Title: Lecture 1: Supersymmetric Field Theory and Topological Twists

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Collection: QFT for Mathematicians

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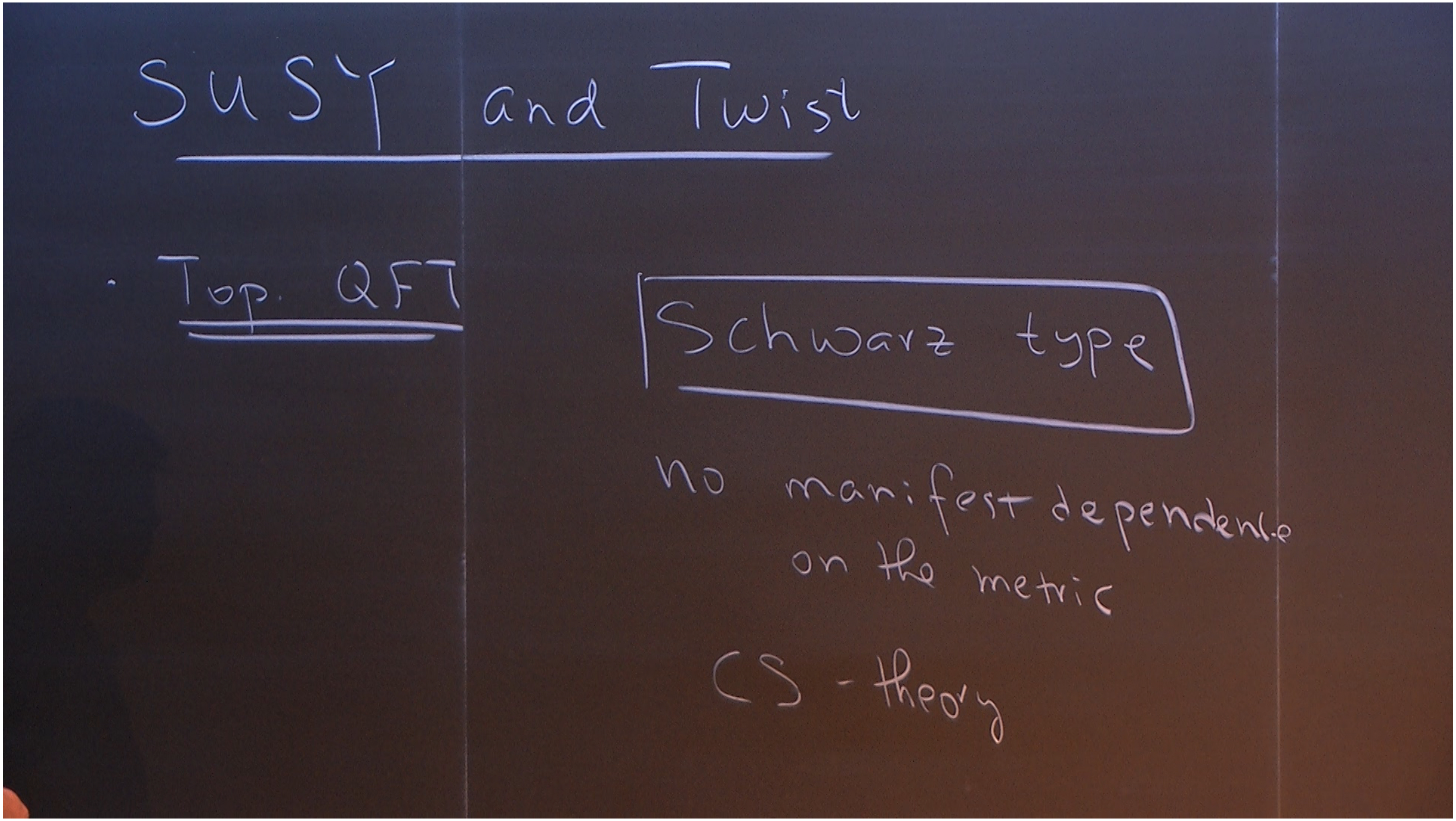

SUSY and Twist
· Top. QFT
Schwarz type
no manifest dependence
on the metric
CS - theory

Twist

Schwarz type

no manifest dependence on the metric

CS - theory

Witten type

Variation of metric = Q-exact

· Q $\qquad Q^2 = 0$

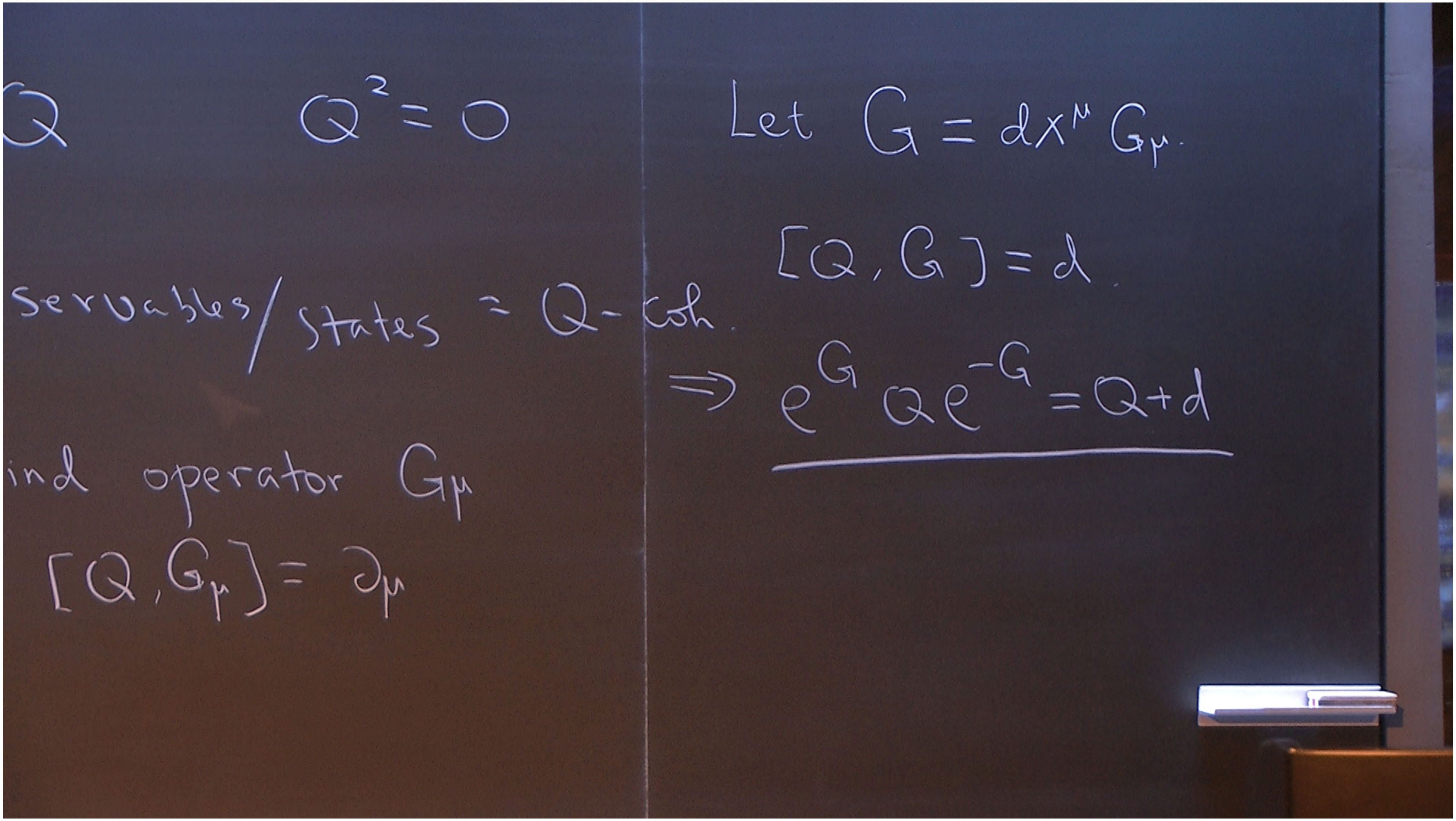
observables/states = Q-coh.

- find operator G_μ

$$[Q, G_\mu] = \partial_\mu$$

Q-exact

theory


Q
Q² = 0
servables / states = Q-coh.
ind operator Gμ
[Q, Gμ] = ∂μ
Let G = dx^μ Gμ.
[Q, G] = d
⇒ e^G Q e^-G = Q + d

