Title: Lecture 2: Boundary Conditions and Extended Defects

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Collection: QFT for Mathematicians

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1d TOPOLOGICAL DEFECTS

$\cap$

2d TQFT

"PHYSICAL" TFT

$$\partial \mathcal{O}_i \equiv 0$$

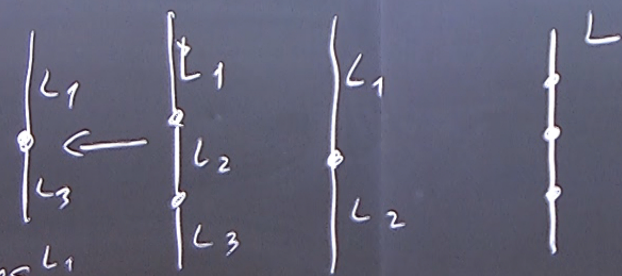
h TFT

$$\partial \mathcal{O}_i = \{Q, \tilde{\mathcal{O}}_i\}$$

$$\{Q^\dagger, Q\} = \partial$$

CATEGORY OF LINE DEFECTS

$\mathrm{HOM}(L_1, L_2) = \mathcal{O}bs^{L_1}_{L_2}$



ALGEBRA OF LOCAL OPS

"PHYSICAL"

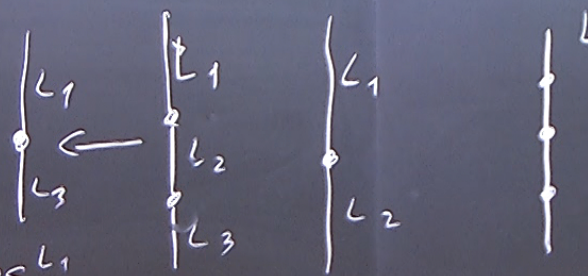
$\partial \mathcal{O}$

h TFT

$\partial \mathcal{O}_i$

$\{Q^\dagger, Q$

CATEGORY
OF LINE
DEFECTS



ALGEBRA
OF LOCAL
OPS

"PHYSICAL"

