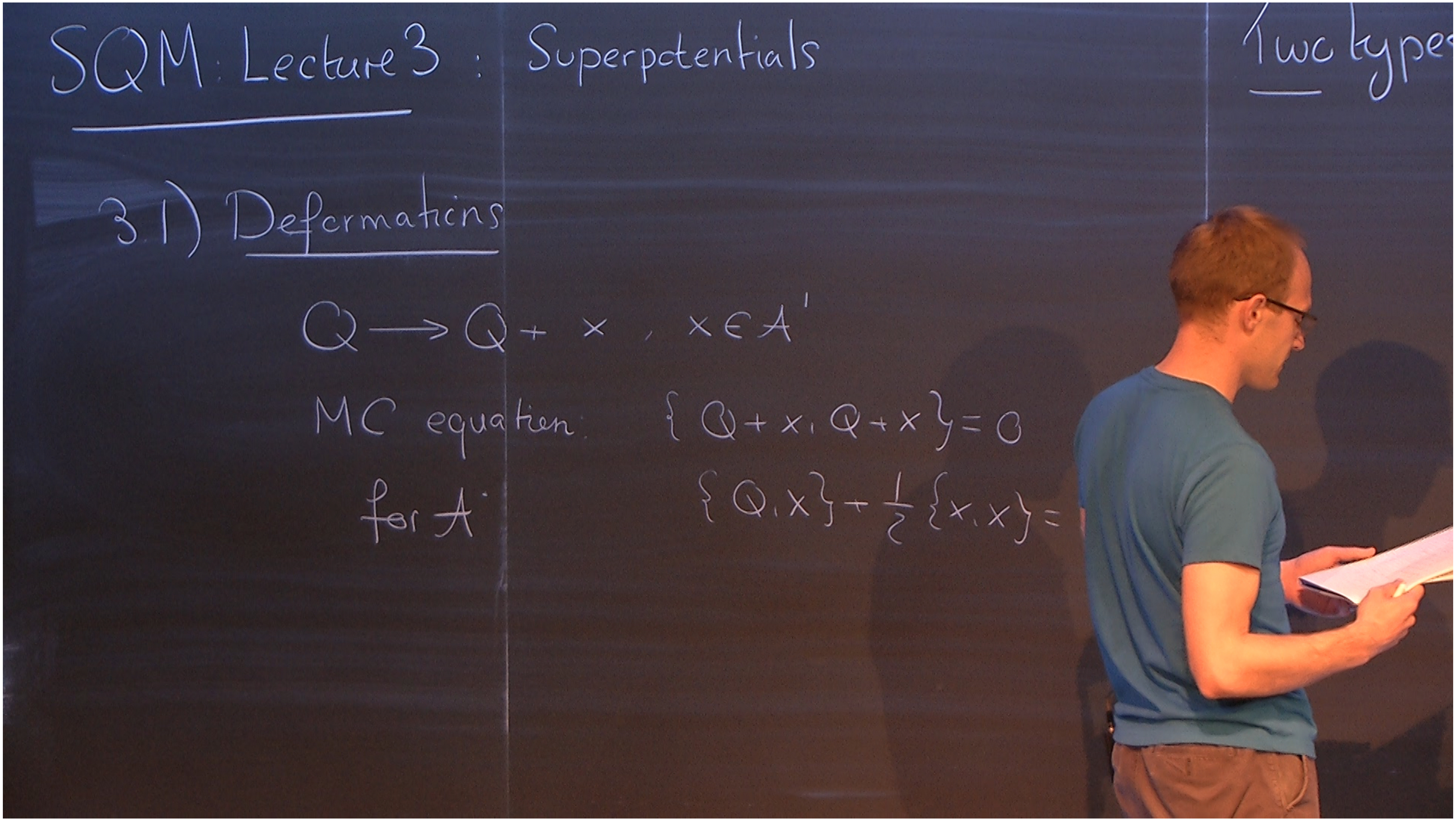
Title: Lecture 3: Supersymmetric Quantum Mechanics and All That

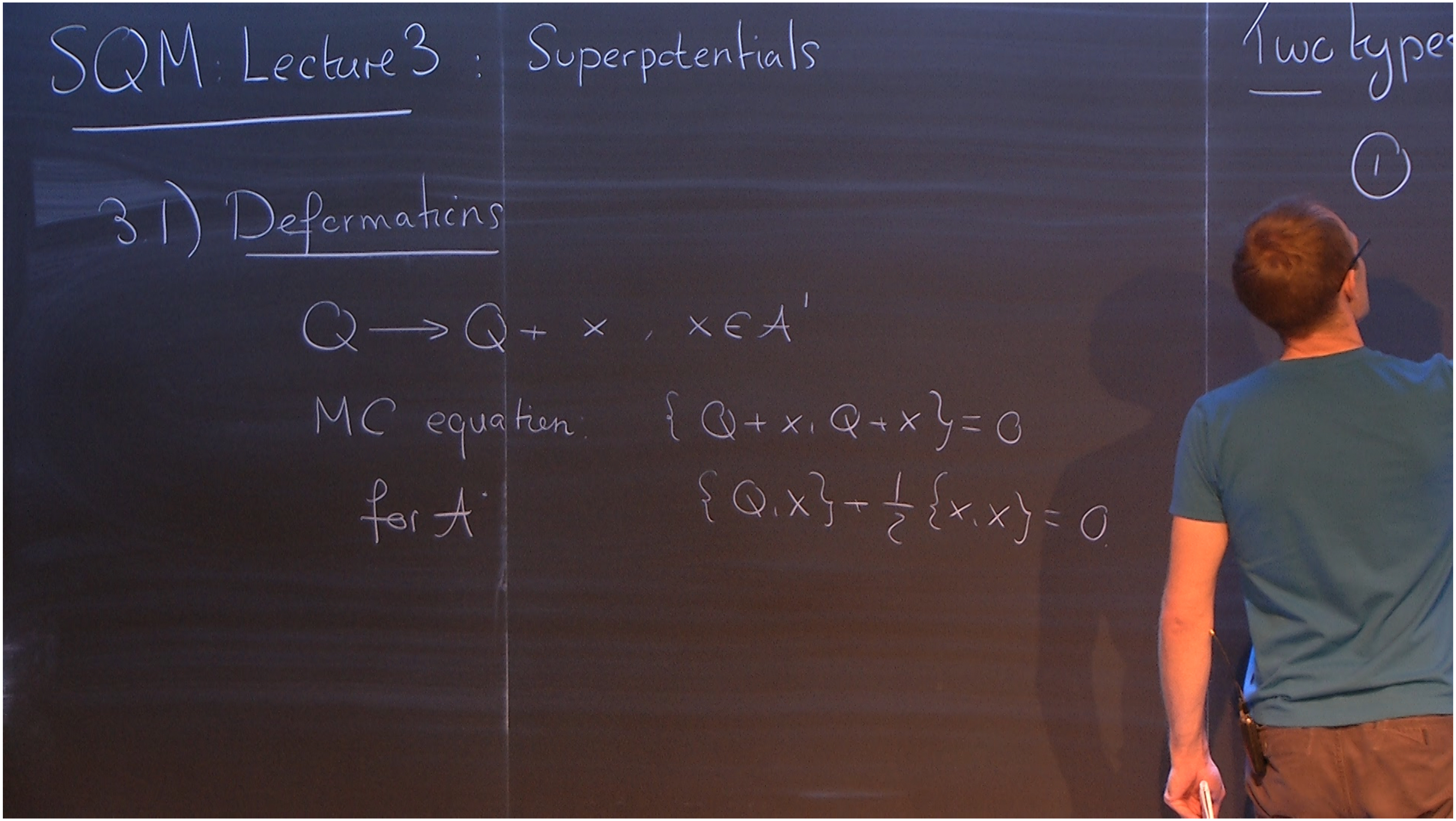
Speakers: Mathew Bullimore

Collection: QFT for Mathematicians

Date: June 21, 2019 - 11:00 AM

URL: http://pirsa.org/19060014


SQM: Lecture 3 : Superpotentials
Two type
3.1) Deformations
Q ⟶ Q + x , x ∈ A'
MC equation: { Q + x, Q + x } = 0
for A'
{ Q, x } + 1/2 { x, x } =


SQM Lecture 3 : Superpotentials
Two types
1
3.1) Deformations
Q ⟶ Q + x , x ∈ A^1
MC equation: {Q + x, Q + x} = 0
for A^•
{Q, x} + 1/2 {x, x} = 0

Two types of finite defs:

(1) Complex param $u \in U$

$$\partial_{\bar{u}} Q(u) = 0$$

- $(\Omega^{\cdot}, Q) \longrightarrow$ complex of hol^c vector bundles on U

- $\mathcal{H}^{\cdot}$. coherent sheaf on U.

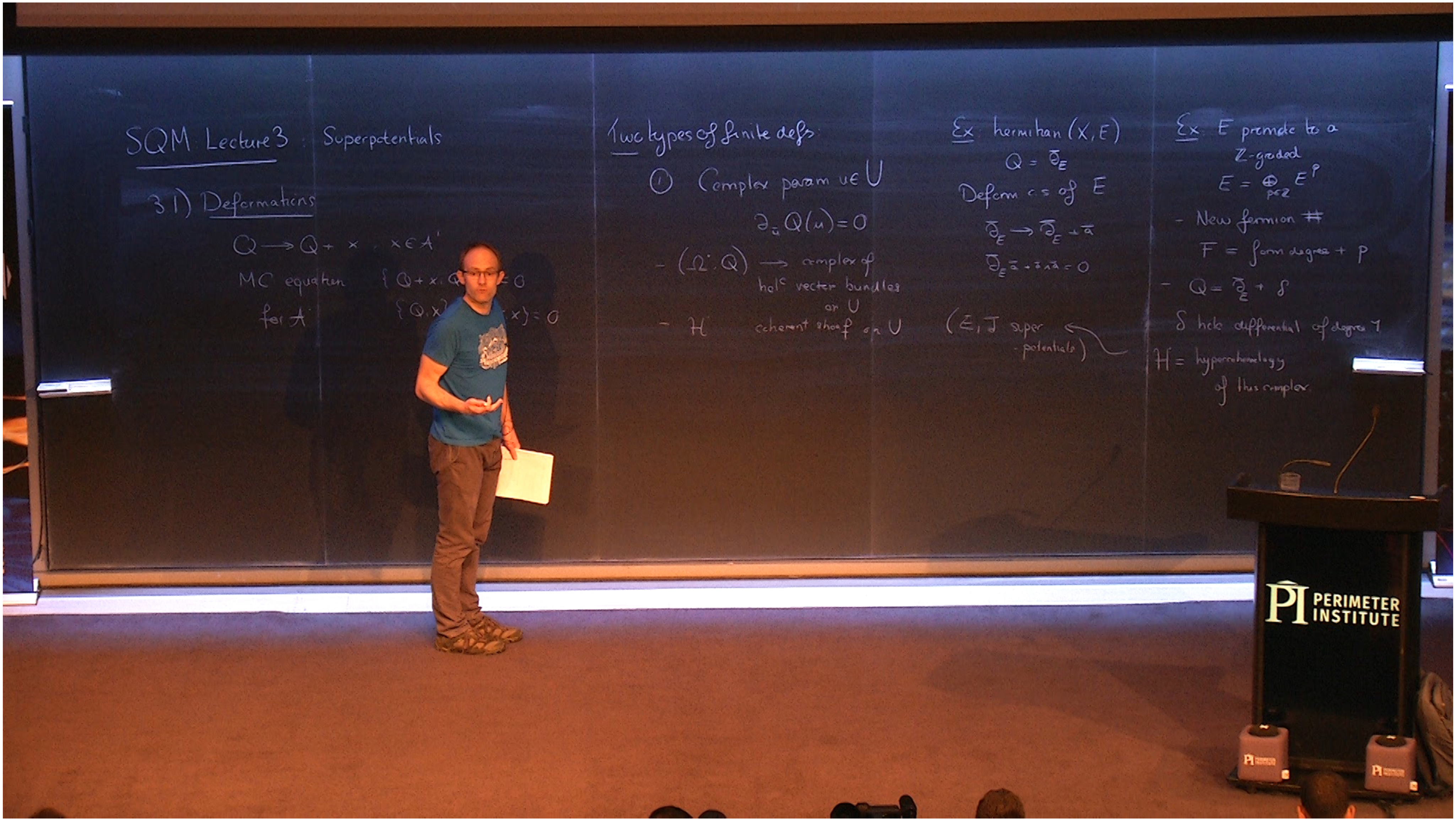
Ex: hermitian (X, E)

$$Q = \bar{\partial}_E$$

Deform c.s. of E.

$$\bar{\partial}_E \longrightarrow \bar{\partial}_E + \bar{a}$$

$$\bar{\partial}_E \bar{a} + \bar{a} \wedge \bar{a} = 0$$


SQM Lecture 3 : Superpotentials
3.1) Deformations
MC equation
Two types of finite defs
① Complex param u ∈ U
complex of
hol^c vector bundles
on U
coherent sheaf on U
Ex hermitian (X, E)
Deform c.s of E
super
potentials
Ex E promote to a
Z-graded
New fermion #
F = form degree + p
δ hol^c differential of degree 1
H = hypercohomology
of this complex
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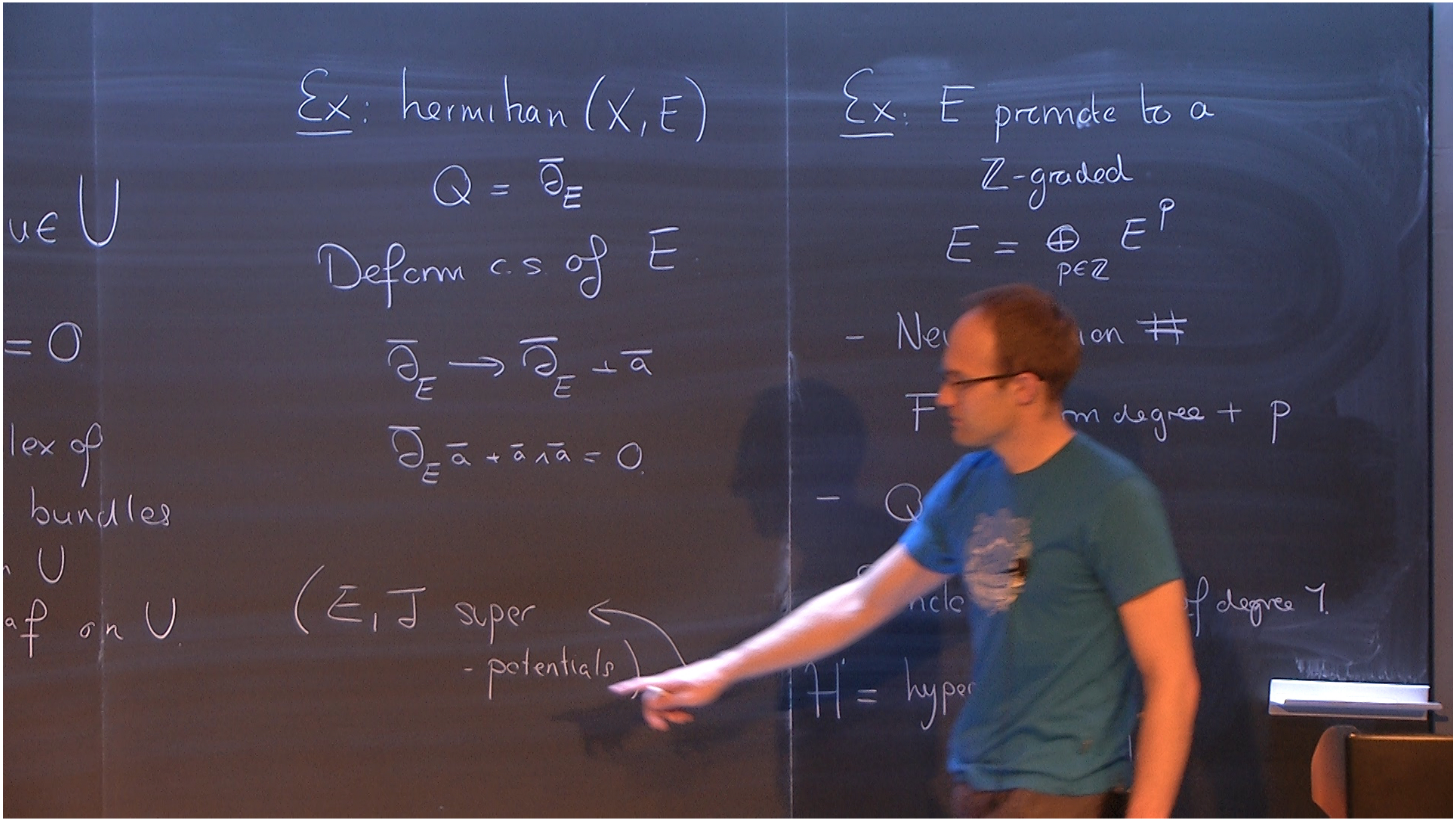
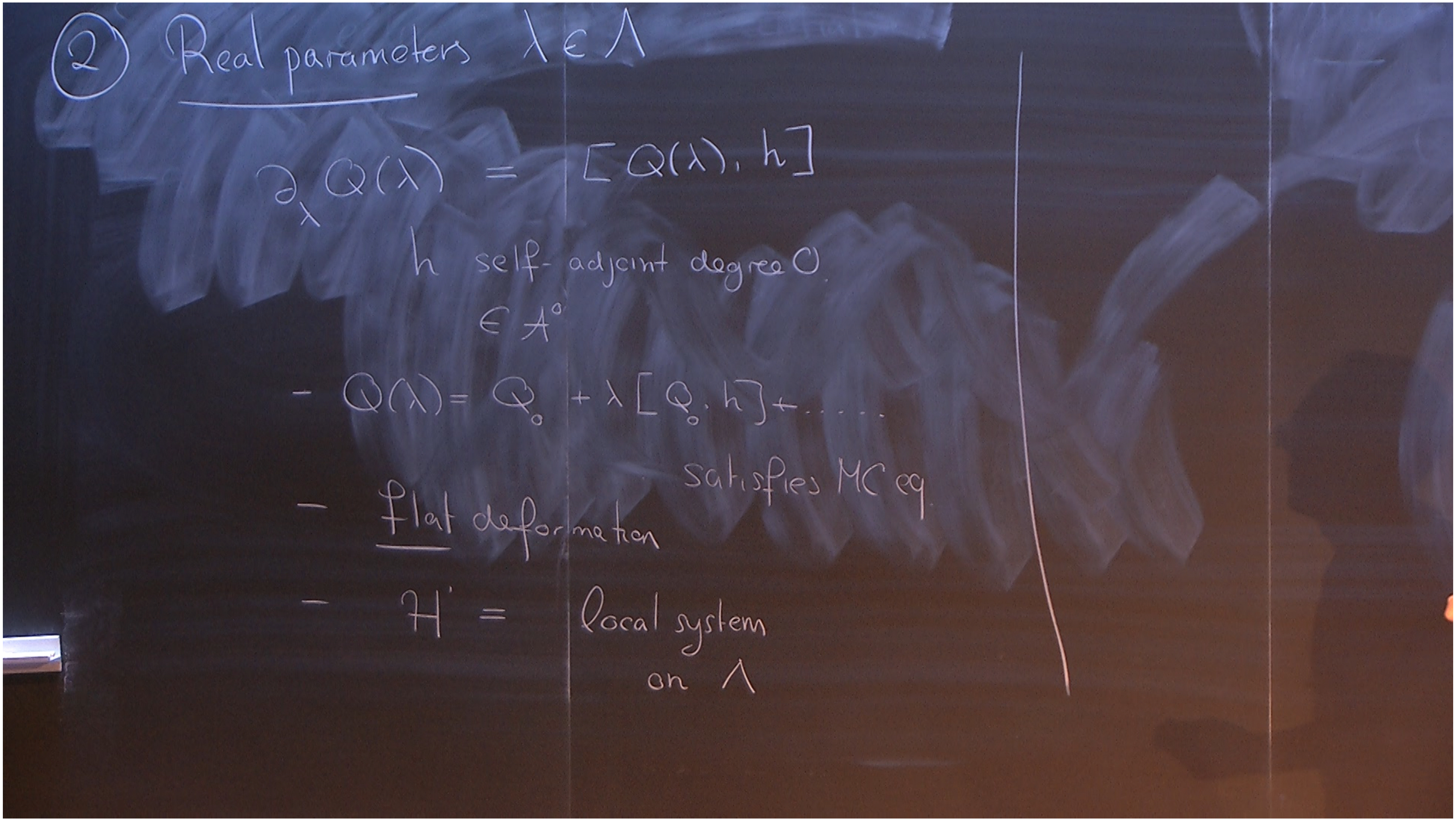

Ex: hermitian (X, E)
Q = ∂̄_E
Deform c.s of E
∂̄_E ⟶ ∂̄_E + ā
∂̄_E ā + ā ∧ ā = 0
(E, J super
- potentials)
Ex: E promote to a
Z-graded
E = ⊕_{p∈Z} E^p
bundles
on U