SUSY and Twist

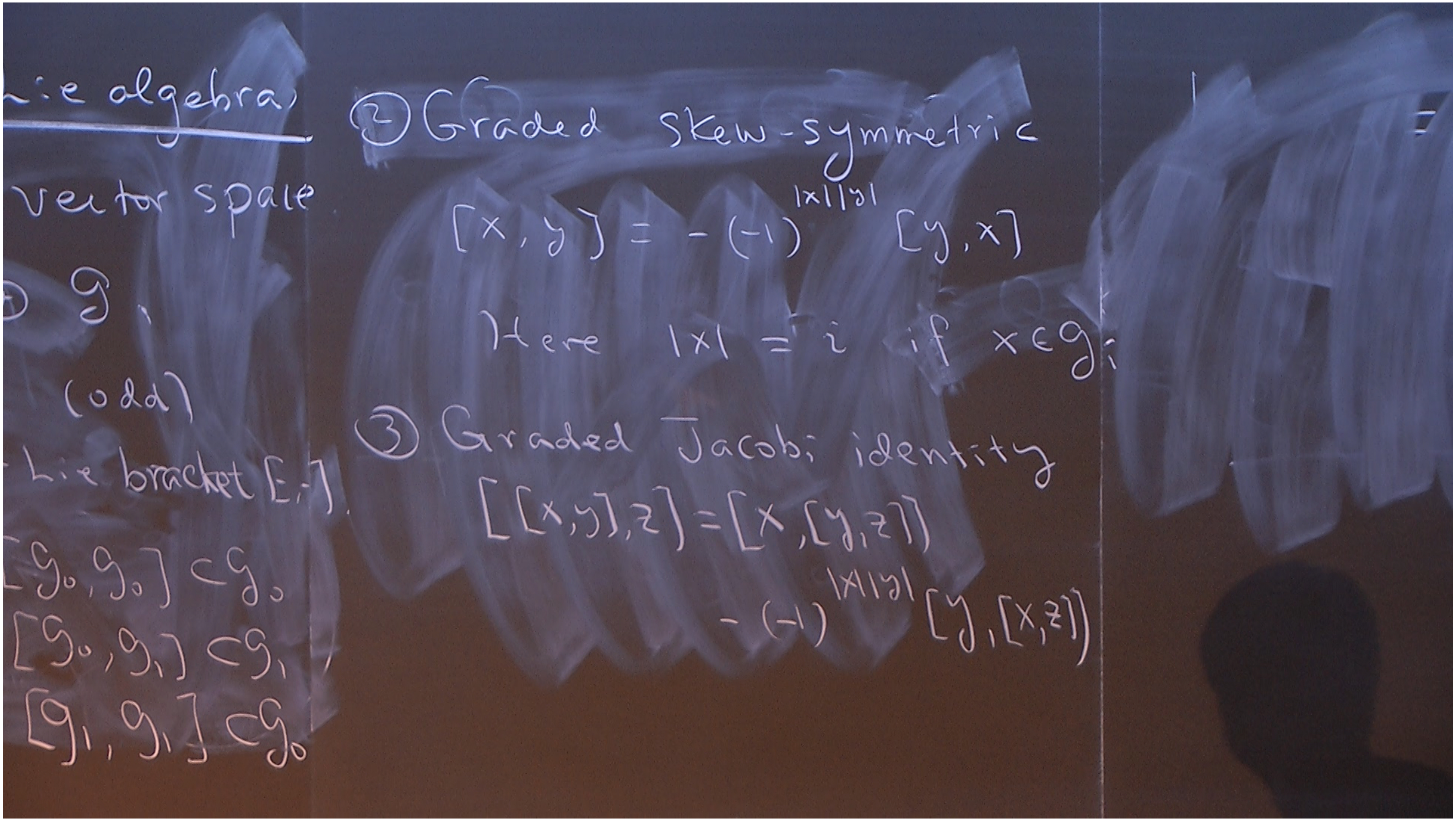
- $\mathcal{O}(x)$ local operator

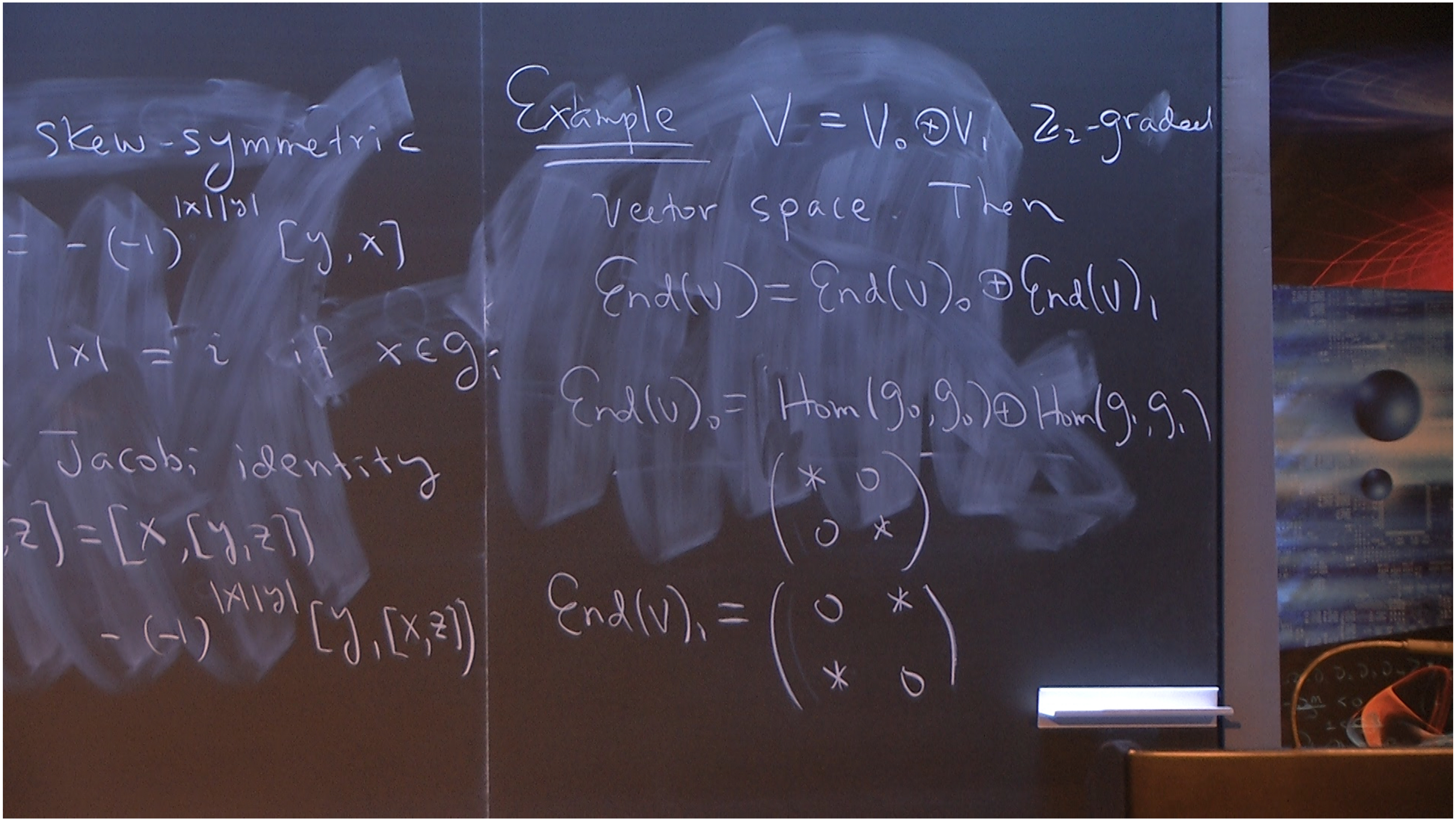
$$Q\,\mathcal{O} = 0$$

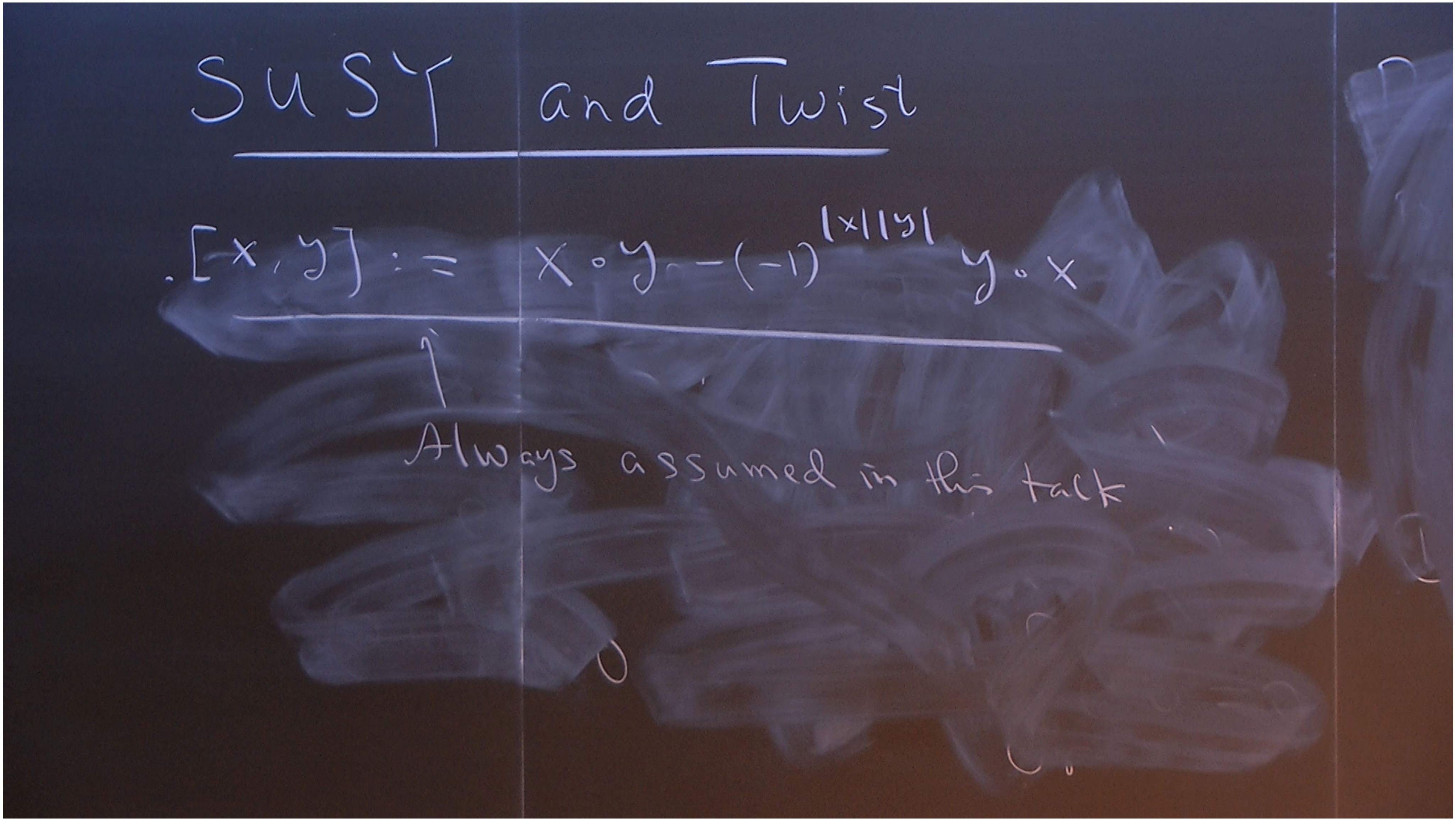
$$\Rightarrow \quad e^{G}\,\mathcal{O} = \mathcal{O} + \mathcal{O}^{(1)} + \mathcal{O}^{(2)} + \cdots \mathcal{O}^{(k)} + \cdots$$

k-form

$$(Q+d)\, e^{G}\mathcal{O} = 0 \quad \Rightarrow \quad Q \int_{\gamma} e^{G}\mathcal{O} = 0$$


Lie algebra
vector space
(odd)
Lie bracket [,]
$[g_0, g_0] \subset g_0$
$[g_0, g_1] \subset g_1$
$[g_1, g_1] \subset g_0$
② Graded skew-symmetric
$[x, y] = -(-1)^{|x||y|} [y, x]$
Here $|x| = i$ if $x \in g_i$
③ Graded Jacobi identity
$[[x,y],z] = [x,[y,z]] - (-1)^{|x||y|} [y,[x,z]]$

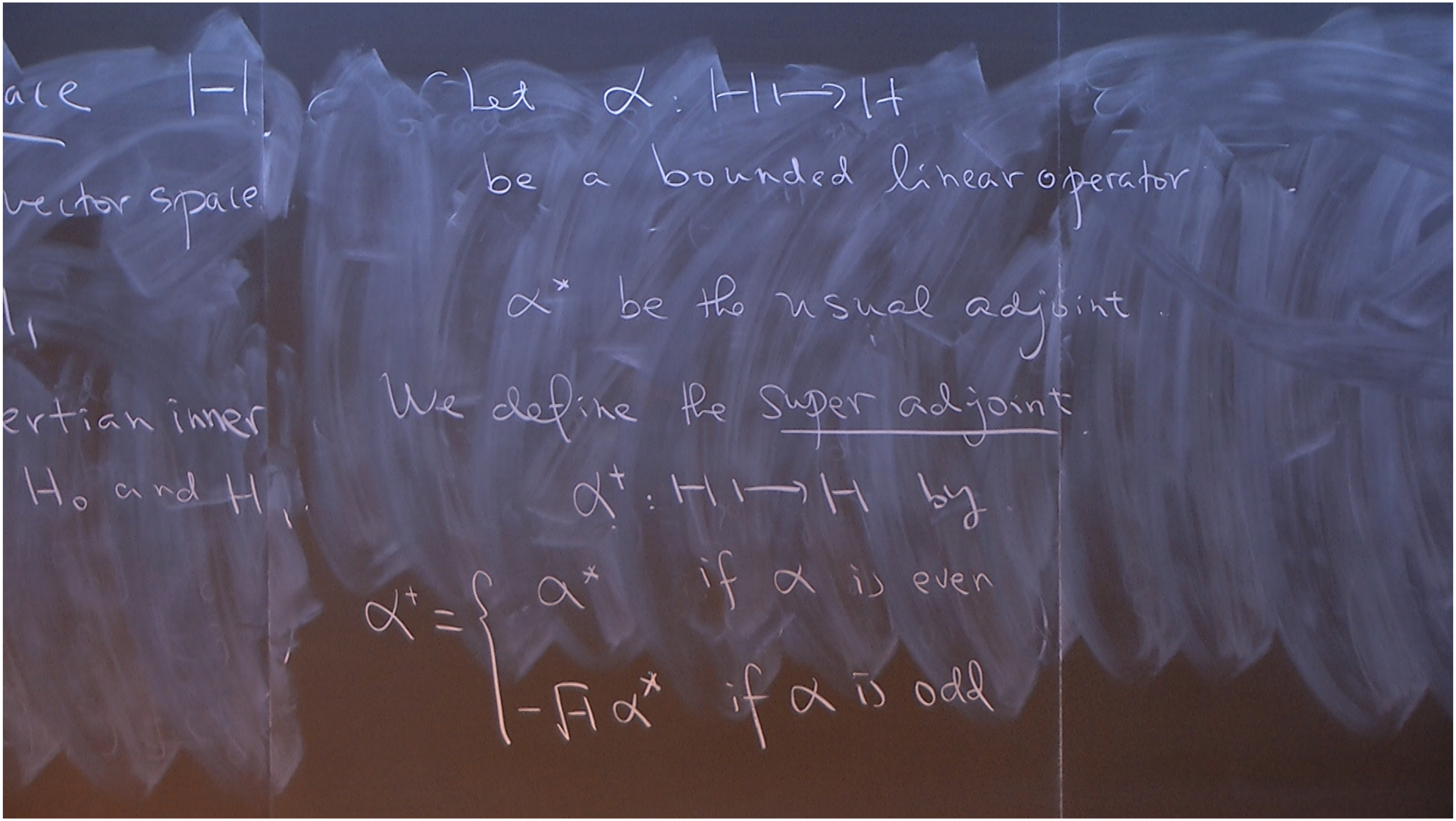

Skew-symmetric
[x, y] = -(-1)^{|x||y|} [y, x]
|x| = i if x ∈ g_i
Jacobi identity
[[x, y], z] = [x, [y, z]] - (-1)^{|x||y|} [y, [x, z]]
Example V = V_0 ⊕ V_1 Z_2-graded
vector space. Then
End(V) = End(V)_0 ⊕ End(V)_1
End(V)_0 = Hom(g_0, g_0) ⊕ Hom(g_1, g_1)
End(V)_1 =

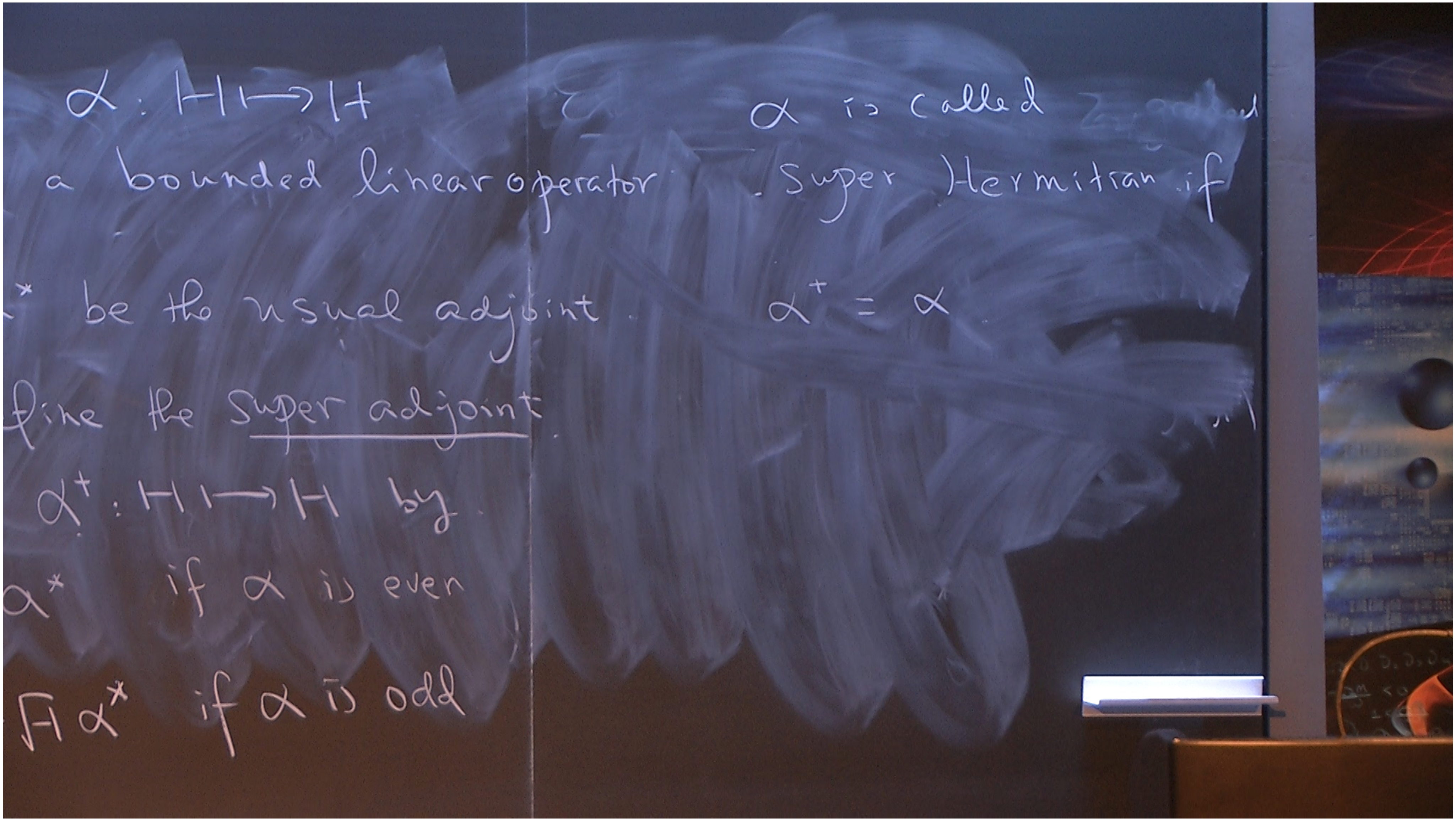

SUSY and Twist
[x, y] := x ∘ y − (−1)^{|x||y|} y ∘ x
Always assumed in this talk

Def'n A super Hilbert space $\mathcal{H}$ is a super $\mathbb{C}$-vector space

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$$

together w/ a Hermitian inner product such that $\mathcal{H}_0$ and $\mathcal{H}_1$ are orthogonal.


vector space
ertian inner
H₀ and H₁
Let α : H ⟶ H
be a bounded linear operator
α* be the usual adjoint.
We define the super adjoint
α⁺ : H ⟶ H by
α⁺ = { α* if α is even
−√−1 α* if α is odd


α : H ⟼ H
a bounded linear operator
α* be the usual adjoint
fine the super adjoint
α† : H ⟼ H by
α* if α is even
√-1 α* if α is odd
α is called
super Hermitian if
α† = α