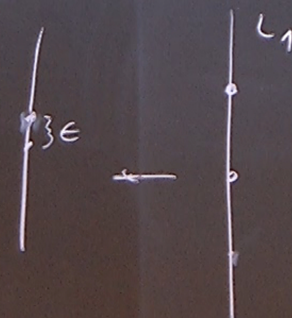
$\partial \mathcal{O}$

$HOM(L_1, L_2) = \mathcal{O}bs^{L_1}_{L_2}$

h TFT

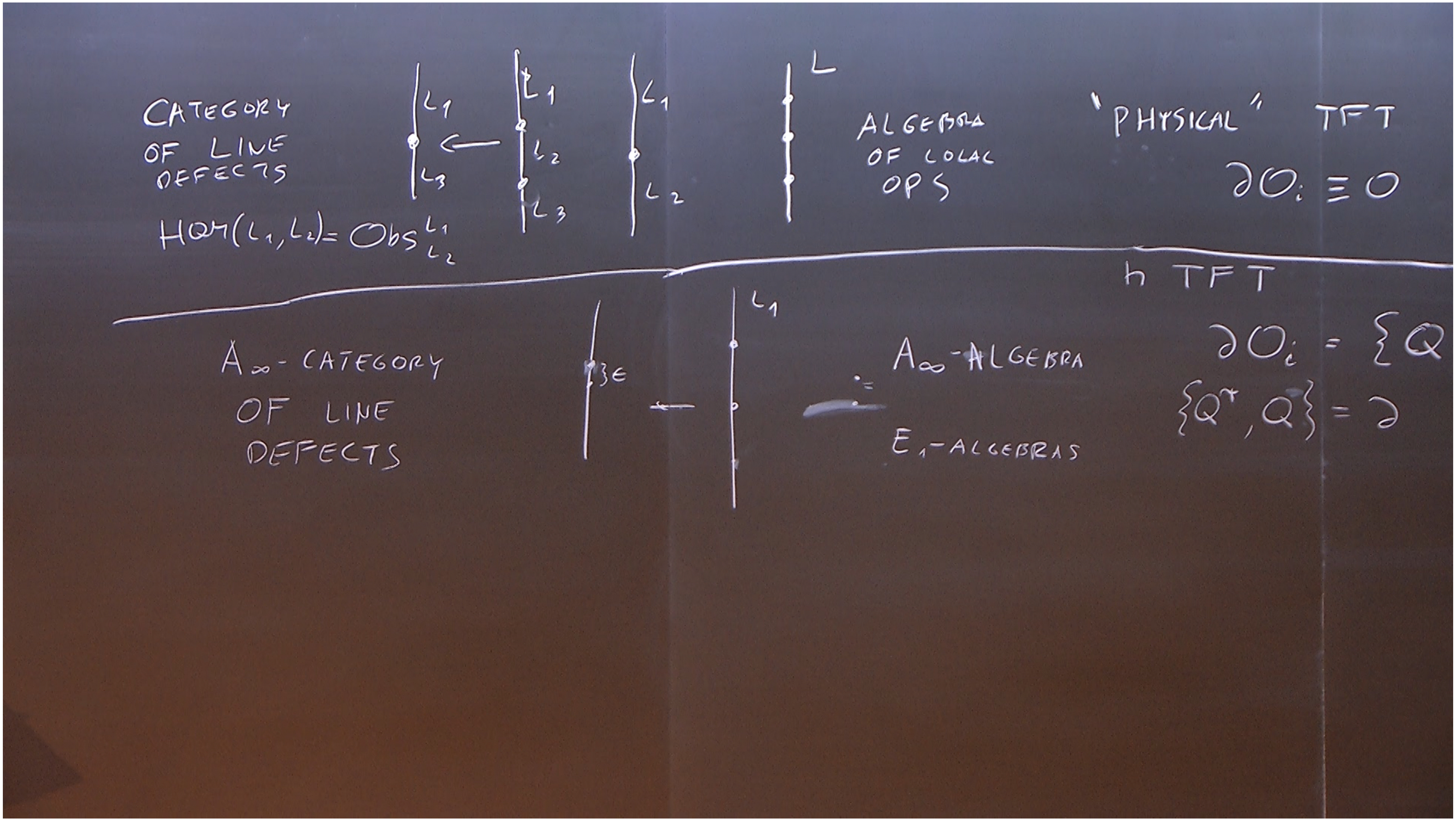
$\partial \mathcal{O}_i$

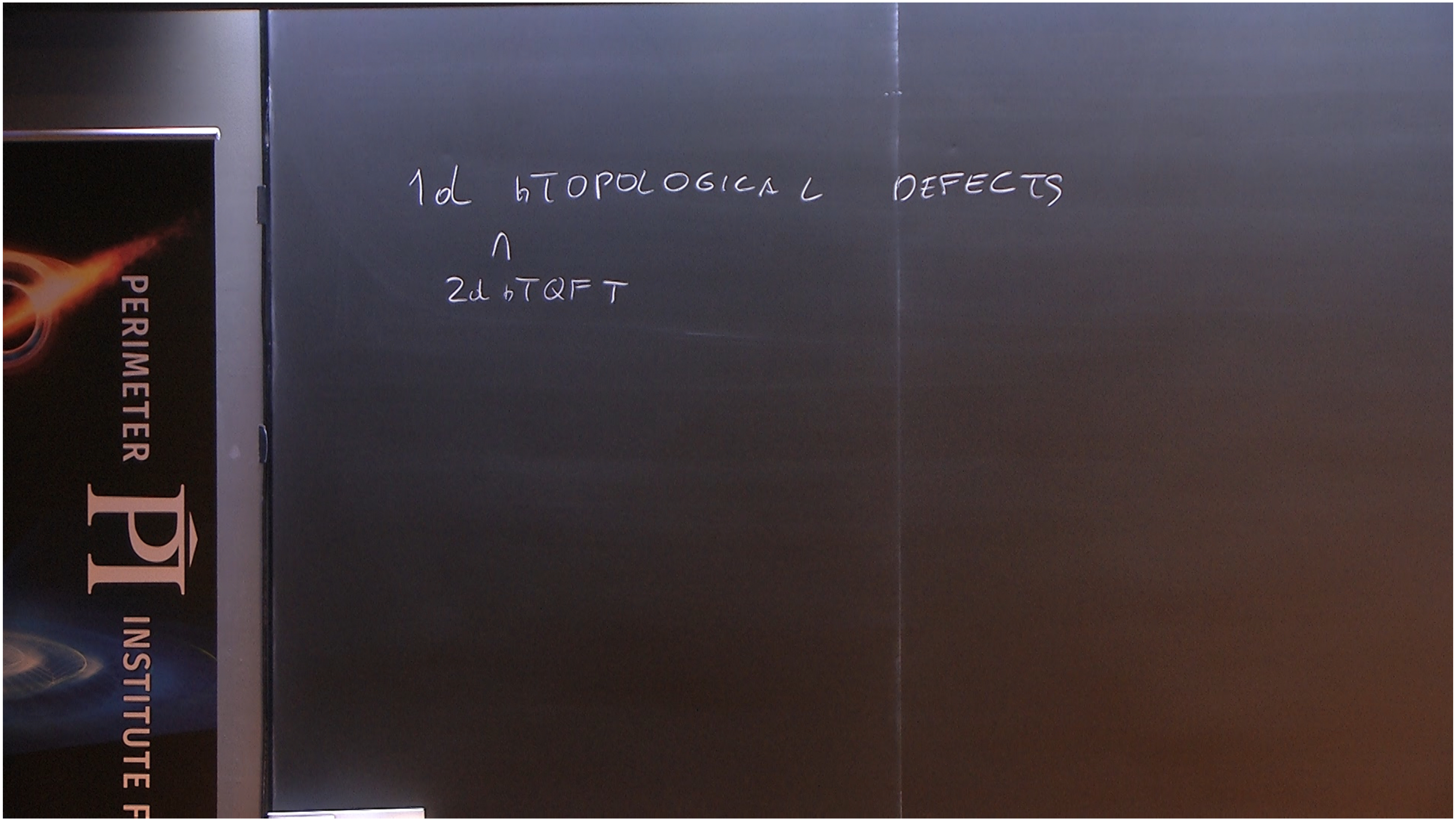
$\{Q^\dagger, Q$



A_∞-ALGEBRA

E_1-ALGEBRAS


CATEGORY OF LINE DEFECTS
L_1
L_3
L_1
L_2
L_3
L_1
L_2
L
ALGEBRA OF LOCAL OPS
"PHYSICAL" TFT
$\partial \mathcal{O}_i \equiv 0$
$HOM(L_1, L_2) = Obs^{L_1}_{L_2}$
h TFT
$\partial \mathcal{O}_i = \{Q$
$\{Q^\dagger, Q\} = \partial$
A_∞- CATEGORY OF LINE DEFECTS
L_1
A_∞-ALGEBRA
E_1-ALGEBRAS


1d hTOPOLOGICAL DEFECTS
∩
2d hTQFT
PERIMETER
PI
INSTITUTE

$$Q^2 = 0 \qquad (Q^\dagger)^2 = 0 \qquad \{Q, Q^\dagger\} = H \cdot \partial_t$$

$$\boxed{Q + x} \qquad \boxed{Qx + x^2 = 0}$$

MC EQUATION FOR dg-ALGEBRA A

$$Q^\dagger$$

$$H + \{Q^\dagger, x\}$$

$$S \to S + \int_{-\infty}^{\infty} dt\, \{Q^\dagger, x\}(t)$$

$$\langle 1 \rangle_x = \left\langle e^{\int_{-\infty}^{\infty} dt \{Q^\dagger, x\}(t)} \right\rangle$$

$$= \left\langle 1 + \int_{-\infty}^{\infty} \{\ \}(t_1)\, dt_1 + \int_{-\infty}^{\infty} \int_{-\infty}^{t_1} \{\ \}(t_1)\ \{\ \}(t_2)\, dt_1\, dt_2 \dots \right\rangle$$

$$Q^2=0 \qquad (Q^\dagger)^2=0 \qquad \{Q,Q^\dagger\}=H=\partial_t$$

$$\boxed{Q^{(x)}=Q+x} \qquad \boxed{Qx+x^2=0}$$ MC EQUATION FOR dg - ALGEBRA A

$$Q^\dagger \qquad H+\{Q^\dagger,x\}$$

$$QQ^\dagger x=-Q^\dagger(Qx)+\partial_t x \qquad S_x=S+\int_{-\infty}^{\infty}dt\,\{Q^\dagger,x\}(t)$$

$$\langle 1 \rangle_x=\left\langle e^{\int_{-\infty}^{\infty}dt\{Q^\dagger,x\}(t)}\right\rangle$$

$$=\left\langle 1+\int_{-\infty}^{\infty}\{Q^\dagger,x\}(t_1)\,dt_1+\int_{-\infty}^{\infty}\int_{-\infty}^{t_1}\{Q^\dagger,x\}(t_1)\,\{Q^\dagger,x\}(t_2)\,dt_1\,dt_2\dots\right\rangle$$

$$\left\langle\int_{-\infty}^{\infty}\{Q^\dagger,(Qx+x^2)\}+\iint\{Q^\dagger,x\}\{Q^\dagger,Q(x)+x^2\}\right. = \left\langle 0+\cancel{\int_{-\infty}^{\infty}\partial_{t_1}x(t_1)}+\iint\partial_{t_1}x(t_1)\{\ \}(t_2)+\iint\{\ \}(t_1)\,\partial_{t_2}x(t_2)\dots\right.$$

$$+\int_{-\infty}^{\infty}\{Q^\dagger,Q(x)\}+\iint\{\ ,x\}\{\ ,Q(x)\}\dots$$

$Q^2 = 0$ $\quad (Q^\dagger)^2 = 0 \quad$ $\{Q, Q^\dagger\} = H = \partial_t$

$\boxed{Q^{(x)} = Q + x}$ $\quad \boxed{Qx + x^2 = 0}$ MC EQUATION FOR dg-ALGEBRA A