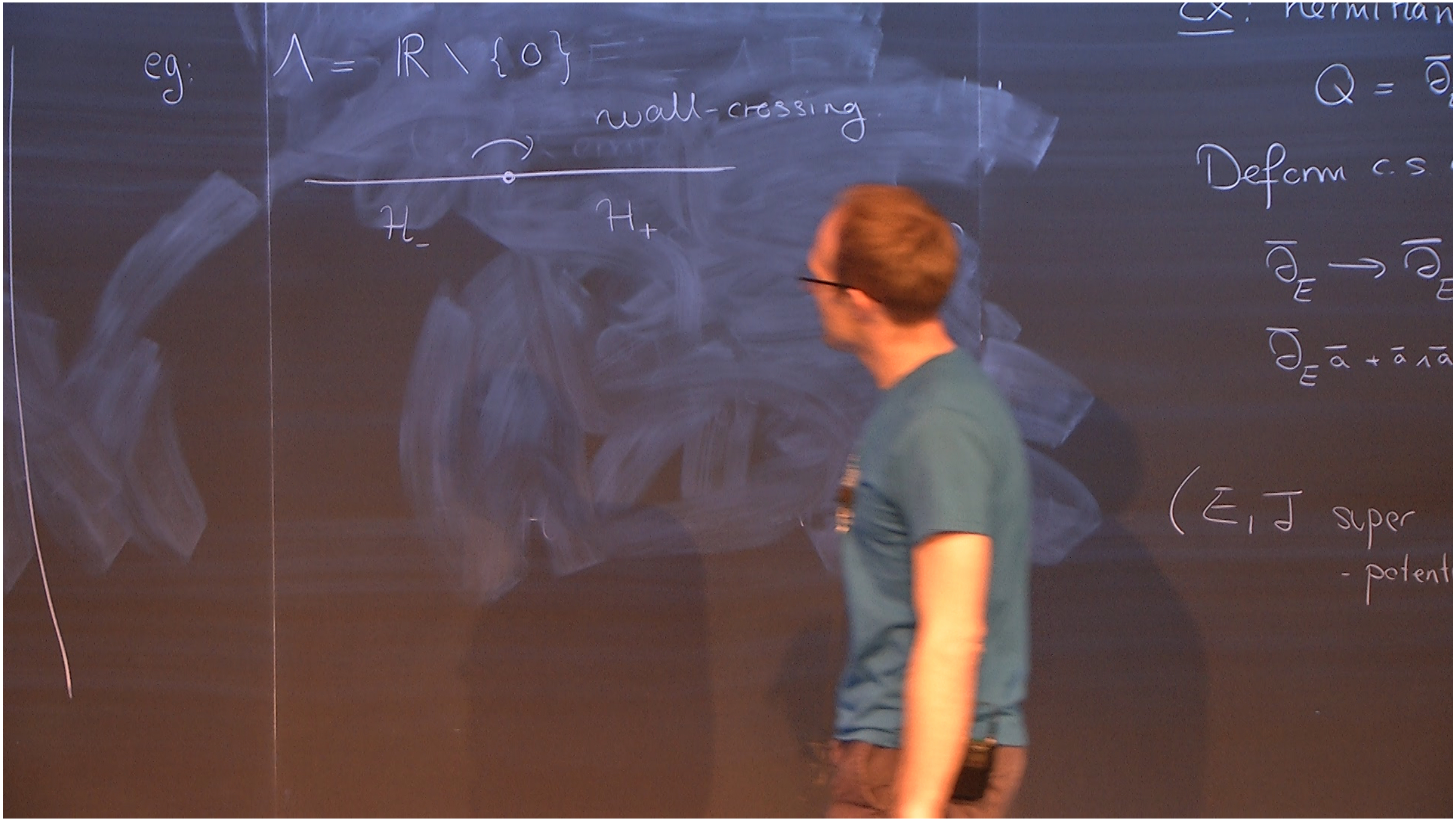
(2) Real parameters $\lambda \in \Lambda$

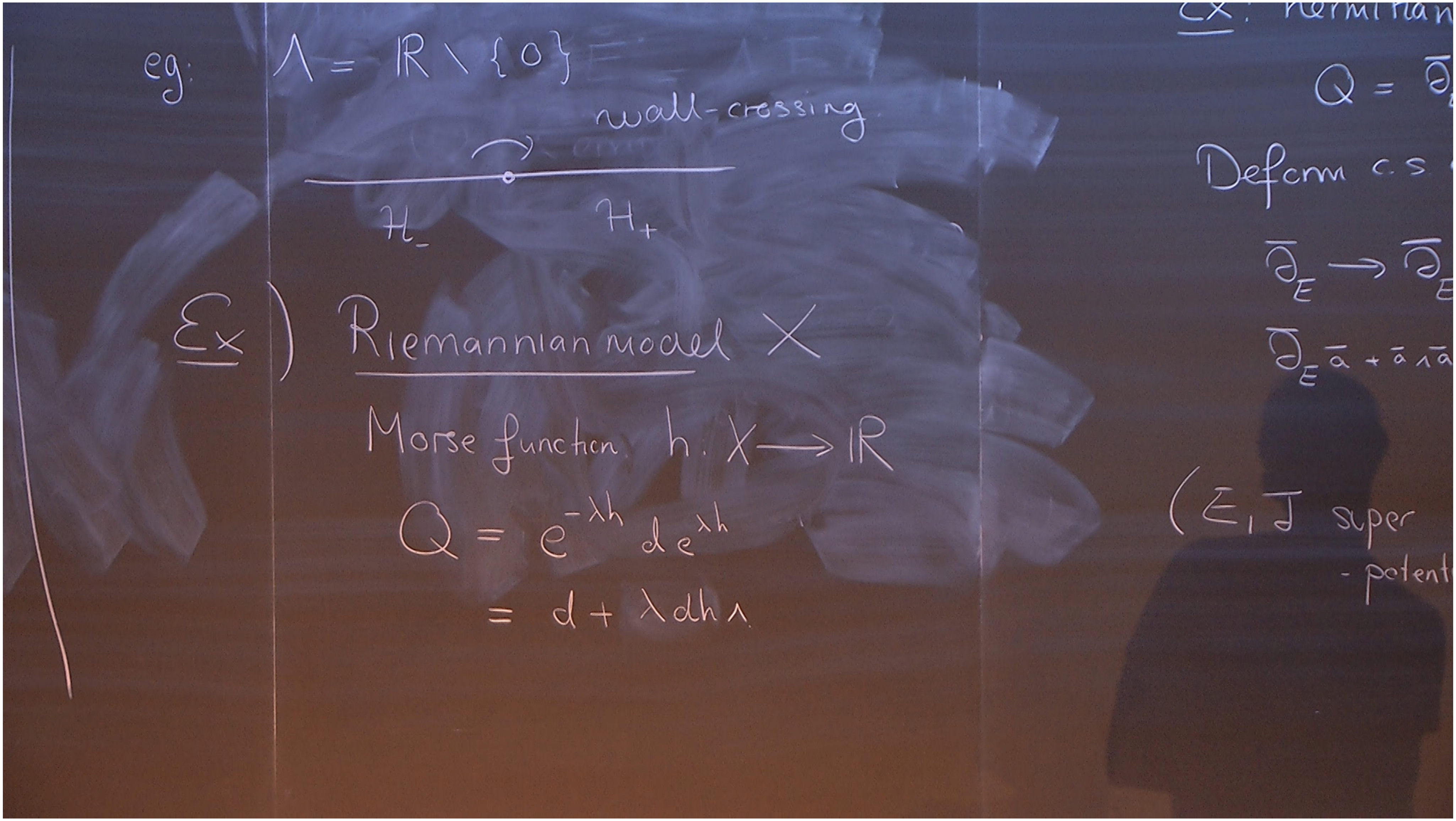
$$\partial_\lambda Q(\lambda) = [Q(\lambda), h]$$

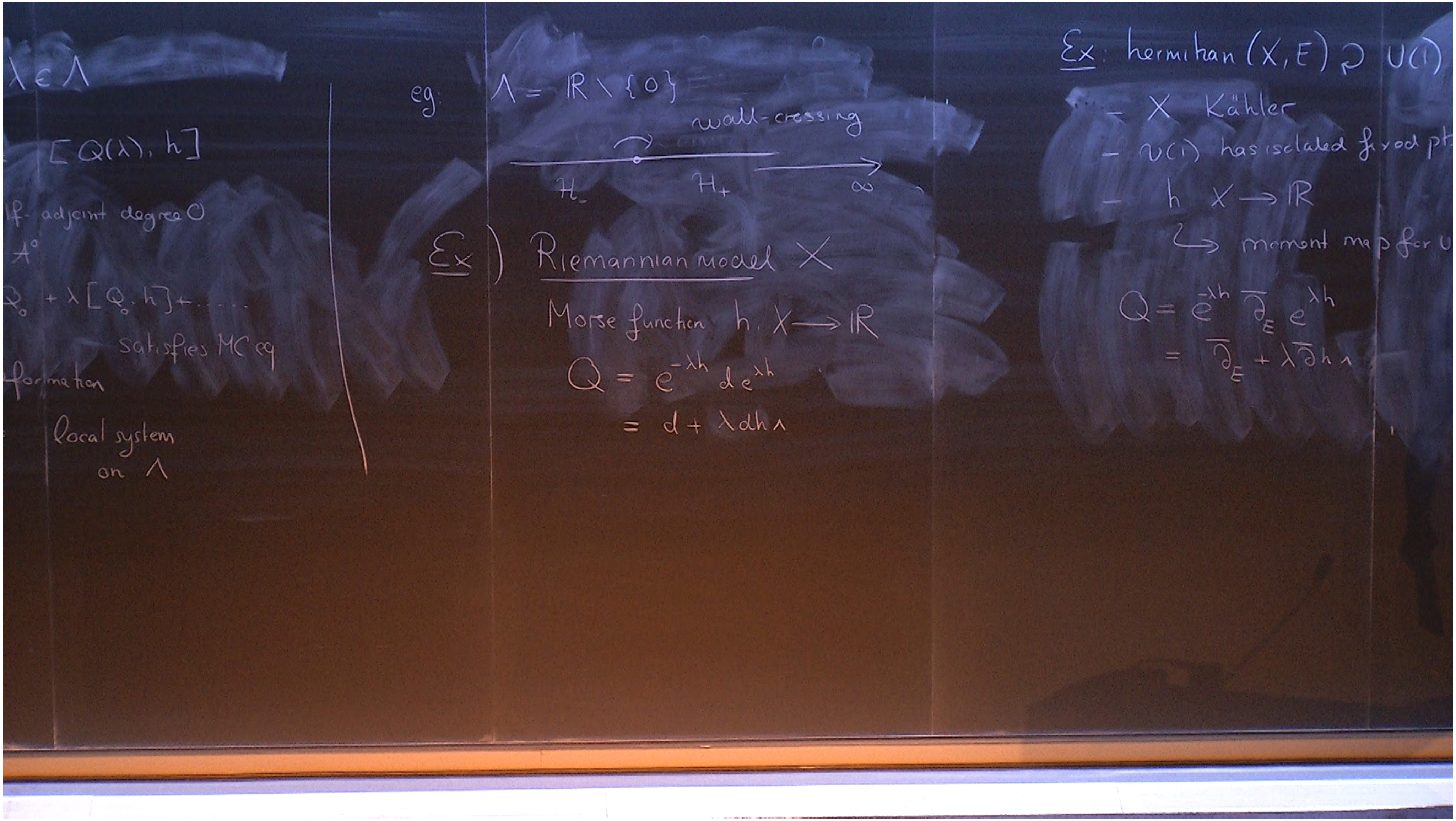
h self-adjoint degree 0.

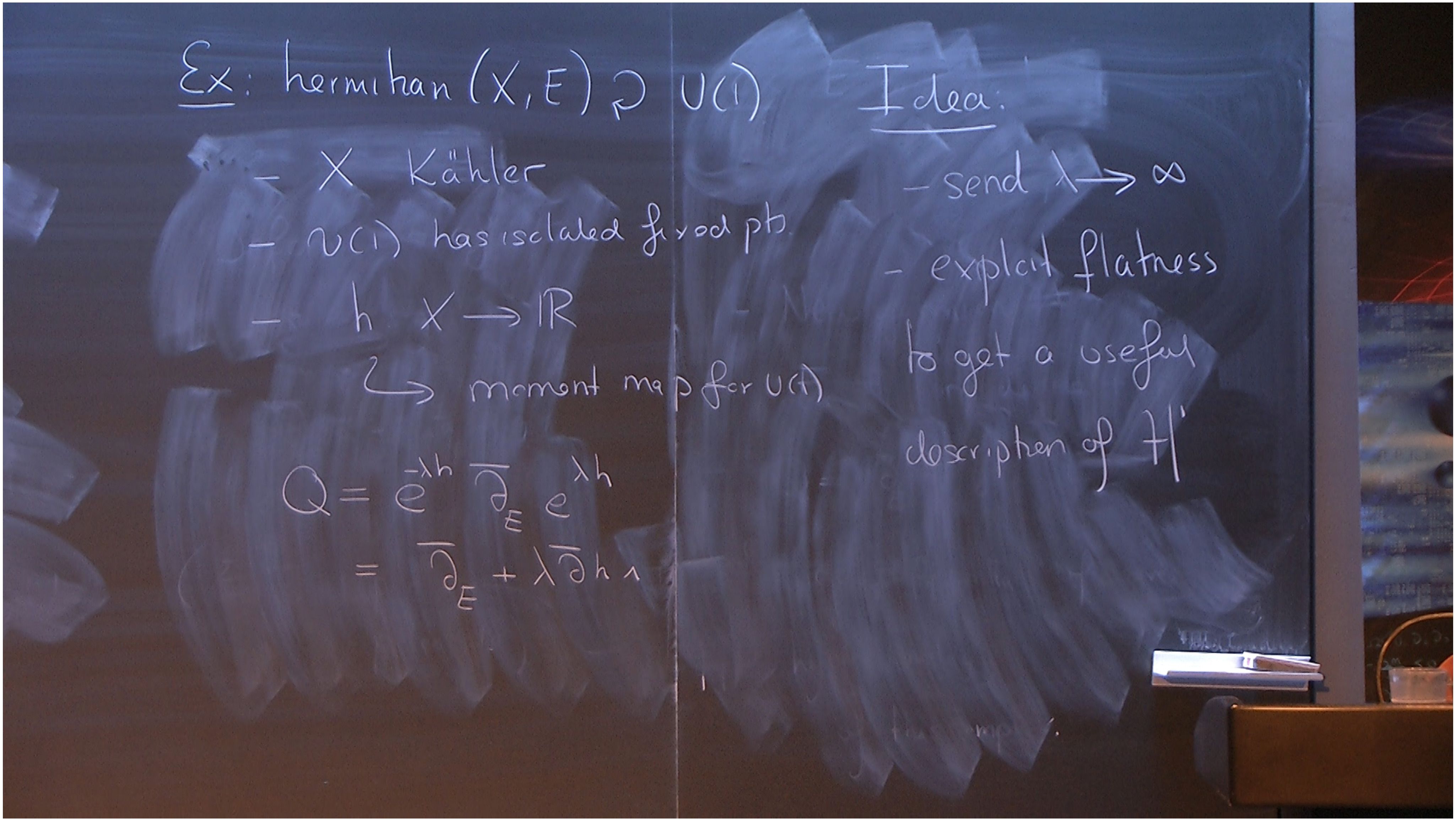
$\in \mathcal{A}^0$

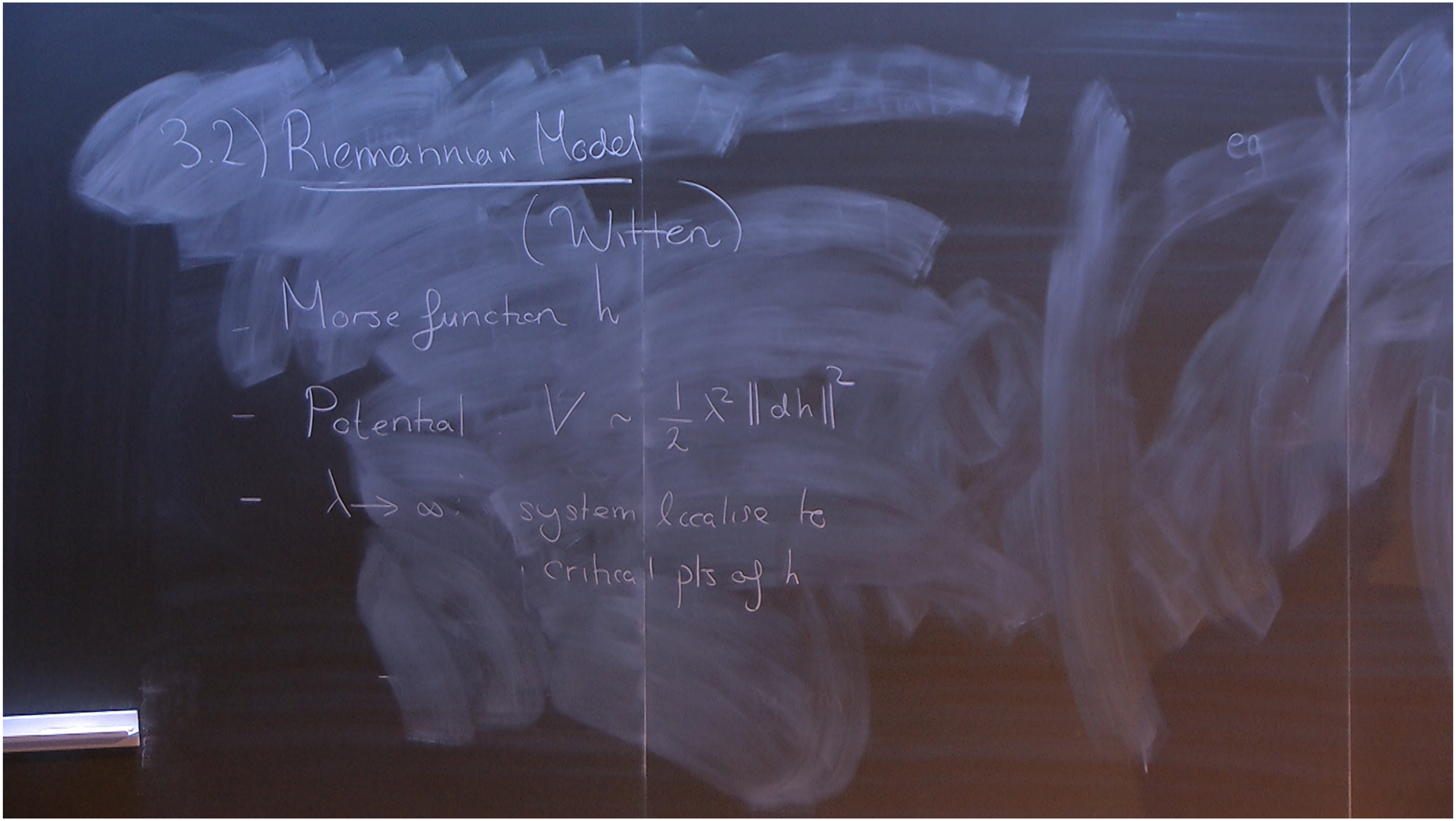

② Real parameters $\lambda \in \Lambda$
$\partial_\lambda Q(\lambda) = [Q(\lambda), h]$
h self-adjoint degree 0
$\in A^0$
- $Q(\lambda) = Q_0 + \lambda [Q_0, h] + \ldots$
satisfies MC eq.
- flat deformation
- $H' =$ local system
on Λ