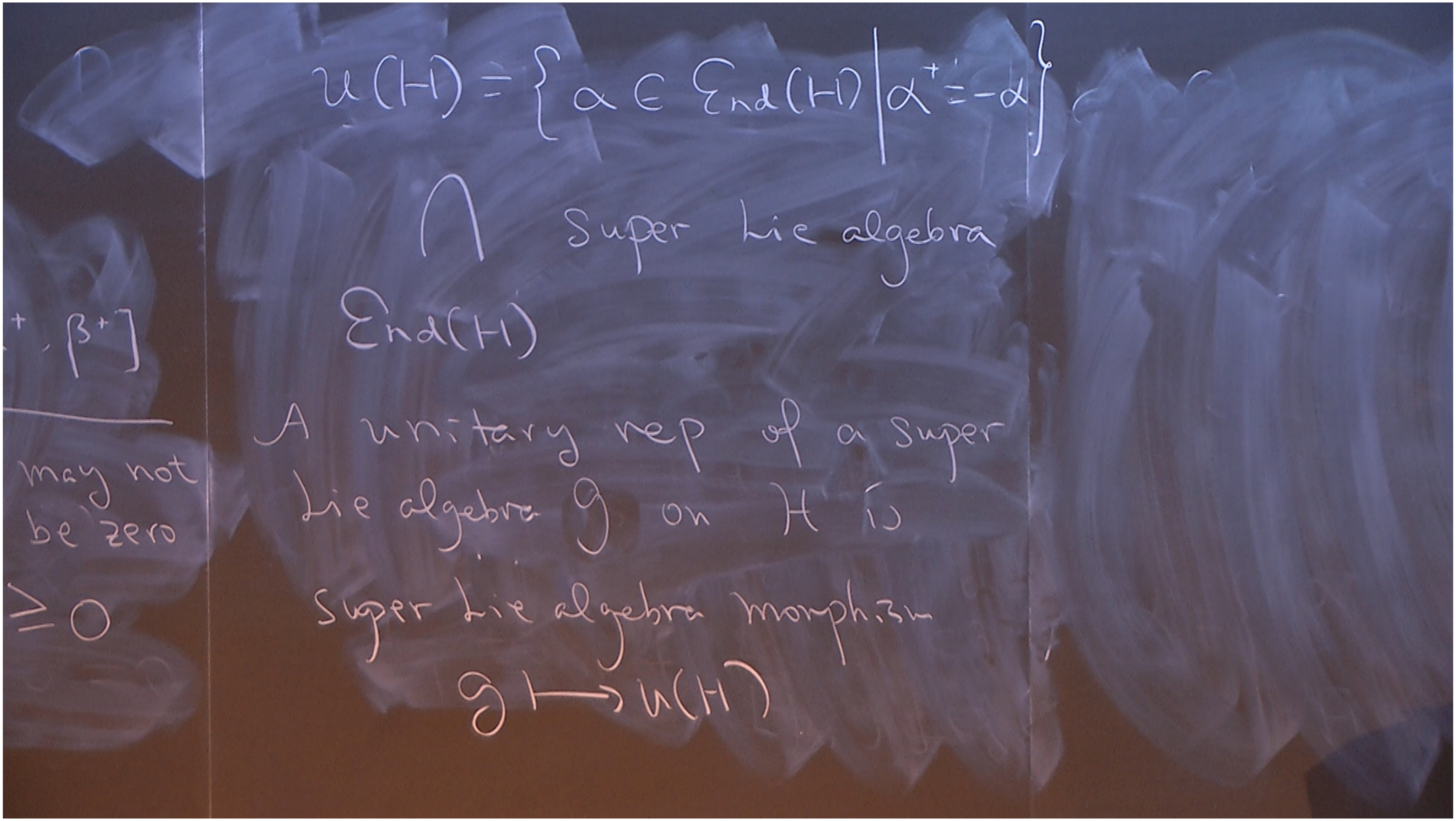
Properties $(\alpha^\dagger)^\dagger = \alpha$ **Def'n**

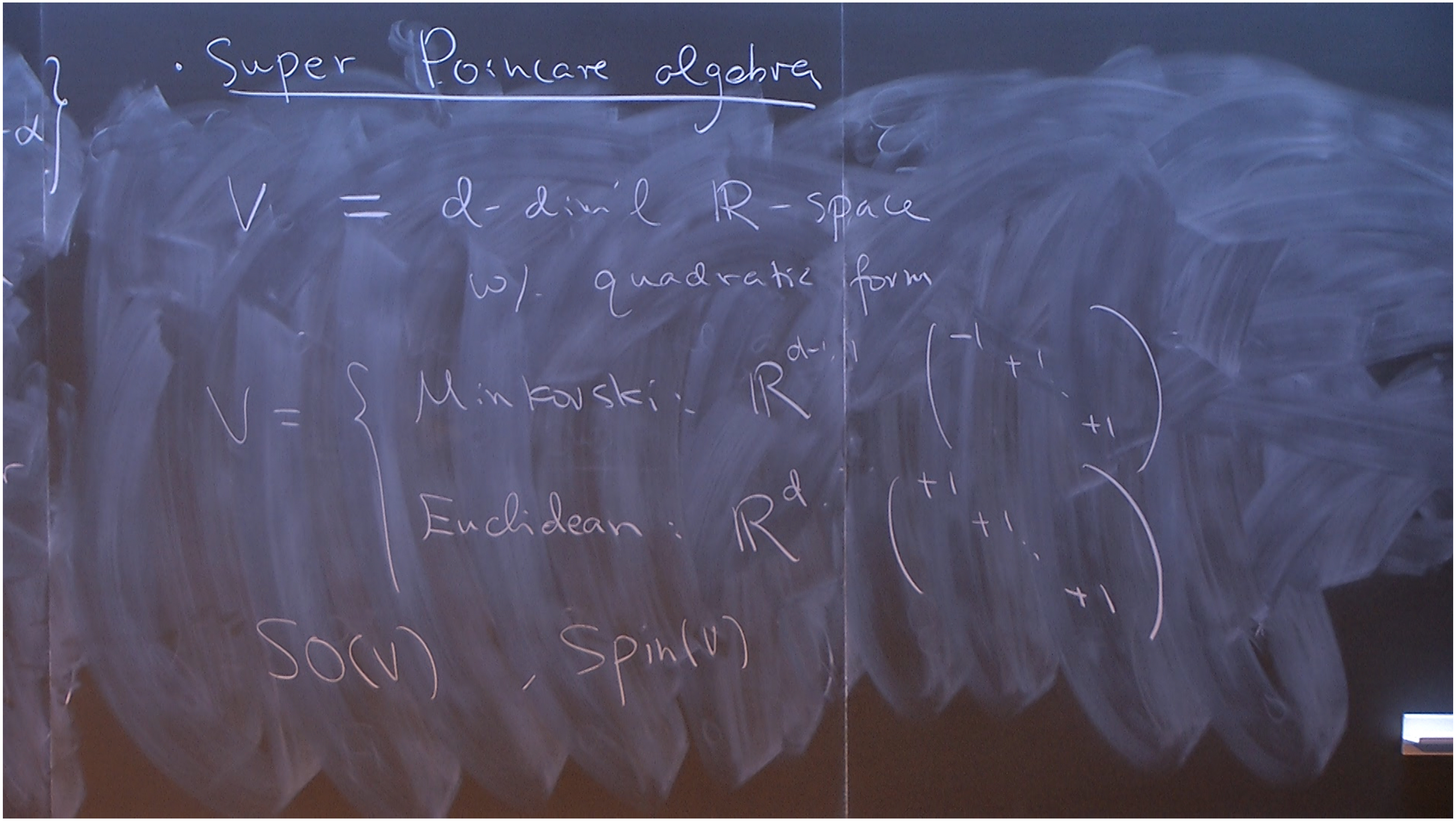
$$(\alpha\beta)^\dagger = (-1)^{|\alpha||\beta|}\beta^\dagger\alpha^\dagger$$

$$[\alpha,\beta]^\dagger = (-1)^{|\alpha||\beta|}[\beta^\dagger,\alpha^\dagger] = -[\alpha^\dagger,\beta^\dagger]$$

- For odd operator Q $\quad [Q,Q] = 2Q^2$ may not be zero

$$\sqrt{-1}\,(QQ^\dagger + Q^\dagger Q) = (QQ^* + Q^*Q) \geq 0$$


u(H) = { α ∈ End(H) | α† = -α }
∩ Super Lie algebra
End(H)
A unitary rep of a super Lie algebra g on H is
super Lie algebra morphism
g ⟼ u(H)
may not be zero
≥ 0


· Super Poincare algebra
V = d-dim'l ℝ-space
w/ quadratic form
V = { Minkowski: $\mathbb{R}^{d-1,1}$ $\begin{pmatrix} -1 & & & \\ & +1 & & \\ & & \ddots & \\ & & & +1 \end{pmatrix}$
Euclidean: $\mathbb{R}^{d}$ $\begin{pmatrix} +1 & & & \\ & +1 & & \\ & & \ddots & \\ & & & +1 \end{pmatrix}$
SO(V), Spin(V)

Let S be a $\mathbb{R}$-rep of $\mathrm{Spin}(V)$ together with a $\mathrm{Spin}(V)$-equivariant symmetric pairing

$$\Gamma : S \otimes S \longrightarrow V$$

$\Longrightarrow$

Define a Super Lie algebra

$$V \oplus S$$

(even) (odd)

$$[V_1 \oplus S_1, V_2 \oplus S_2] = -2\Gamma(S_1, S_2)$$

i.e. $[V, V] = [V, S] = 0$

$$[S, S] \xrightarrow{\Gamma} V$$

$\Longrightarrow$ it generates a super Lie group

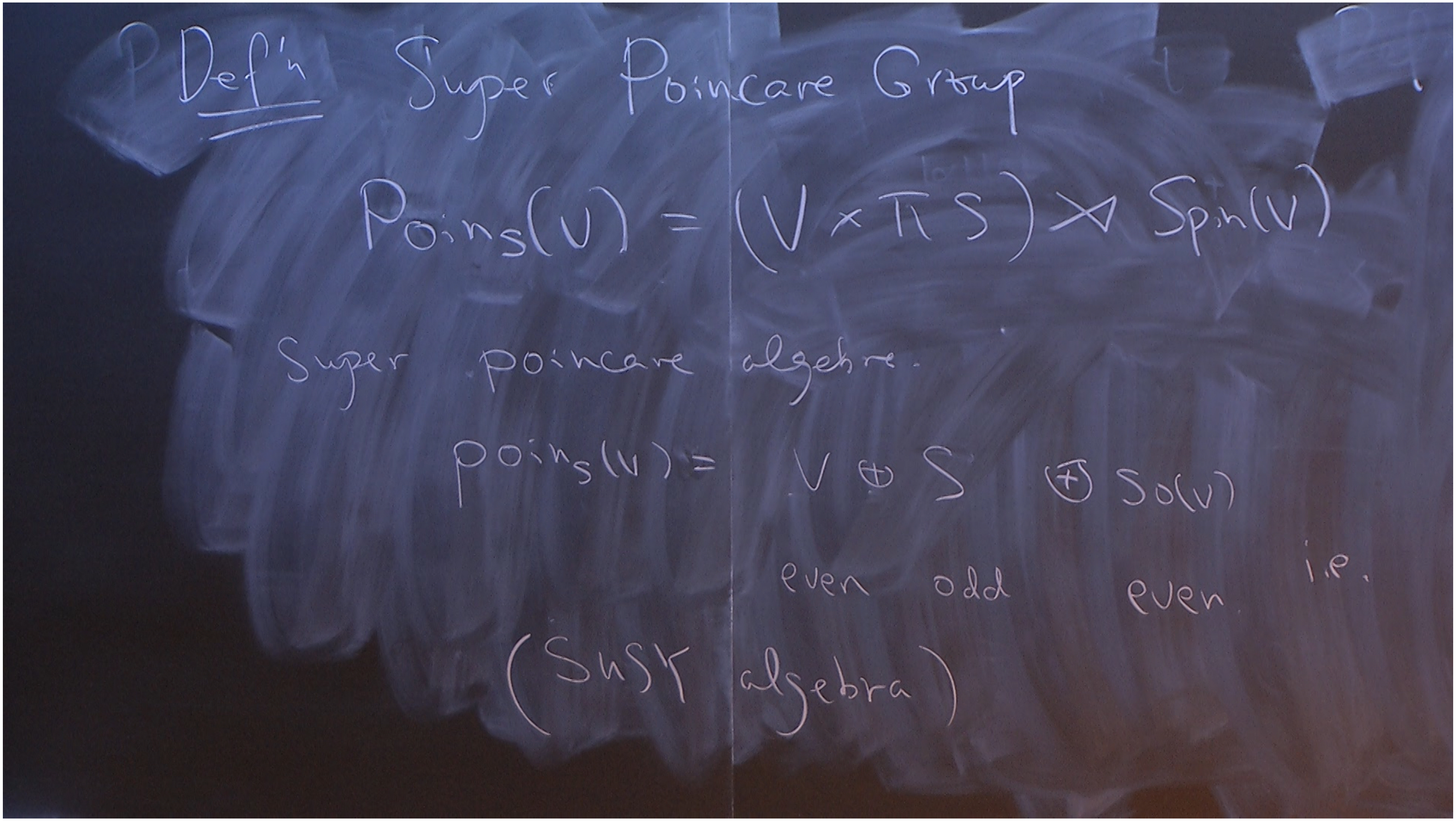
$$V \times \Pi S$$

Π: parity changing operator

(so ΠS is odd)

w/ Spin(V)-equivariant group law

$$(v_1, s_1) \cdot (v_2, s_2) = \left(v_1 + v_2 + \tfrac{1}{2}[s_1, s_2],\ s_1 + s_2\right)$$

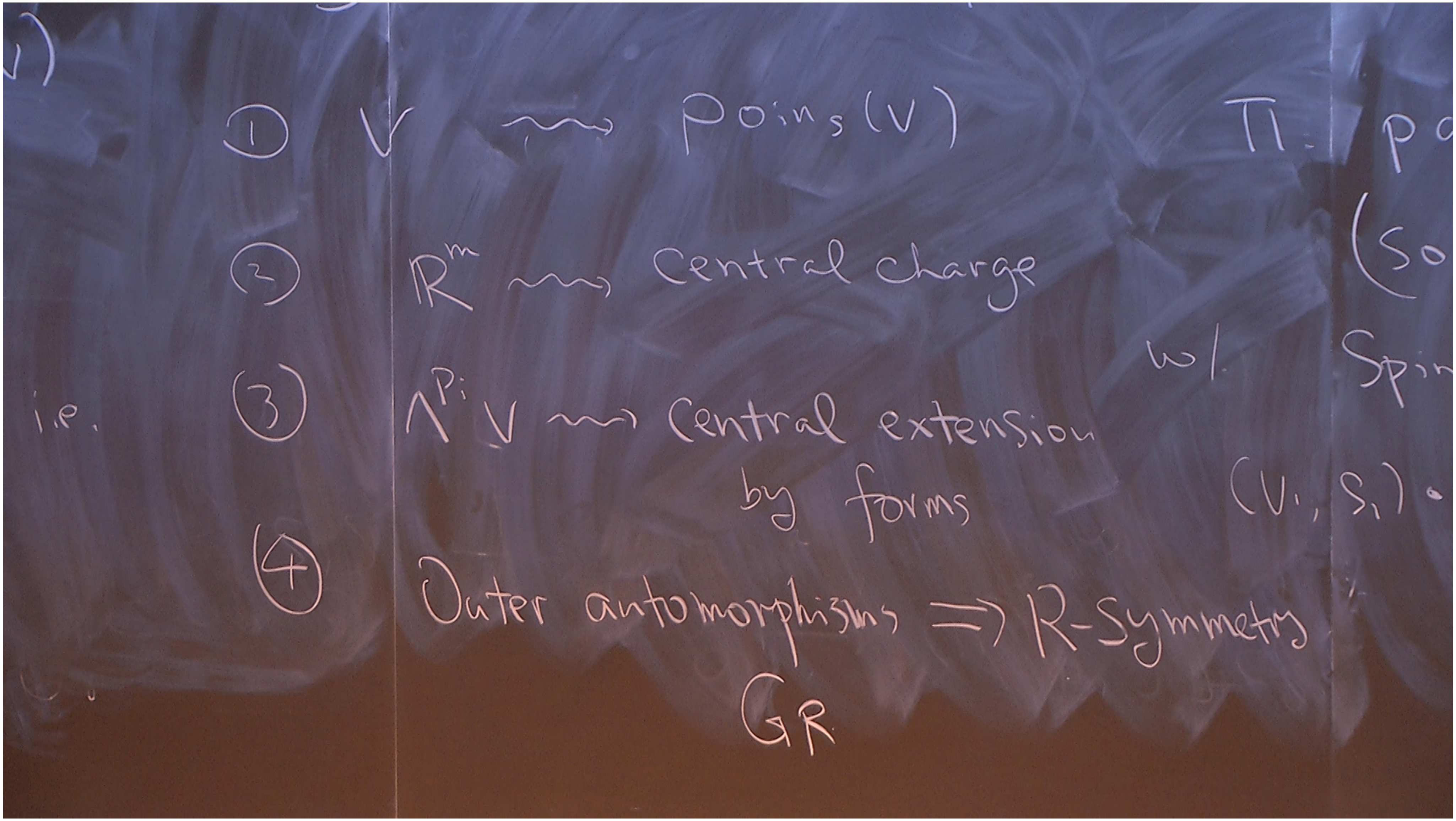

Def'n Super Poincare Group
Poin_S(V) = (V × ΠS) ⋊ Spin(V)
Super poincare algebra.
poin_S(V) = V ⊕ S ⊕ so(V)
even odd even i.e.
(SUSY algebra)