A_∞-ALGEBRA : $\mu_n(\cdots)$ $\quad A^{\otimes n} \longrightarrow A[2-n]$

MC

$$\boxed{\mu_1(x) + \mu_2(x,x) + \mu_3(x,x,x) \cdots = 0}$$

$Q^2 = 0$ $\qquad (Q^\dagger)^2 = 0$ $\qquad \{Q, Q^\dagger\} = H = \partial_t$

$\boxed{Q^{(x)} = Q + x}$ $\qquad \boxed{Qx + x^2 = 0}$ MC EQUATION FOR dg - ALGEBRA A

A_∞ - ALGEBRA : $\mu_n(\cdots) \quad A^{\otimes n} \longrightarrow A[2-n]$

MC

$$\boxed{\mu_1(x) + \mu_2(x,x) + \mu_3(x,x,x) \cdots = 0}$$

$A^{(x)} = A$

$$\mu_1^x(\cdot) = \mu_1(\cdot) + \mu_2(\cdot, x) + \mu_2(x, \cdot) + \mu_3(\cdot, x, x) + \cdots$$

$Q^2 = 0$ $\quad (Q^\dagger)^2 = 0$ $\quad \{Q, Q^\dagger\} = H \cdot \partial_t$

$\boxed{Q^{(x)} = Q + x}$ $\quad \boxed{Qx + x^2 = 0}$ MC EQUATION FOR dg-ALGEBRA A

A_∞-ALGEBRA : $\mu_n(\cdots)$ $\quad A^{\otimes n} \longrightarrow A[2-n]$

MC

$$\boxed{\mu_1(x) + \mu_2(x,x) + \mu_3(x,x,x) \cdots = 0}$$

$A^{(x)} = A$

$$\mu_1^x(\cdot) = \mu_1(\cdot) + \mu_2(\cdot, x) + \mu_2(x, \cdot) + \mu_3(\cdot, x, x) + \cdots$$

$$\mu_2^x(\cdot,\cdot) = \mu_2(\cdot,\cdot) + \mu_3(x,\cdot,\cdot) + \mu_3(\cdot,x,\cdot) + \mu_3(\cdot,\cdot,x) + \cdots$$