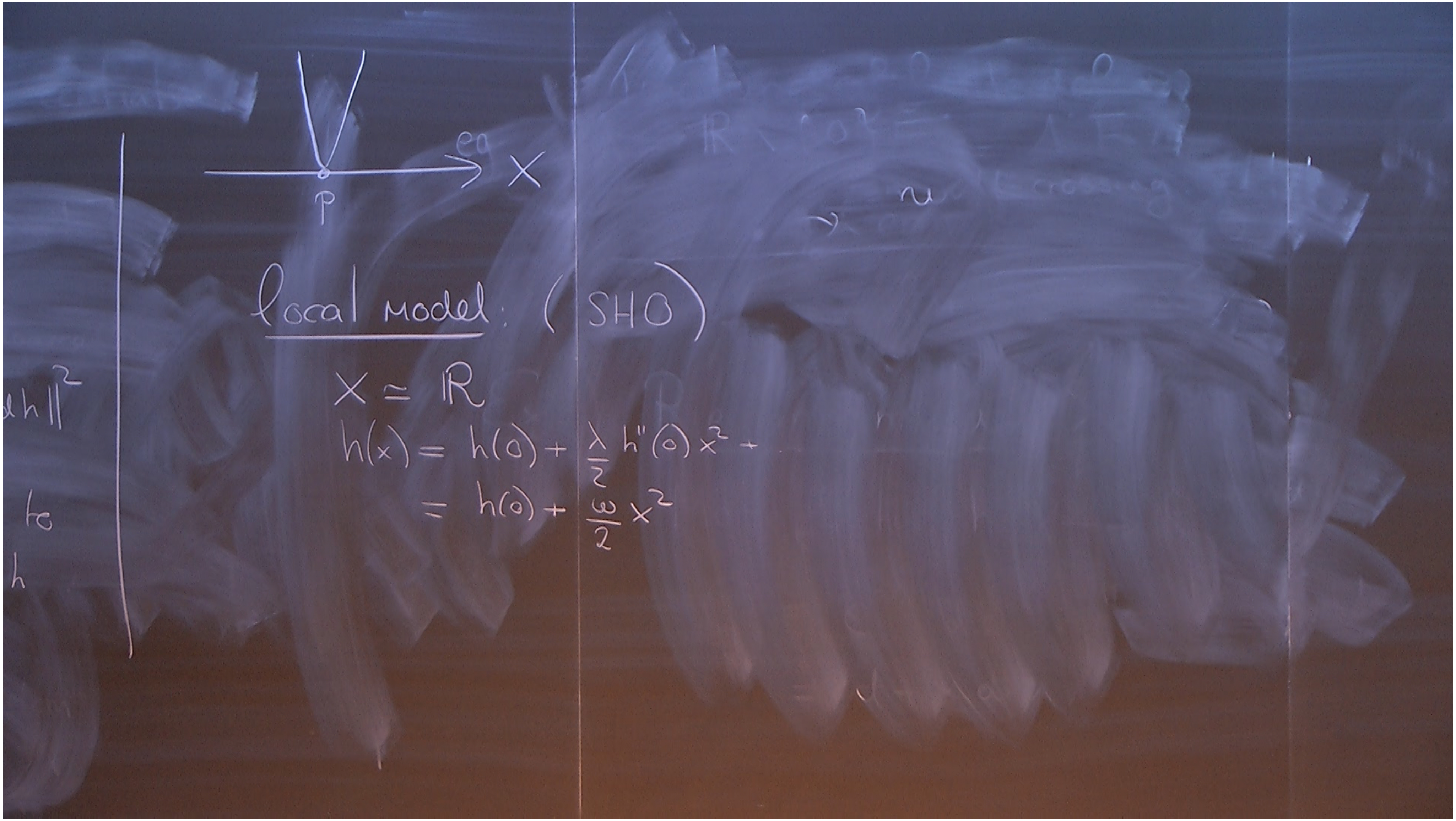

eg: Λ = ℝ \ {0}
wall-crossing
H₋
H₊
Ex: hermitian
Q = ∂̄
Deform c.s.
(E, J super
- potent

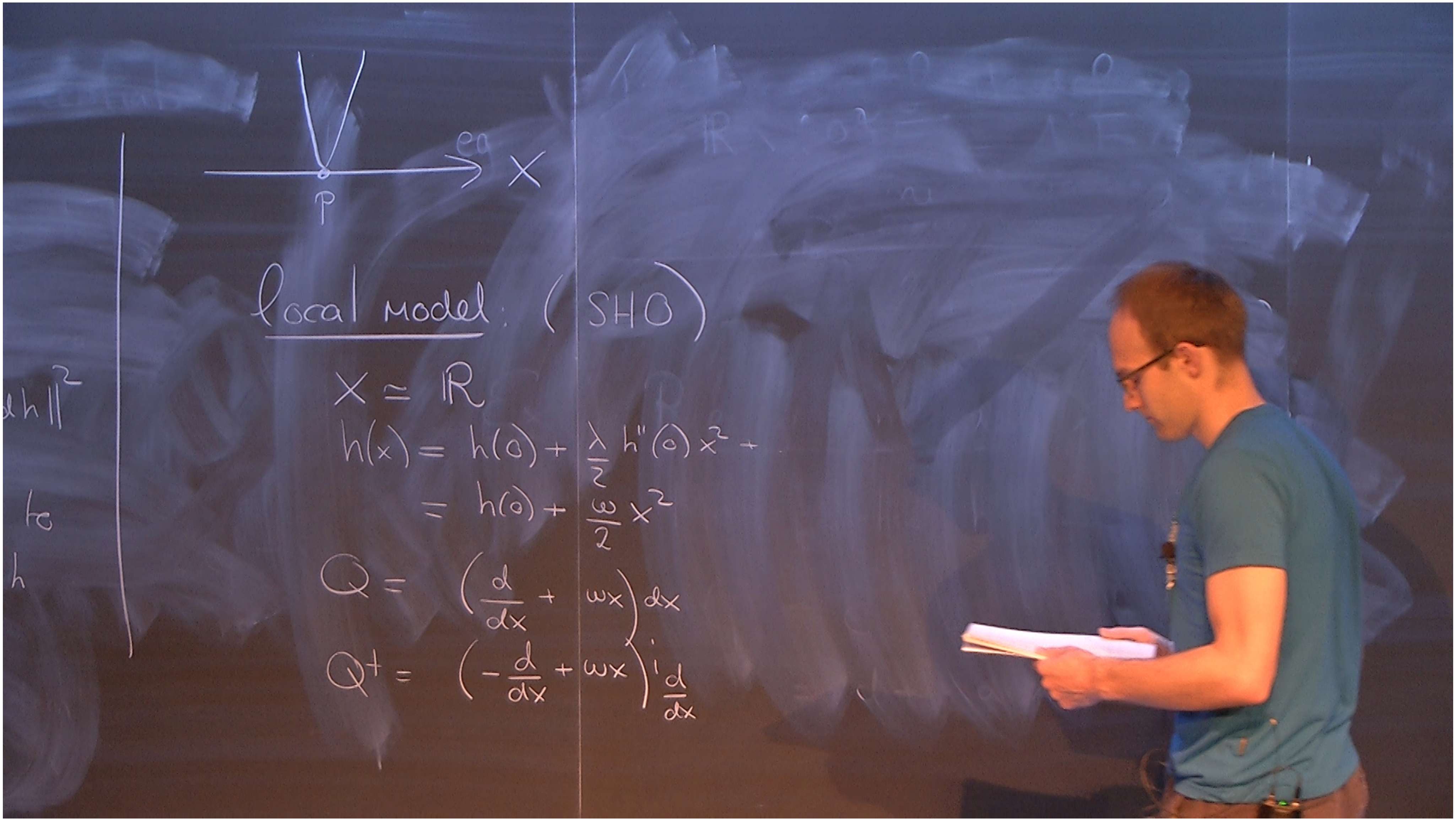

eg: $\Lambda = \mathbb{R} \setminus \{0\}$
wall-crossing
$\mathcal{H}_-$
$\mathcal{H}_+$
Ex) Riemannian model X
Morse function $h: X \longrightarrow \mathbb{R}$
$Q = e^{-\lambda h} d e^{\lambda h}$
$= d + \lambda dh \wedge$
Ex: hermitian
$Q = \bar{\partial}$
Deform c.s
$\bar{\partial}_E \longrightarrow \bar{\partial}_E$
$\bar{\partial}_E \bar{a} + \bar{a} \wedge \bar{a}$
$(E, J$ super
- potent

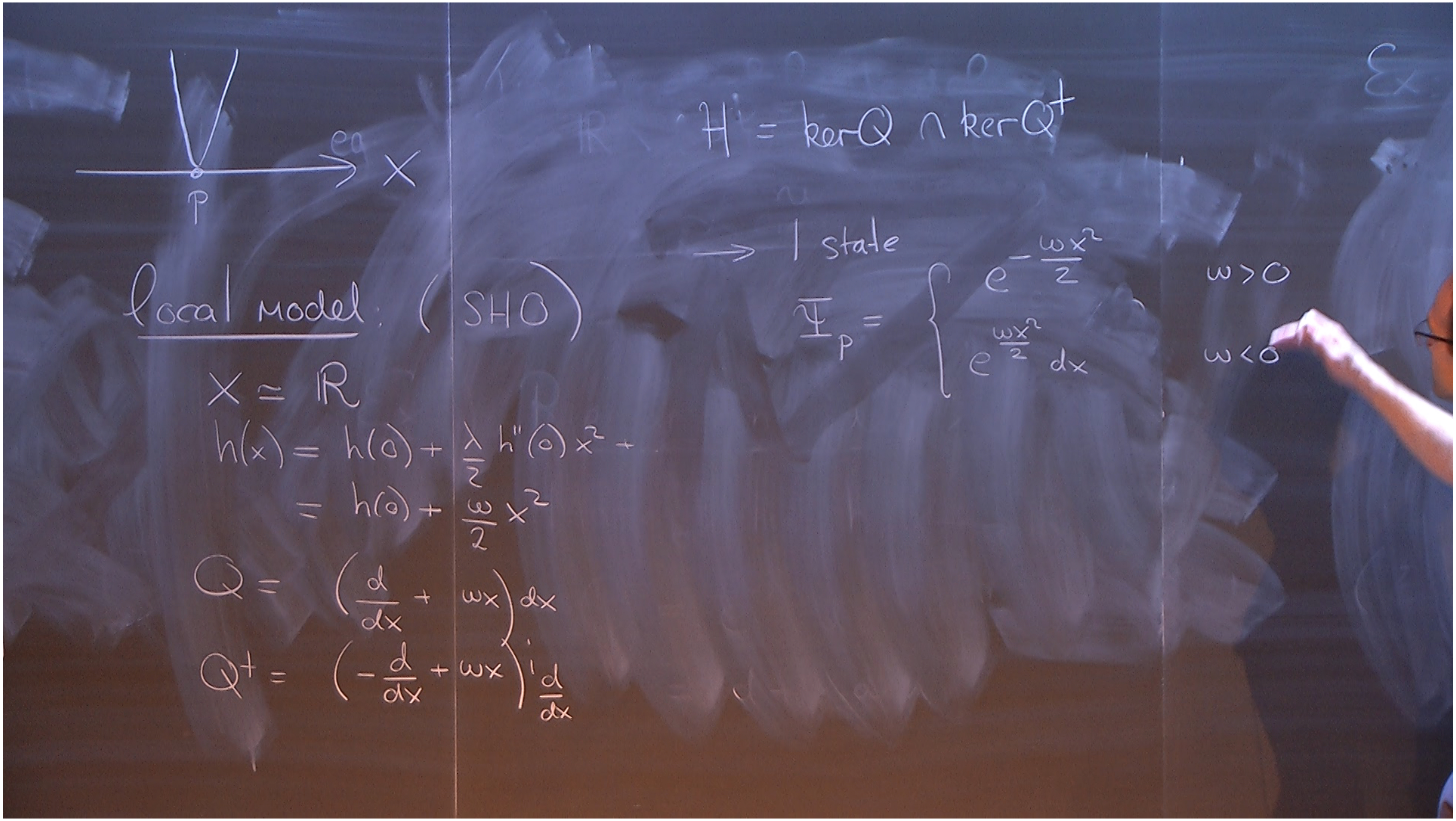

λ ∈ Λ
[Q(λ), h]
lf- adjoint degree 0
A⁰
Q₀ + λ[Q₀, h] + ...
satisfies MC eq
formation
local system on Λ
eg: Λ = ℝ \ {0}
wall-crossing
H₋
H₊
∞
Ex) Riemannian model X
Morse function h: X → ℝ
Q = e^{-λh} d e^{λh}
= d + λ dh ∧
Ex: hermitian (X, E) ⟲ U(1)
– X Kähler
– U(1) has isolated fixed pts
– h: X → ℝ
↪ moment map for U
Q = e^{-λh} ∂̄_E e^{λh}
= ∂̄_E + λ ∂̄h ∧


Ex: hermitian (X, E) ↻ U(1)
– X Kähler
– U(1) has isolated fixed pts
– h X → ℝ
↳ moment map for U(1)
Q = e^{-λh} ∂̄_E e^{λh}
= ∂̄_E + λ ∂̄h ∧
Idea:
– send λ → ∞
– exploit flatness
to get a useful
description of H'

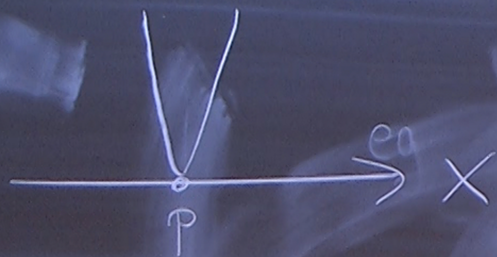

3.2) Riemannian Model
(Witten)
- Morse function h
- Potential $V \sim \frac{1}{2}\lambda^2 \|dh\|^2$
- $\lambda \to \infty$: system localise to critical pts of h
eg


X
p
local model: (SHO)
X ≃ R
h(x) = h(0) + λ/2 h''(0) x² +
= h(0) + ω/2 x²


X
p
local model: (SHO)
X ≃ R
h(x) = h(0) + λ/2 h''(0) x² + ...
= h(0) + ω/2 x²
Q = (d/dx + ωx) dx
Q† = (−d/dx + ωx) i d/dx



X
P
local model: (SHO)
$X \simeq \mathbb{R}$
$h(x) = h(0) + \frac{\lambda}{2} h''(0) x^2 + \cdots$
$= h(0) + \frac{\omega}{2} x^2$
$Q = \left(\frac{d}{dx} + \omega x\right) dx$
$Q^+ = \left(-\frac{d}{dx} + \omega x\right) \iota_{\frac{d}{dx}}$
$\mathcal{H}^i = \ker Q \cap \ker Q^+$
→ 1 state
$\Psi_p = \begin{cases} e^{-\frac{\omega x^2}{2}} & \omega > 0 \\ e^{\frac{\omega x^2}{2}} dx & \omega < 0 \end{cases}$
Ex



$$\mathcal{H} = \ker Q \cap \ker Q^{\dagger}$$

$\longrightarrow$ 1 state

$$\Psi_p = \begin{cases} e^{-\frac{\omega x^2}{2}} & \omega > 0 \\ e^{\frac{\omega x^2}{2}}\, dx & \omega < 0 \end{cases}$$

<u>local model</u>: (SHO)

$$X \simeq \mathbb{R}$$

$$h(x) = h(0) + \frac{\lambda}{2} h''(0)\, x^2 + \cdots$$

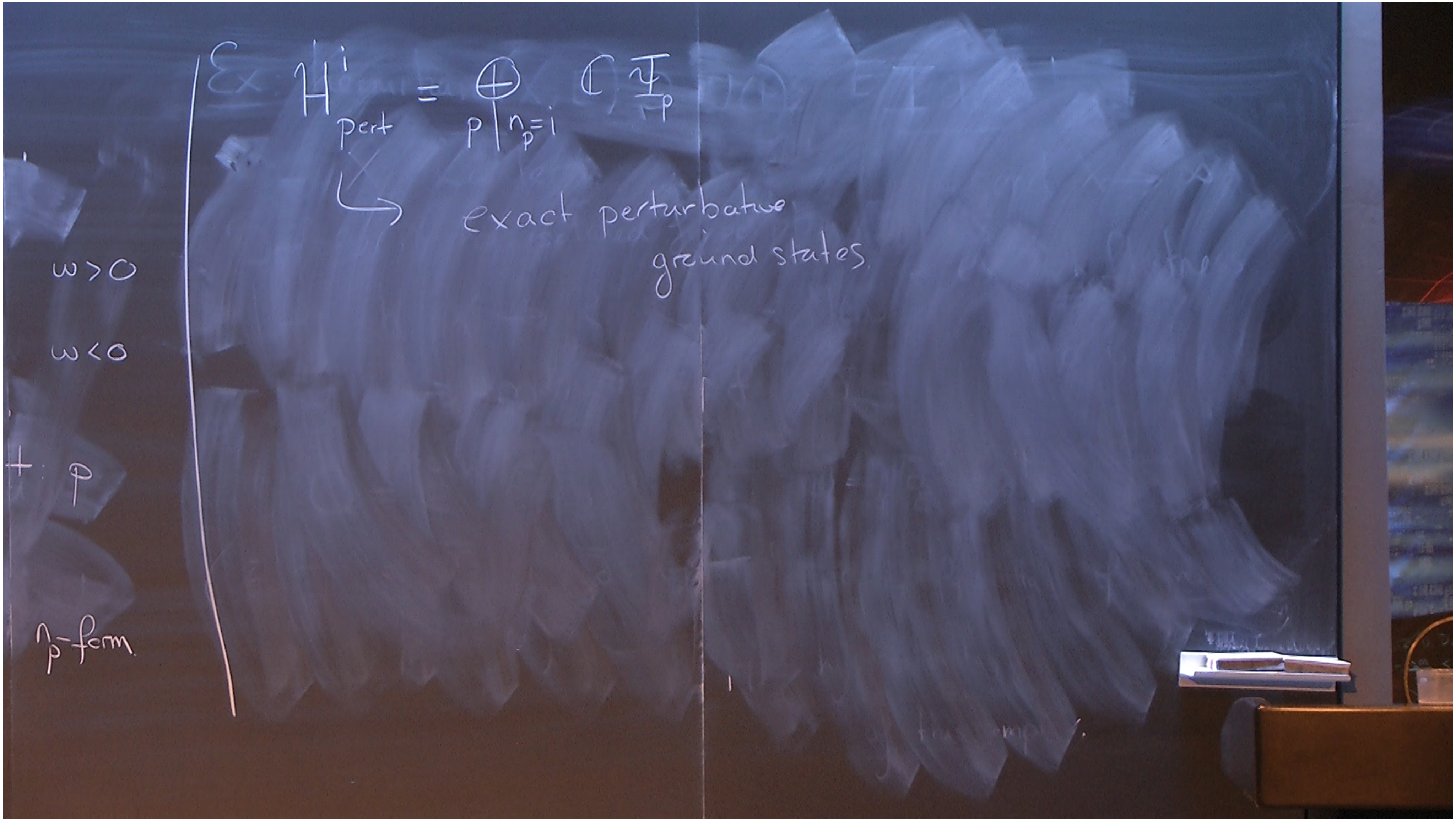
$$= h(0) + \frac{\omega}{2} x^2$$

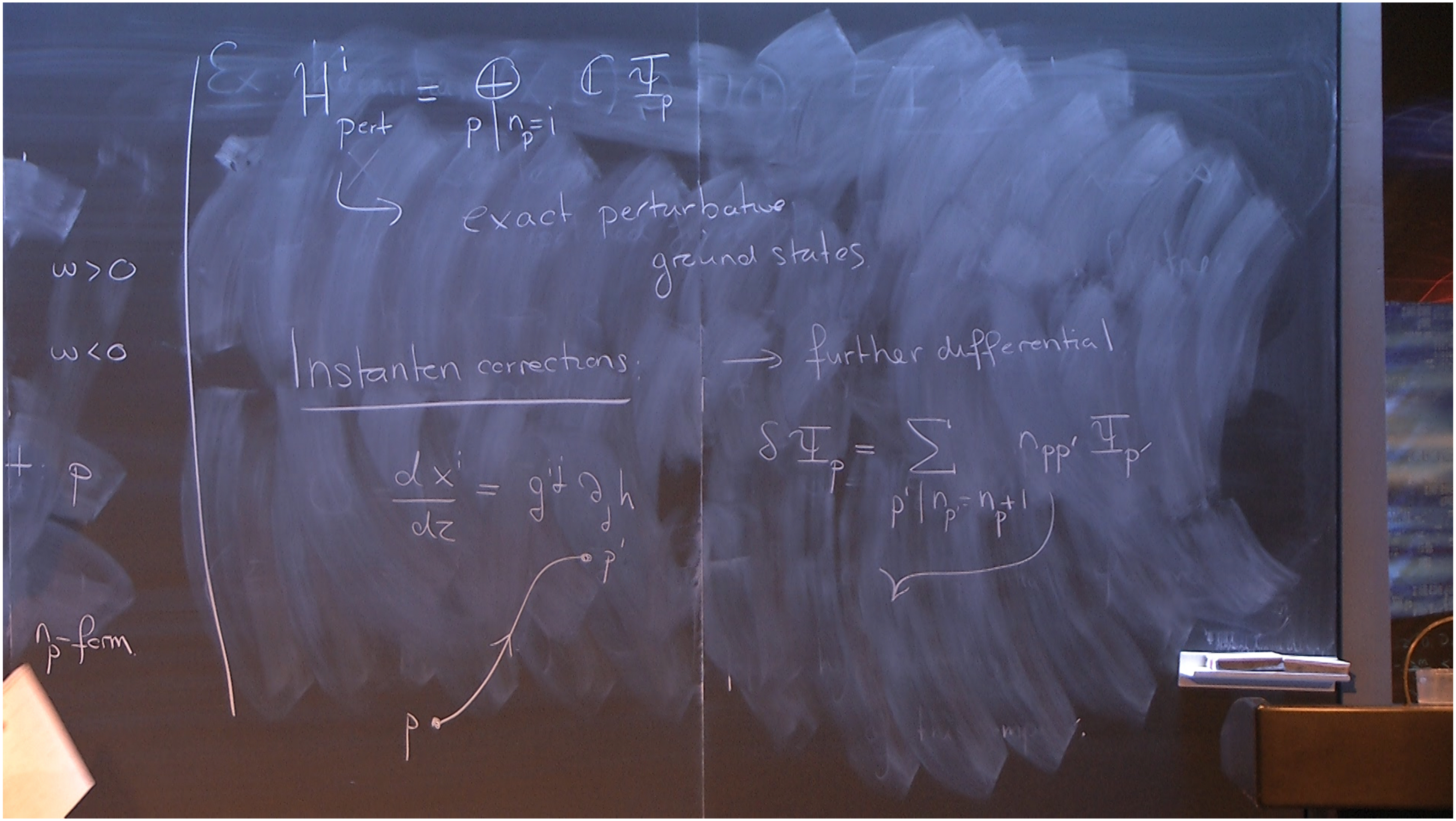
$$Q = \left(\frac{d}{dx} + \omega x\right) dx$$