RK $\mathrm{Sym}^2(S) = V \oplus \mathbb{R}^m \oplus \bigoplus_i \Lambda^{p_i} V$

① $V \rightsquigarrow$ $\mathfrak{poin}_s(V)$

② $\mathbb{R}^m \rightsquigarrow$ central charge

③ $\Lambda^{p_i} V \rightsquigarrow$ central extension by forms


① V ⇝ Poin_s(V)
② R^m ⇝ central charge
w/
Spin
i.e.
③ Λ^{p_i} V ⇝ central extension by forms
(V_1, S_1)
④ Outer automorphisms ⇒ R-Symmetry
G_R

Superspace: Let's denote

$$V_S = V \times \Pi S$$

$$\mathcal{O}(V_S) = C^{\infty}(V) \otimes \Lambda^{\cdot}(S^{\vee})$$

Choose linear coordinates X^{μ}, θ^{α}

$$X^{\mu} X^{\nu} = X^{\nu} X^{\mu} \qquad \theta^{\alpha}\theta^{\beta} = -\theta^{\beta}\theta^{\alpha}.$$

$$V_S = V \times \Pi S$$

$$\mathcal{O}(V_S) = C^\infty(V) \otimes \Lambda^\cdot(S^\vee)$$

Choose linear coordinates x^μ, θ^α

$$x^\mu x^\nu = x^\nu x^\mu \qquad \theta^\alpha \theta^\beta = -\theta^\beta \theta^\alpha .$$

$$f(x,\theta) = \sum_I f_I(x)\,\theta^I , \quad \theta^I = \theta^{i_1} \cdots \theta^{i_k}$$

$$I = \{ i_1 < \dots < i_k \}$$

Let's denote

$\times \Pi S$

$(V) \otimes \Lambda^{\cdot}(S^{\vee})$

Coordinates x^{μ}, θ^{α}

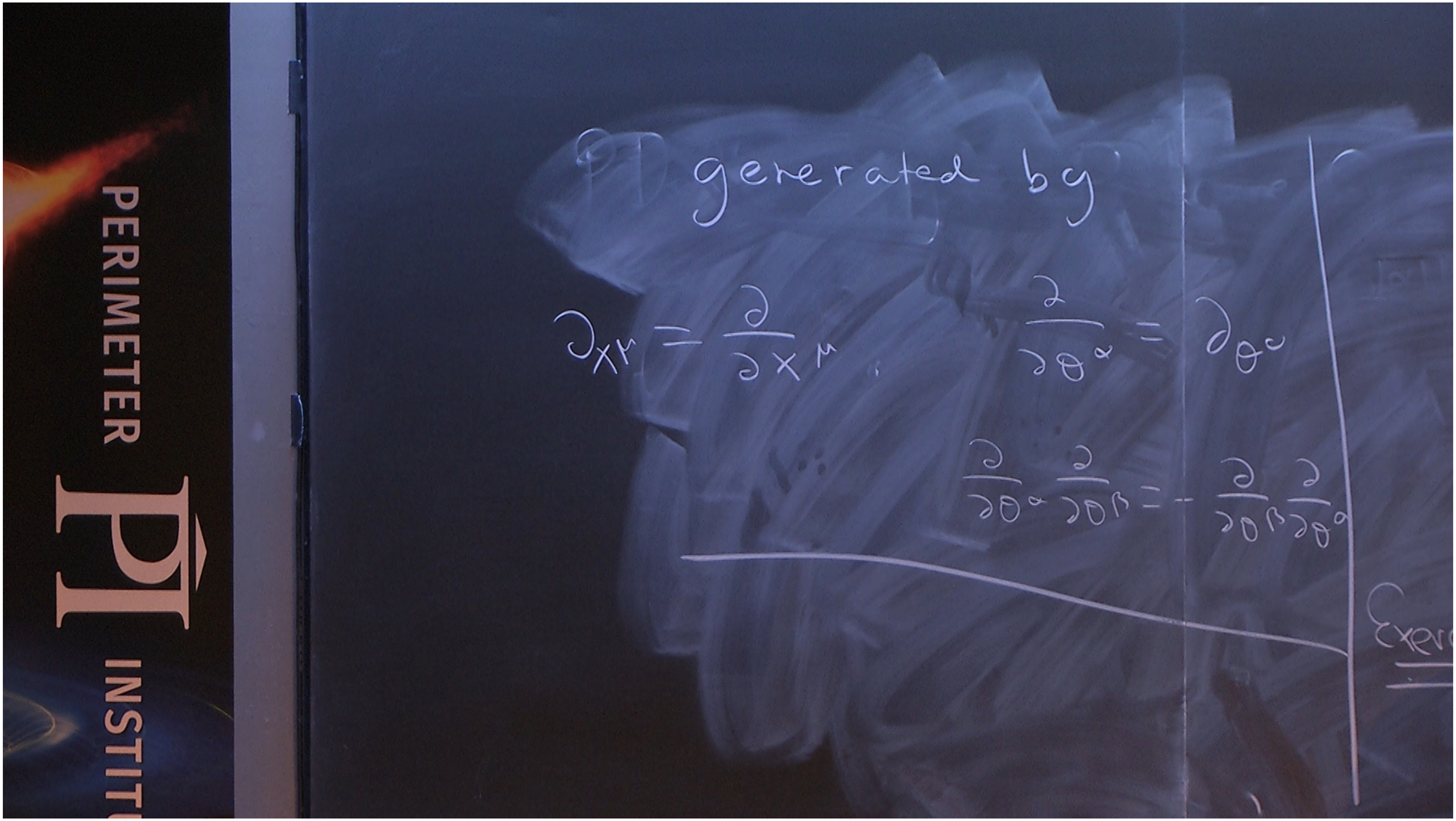
$x^{\nu} x^{\mu}$ $\quad \theta^{\alpha}\theta^{\beta} = -\theta^{\beta}\theta^{\alpha}$

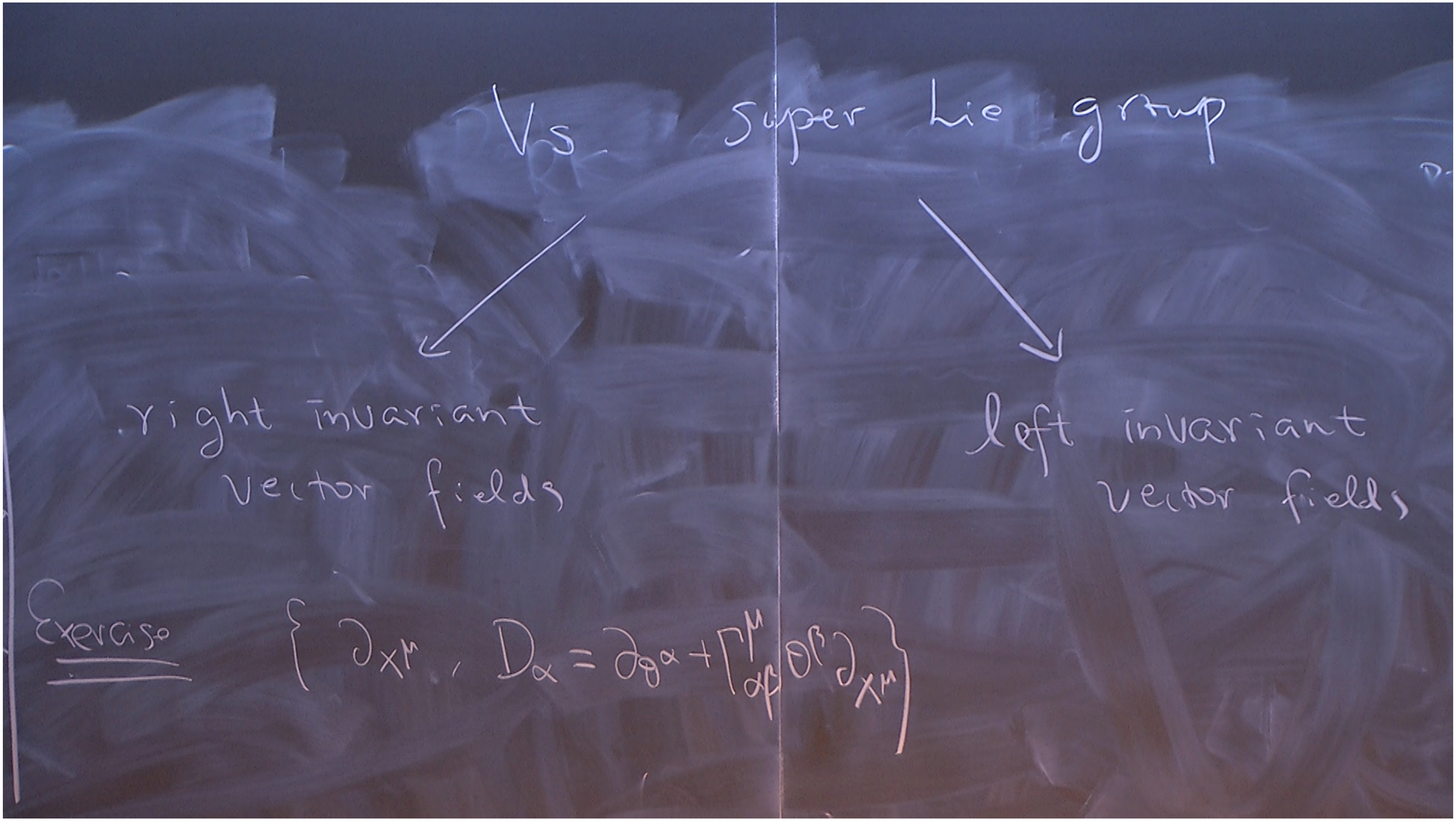
$f_I(x)\theta^I, \quad \theta^I = \theta^{i_1} \cdots \theta^{i_k}$

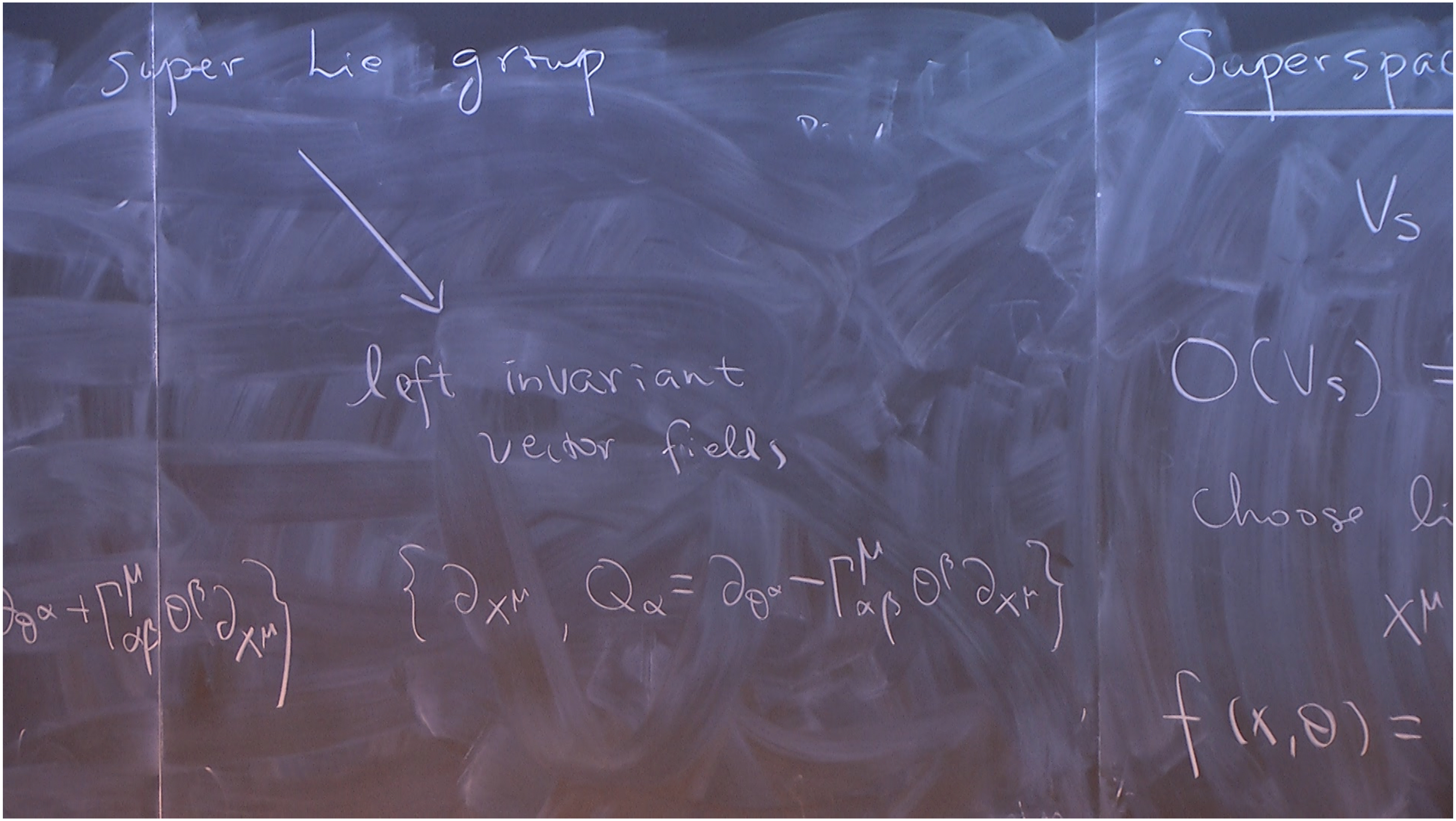
$I = \{i_1 < \cdots < i_k\}$

$\mathrm{Der}(V_s)$ = Super derivations

$$= \left\{ D \in \mathrm{End}(V_s) \;\middle|\; D(f \cdot g) = D(f)\cdot g + (-1)^{|D||f|} f\, Dg \right\}$$


PERIMETER
PI
generated by
$\partial_{x^\mu} = \frac{\partial}{\partial x^\mu}$
$\frac{\partial}{\partial \theta^\alpha} = \partial_{\theta^\alpha}$
$\frac{\partial}{\partial \theta^\alpha}\frac{\partial}{\partial \theta^\beta} = -\frac{\partial}{\partial \theta^\beta}\frac{\partial}{\partial \theta^\alpha}$


∀s super Lie group
right invariant vector fields
left invariant vector fields
Exercise
{ ∂_{x^μ} , D_α = ∂_{θ^α} + Γ^μ_{αβ} θ^β ∂_{x^μ} }


super Lie group
· Superspac
left invariant vector fields
V_s
$O(V_s) =$
Choose li
x^μ
$f(x,\theta) =$
$\partial_{\theta^\alpha} + \Gamma^\mu_{\alpha\beta}\theta^\beta\partial_{x^\mu})$
$\{\partial_{x^\mu}, Q_\alpha = \partial_{\theta^\alpha} - \Gamma^\mu_{\alpha\beta}\theta^\beta\partial_{x^\mu}\}$