$$x_A \to \sum_n \varphi_n(x_A^{\otimes n}) = x_B$$

$$Q^2 = 0 \qquad (Q^x)^2 = 0$$

$$\boxed{Q^{(x)} = Q + x} \qquad \boxed{Qx + x^2 = 0}$$

A_∞ MORPHISM

$$A \xrightarrow{\varphi} B$$

$$\varphi_n : A^{\otimes n} \longrightarrow B[1-n]$$

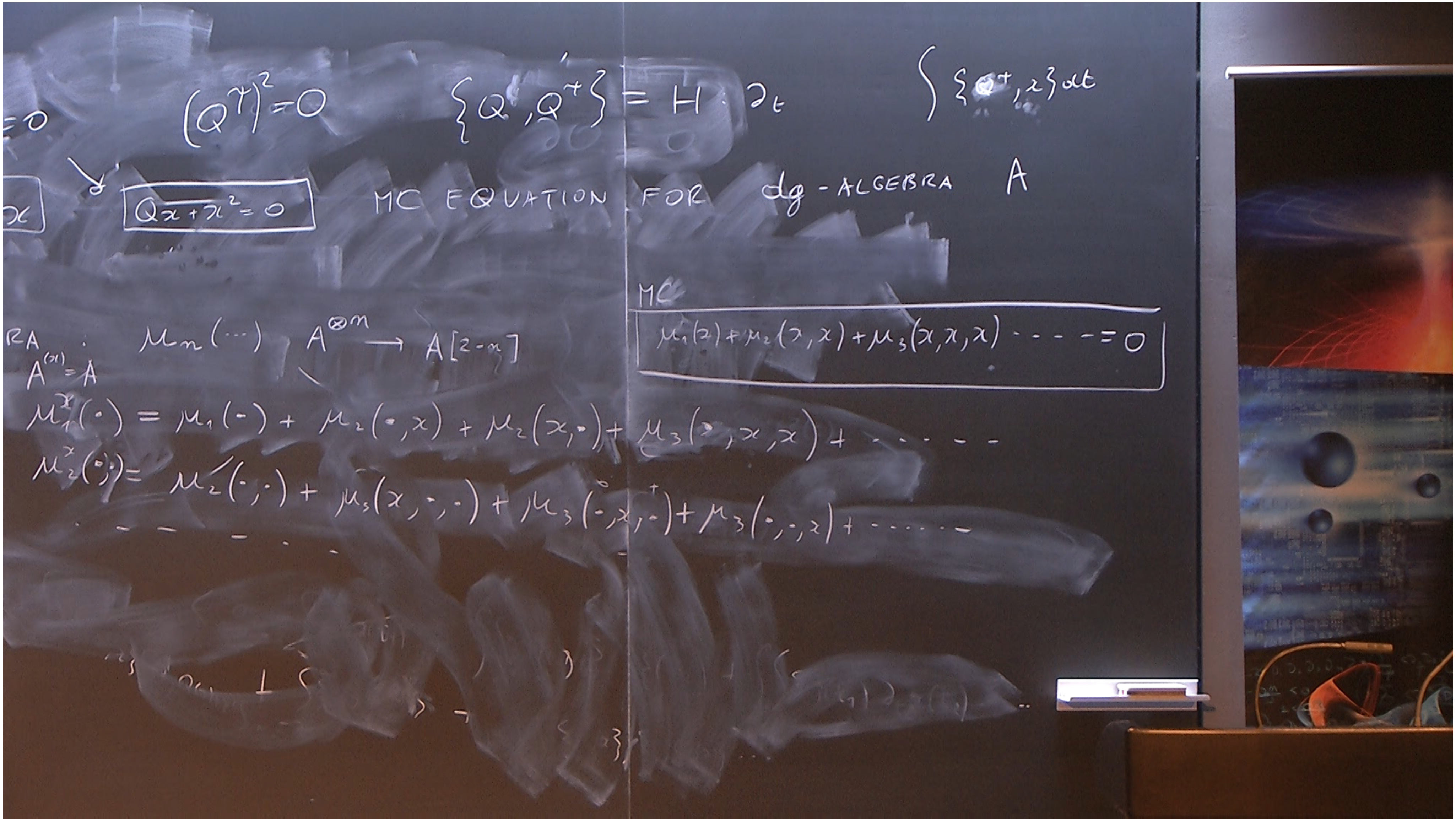
A_∞ QUASI-ISO

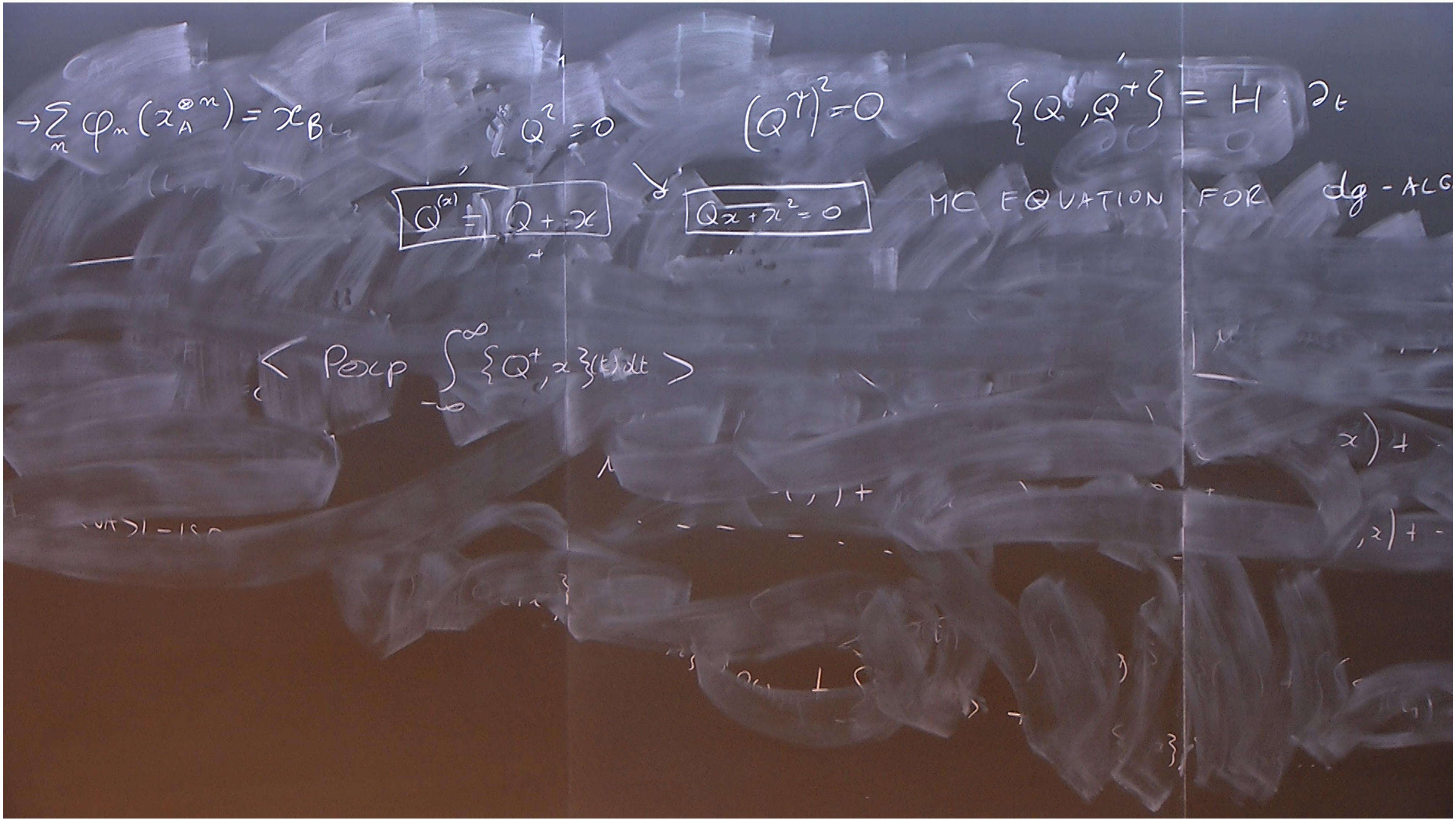
A_∞ - ALGEBRA : $\mu_n(\cdots)$ $A^{\otimes n} \longrightarrow$

$$A^{(x)} = A$$

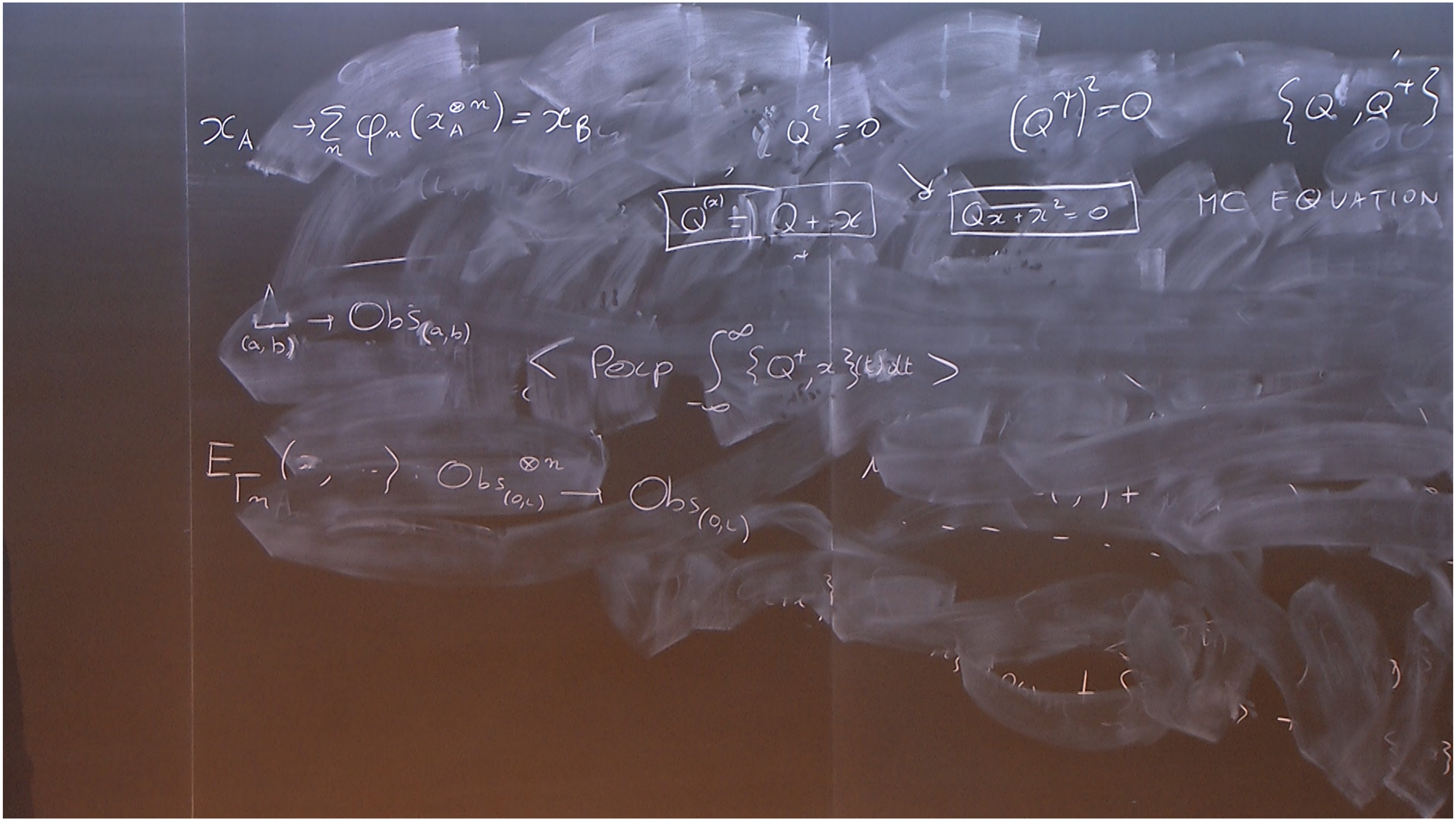
$$\mu_1^x(\cdot) = \mu_1(\cdot) + \mu_2(\cdot$$