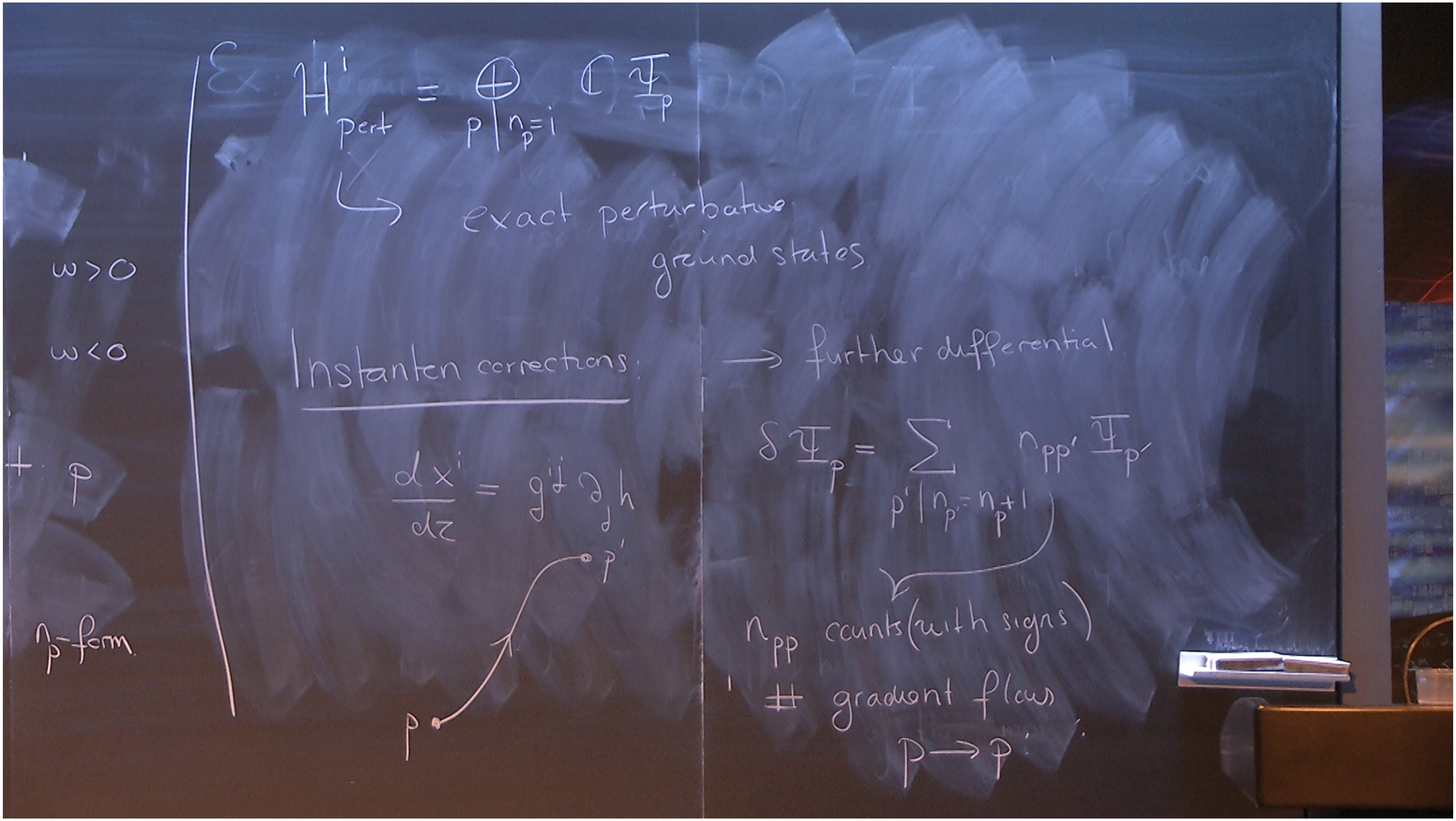
$$Q^{\dagger} = \left(-\frac{d}{dx} + \omega x\right) \iota_{\frac{d}{dx}}$$

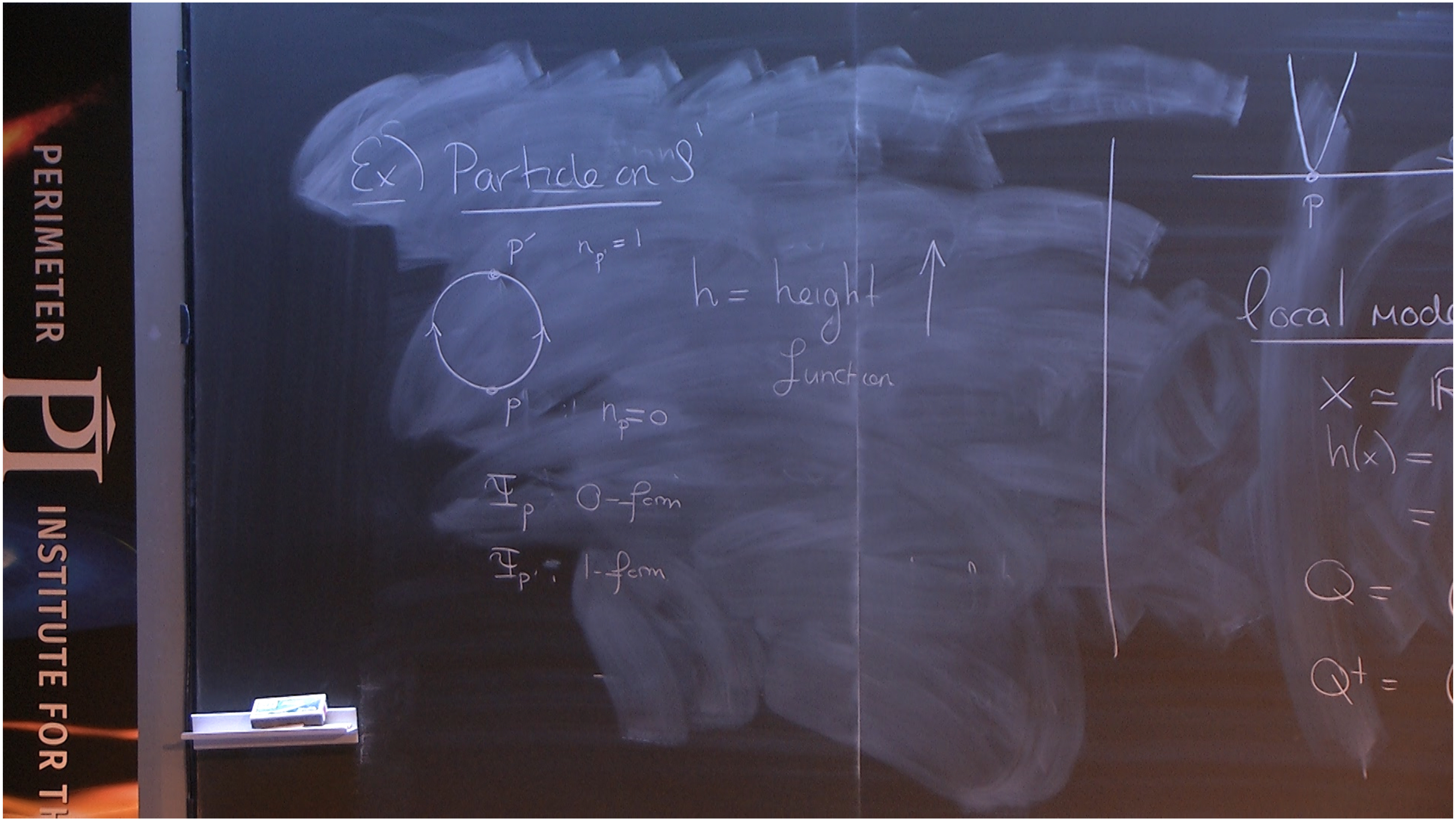
At general critical pt: p

- Morse index n_p
- 1 state Ψ_p: n_p-form


Ex: $\mathcal{H}^i_{pert} = \bigoplus_{p \,|\, n_p = i} \mathbb{C}\, \Psi_p$
exact perturbative ground states
$\omega > 0$
$\omega < 0$
$+$: p
n_p-form


Ex: $H^i_{pert} = \bigoplus_{p | n_p = i} \mathbb{C}\, \Psi_p$
exact perturbative ground states
Instanton corrections:
→ further differential
$\delta \Psi_p = \sum_{p' | n_{p'} = n_p + 1} n_{pp'} \Psi_{p'}$
$\frac{dx^i}{dz} = g^{ij} \partial_j h$
p
p'
ω > 0
ω < 0
n_p-form