$\Gamma^{\mu}_{\alpha\beta} \rightsquigarrow \Gamma : S \otimes S \longmapsto V$

V_S super Lie group

by

$\frac{\partial}{\partial\theta^\alpha} = \partial_{\theta^\alpha}$

$\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta} = -\frac{\partial}{\partial\theta^\beta}\frac{\partial}{\partial\theta^\alpha}$

right invariant vector fields

left invariant vector fields

Exercise

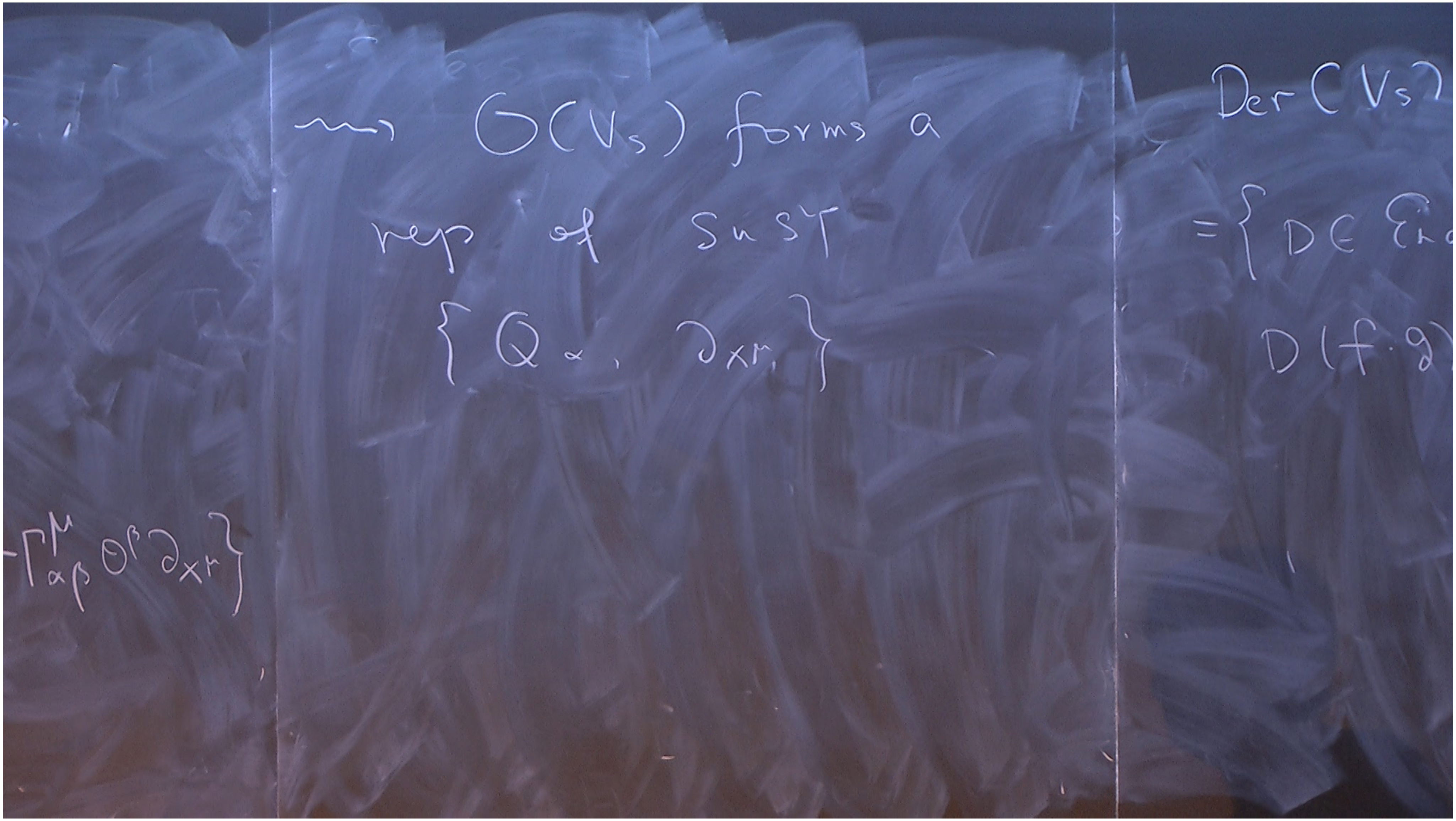
$\{ \partial_{x^\mu} , D_\alpha = \partial_{\theta^\alpha} + \Gamma^{\mu}_{\alpha\beta}\theta^\beta \partial_{x^\mu} \}$

$\{ \partial_{x^\mu} , Q_\alpha = \partial_{\theta^\alpha}$

Satisfying:

$[Q_\alpha , Q_\beta] = -2\Gamma^{\mu}_{\alpha\beta}\partial_{x^\mu}$

$[D_\alpha , D_\beta] = 2\Gamma^{\mu}_{\alpha\beta}\partial_{x^\mu}$


$-\Gamma^{\mu}_{\alpha\beta}\theta^{\beta}\partial_{x^{\mu}}\}$
⇝ G(V_s) forms a
rep of SUSY
$\{Q_\alpha, \partial_{x^\mu}\}$
Der(V_s)
= { D ∈ En
D(f·g)

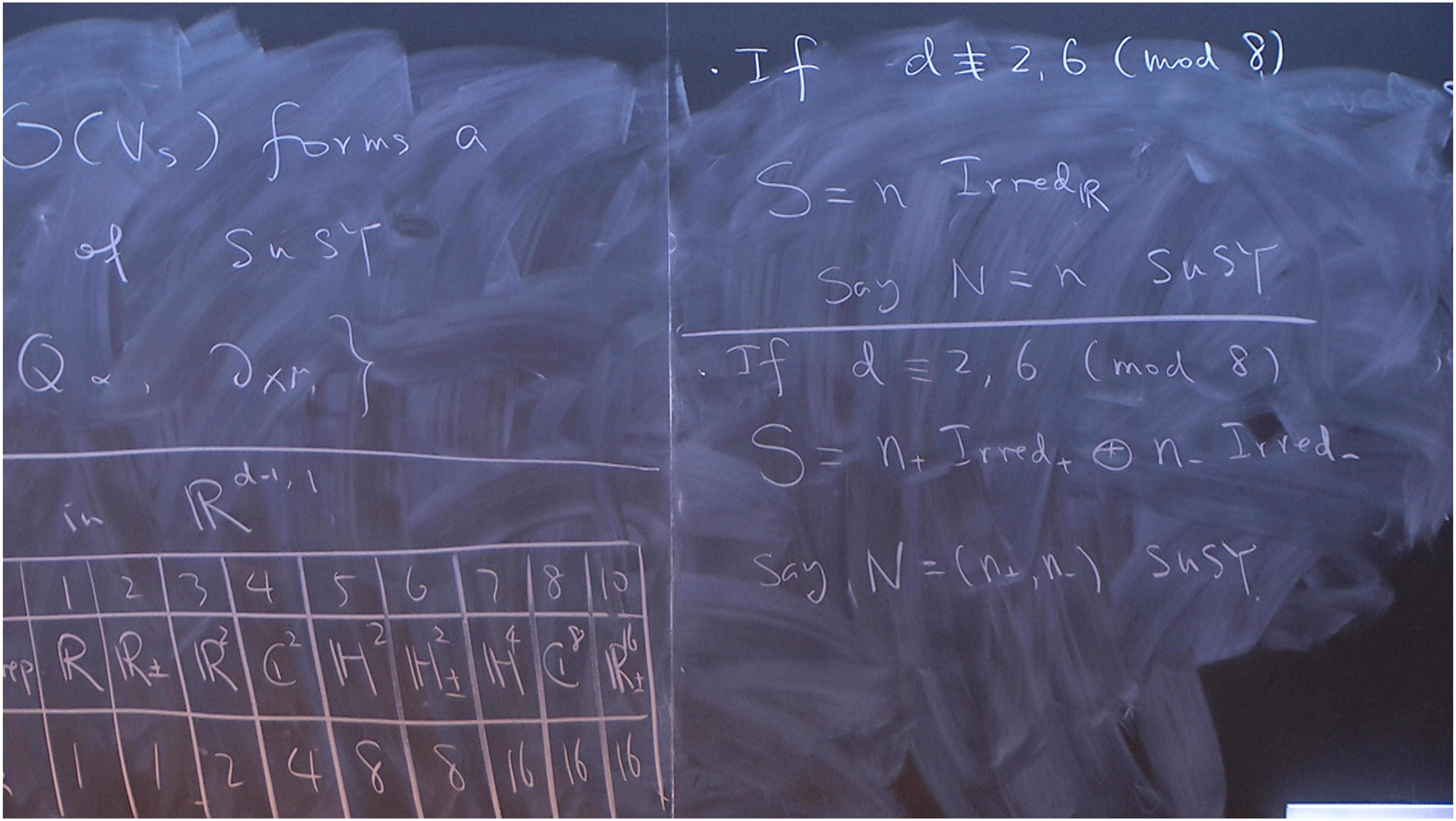
$-\Gamma^{\mu}_{\alpha\beta}\,\theta^{\beta}\,\partial_{x^{\mu}}\}$

$\rightsquigarrow$ $G(V_s)$ forms a rep of SUSY

$$\{Q_\alpha, \partial_{x^\mu}\}$$

SUSY in $\mathbb{R}^{d-1,1}$

d	1	2	3	4	5	6	7	8	10
Irred $\mathbb{R}$-rep	$\mathbb{R}$	$\mathbb{R}_\pm$	$\mathbb{R}^2$	$\mathbb{C}^2$	$\mathbb{H}^2$	$\mathbb{H}^2_\pm$	$\mathbb{H}^4$	$\mathbb{C}^8$	$\mathbb{R}^{16}_\pm$
$\dim_{\mathbb{R}}$	1	1	2	4	8	8	16	16	16


S(V_s) forms a
of SUSY
Q_α , ∂_{x^μ} }
in R^{d-1,1}
1 2 3 4 5 6 7 8 10
rep R R± R² C² H² H²± H⁴ C⁸ R¹⁶±
1 1 2 4 8 8 16 16 16
· If d ≢ 2, 6 (mod 8)
S = n Irred_R
Say N = n SUSY
· If d ≡ 2, 6 (mod 8)
S = n₊ Irred₊ ⊕ n₋ Irred₋
Say N = (n₊, n₋) SUSY

Example: $d=2$ $\quad \mathbb{R}^{1,1}$

$$SO(1,1) = \left\{ e^{\alpha J} = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \,\middle|\, \alpha \in \mathbb{R},\ J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

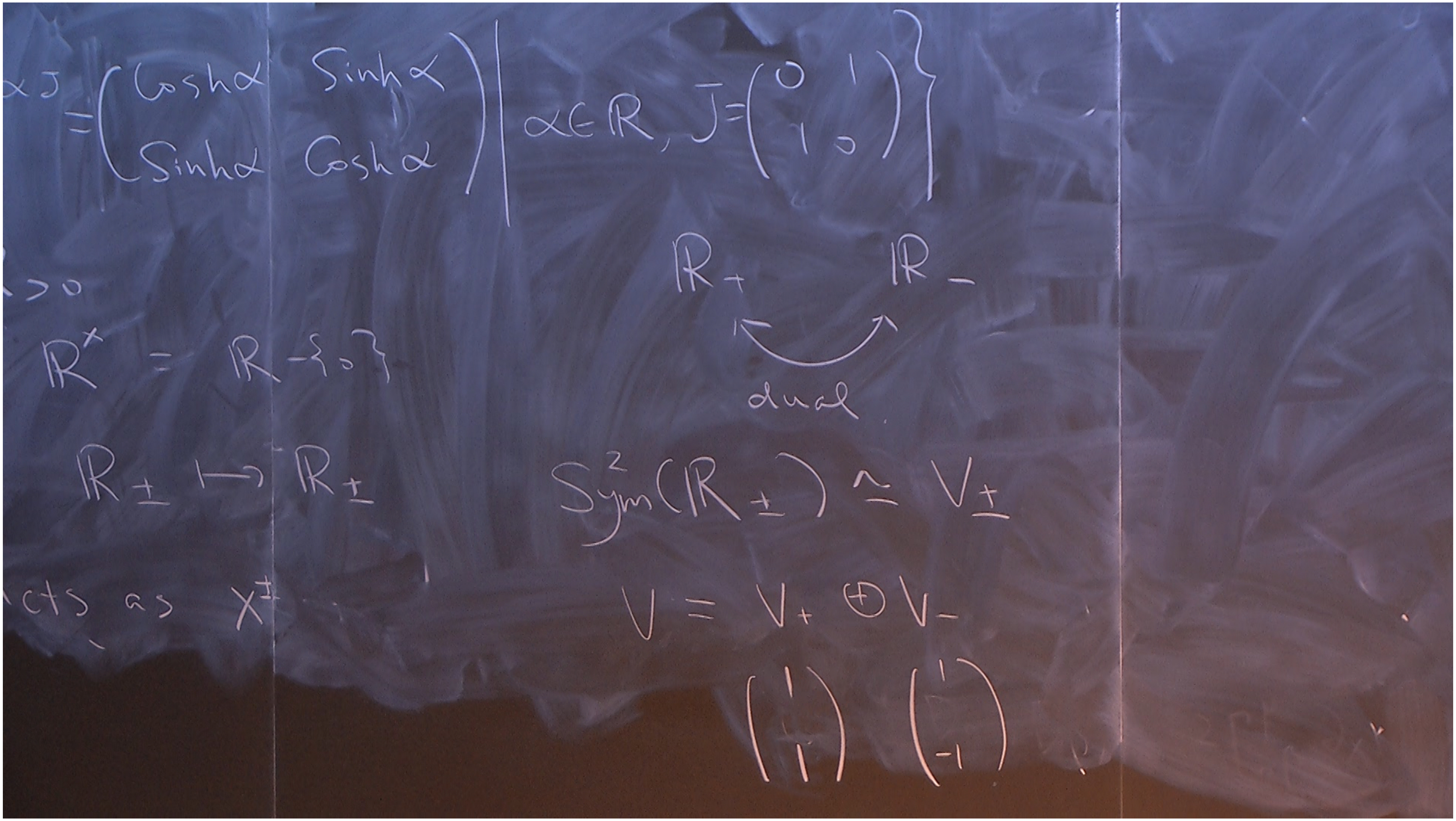
$$\simeq \mathbb{R}_{>0}$$

$$Spin(1,1) \simeq \mathbb{R}^{\times} = \mathbb{R} - \{0\}$$

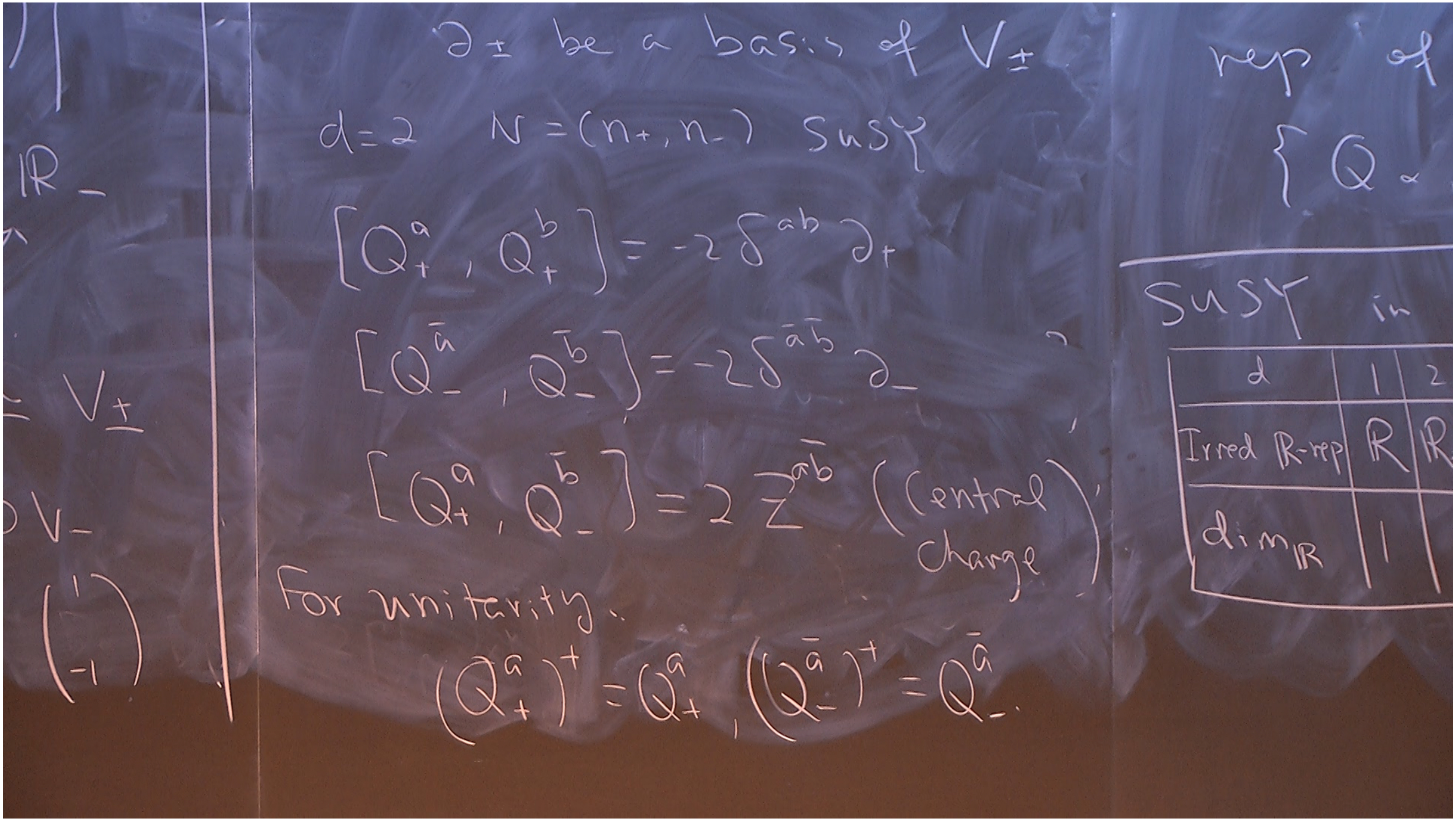
$$\mathbb{R}^{\times} : \quad \mathbb{R}_{\pm} \longmapsto \mathbb{R}_{\pm}$$