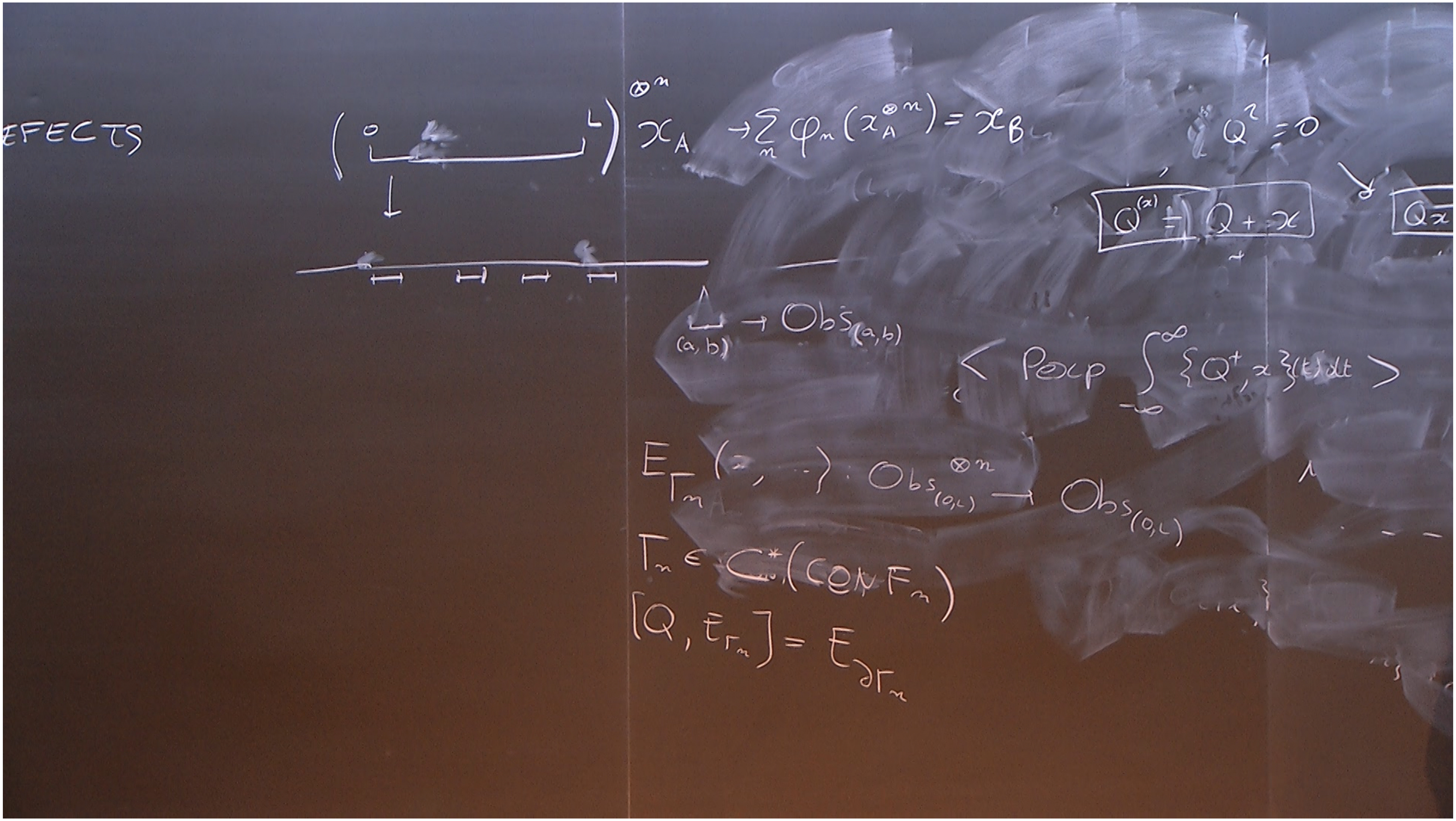
$$\mu_2^x(\cdot,\cdot) = \mu_2(\cdot,\cdot) + \mu_3(x$$


$(Q^+)^2 = 0$
$\{Q, Q^+\} = H : \partial_t$
$\int \{Q^+, x\} dt$
$\boxed{Qx + x^2 = 0}$
MC EQUATION FOR dg-ALGEBRA A
MC
$\boxed{\mu_1(x) + \mu_2(x,x) + \mu_3(x,x,x) \cdots = 0}$
RA : $\mu_n(\cdots)$ $A^{\otimes n} \longrightarrow A[2-n]$
$A^{(x)} = A$
$\mu_1^x(\cdot) = \mu_1(\cdot) + \mu_2(\cdot, x) + \mu_2(x, \cdot) + \mu_3(\cdot, x, x) + \cdots$
$\mu_2^x(\cdot,\cdot) = \mu_2(\cdot,\cdot) + \mu_3(x,\cdot,\cdot) + \mu_3(\cdot,x,\cdot) + \mu_3(\cdot,\cdot,x) + \cdots$



$\to \sum_n \varphi_n(x_A^{\otimes n}) = x_B$
$Q^2 = 0$
$(Q^\dagger)^2 = 0$
$\{Q, Q^\dagger\} = H \cdot \partial_t$
$Q^{(x)} = Q + x$
$Qx + x^2 = 0$
MC EQUATION FOR dg - ALG
$\langle \, \mathrm{Pexp} \int_{-\infty}^{\infty} \{Q^\dagger, x\}(t)\, dt \, \rangle$


$x_A \to \sum_n \varphi_n(x_A^{\otimes n}) = x_B$
$Q^2 = 0$
$(Q^\dagger)^2 = 0$
$\{Q, Q^\dagger\}$
$\boxed{Q^{(x)} = Q + x}$
$\boxed{Qx + x^2 = 0}$
MC EQUATION
$\Delta_{(a,b)} \to Obs_{(a,b)}$
$\langle P\exp \int_{-\infty}^{\infty} \{Q^\dagger, x\}(t)\,dt \rangle$
$E_{\Gamma_n}(\cdot, \ldots) : Obs_{(0,L)}^{\otimes n} \to Obs_{(0,L)}$