Ex: H^i_pert = ⊕_{p | n_p = i} ℂ Ψ_p
exact perturbative ground states
ω > 0
ω < 0
p
n_p-form
Instanton corrections:
→ further differential
δΨ_p = Σ_{p' | n_{p'} = n_p + 1} n_{pp'} Ψ_{p'}
dx^i/dz = g^{ij} ∂_j h
p
p'
n_pp counts (with signs)
gradient flows
p → p'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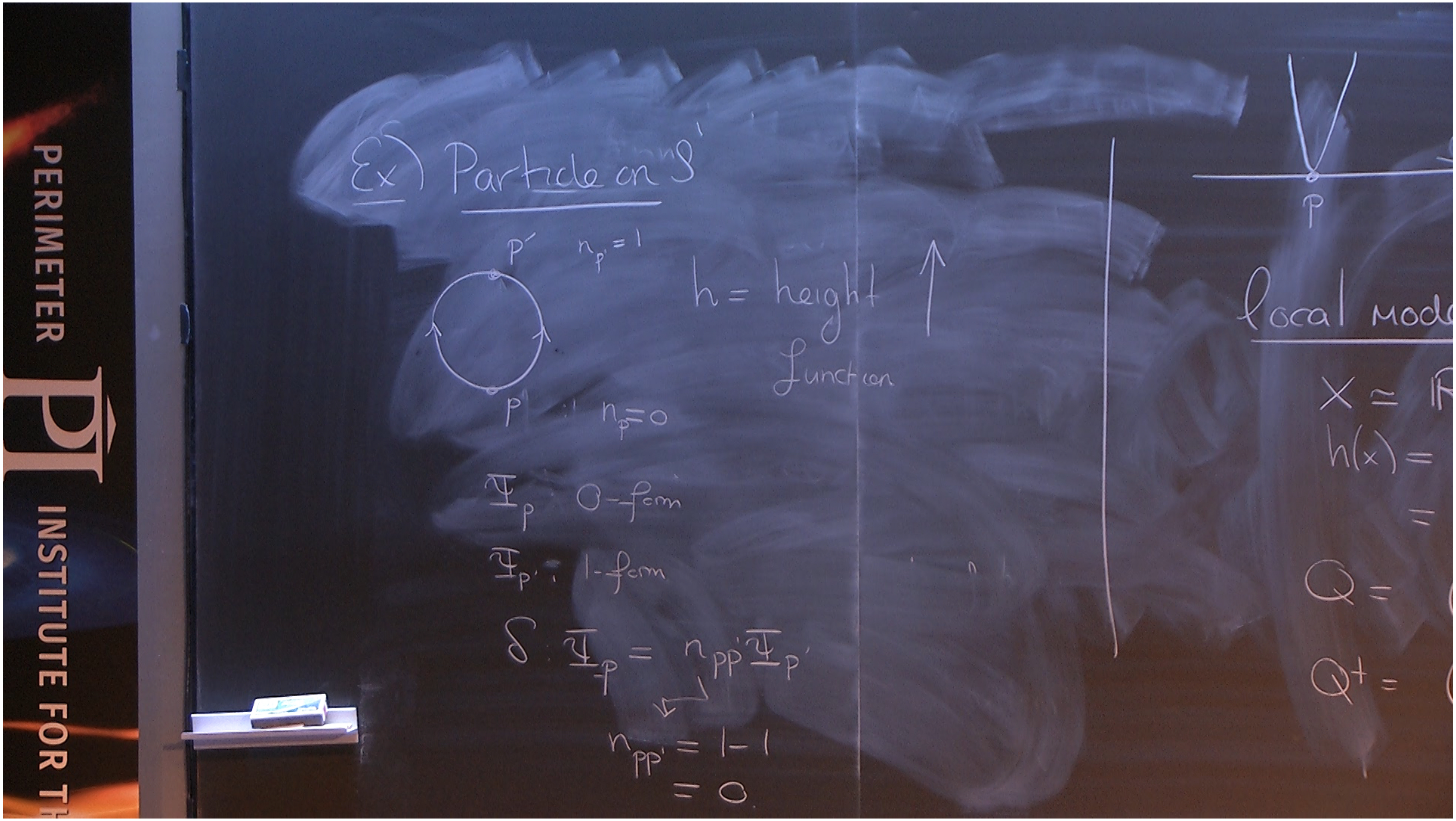
Ex) Particle on S1
p'
np' = 1
h = height
function
p
np = 0
Ψp : 0-form
Ψp' : 1-form
p
local mode
X ≅ R
h(x) =
Q =
Q† =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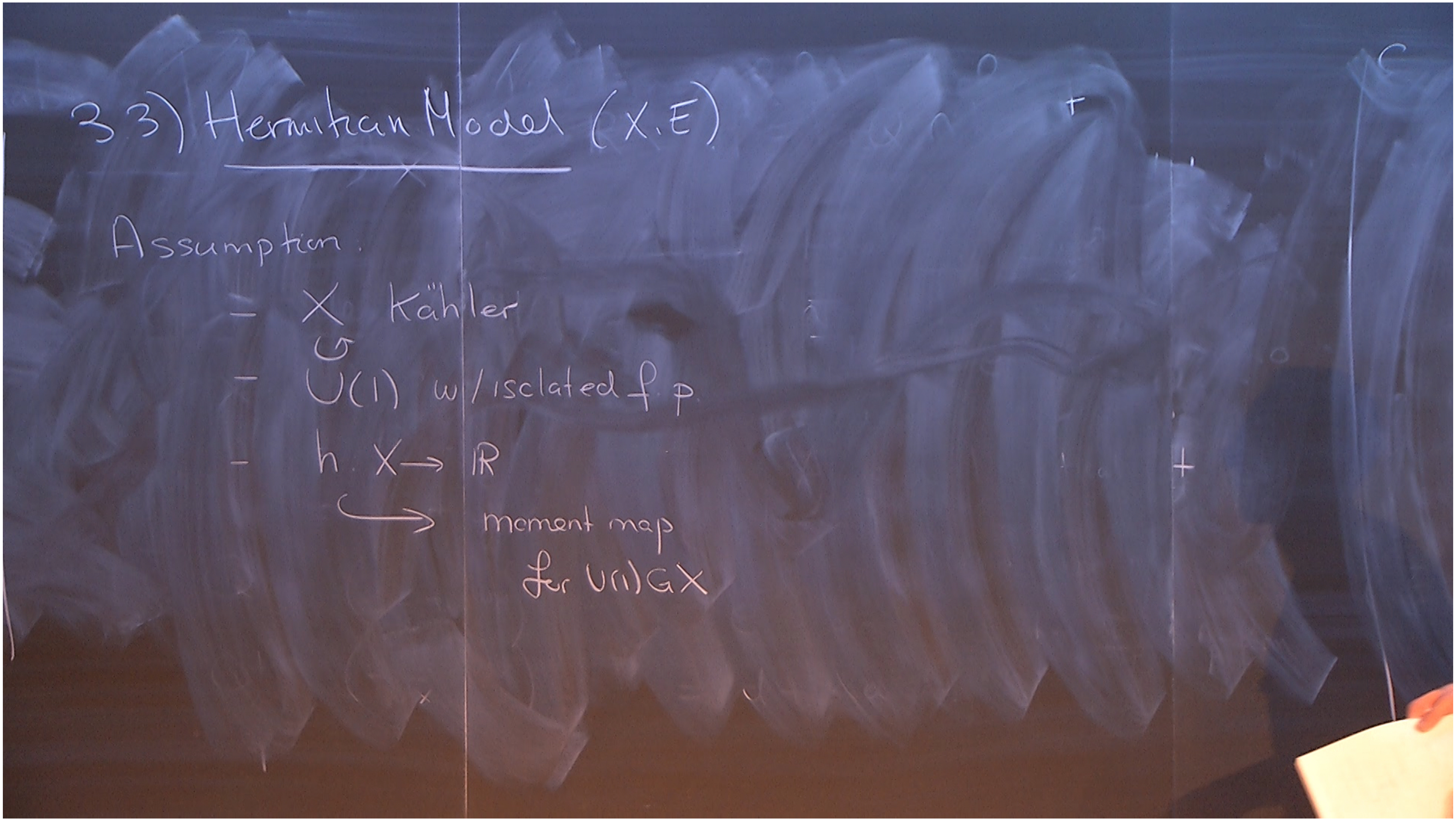
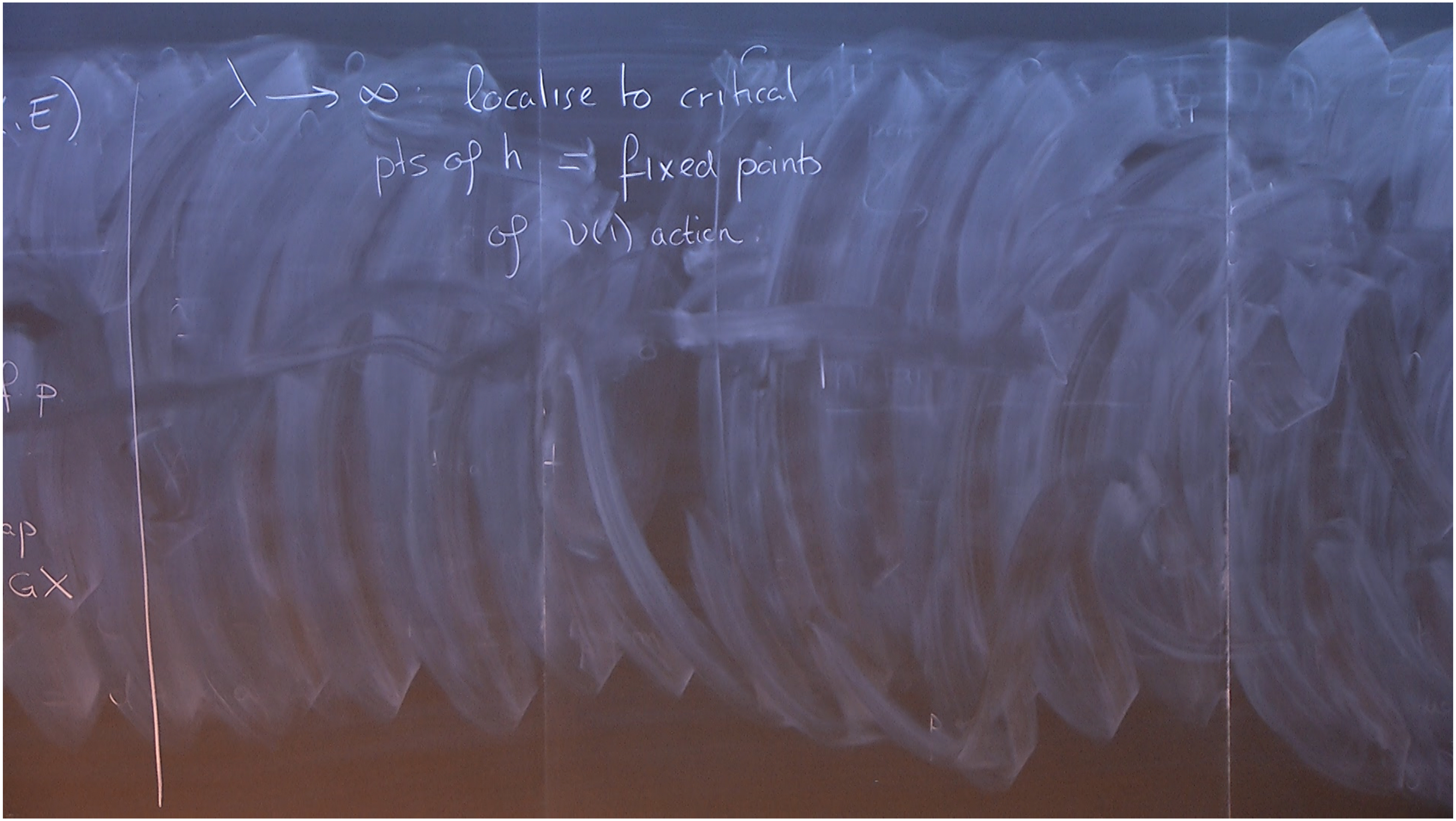
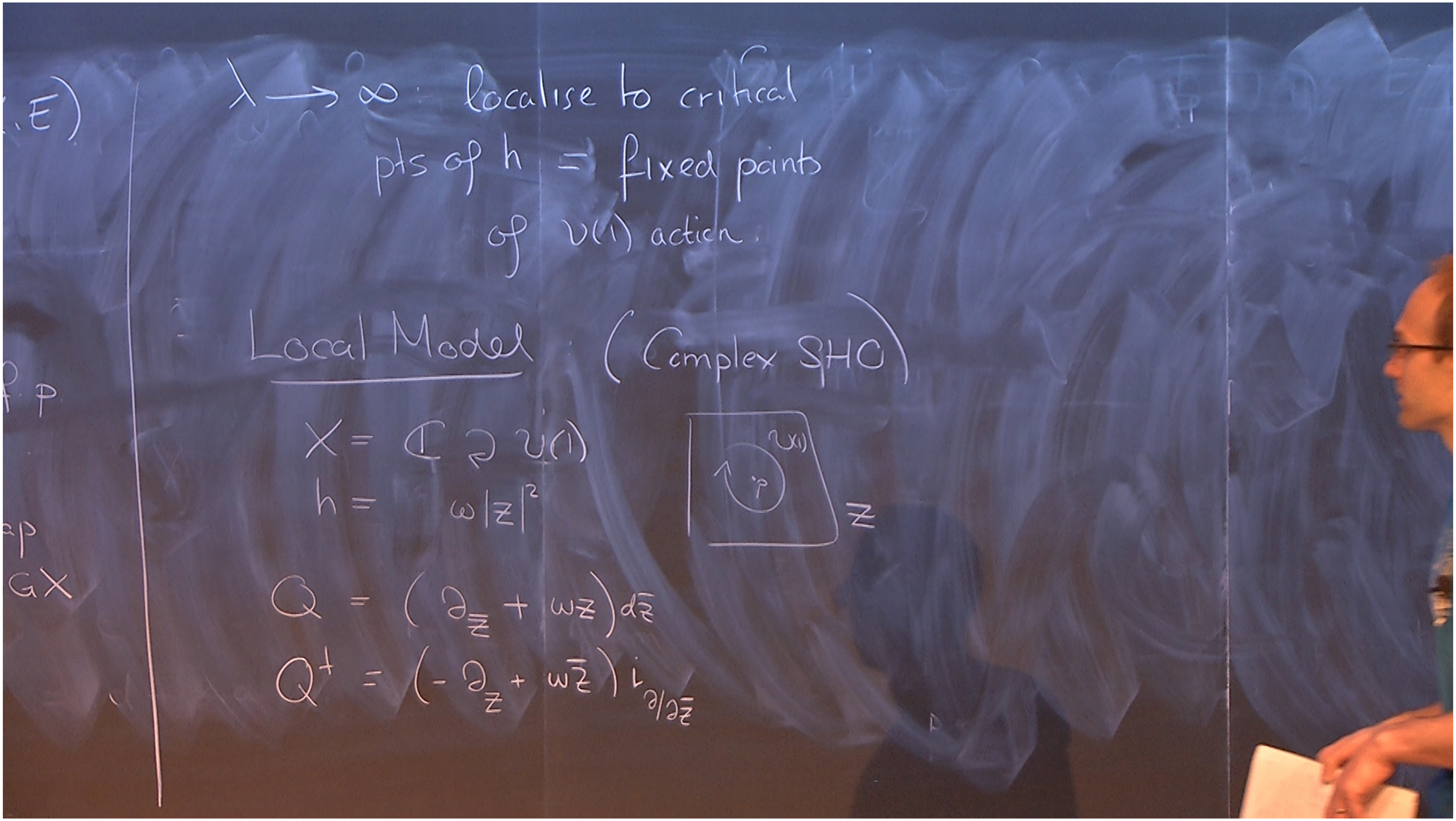
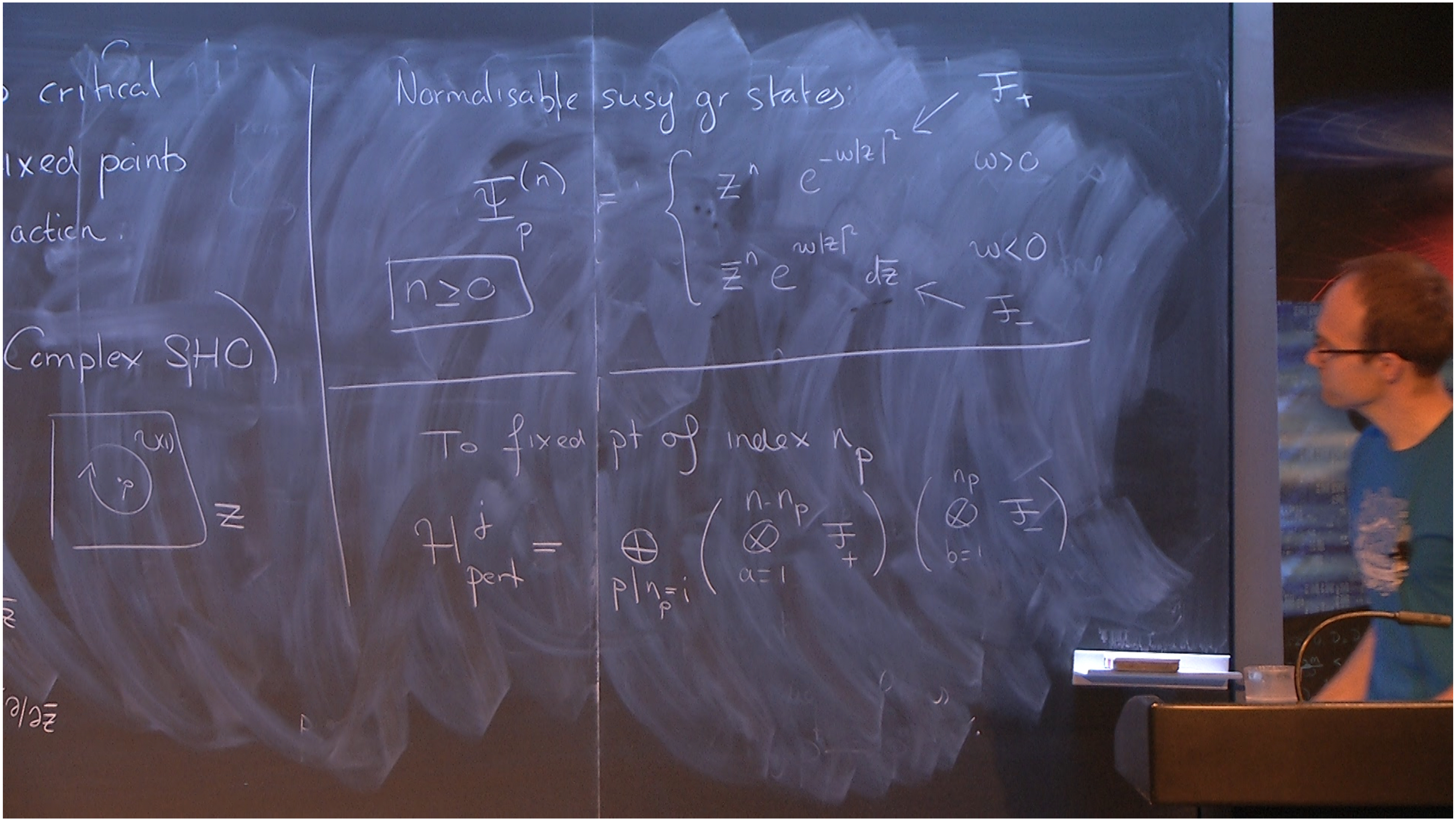
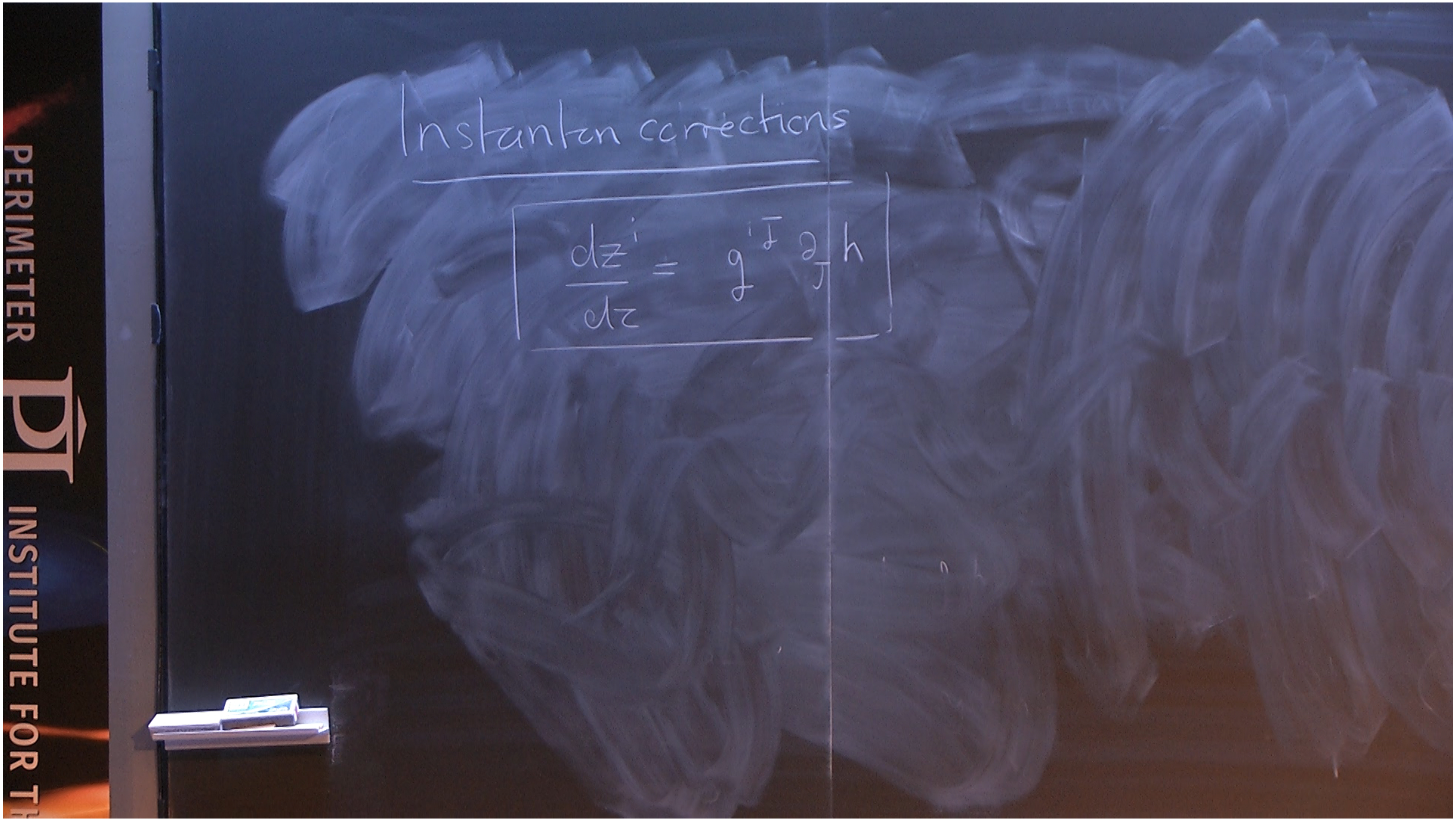
Ex) Particle on S^1
p' n_p' = 1
h = height function
p n_p = 0
Ψ_p : 0-form
Ψ_p' : 1-form
δ : Ψ_p = n_pp' Ψ_p'
n_pp' = 1-1
= 0
p
local model
X ≃ R
h(x) =
=
Q =
Q† =


3.3) Hermitian Model (X, E)
Assumption:
- X Kähler
- U(1) w/ isolated f.p.
- h: X → ℝ
moment map
for U(1) ↻ X


λ → ∞ : localise to critical
pts of h = fixed points
of U(1) action.


$\lambda \to \infty$: localise to critical
pts of h = fixed points
of $U(1)$ action.
Local Model : (Complex SHO)
$X = \mathbb{C} \circlearrowleft U(1)$
$h = \omega |z|^2$
$U(1)$
p
z
$Q = (\partial_{\bar{z}} + \omega z) d\bar{z}$
$Q^\dagger = (-\partial_z + \omega \bar{z}) \iota_{\partial/\partial \bar{z}}$


critical
fixed points
action:
Complex SHO
Normalisable susy gr states:
n ≥ 0
ω>0
ω<0
To fixed pt of index n_p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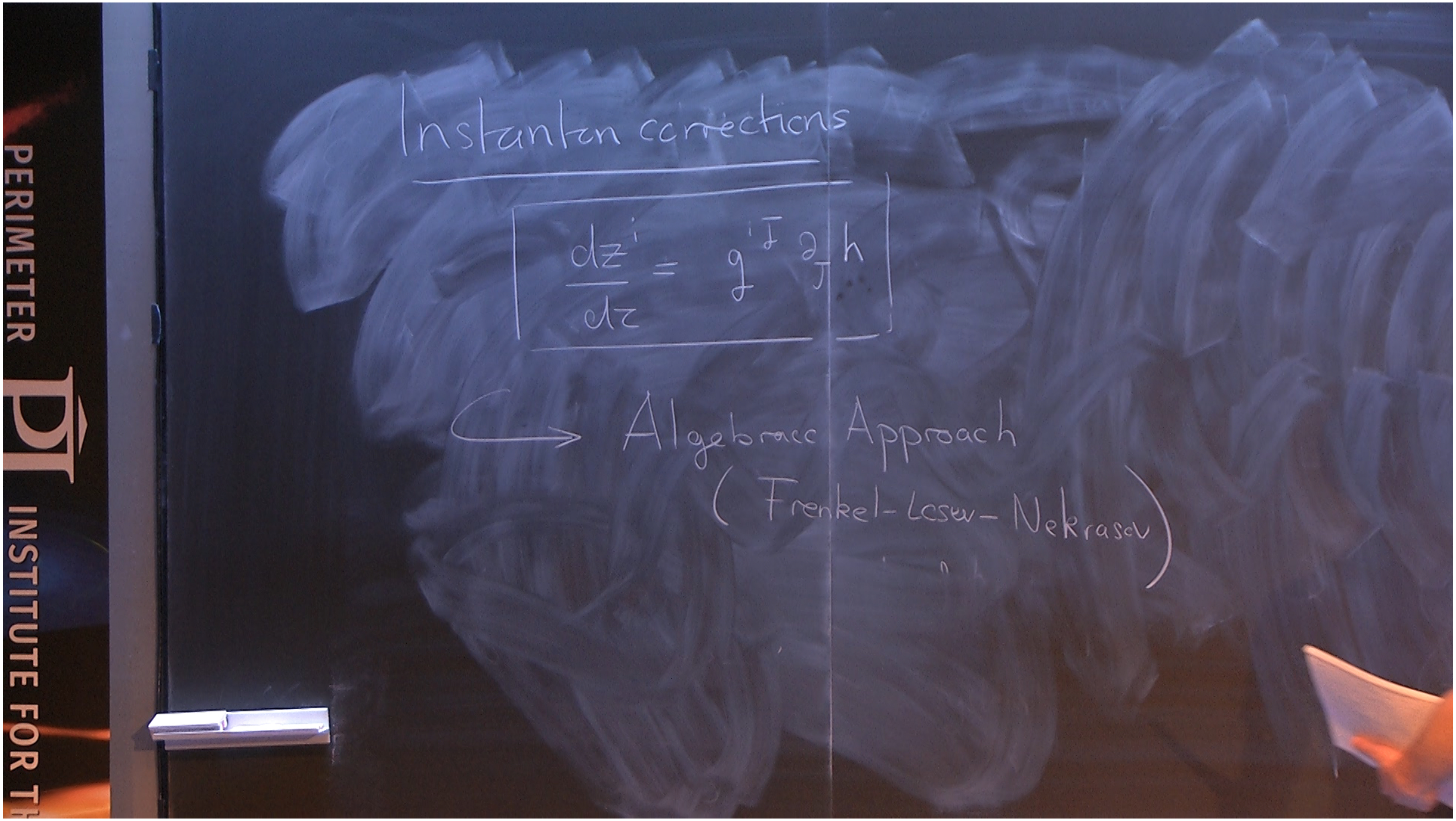
Instanton corrections
$\frac{dz^i}{d\tau} = g^{i\bar{J}} \partial_{\bar{J}} h$


Instanton corrections
$\frac{dz^i}{dz} = g^{i\bar{J}} \partial_{\bar{J}} h$
Algebraic Approach
(Frenkel-Losev-Nekrasov)
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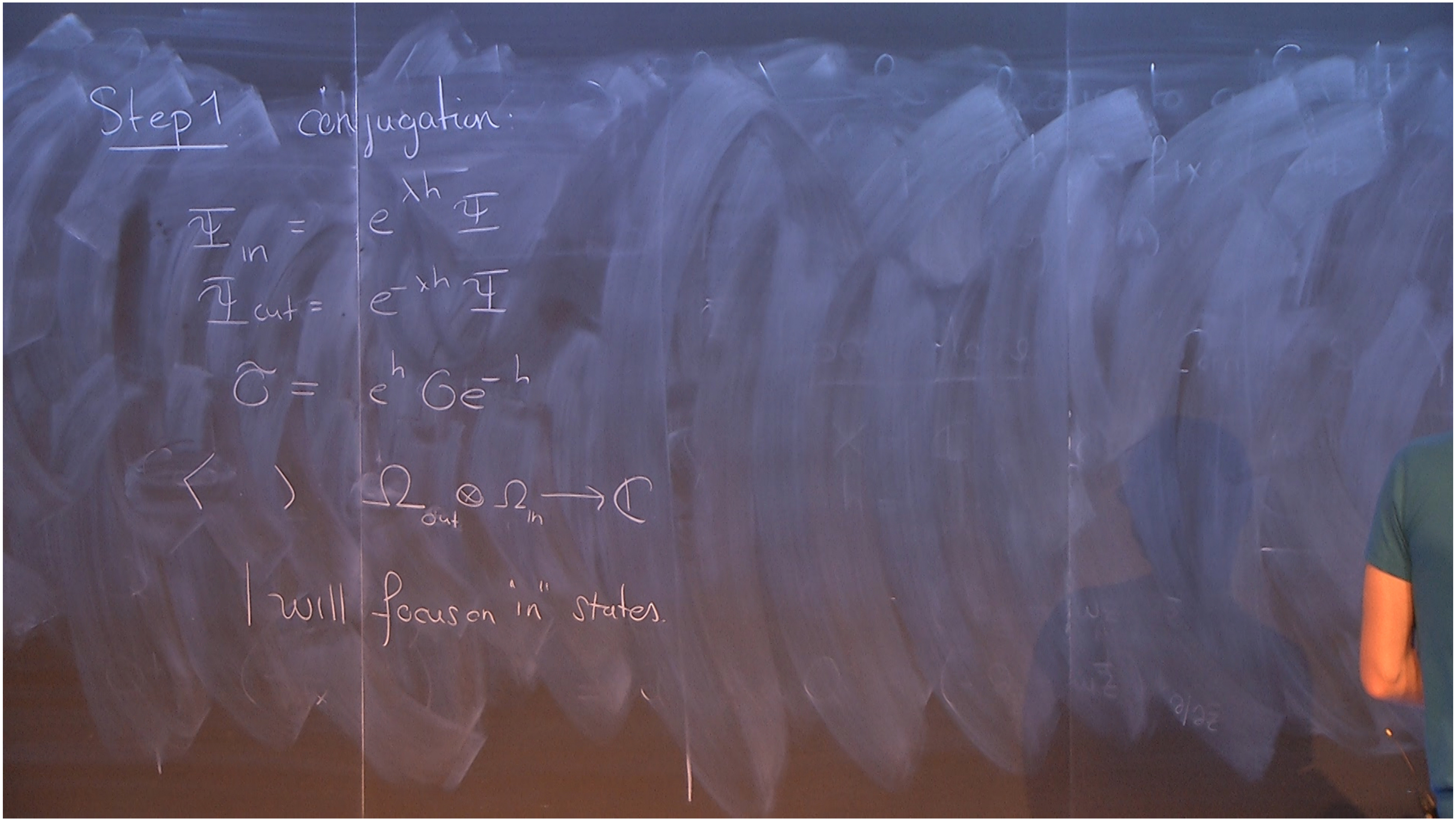
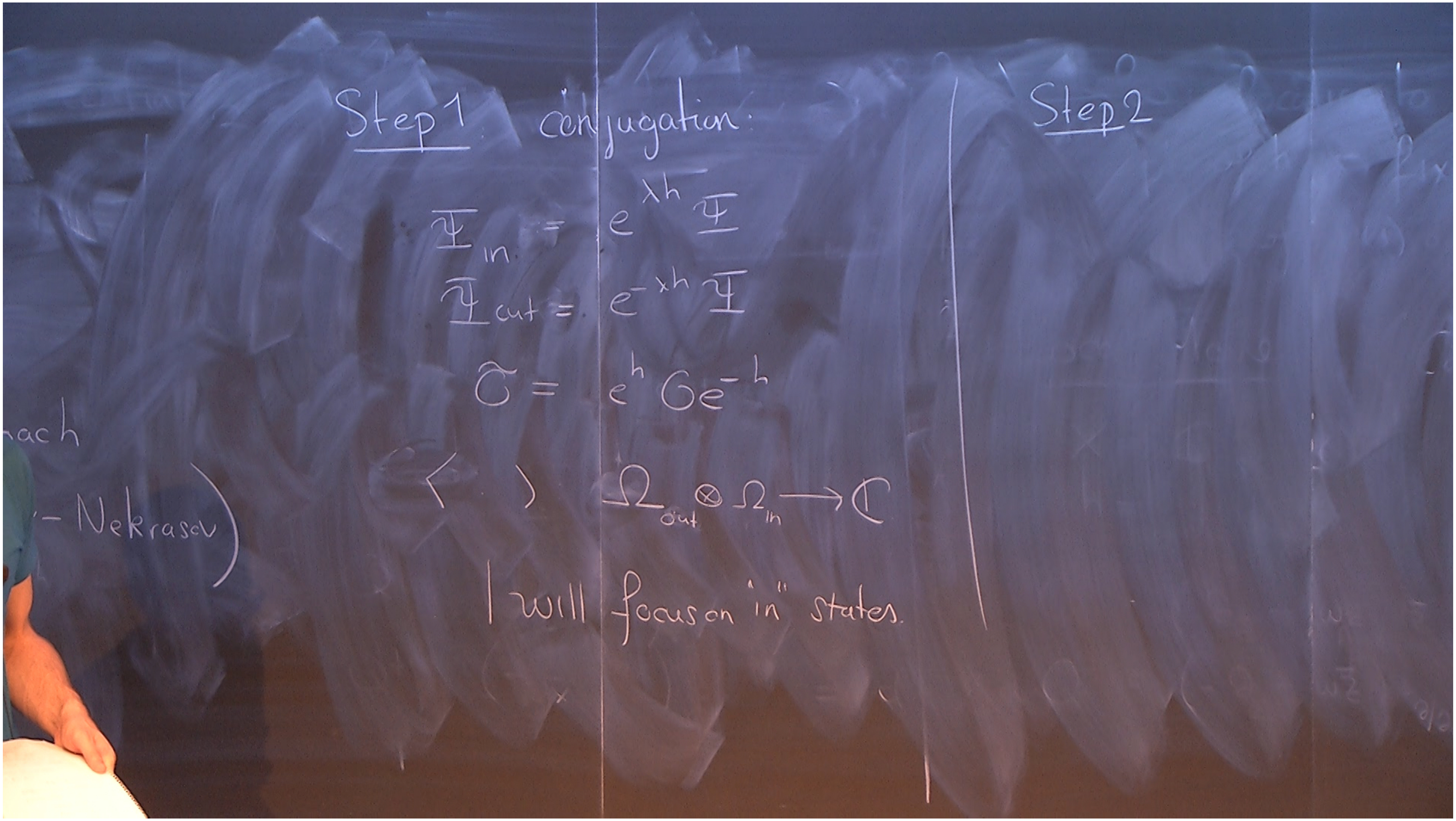