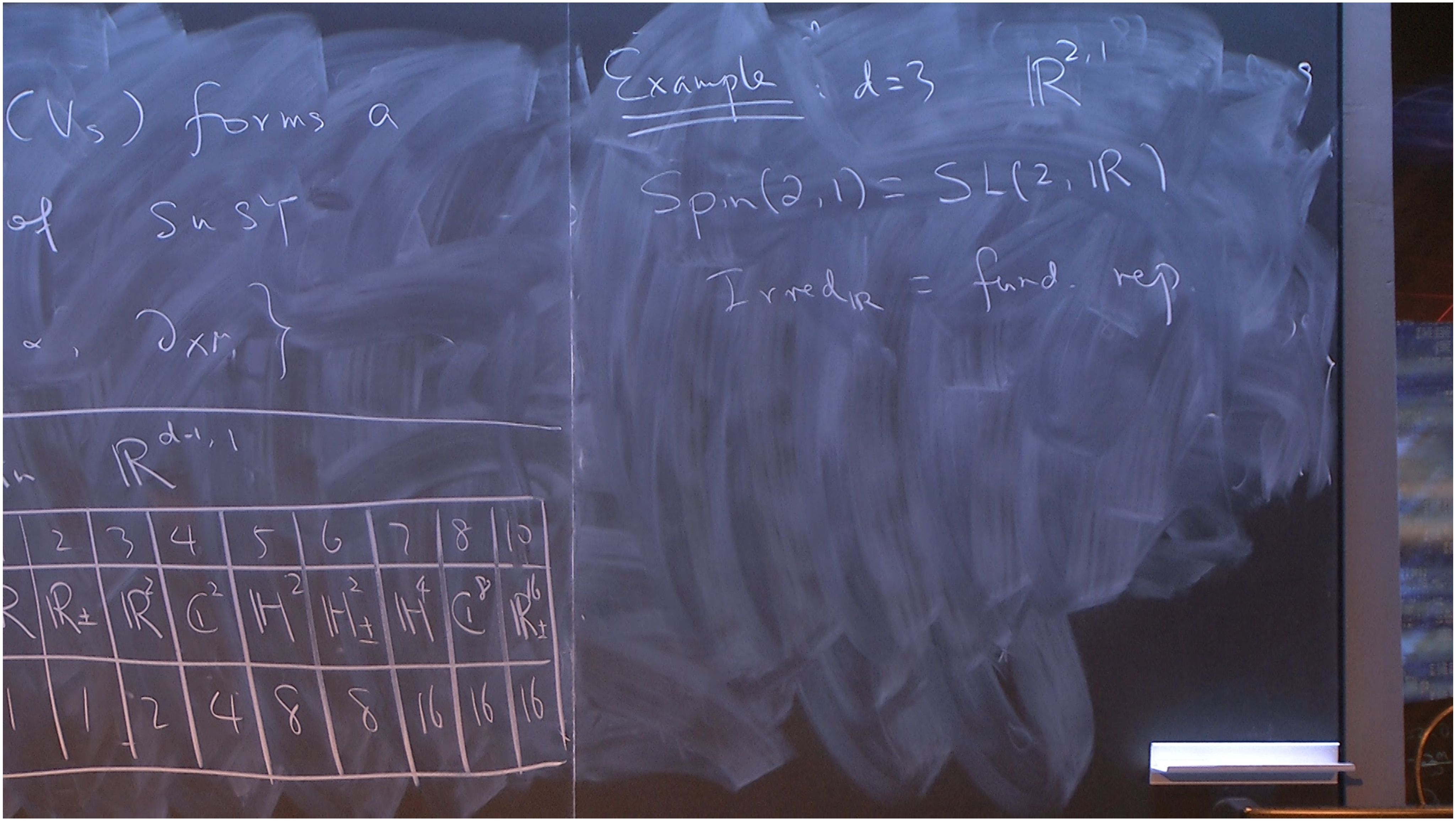
$\rotatebox{90}{$\in$}$

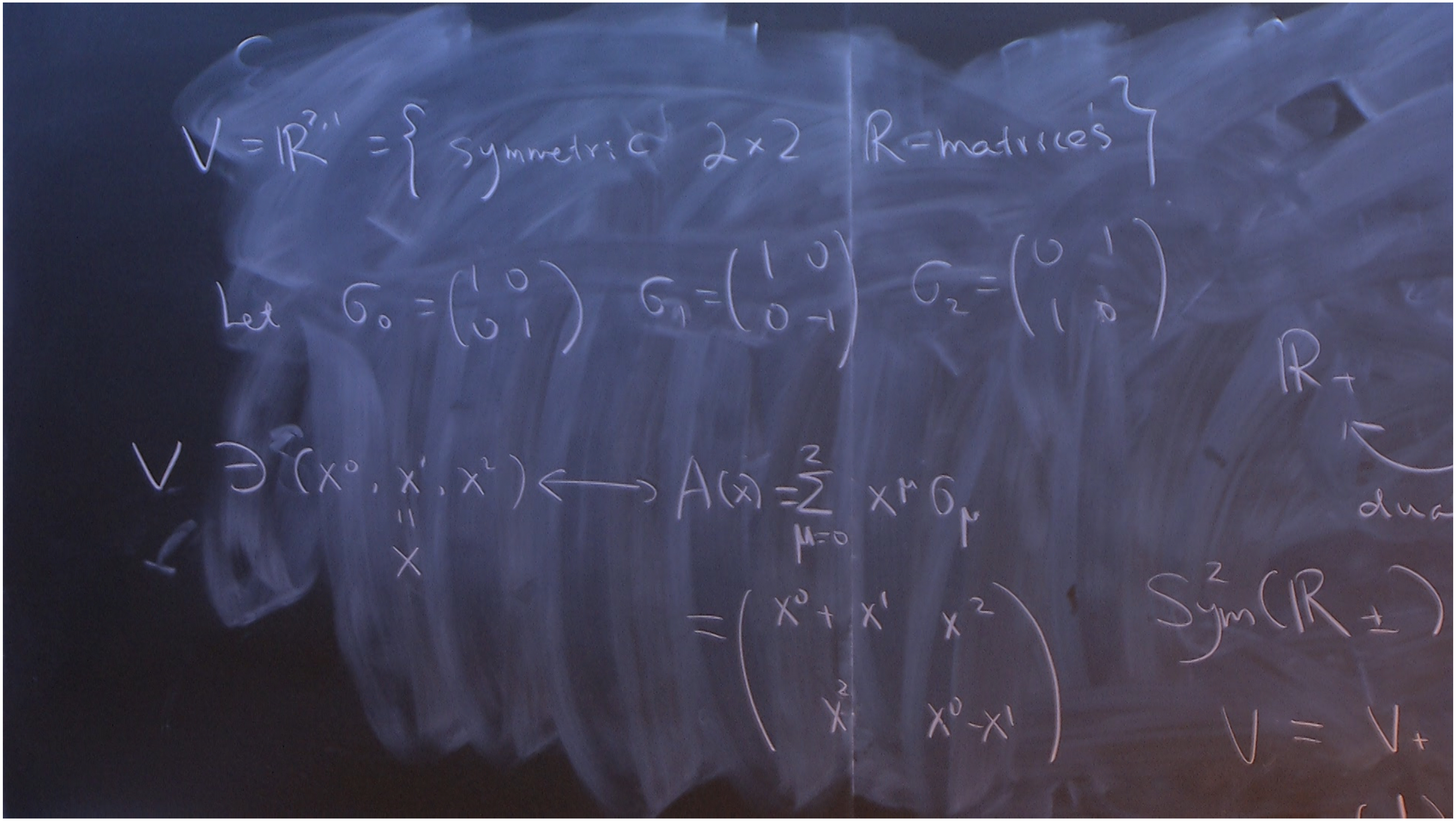
x acts as $x^{\pm}$ ...



$= \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \Big| \alpha \in \mathbb{R}, \bar{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Big\}$
$\mathbb{R}_+$ $\mathbb{R}_-$
dual
$\mathbb{R}^\times = \mathbb{R} - \{0\}$
$\mathbb{R}_\pm \longmapsto \mathbb{R}_\pm$
$Sym^2(\mathbb{R}_\pm) \simeq V_\pm$
cts as $X^\pm$
$V = V_+ \oplus V_-$


∂± be a basis of V±
d=2 N=(n+, n-) SUSY
[Q+^a, Q+^b] = -2δ^ab ∂+
[Q-^ā, Q-^b̄] = -2δ^āb̄ ∂-
[Q+^a, Q-^b̄] = 2 Z^ab̄ (central charge)
For unitarity:
(Q+^a)† = Q+^a, (Q-^ā)† = Q-^ā
rep of
SUSY in
d
Irred R-rep
dim_R


(V_s) forms a
of SuSY
α, ∂x^μ }
in R^{d-1,1}
2 3 4 5 6 7 8 10
R R± R² C² H² H²± H⁴ C⁸ R¹⁶±
1 1 2 4 8 8 16 16 16
Example : d=3 R^{2,1}
Spin(2,1) = SL(2, R)
Irred_R = fund. rep.


$V = \mathbb{R}^{2,1} = \{$ symmetric 2×2 $\mathbb{R}$-matrices $\}$
Let $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$V \ni (x^0, x^1, x^2) \longleftrightarrow A(x) = \sum_{\mu=0}^{2} x^\mu \sigma_\mu$
x
$= \begin{pmatrix} x^0 + x^1 & x^2 \\ x^2 & x^0 - x^1 \end{pmatrix}$
$\mathbb{R}_+$
dua
$Sym^2(\mathbb{R}_\pm)$
$V = V_+$

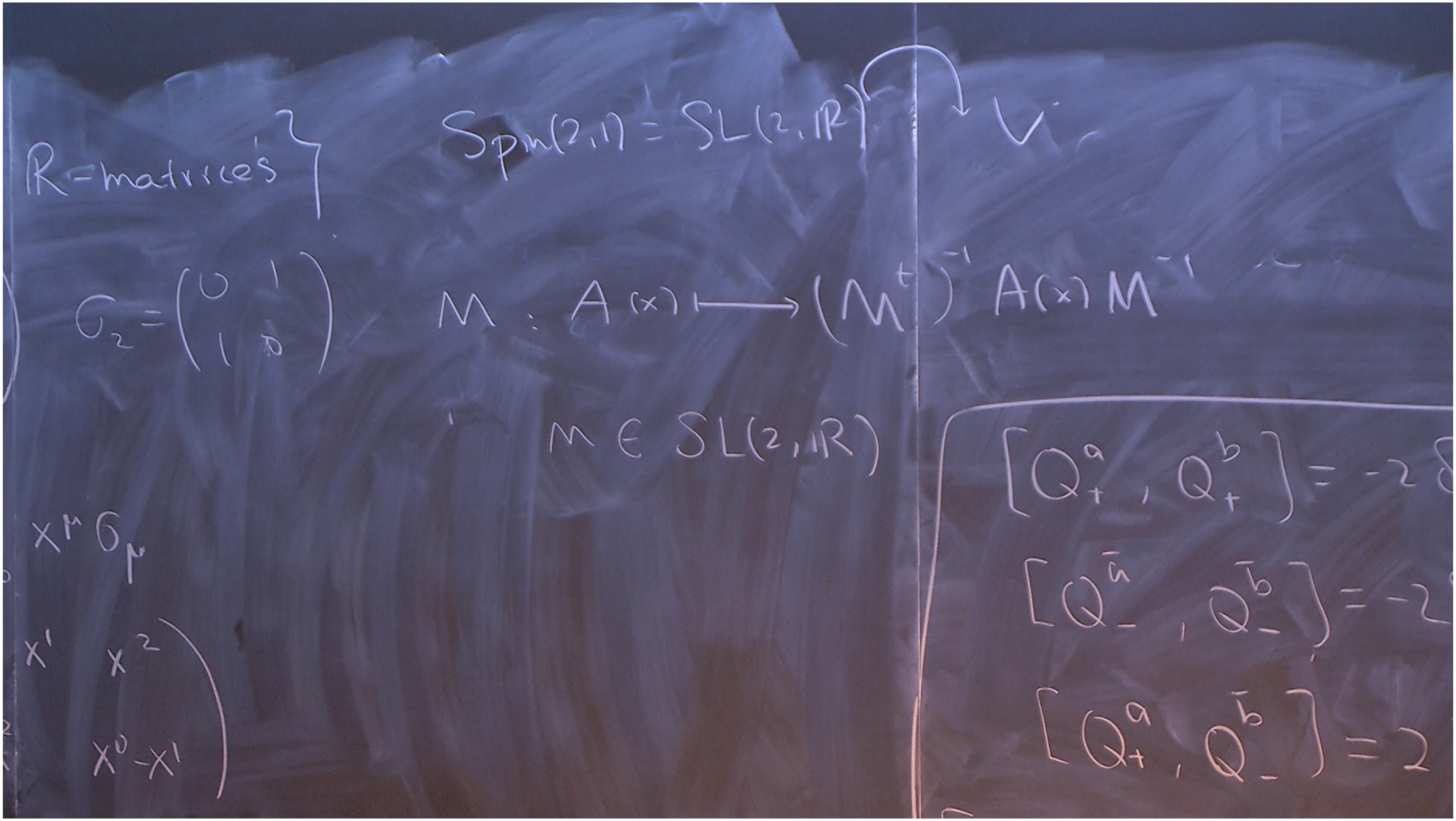
Let $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