EFECTS
$\left(\overset{0}{\llcorner}\!\!\_\_\_\_\_\!\!\lrcorner^{L}\right)^{\otimes n} x_A \to \sum_n \varphi_n(x_A^{\otimes n}) = x_B$
$Q^2 = 0$
$\boxed{Q^{(x)} = Q + x}$
$\to \mathrm{Obs}_{(a,b)}$
(a,b)
$\langle \mathrm{Pexp} \int_{-\infty}^{\infty} \{Q^+, x\}(t)\,dt \rangle$
$E_{\Gamma_n}(\ ,\ \dots): \mathrm{Obs}_{(0,L)}^{\otimes n} \to \mathrm{Obs}_{(0,L)}$
$\Gamma_n \in C_*(\mathrm{CONF}_n)$
$[Q, E_{\Gamma_n}] = E_{\partial\Gamma_n}$

$) = x_B$ $Q^2 = 0$ $(Q^\dagger)^2 = 0$ $\{Q, Q^\dagger\} = H : \partial_t$ $\int \{Q^\dagger, x\} dt$

$\boxed{Q^{(x)} = Q + x}$ $\boxed{Qx + x^2 = 0}$ MC EQUATION FOR dg - ALGEBRA A

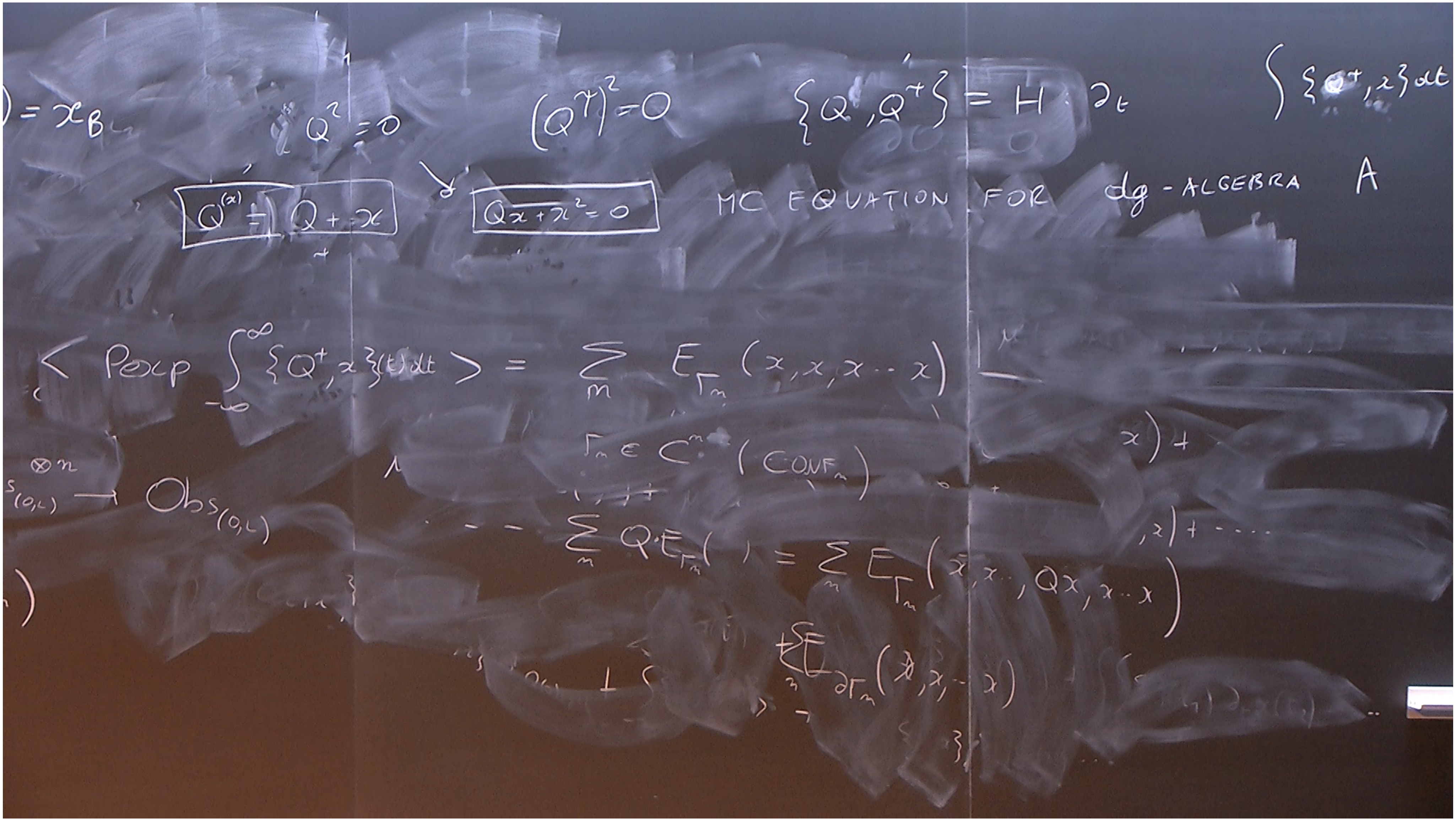
$$\langle \operatorname{Pexp} \int_{-\infty}^{\infty} \{Q^\dagger, x\}(t)\, dt \rangle = \sum_n E_{\Gamma_n}(x, x, x \cdots x)$$