Step 1 conjugation:
$\Psi_{in} = e^{\lambda h} \Psi$
$\Psi_{out} = e^{-\lambda h} \Psi$
$\tilde{G} = e^{h} G e^{-h}$
$\langle \ \rangle \ \Omega_{out} \otimes \Omega_{in} \longrightarrow \mathbb{C}$
I will focus on "in" states.


Step 1 conjugation:
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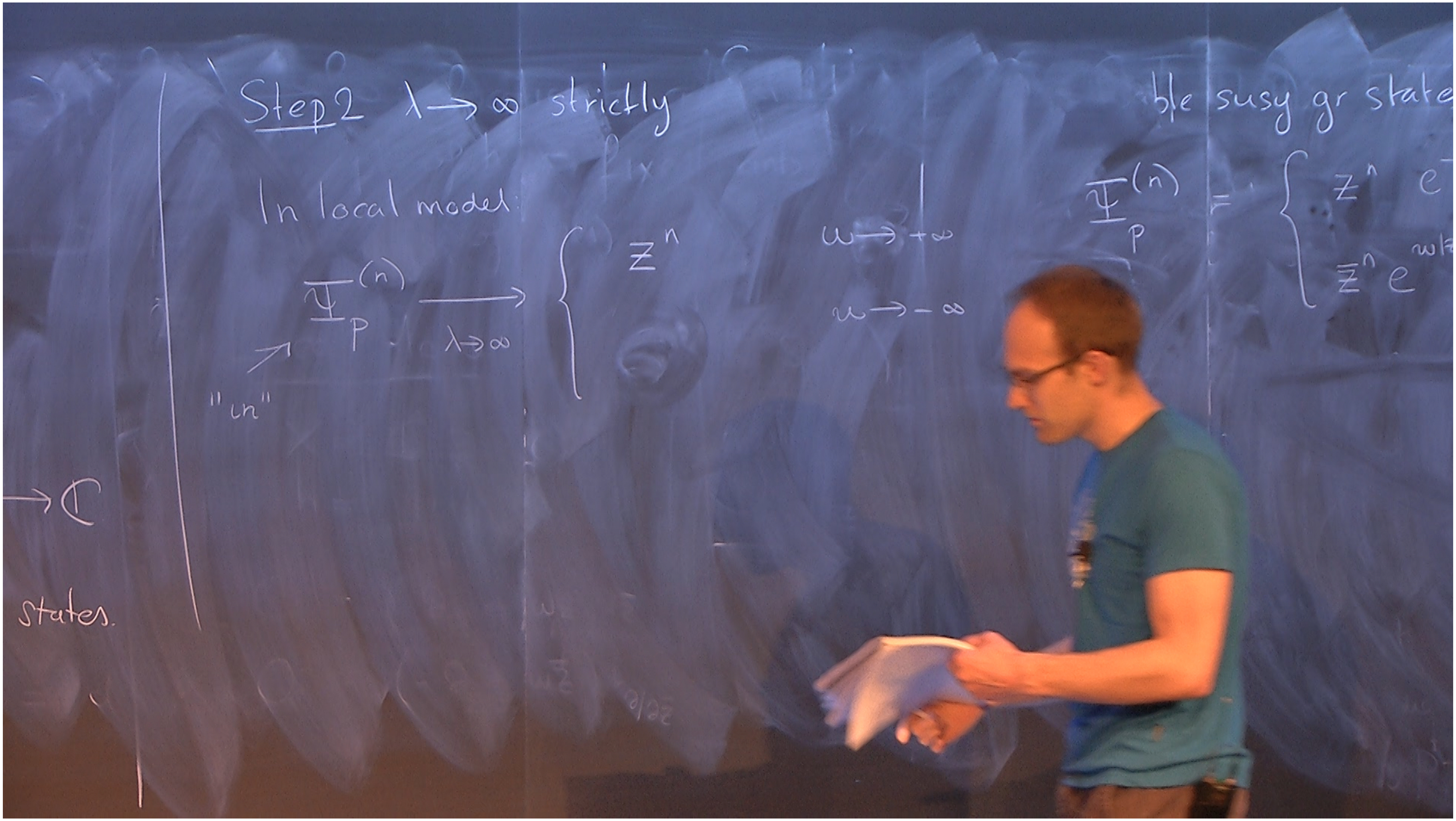
$\tilde{G} = e^{h} G e^{-h}$
$\langle\ \rangle\ \Omega_{out}\otimes\Omega_{in}\longrightarrow \mathbb{C}$
I will focus on "in" states.
Step 2
ach
-Nekrasov)


Step 2 λ → ∞ strictly
ble susy gr state
In local model:
λ→∞
"in"
ω → +∞
ω → −∞
states.

Step 2 $\lambda \to \infty$ strictly

...ble susy gr states: $\swarrow \mathcal{F}_+$

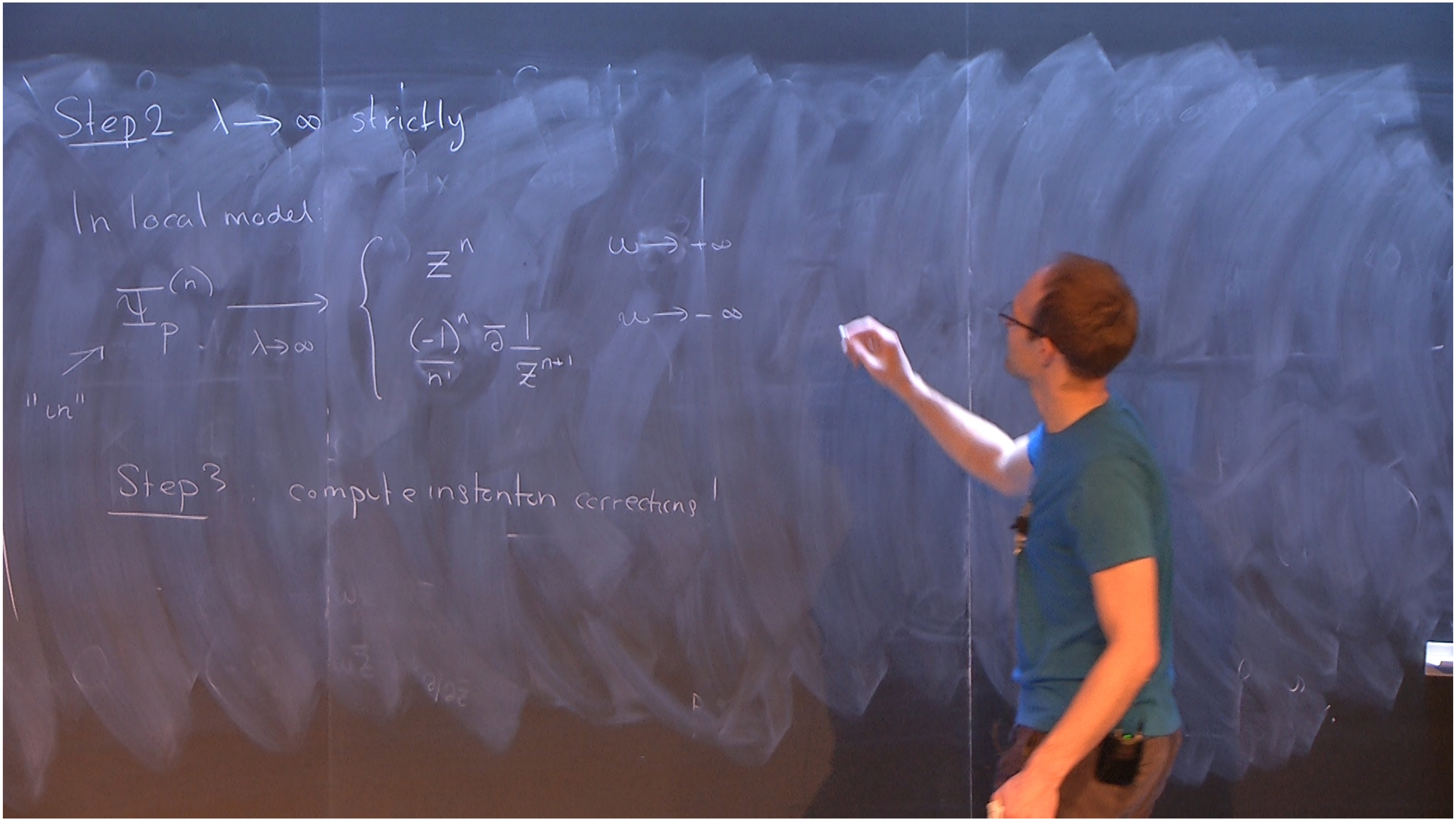
In local model:

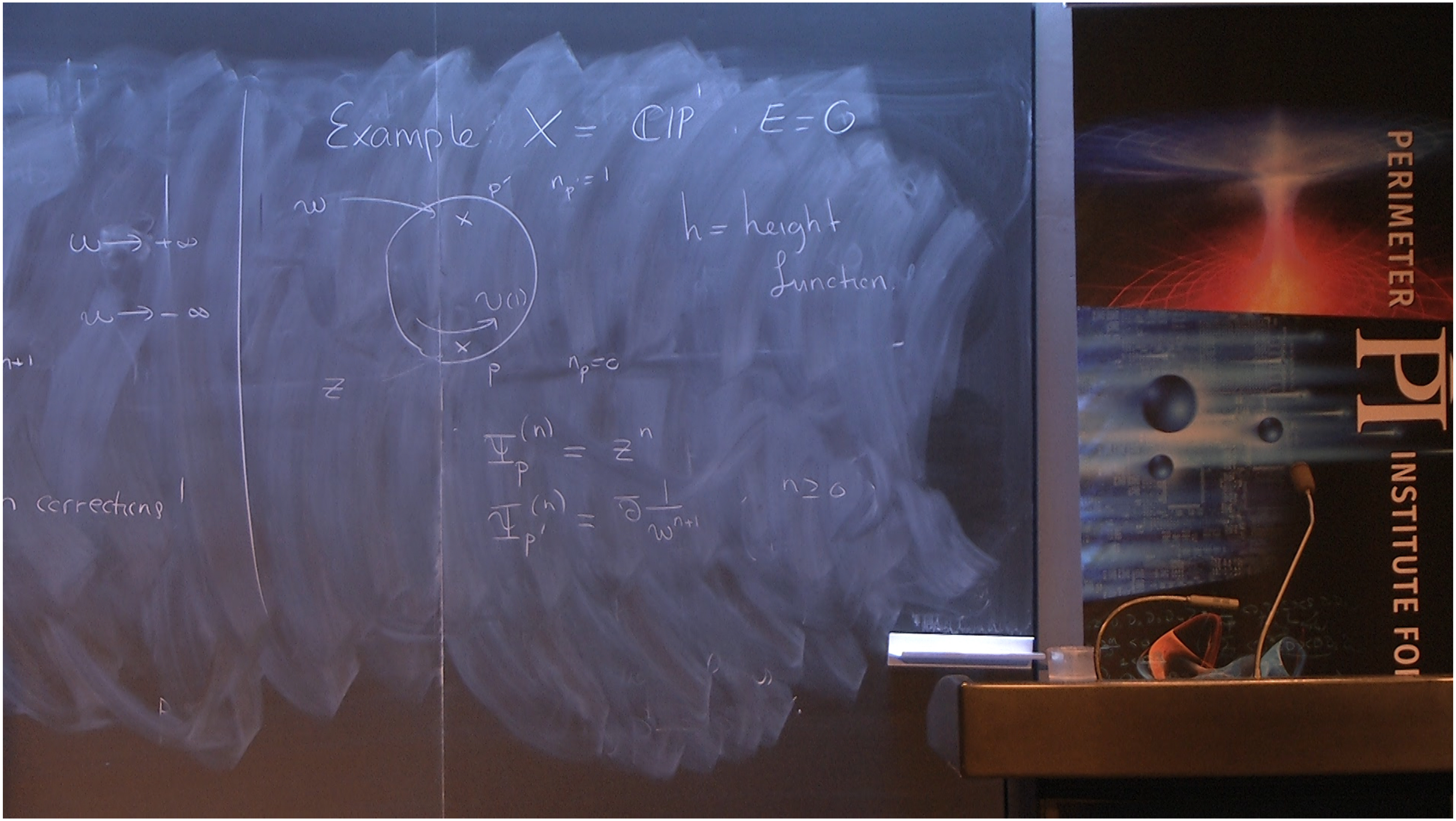
$$\Psi_p^{(n)} \xrightarrow[\lambda \to \infty]{} \begin{cases} z^n & \omega \to +\infty \\ \dfrac{(-1)^n}{n!} \bar{\partial} \dfrac{1}{z^{n+1}} & \omega \to -\infty \end{cases}$$

$\nearrow$ "in"

$$\Psi_p^{(n)} = \begin{cases} z^n \, e^{-\omega |z|^2} & \omega > 0 \\ \bar{z}^n \, e^{\omega |z|^2} \, d\bar{z} & \omega < 0 \end{cases}$$

$\leftarrow \mathcal{F}_-$


Step 2 λ → ∞ strictly
In local model:
Ψ_P^(n) → (λ→∞) { z^n ; (-1)^n/n! ∂̄ 1/z^(n+1) }
ω → +∞
ω → −∞
"in"
Step 3: compute instanton corrections!