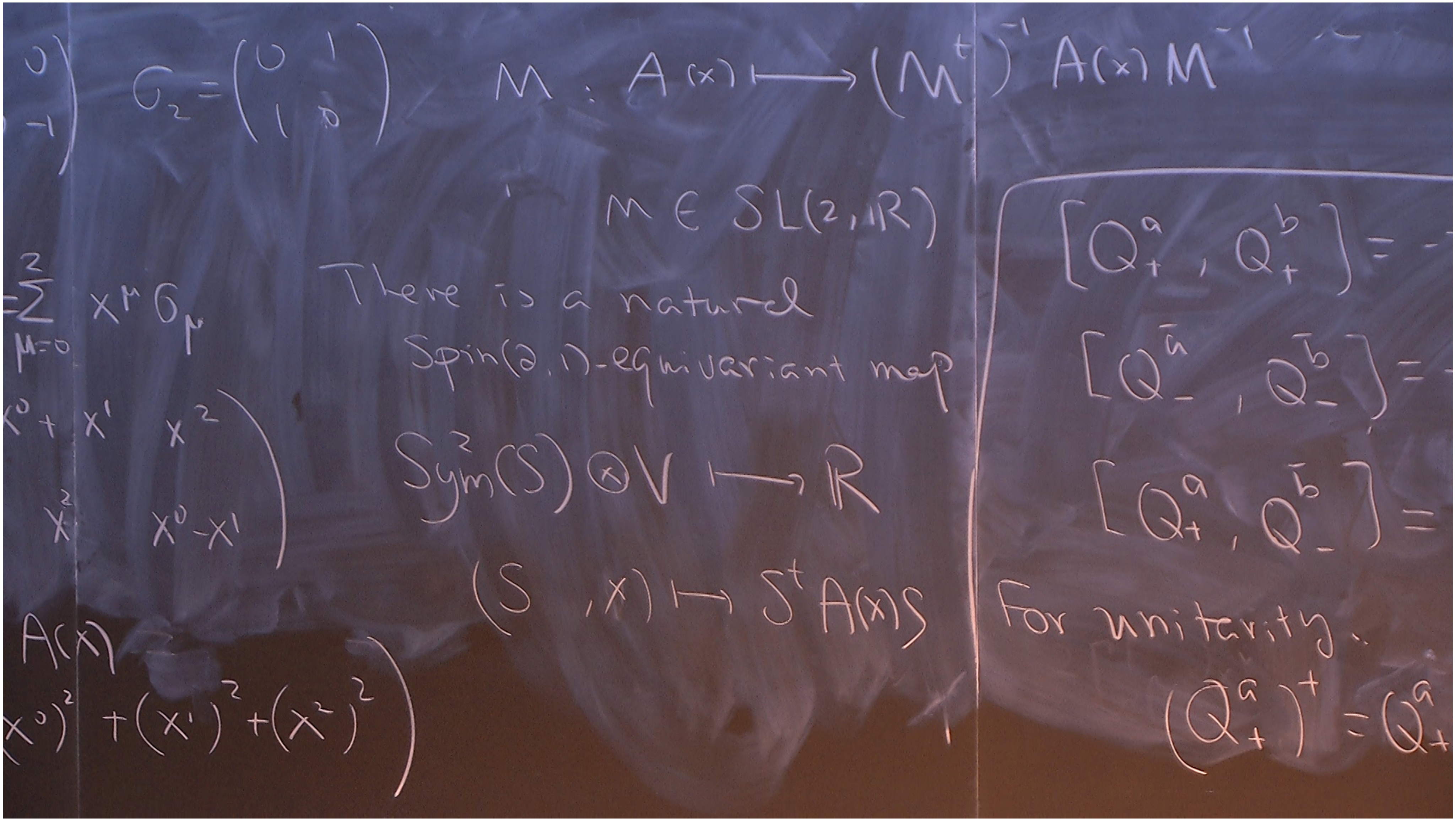
$$V \ni \underbrace{(x^0, x^1, x^2)}_{= \, x} \longleftrightarrow A(x) = \sum_{\mu=0}^{2} x^\mu \sigma_\mu$$

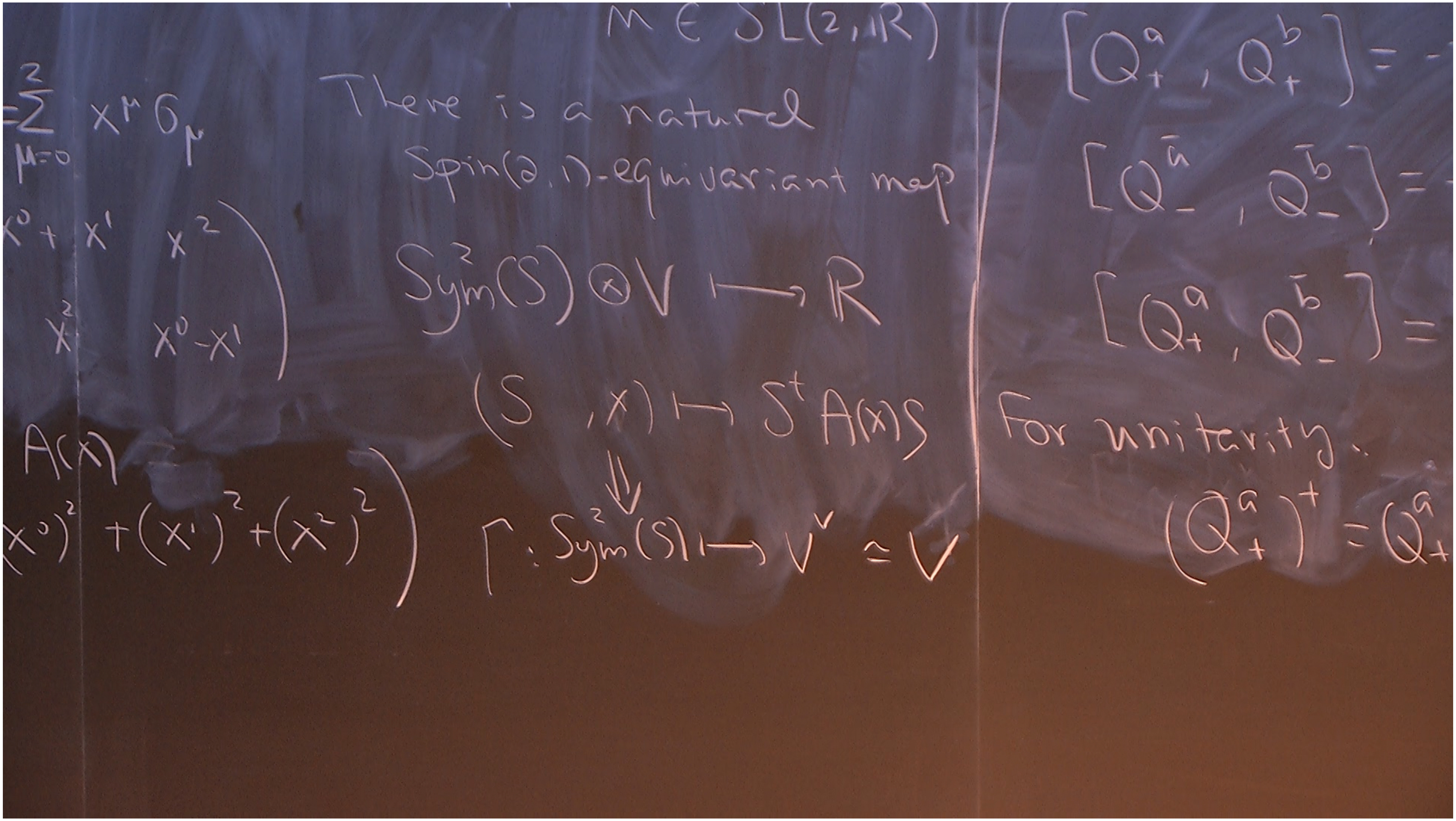
$$= \begin{pmatrix} x^0 + x^1 & x^2 \\ x^2 & x^0 - x^1 \end{pmatrix}$$

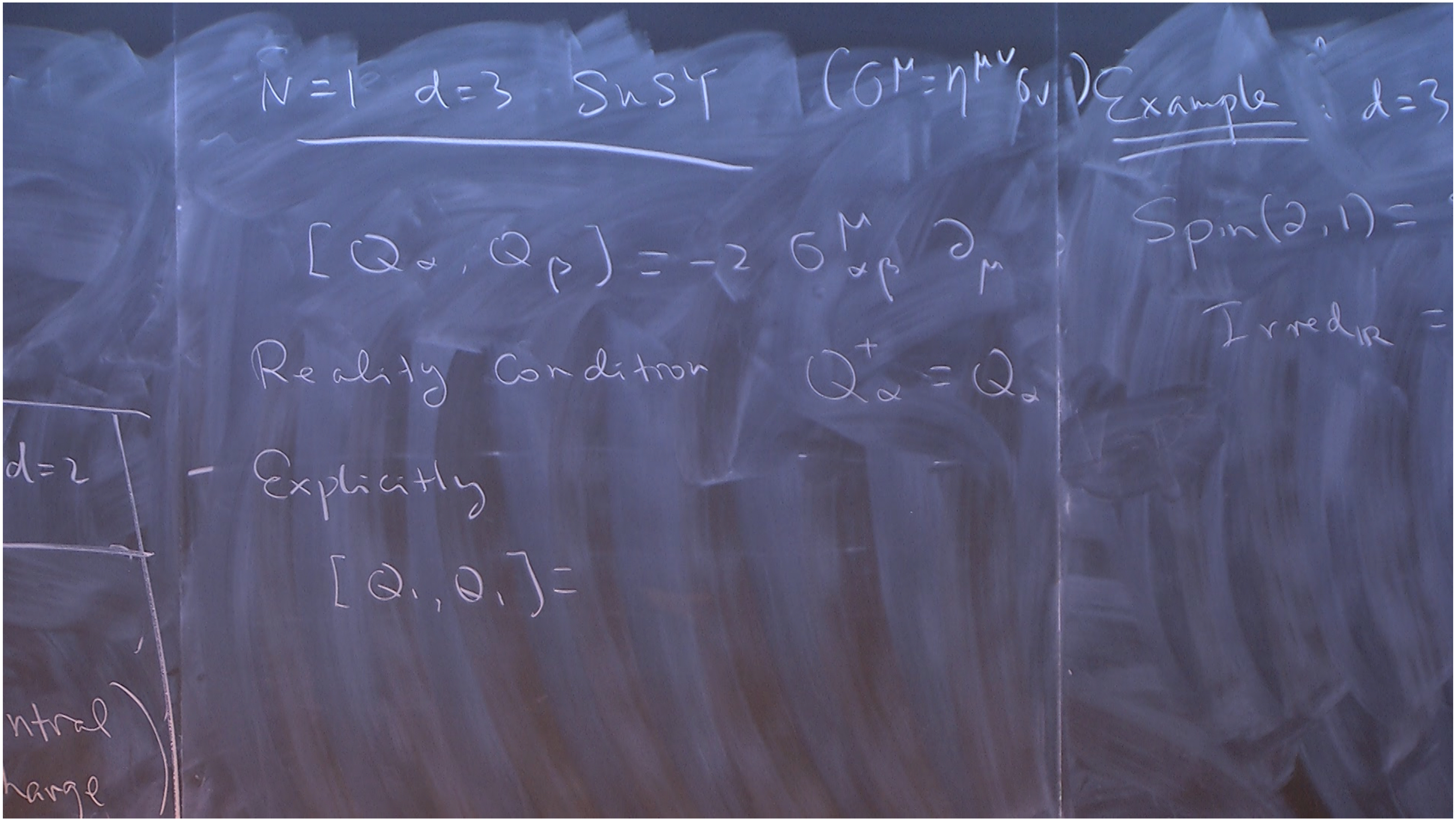
$$|x|^2 = -\det A(x)$$

$$\left(= -(x^0)^2 + (x^1)^2 + (x^2)^2 \right)$$


R-matrices
Spin(2,1) = SL(2,R)
σ2 = (0 1 / 1 0)
M : A(x) ⟼ (M^t)^-1 A(x) M^-1
M ∈ SL(2,R)
x^μ σ_μ
x^1 x^2
x^0 - x^1
[Q^a_+ , Q^b_+] = -2
[Q^ā_- , Q^b̄_-] = -2
[Q^a_+ , Q^b̄_-] = 2


M : A(x) ⟼ (M^t)^-1 A(x) M^-1
M ∈ SL(2, R)
There is a natural
Spin(2,1)-equivariant map
Sym^2(S) ⊗ V ⟼ R
(S, x) ⟼ S^t A(x) S
For unitarity:


M ∈ SL(2, ℝ)
There is a natural
Spin(2,1)-equivariant map
Sym²(S) ⊗ V ⟶ ℝ
(S , x) ⟼ Sᵗ A(x) S
⇓
Γ : Sym²(S) ⟶ V^∨ ≃ V
A(x)
For unitarity:


N=1 d=3 SUSY
(σ^μ = η^{μν} σ_ν)
[Q_α , Q_β] = −2 σ^μ_{αβ} ∂_μ
Reality Condition Q^+_α = Q_α
Explicitly
[Q_1 , Q_1] =
Example : d=3
Spin(2,1) =
Irred_IR =
d=2
ntral
harge

d=2

ntral
harge

$Q^{\bar{a}}$

$$[Q_\alpha, Q_\beta] = -2\,\sigma^\mu_{\alpha\beta}\,\partial_\mu$$

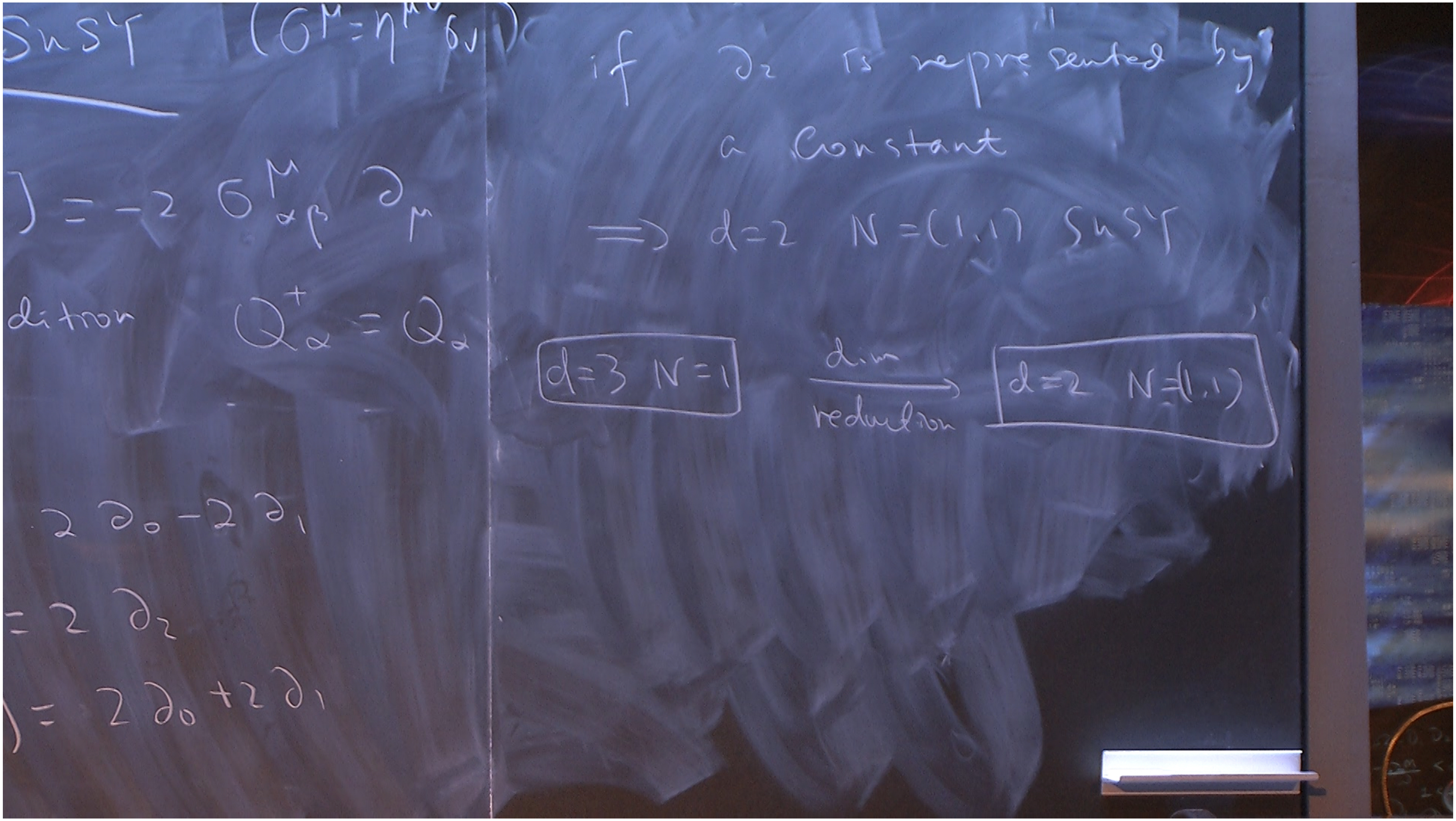
Reality condition $Q_\alpha^\dagger = Q_\alpha$