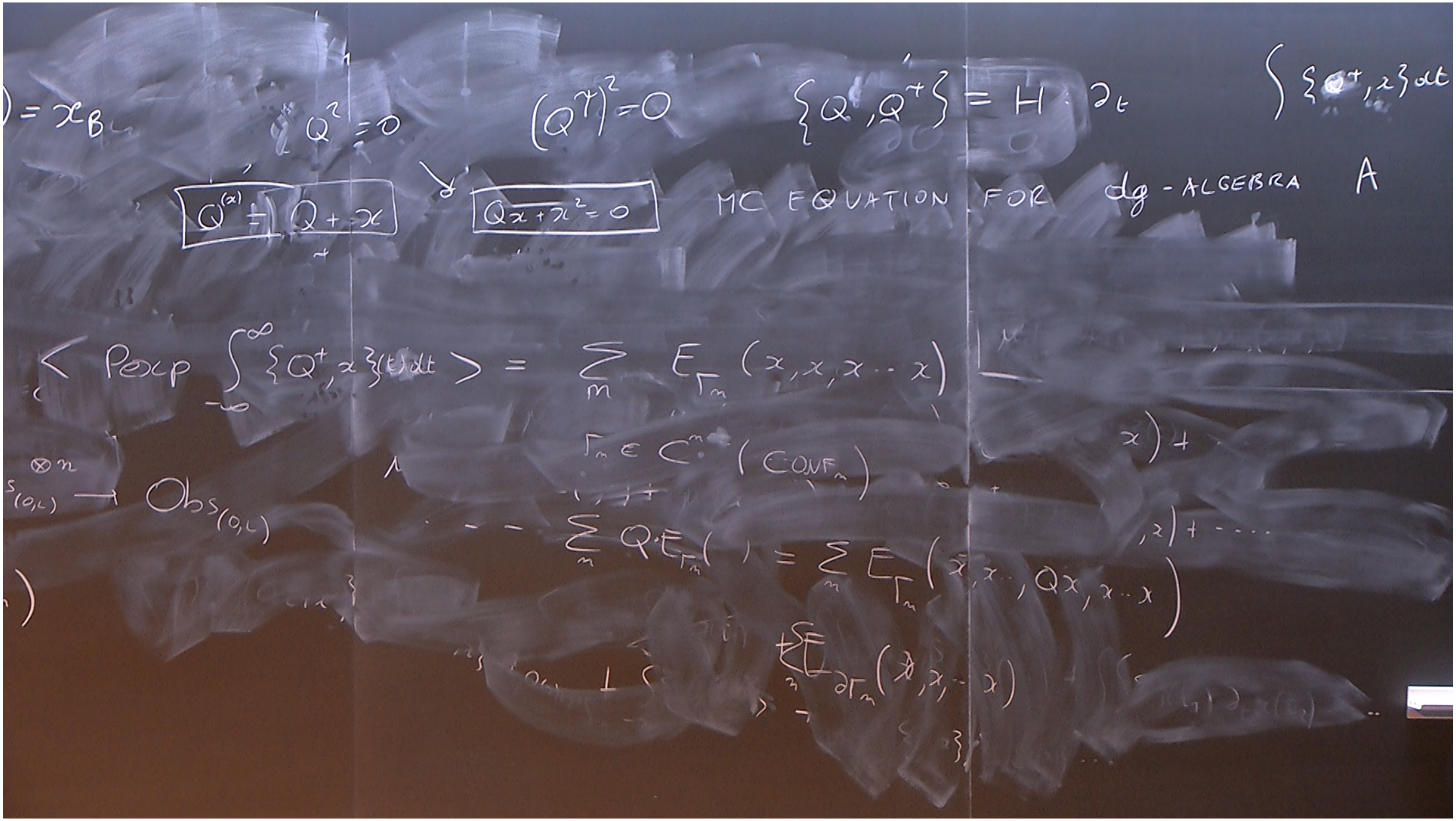
$\Gamma_n \in C^{n-1}($ $x) +$

$\otimes n$

$s_{(0,L)} \longrightarrow \mathrm{Obs}_{(0,L)}$ $, x) + \cdots$

$)$


= x_B
Q^2 = 0
(Q^+)^2 = 0
{Q, Q^+} = H ∂_t
∫ {Q^+, x} dt
Q^(x) = Q + x
Qx + x^2 = 0
MC EQUATION FOR dg-ALGEBRA A
< Pexp ∫_{-∞}^{∞} {Q^+, x}(t) dt > = Σ_n E_{Γ_n}(x, x, x .. x)
Γ_n ∈ C^n(CONF_n)
⊗n
Obs_(0,L)
Σ_n Q·E_{Γ_n}( ) = Σ_n E_{Γ_n}(x, x.., Qx, x..x)
+ Σ E_{∂Γ_n}(x, x, .. x)


$= x_B$
$Q^2 = 0$
$(Q^+)^2 = 0$
$\{Q, Q^+\} = H : \partial_t$
$\int \{Q^+, x\} dt$
$\boxed{Q^{(x)} = Q + x}$
$\boxed{Qx + x^2 = 0}$
MC EQUATION FOR dg-ALGEBRA A
$\langle P\exp \int_{-\infty}^{\infty} \{Q^+, x\}(t)\,dt \rangle = \sum_m E_{\Gamma_m}(x, x, x \cdots x)$
$\Gamma_m \in C^m(\mathrm{CONF}_m)$
$\otimes n$
$\to \mathrm{Obs}_{(0,L)}$
$\sum_m Q \cdot E_{\Gamma_m}(\ ) = \sum_m E_{\Gamma_m}(x, x \cdots, Qx, x \cdots x)$
$+ \sum E_{\partial\Gamma_m}(x, x, \cdots x)$

OGICAL DEFECTS

$\left({}_{0}\llcorner\!\!\_\_\_\_\!\!\lrcorner^{L} \right)^{\otimes n} x_A \to \sum_n \varphi_n(x_A^{\otimes n}) = x_B$

$\downarrow$

$M_3 \in C^1(\mathrm{CONF}^3_{[0,L]}) \sim L$

$\partial \Gamma_n = \Gamma_{n-1} \circ M_2 + \Gamma_{n-2} \circ M_3 + \cdots \sum \Gamma_{n-k} \circ M_k$

$M_2 = \underline{|\vdash\!\dashv \;\; \vdash\!\dashv|}_{0 \qquad L}$

$|\text{⊢⊣} \;\; \vdash\!\dashv| \xrightarrow[M_3]{} |\vdash\!\dashv \;\; \text{⊢⊣}|$

$(a,b) \to \mathrm{Obs}_{(a,b)}$

$E_{\Gamma_n}(\cdot, \ldots) : \mathrm{Obs}_{(0,L)}^{\otimes n} \longrightarrow \mathrm{Obs}_{(0,L)}$

$\Gamma_n \in C^*(\mathrm{CONF}_n)$

$[Q, E_{\Gamma_n}] = E_{\partial \Gamma_n}$

$\langle \; \mathrm{P\!\exp} \int_{-\infty}^{\infty}$

$\boxed{Q^{(x)} =}$

$x_A \to \sum_n \varphi_n(x_A^{\otimes n}) = x_B$

$Q^2 = 0$ $\quad (Q^{\gamma})^2 = 0$ $\quad \{Q^{+}, Q^{+}\} =$

$\boxed{Q^{(x)} = Q + x}$ $\quad \boxed{Qx + x^2 = 0}$ MC EQUATION

$\to \mathrm{Obs}_{(a,b)}$

(a,b)

$\left\langle \mathrm{Pexp} \int_{-\infty}^{\infty} \{Q^{+}, x\}(t)\,dt \right\rangle = \sum_n E_{\Gamma_n}(x, x, x \cdots x)$

$+ \Gamma_{n-2} \circ M_3 + \cdots \sum F_{n-k} \circ M_k$

$E_{\Gamma_n}(x, \ldots) : \mathrm{Obs}_{(0,L)}^{\otimes n} \to \mathrm{Obs}_{(0,L)}$

$\Gamma_n \in C^{*}(\mathrm{CONF}_n)$

$\Gamma_n \in C^{*}(\mathrm{CONF}_n)$

$[Q, E_{\Gamma_n}] = E_{\partial \Gamma_n}$

$\sum_n Q \cdot E_{\Gamma_n}(\;) = \sum_n E_{\Gamma_n}(x, \ldots$

Γ_1

Γ_2

$+ \sum_n E_{\partial\Gamma_n}(x, x, \ldots$

NEAR BOUNDARY

LOGICAL DEFECTS

$\left(\overset{\circ}{\llcorner} \qquad \overset{L}{\lrcorner} \right)$

$x_A^{\otimes n} \to \sum_n \varphi_n (x_A^{\otimes n}) = x_B$

$M_3 \in C^1(\mathrm{CONF}_3)\sim L$

$\partial \Gamma_n = \Gamma_{n-1} \circ M_2 + \Gamma_{n-2} \circ M_3 + \cdots \sum \Gamma_{n-k} \circ M_k$