Example: X = CP^1, E = O
ω → +∞
ω → −∞
n corrections
h = height function
p' n_p' = 1
p n_p = 0
Ψ_p^(n) = z^n
Ψ_p'^(n) = ∂ 1/ω^(n+1)
n ≥ 0
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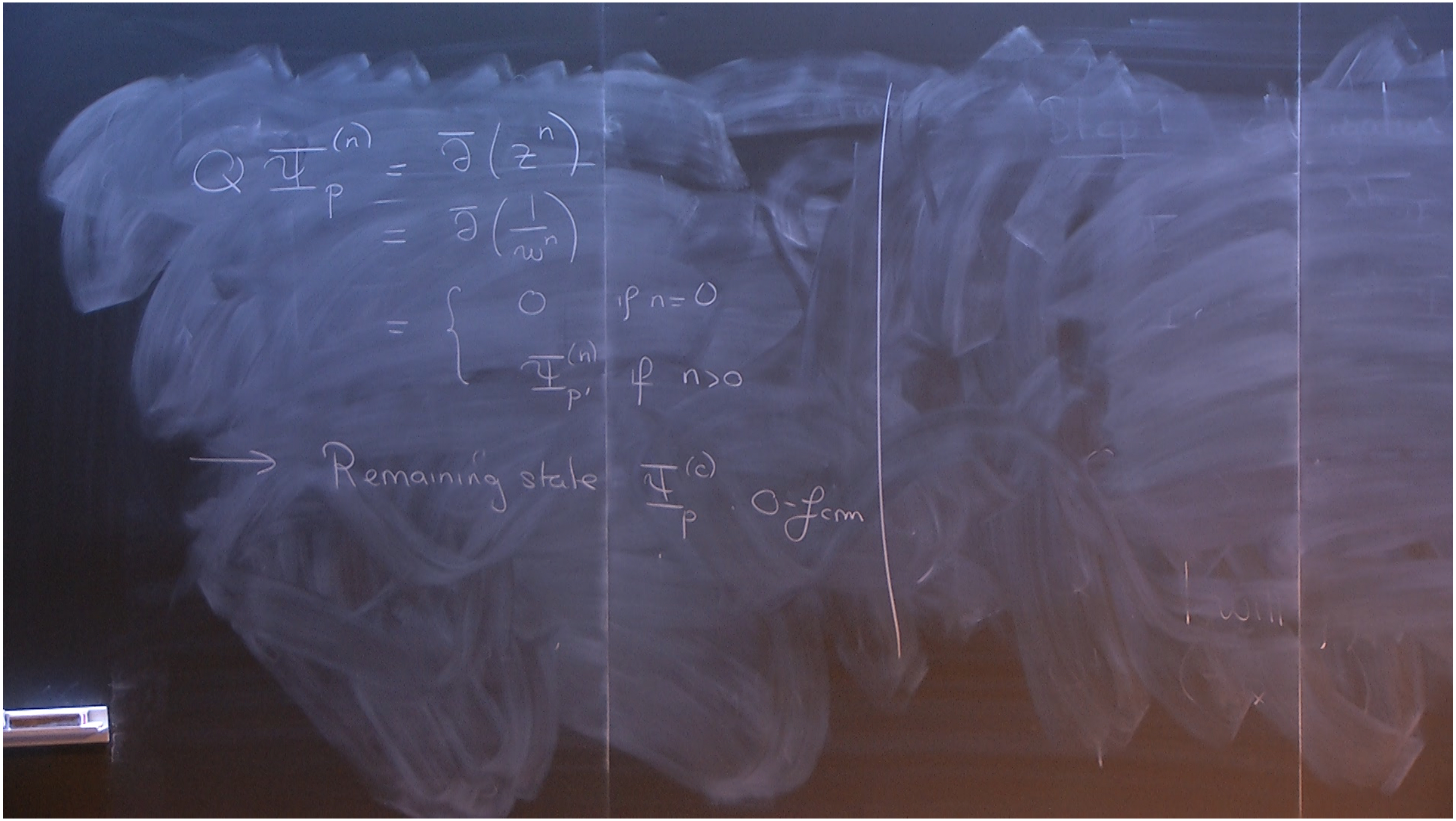
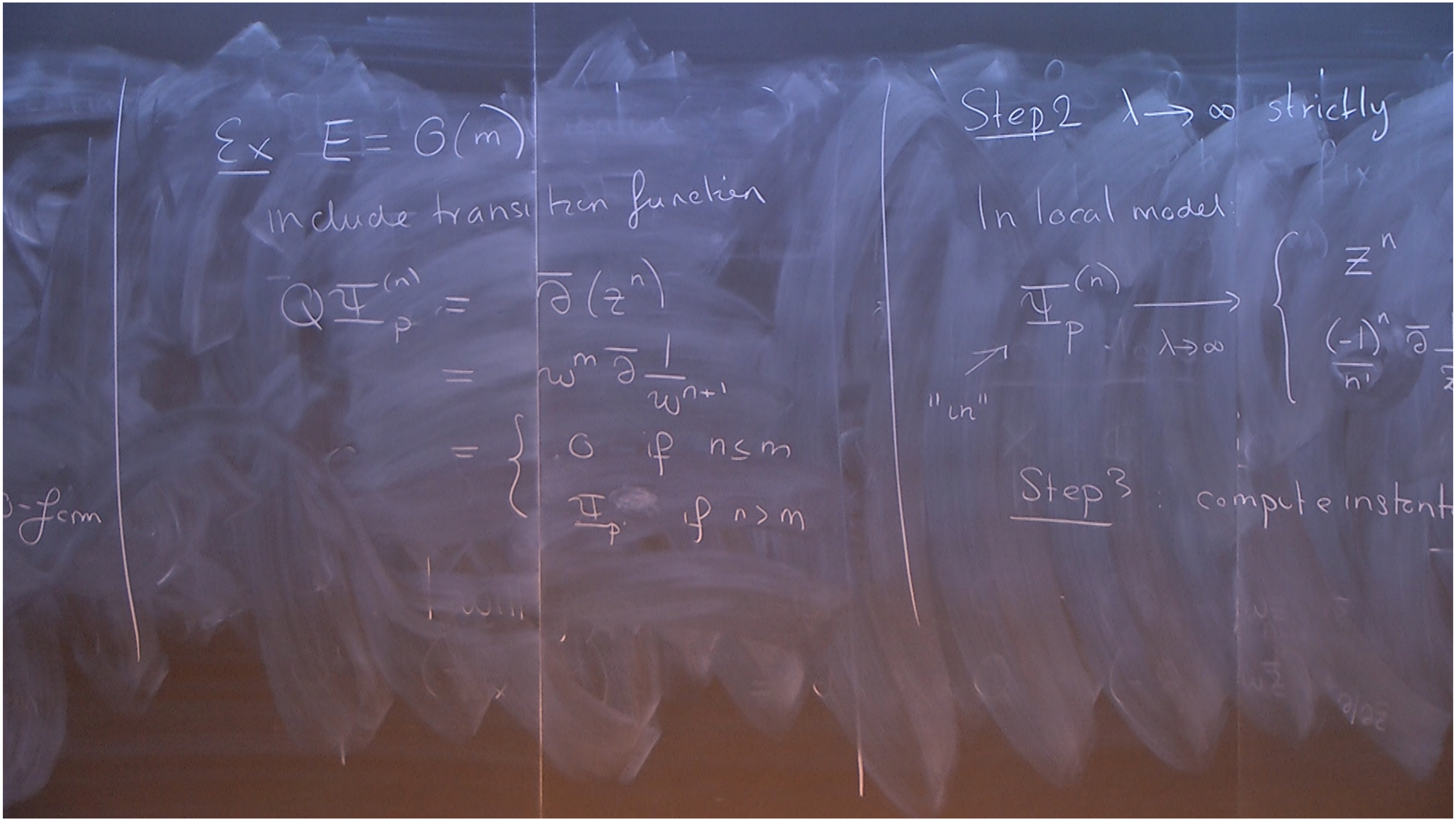
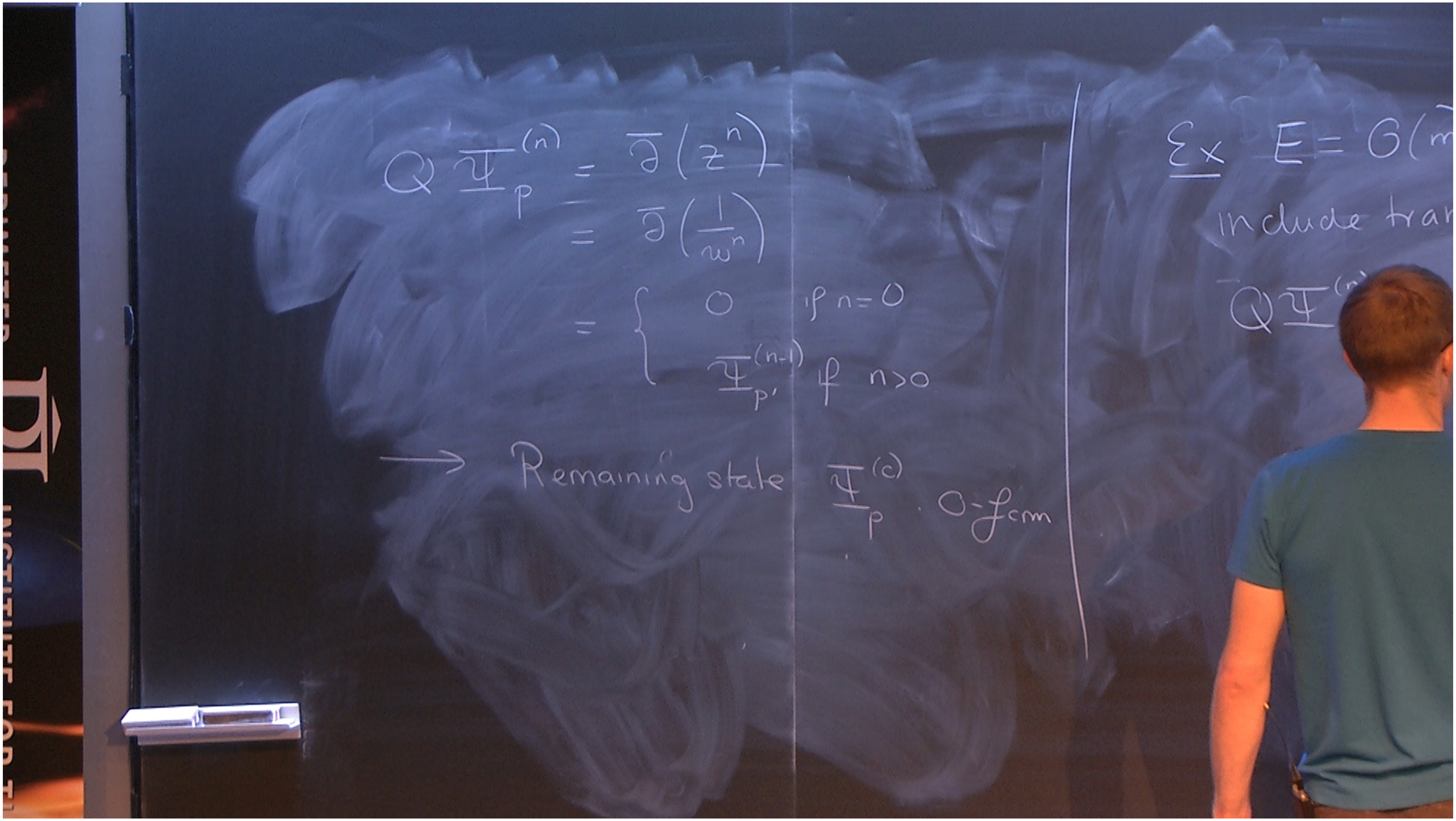
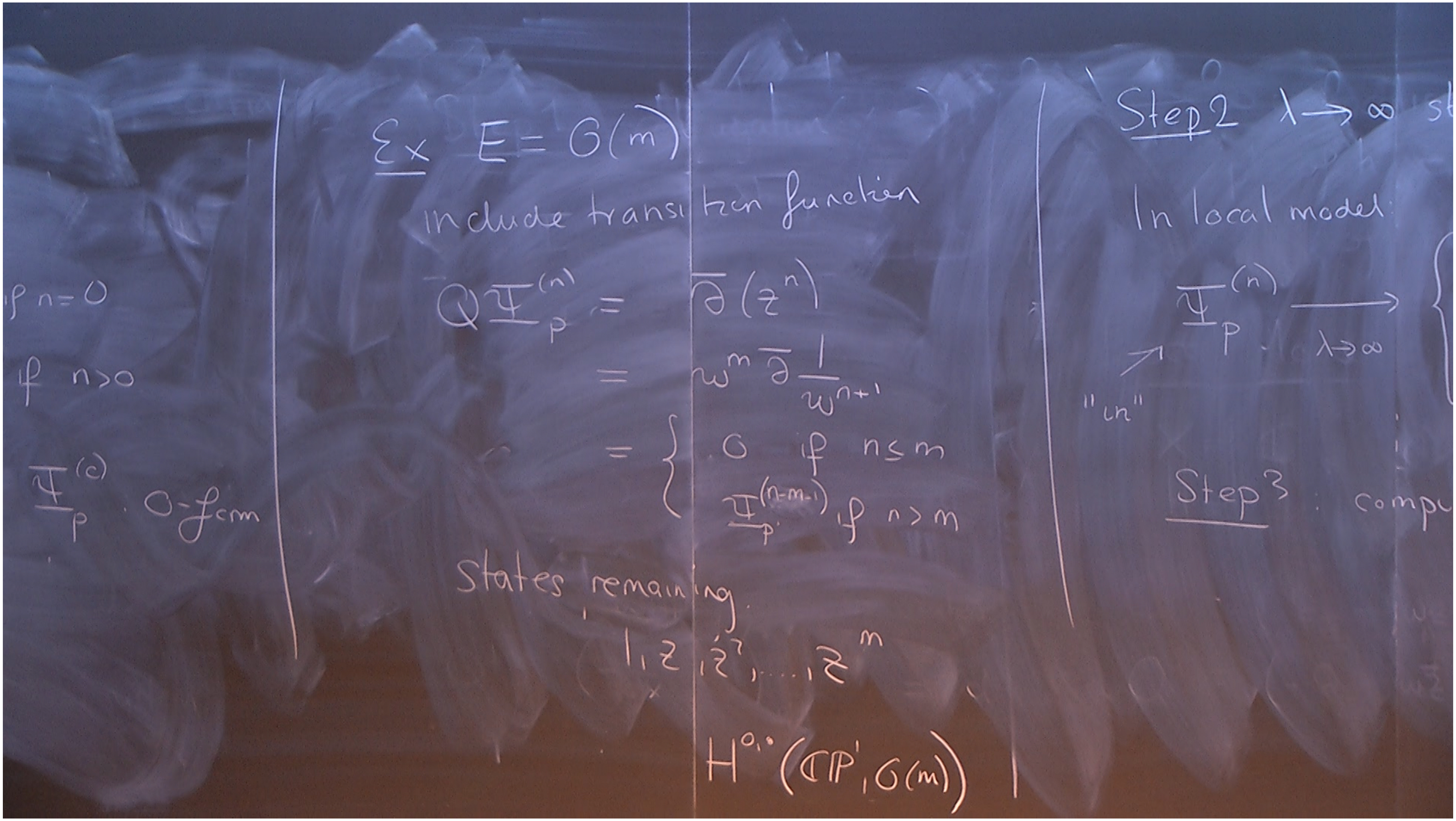
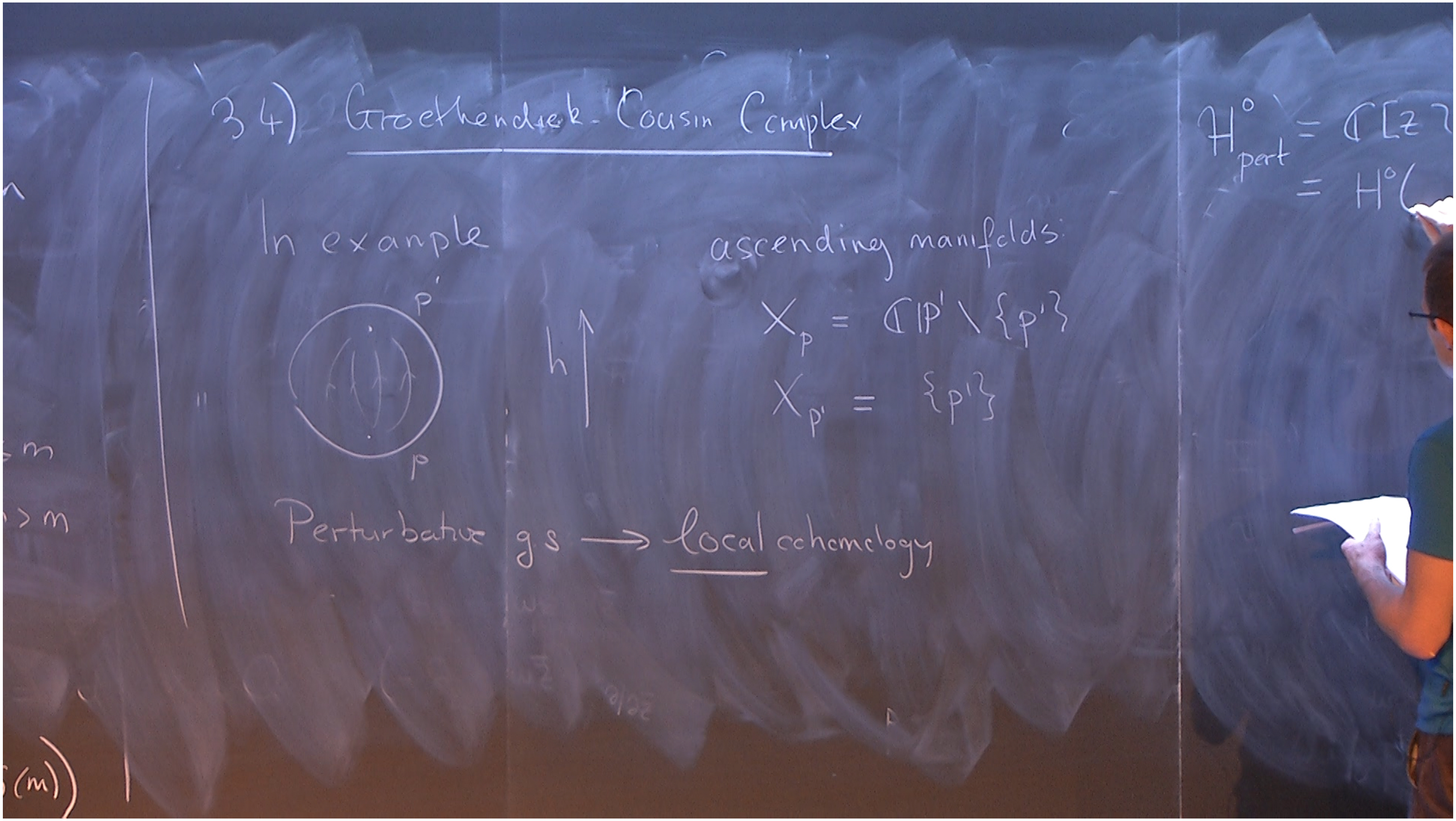
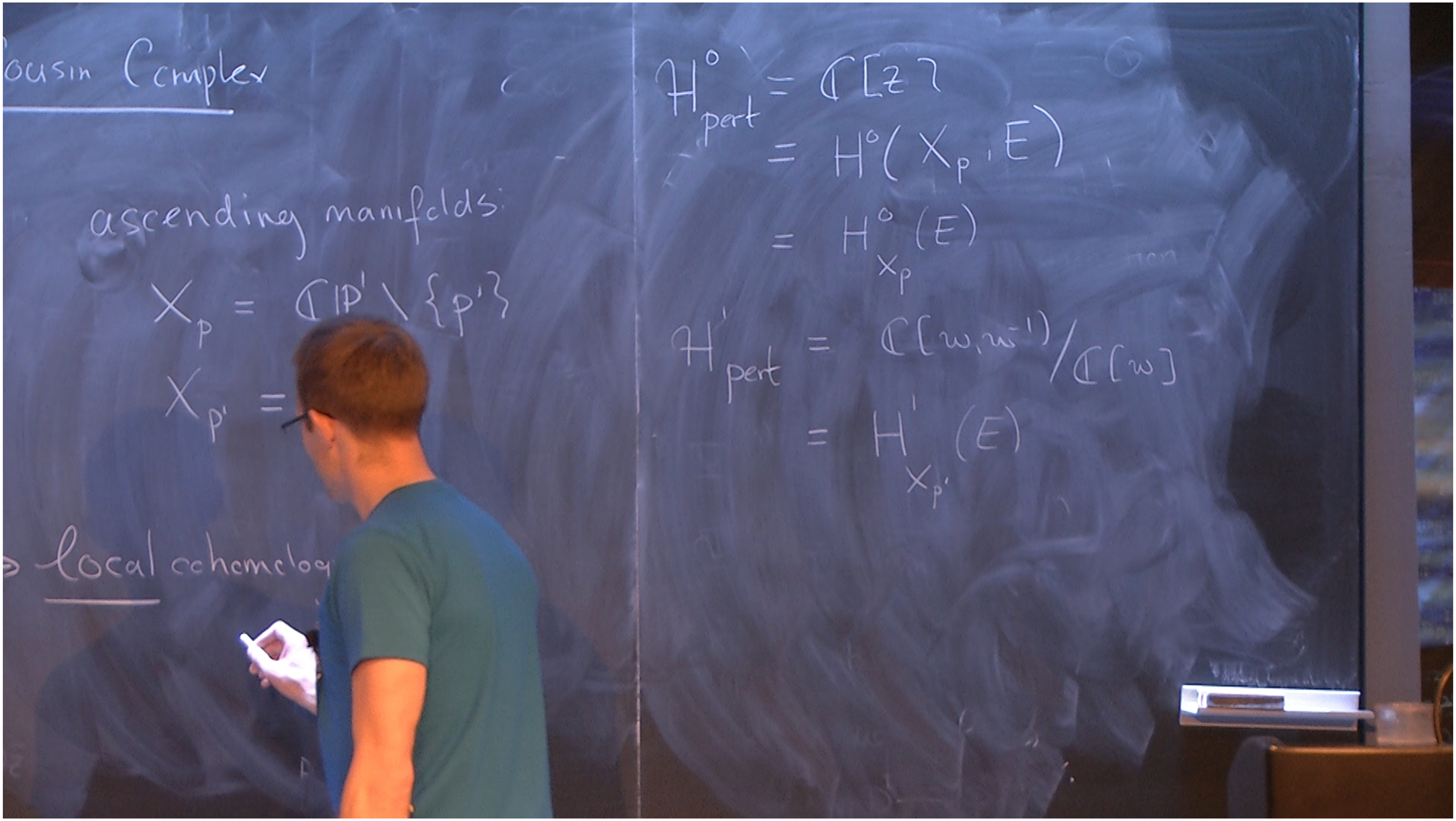

$Q \Psi_p^{(n)} = \bar{\partial}(z^n)$
$= \bar{\partial}\left(\frac{1}{w^n}\right)$
$= \begin{cases} 0 & \text{if } n=0 \\ \Psi_{p'}^{(n)} & \text{if } n>0 \end{cases}$
→ Remaining state $\Psi_p^{(c)}$ 0-form