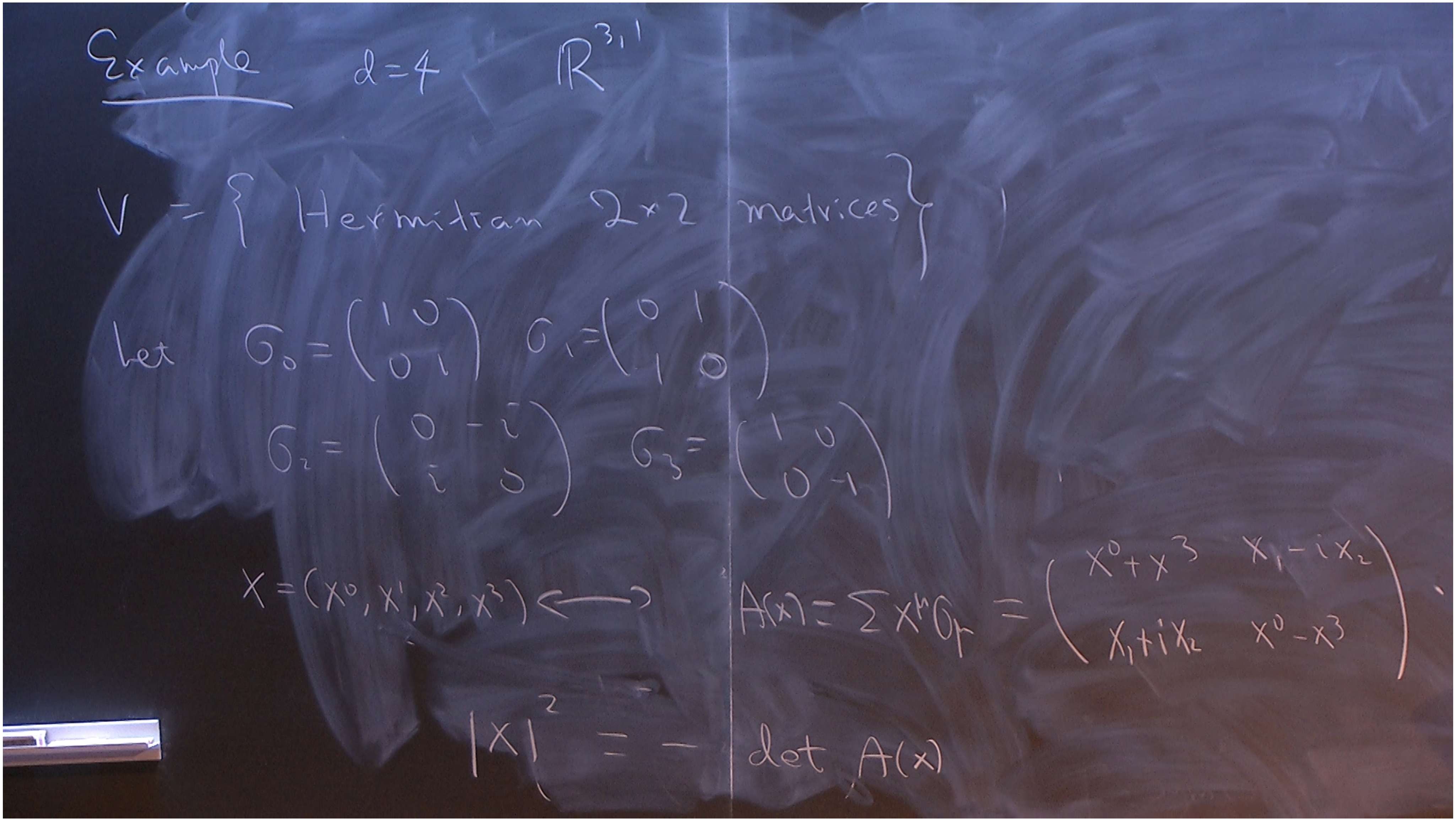
Explicitly

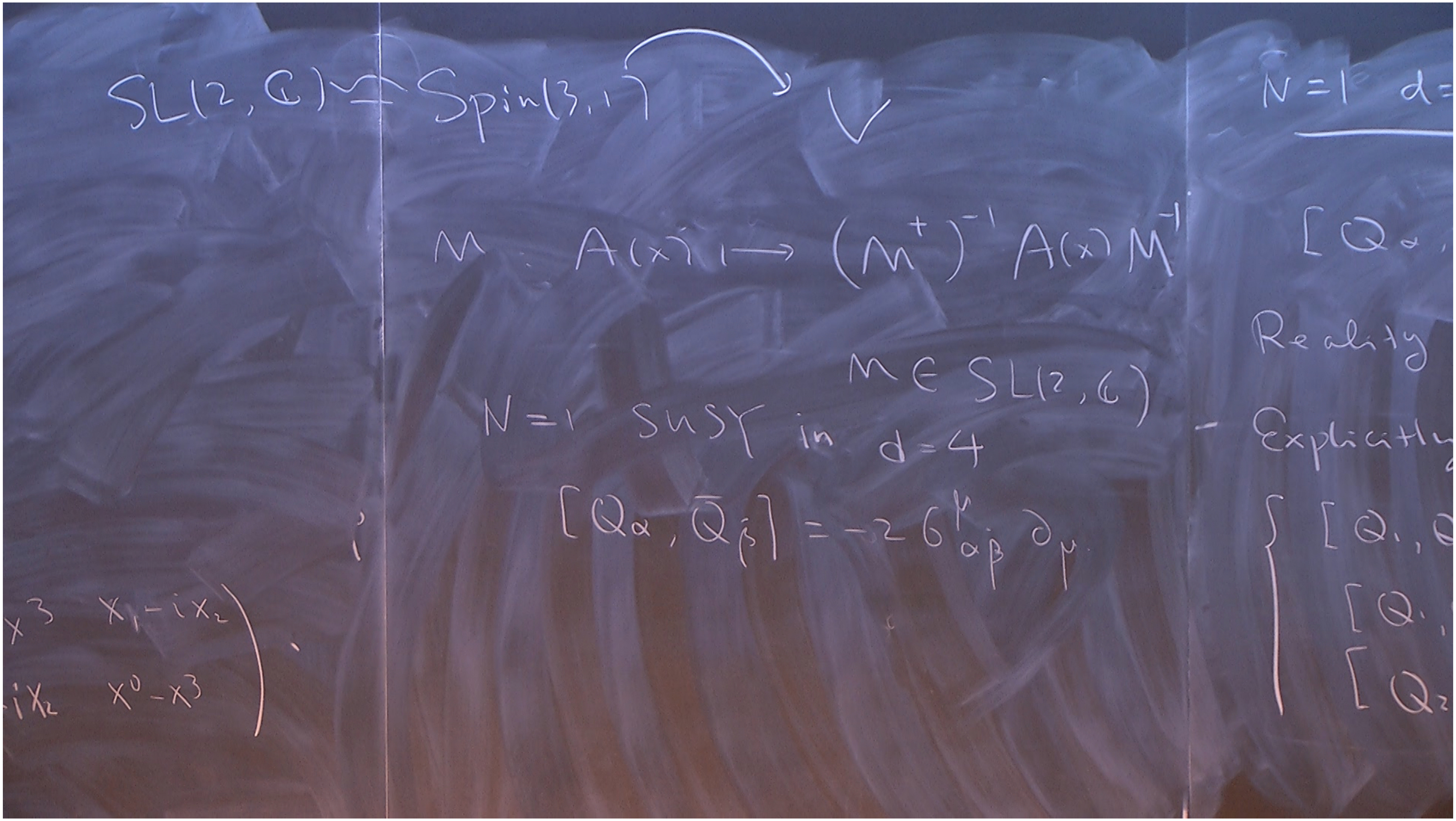
$$\begin{cases} [Q_1, Q_1] = 2\partial_0 - 2\partial_1 \\ [Q_1, Q_2] = 2\partial_2 \\ [Q_2, Q_2] = 2\partial_0 + 2\partial_1 \end{cases}$$

$Spin(2,1) =$

$Irred_{IR} =$


SUSY
if ∂2 is represented by
a Constant
⟹ d=2 N=(1,1) SUSY
d=3 N=1
dim reduction
d=2 N=(1,1)


Example d=4 $\mathbb{R}^{3,1}$
V = { Hermitian 2×2 matrices }
Let $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$x = (x^0, x^1, x^2, x^3) \longleftrightarrow A(x) = \sum x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x_1 - i x_2 \\ x_1 + i x_2 & x^0 - x^3 \end{pmatrix}$.
$|x|^2 = -\det A(x)$


SL(2, C) ≃ Spin(3,1)
M: A(x) ↦ (M†)^-1 A(x) M^-1
M ∈ SL(2, C)
N=1 SUSY in d=4
[Qα, Q̄β̇] = -2 σ^μ_αβ̇ ∂_μ
N=1 d=
Reality
Explicitly

$$\begin{pmatrix} x^3 & x_1 - i x_2 \\ i x_2 & x^0 - x^3 \end{pmatrix}$$

$$M:\ A(x) \longmapsto (M^\dagger)^{-1} A(x) M^{-1}, \qquad M \in SL(2,\mathbb{C})$$

$N=1$ SUSY in $d=4$

$$\left\{ \begin{array}{l} [Q_\alpha, \bar{Q}_{\dot\beta}] = -2\,\sigma^\mu_{\alpha\dot\beta}\,\partial_\mu \\ {[Q_1, \bar{Q}_1]} = -2(\partial_0 + \partial_3) \\ {[Q_2, \bar{Q}_2]} = -2(\partial_0 - \partial_3) \\ {[Q_1, \bar{Q}_2]} = -2(\partial_1 - i\partial_2) \\ {[Q_2, \bar{Q}_1]} = -2(\partial_1 + i\partial_2) \end{array} \right.$$

$[Q_\alpha,$

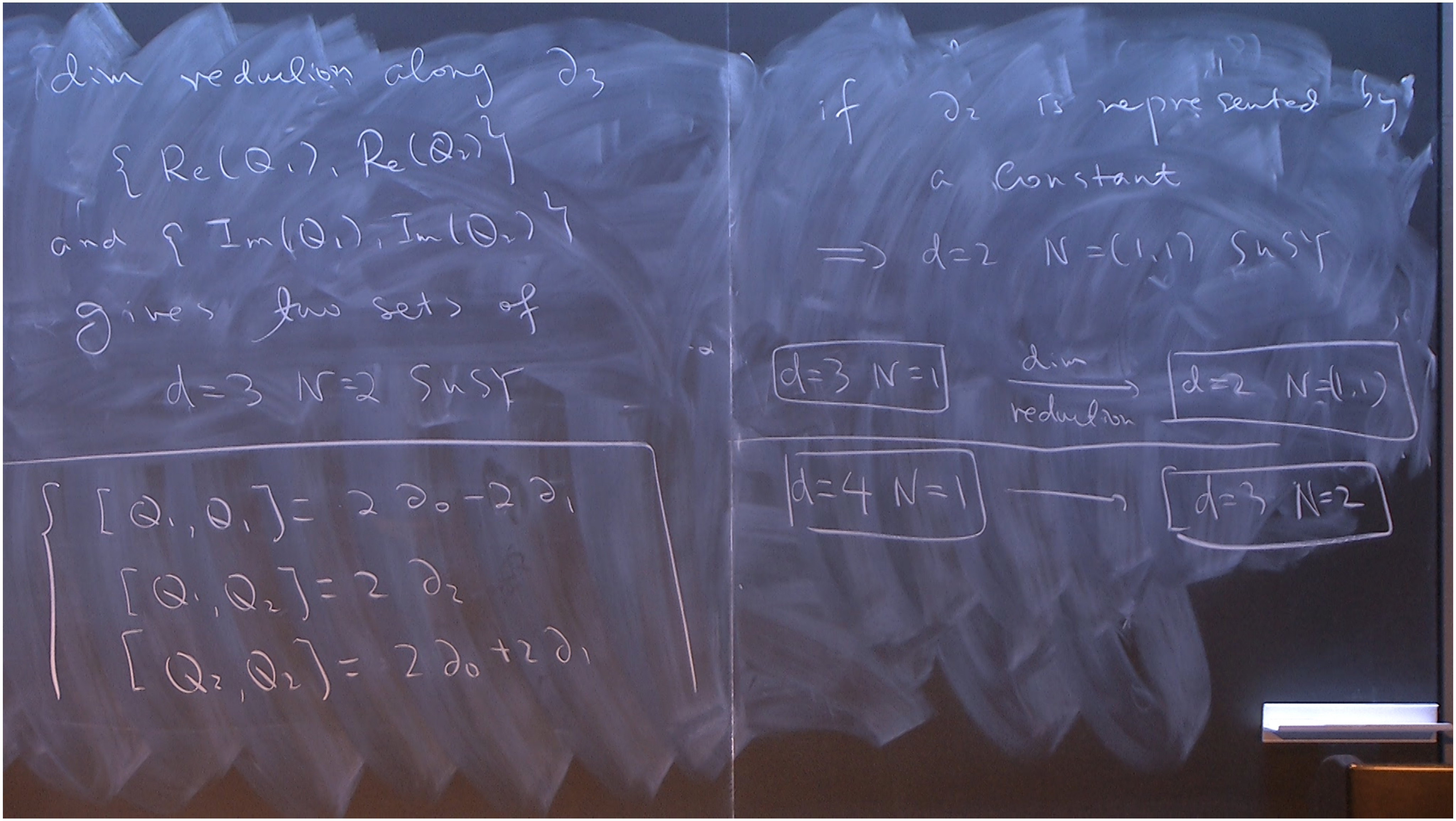
Reality

\- Explicitly

$[Q_1,$

$[Q_1,$

$[Q_2$


dim reduction along ∂_3
$\{Re(Q_1), Re(Q_2)\}$
and $\{Im(Q_1), Im(Q_2)\}$
gives two sets of
d=3 N=2 SUSY
$[Q_1, Q_1] = 2\partial_0 - 2\partial_1$
$[Q_1, Q_2] = 2\partial_2$
$[Q_2, Q_2] = 2\partial_0 + 2\partial_1$
if ∂_2 is represented by
a Constant
$\Longrightarrow$ d=2 N=(1,1) SUSY
d=3 N=1
dim reduction
d=2 N=(1,1)
d=4 N=1
d=3 N=2