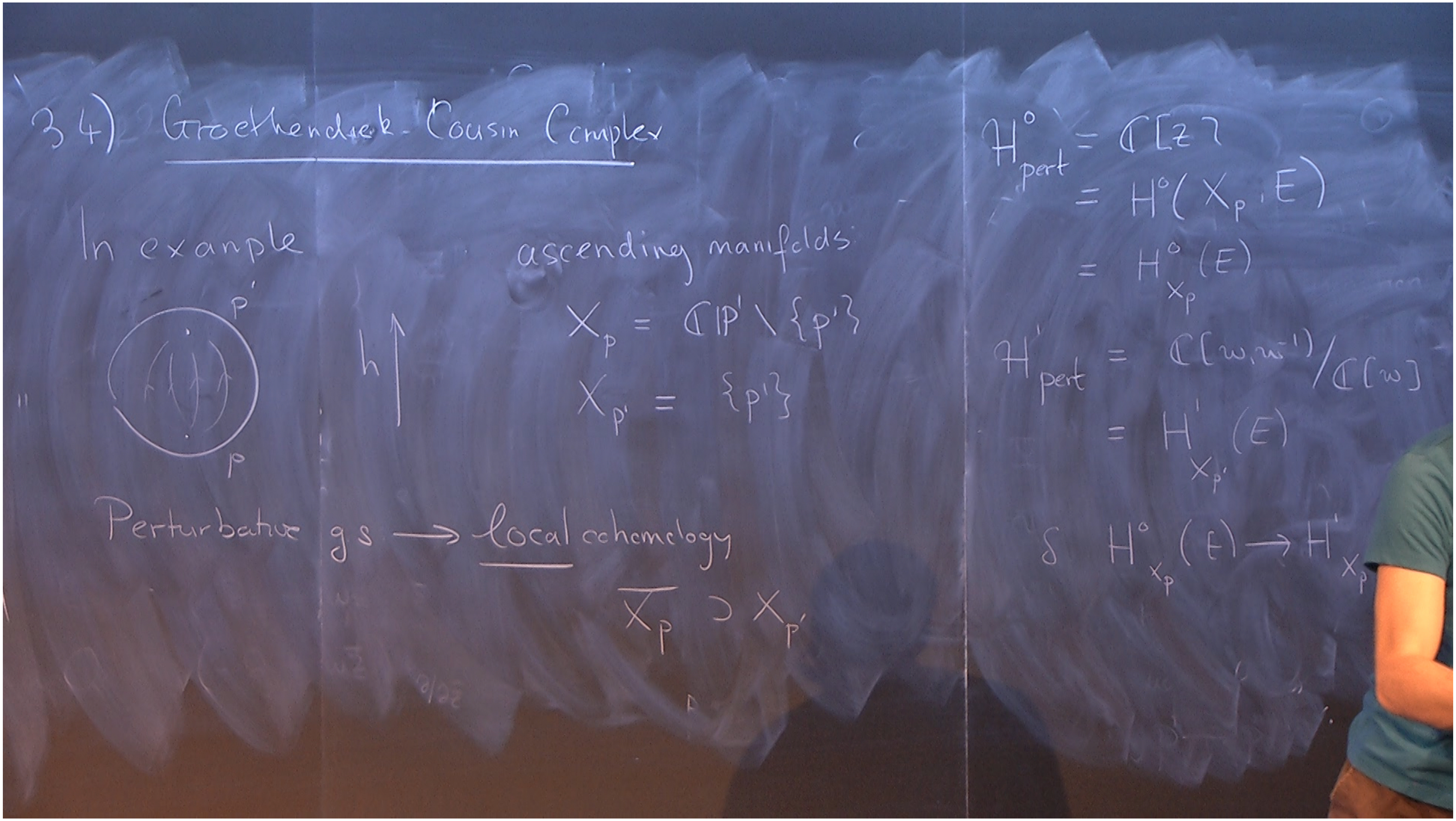

-form
Ex E = O(m)
include transition function
$Q\Psi_p^{(n)} = \bar\partial(z^n)$
$= w^m \bar\partial \frac{1}{w^{n+1}}$
$= \begin{cases} 0 & \text{if } n \le m \\ \Psi_p & \text{if } n > m \end{cases}$
Step 2 $\lambda \to \infty$ strictly
In local model:
$\Psi_p^{(n)} \xrightarrow{\lambda\to\infty}$
"in"
z^n
$\frac{(-1)^n}{n!} \bar\partial$
Step 3 : compute instant

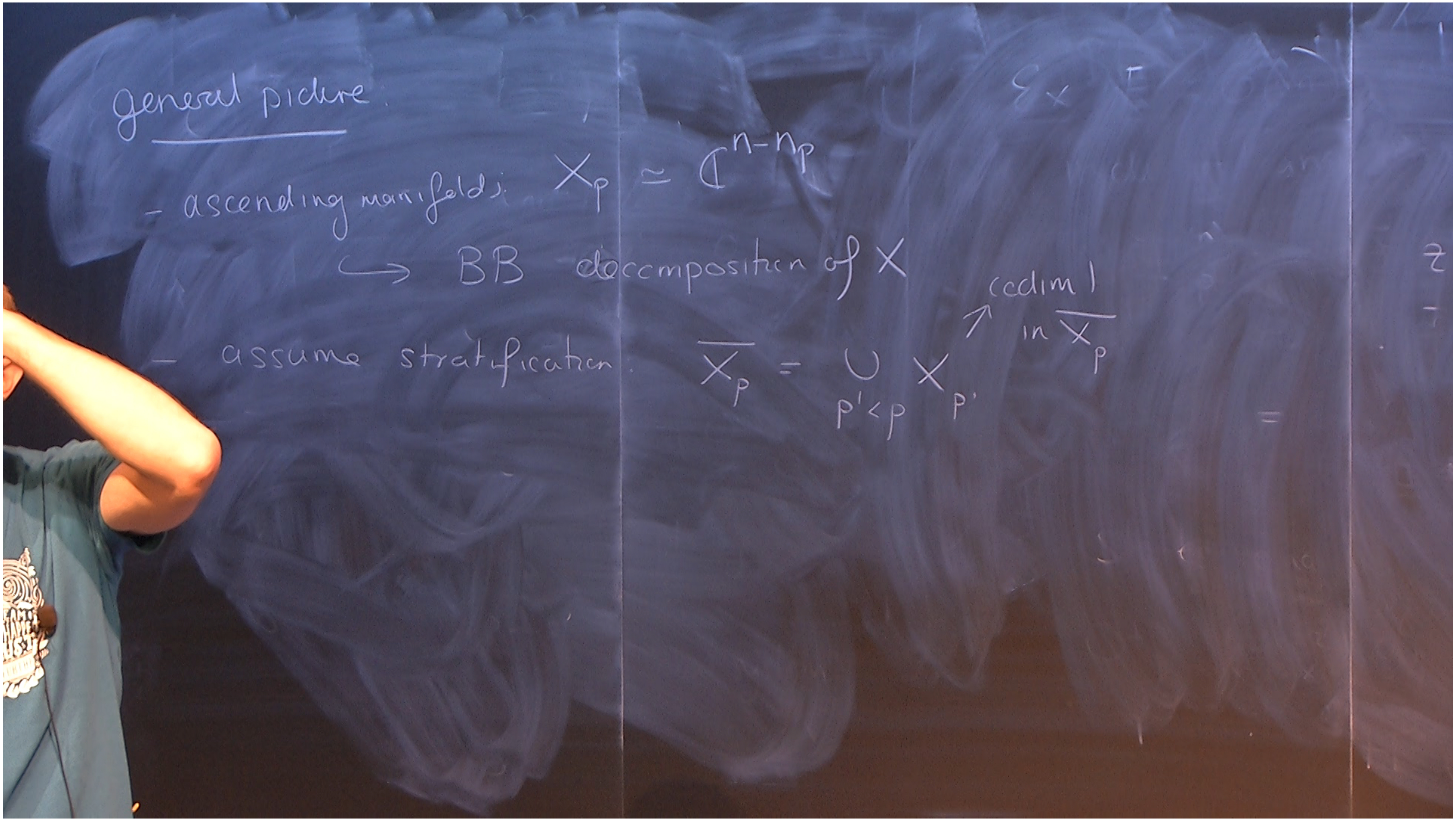

Q Ψ_p^(n) = ∂̄(z^n)
= ∂̄(1/w^n)
= { 0 if n=0
Ψ_p'^(n-1) if n>0
→ Remaining state Ψ_p^(0) 0-form
Ex E = G(m
include tra
Q Ψ^(n)

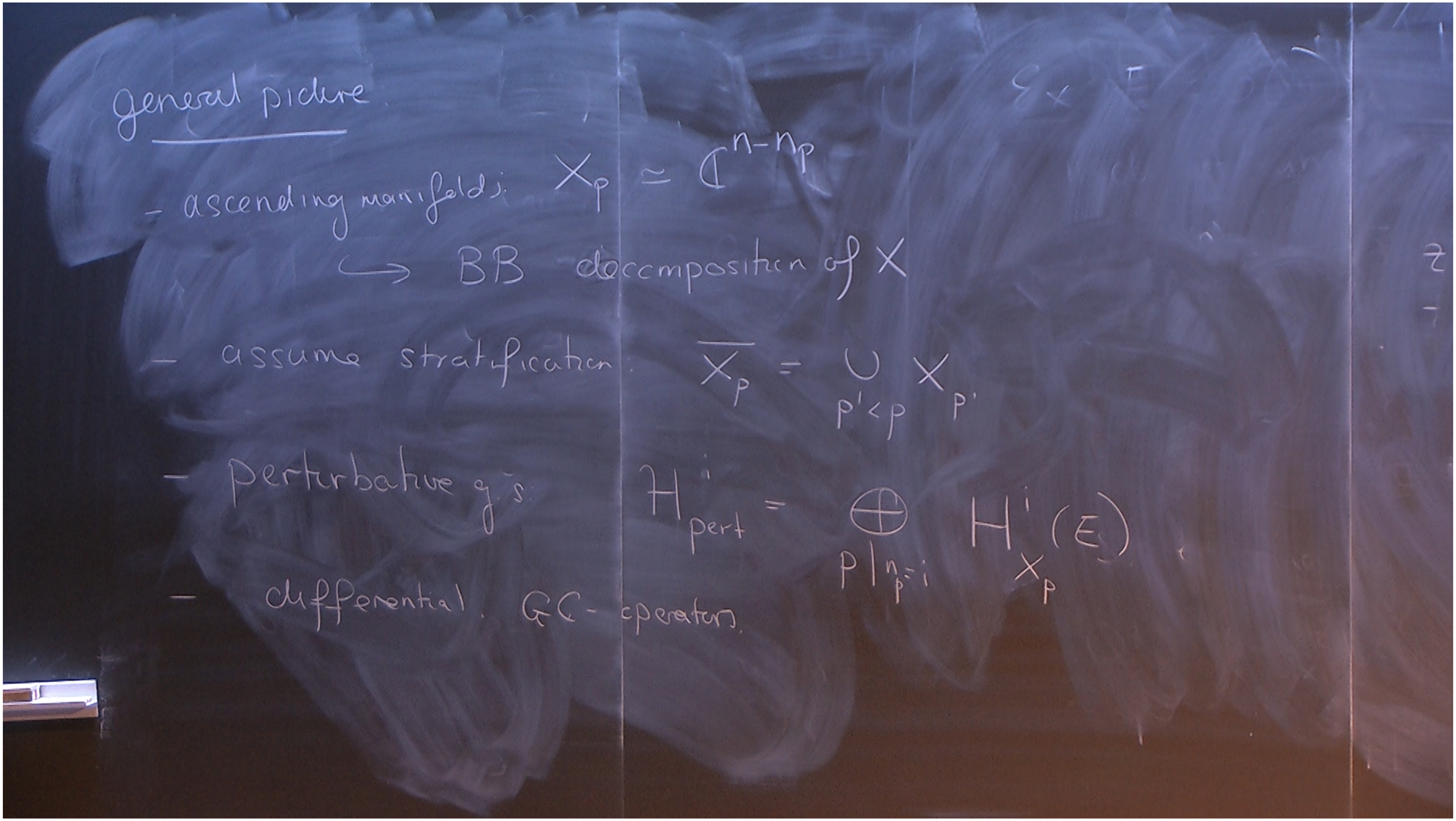

if n=0
if n>0
Ψ_p^(c) : 0-form
Ex E = O(m)
include transition function
QΨ_p^(n) = ∂̄(z^n)
= w^m ∂̄ 1/w^(n+1)
= { 0 if n≤m ; Ψ_p^(n-m-1) if n>m
States remaining
1, z, z², ..., z^m
H^{0,0}(CP^1, O(m))
Step 2 λ→∞
In local model:
"in"
Ψ_p^(n) →(λ→∞)
Step 3 : compu

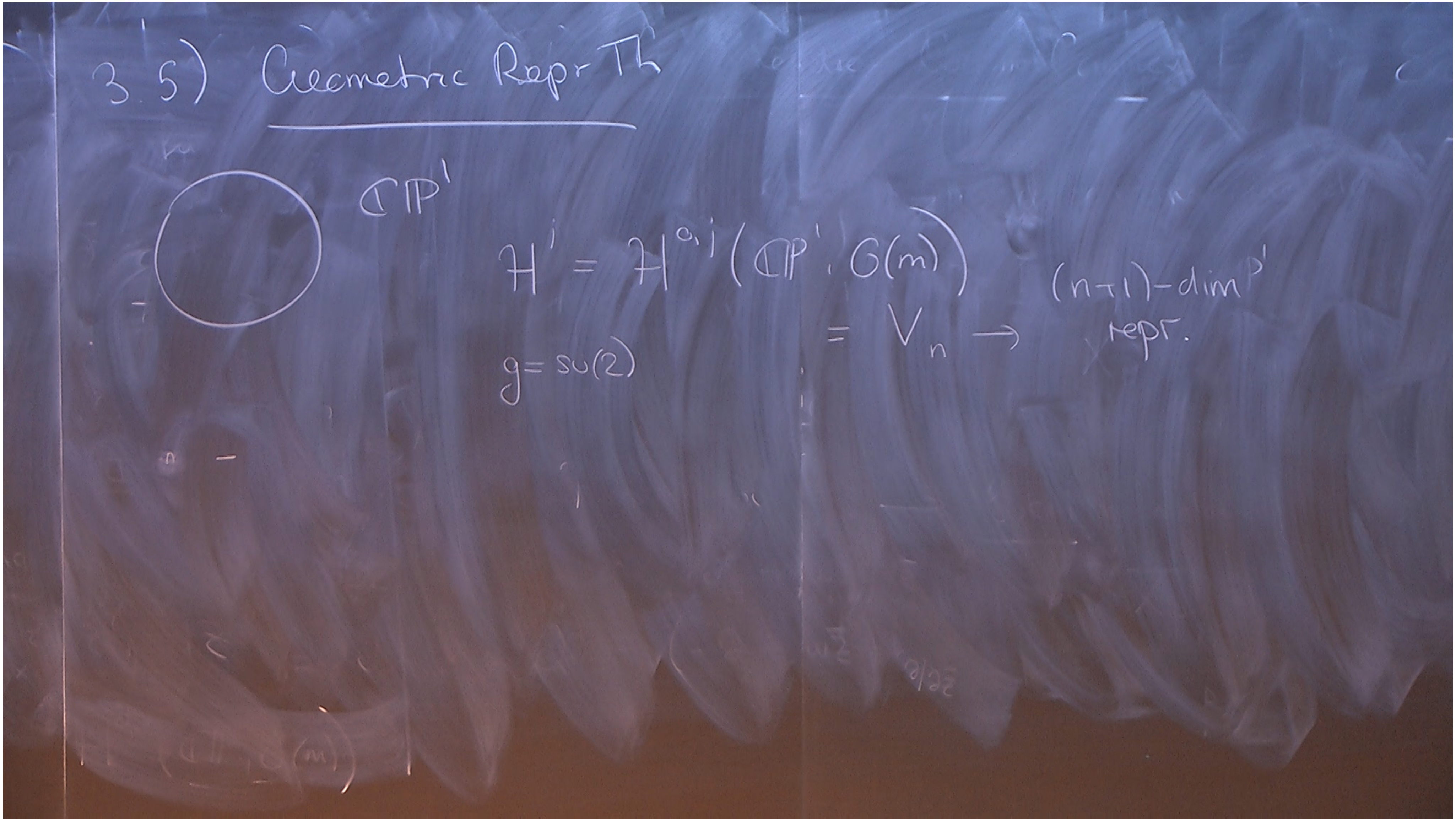

3.4) Grothendieck-Cousin Complex
In example
ascending manifolds:
p'
p
h
X_p = CP^1 \ {p'}
X_p' = {p'}
Perturbative g.s → local cohomology
H^0_pert = C[z
= H^0(


ousin Complex
ascending manifolds:
X_p = ℂP¹ \ {p'}
X_p' =
local cohomolog
H⁰_pert = ℂ[z]
= H⁰(X_p, E)
= H⁰_X_p (E)
H¹_pert = ℂ[w, w⁻¹]/ℂ[w]
= H¹_X_p' (E)


3.4) Grothendieck-Cousin Complex
In example
p'
p
h
ascending manifolds:
X_p = CP^1 \ {p'}
X_p' = {p'}
Perturbative gs → local cohomology
X_p ⊃ X_p'
H^0_pert = C[z]
= H^0(X_p, E)
= H^0_{X_p}(E)
H^1_pert = C[w,w^-1]/C[w]
= H^1_{X_p'}(E)
δ H^0_{X_p}(E) → H^1_{X_p'}


general picture:
- ascending manifolds: $X_p \simeq \mathbb{C}^{n-n_p}$
↪ BB decomposition of X
- assume stratification: $\overline{X_p} = \bigcup_{p'<p} X_{p'}$
↗ (codim 1 in $\overline{X_p}$)


general picture
- ascending manifolds: X_p ≃ ℂ^(n−n_p)
↪ BB decomposition of X
- assume stratification: X̄_p = ∪_(p'<p) X_p'
- perturbative g's: H^i_pert = ⊕_(p | n_p = i) H^i_(X_p)(E)
- differential: GC-operators


3.5) Geometric Repr Th
$\mathbb{CP}^1$
$\mathcal{H}^j = \mathcal{H}^{0,j}(\mathbb{CP}^1, \mathcal{O}(n))$
$= V_n \rightarrow$ (n+1)-dim'l repr.
g = su(2)

3.5) Geometric Repr Th

$\mathbb{CP}^1$

$$\mathcal{H}^i = \mathcal{H}^{0,i}(\mathbb{CP}^1, \mathcal{O}(m))$$

$$= V_m \longrightarrow \quad (n+1)\text{-dim repr.}$$

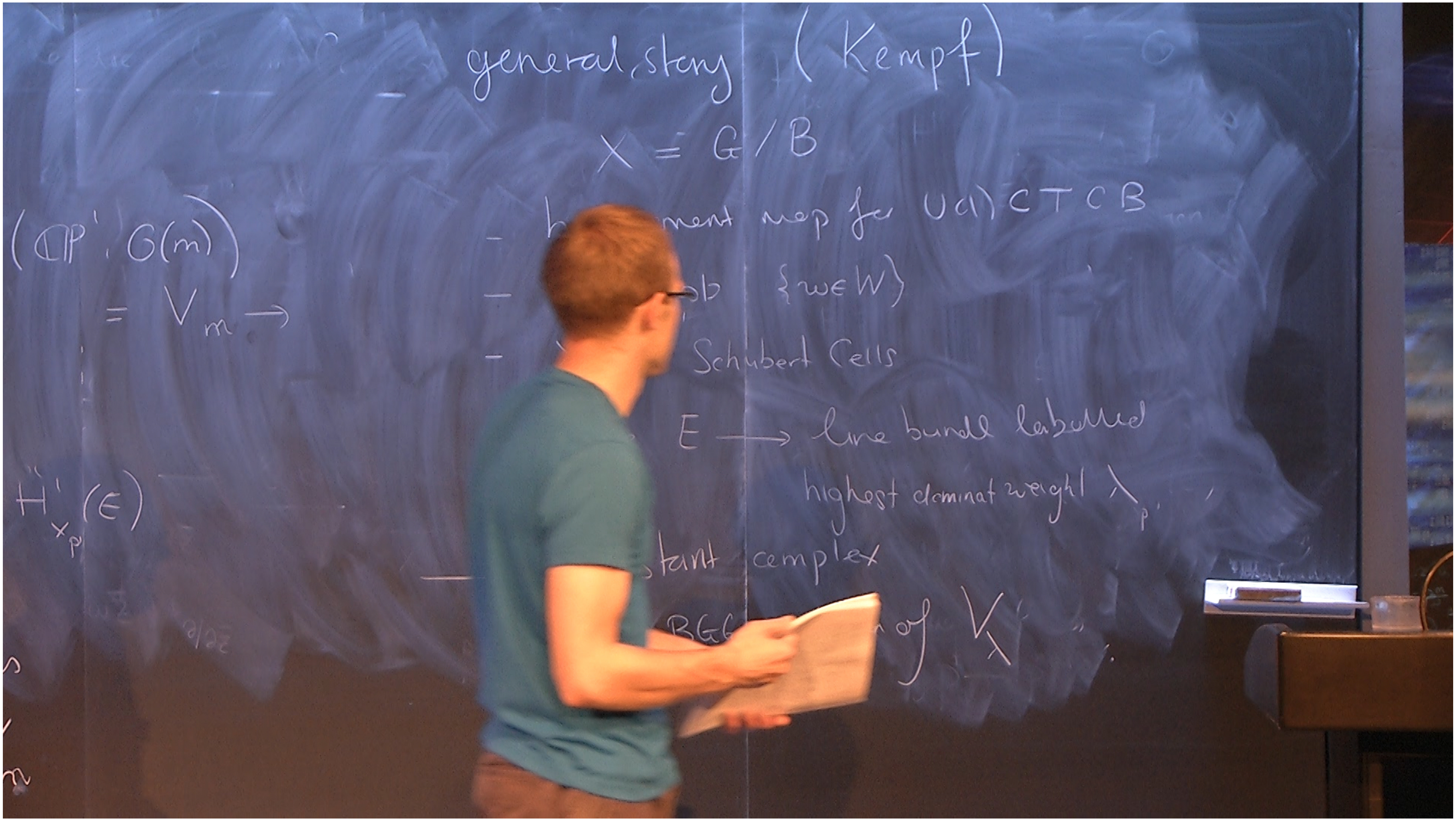
$g = su(2)$

- instanton complex

$$\delta : H^0_{X_p}(E) \longrightarrow H^1_{X_p}(E)$$

- $H^0_{X_p}(E)$, $H^1_{X_p}(E)$ $\longrightarrow$ Verma modules

- BGG resolution of V_m


general story (Kempf)
X = G/B
Schubert Cells
E ⟶ line bundle labelled
highest dominant weight
complex