$\partial^2 \Gamma_n = 0 \qquad \sum M_k \circ M_{n-k} = 0$

$M_2 = \ldots$

M_3

$\Delta_{(a,b)} \to \mathrm{Obs}_{(a,b)}$

$E_{\Gamma_n}(\ldots, \ldots) : \mathrm{Obs}_{(0,L)}^{\otimes n} \to \mathrm{Obs}_{(0,L)}$

$\Gamma_n \in C_*(\mathrm{CONF}_n)$

$[Q, E_{\Gamma_n}] = E_{\partial \Gamma_n}$

$\boxed{Q^{(x)} \ldots}$

$< \mathrm{Pexp} \int_{-\infty}^{\infty}$

NEA

TOPOLOGICAL DEFECTS

Λ

TQFT

$\left(\underset{0 \qquad\qquad L}{\rule{3cm}{0.4pt}} \right)^{\otimes n} x_A \to \sum_n \varphi_n (x_A^{\otimes n}) = x$

$M_3 \in C^1(\mathrm{CONF}^3_{[0,L]}) \sim L$

$\Lambda_{(a,b)} \to \mathrm{Obs}_{(a,b)}$

$\partial \Gamma_n = \Gamma_{n-1} \circ M_2 + \Gamma_{n-2} \circ M_3 + \cdots \sum \Gamma_{n-k} \circ M_k$

$\partial^2 \Gamma_n = 0 \qquad \sum M_k \circ M_{n-k} = 0$

$M_2 = \underset{0 \qquad L}{|\mapsto \ \mapsto|}$

$|\mapsto \ \mapsto| \xrightarrow[M_3]{} |\mapsto \ \mapsto|$

$E_{\Gamma_n}(\cdot, \ldots) : \mathrm{Obs}_{(0,L)}^{\otimes n} \to$

$\Gamma_n \in C_*(\mathrm{CONF}_n)$

$[Q, E_{\Gamma_n}] = E_{\partial \Gamma_n}$

$Q^2 = 0$ $\quad (Q^{+})^2 = 0 \quad \{Q, Q^{+}\} = H \cdot \partial_t \quad \int \{Q^{+}, x\}\, dt$

$\boxed{Q^{(x)} = Q + x}$ $\quad \boxed{Qx + x^2 = 0}$ MC EQUATION FOR dg - ALGEBRA A

$$\mu_m(\cdot\cdot) = E_{M_m}(x \cdots x)$$

$$\langle P\exp \int_{-\infty}^{\infty} \{Q^{+}, x\}(t)\, dt \rangle = \sum_m E_{\Gamma_m}(x, x, x \cdots x)$$

$$\sum \mu_m(x^{\otimes m}) = 0$$

$\Gamma_m \in C^{m}(\mathrm{CONF}_m)$

$\to \mathrm{Obs}_{(0,L)}$

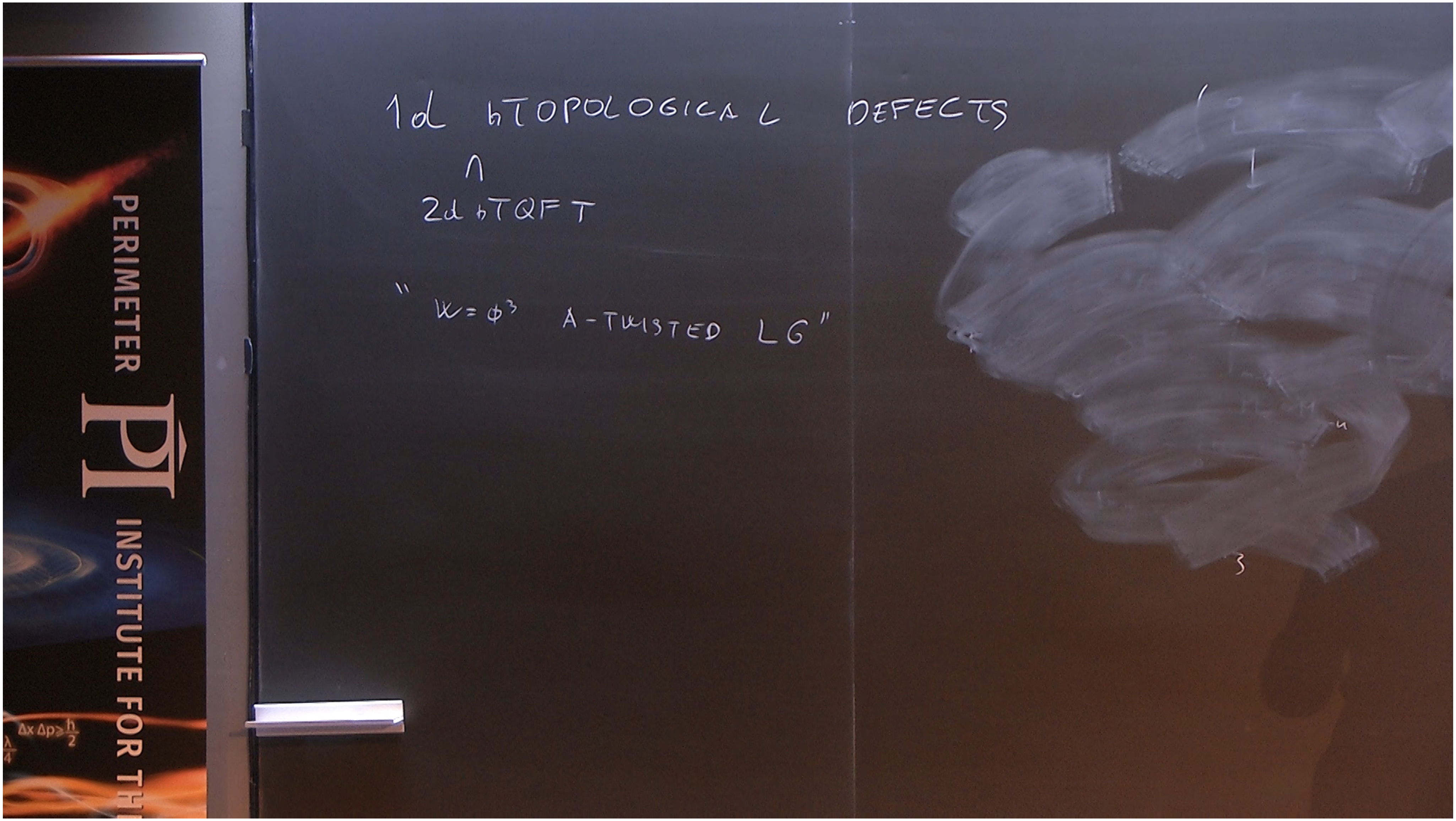
$$\sum_m Q \cdot E_{\Gamma_m}(\ ) = \sum_m E_{\Gamma_m}(x, x \cdot\cdot, Qx, x \cdot\cdot x)$$

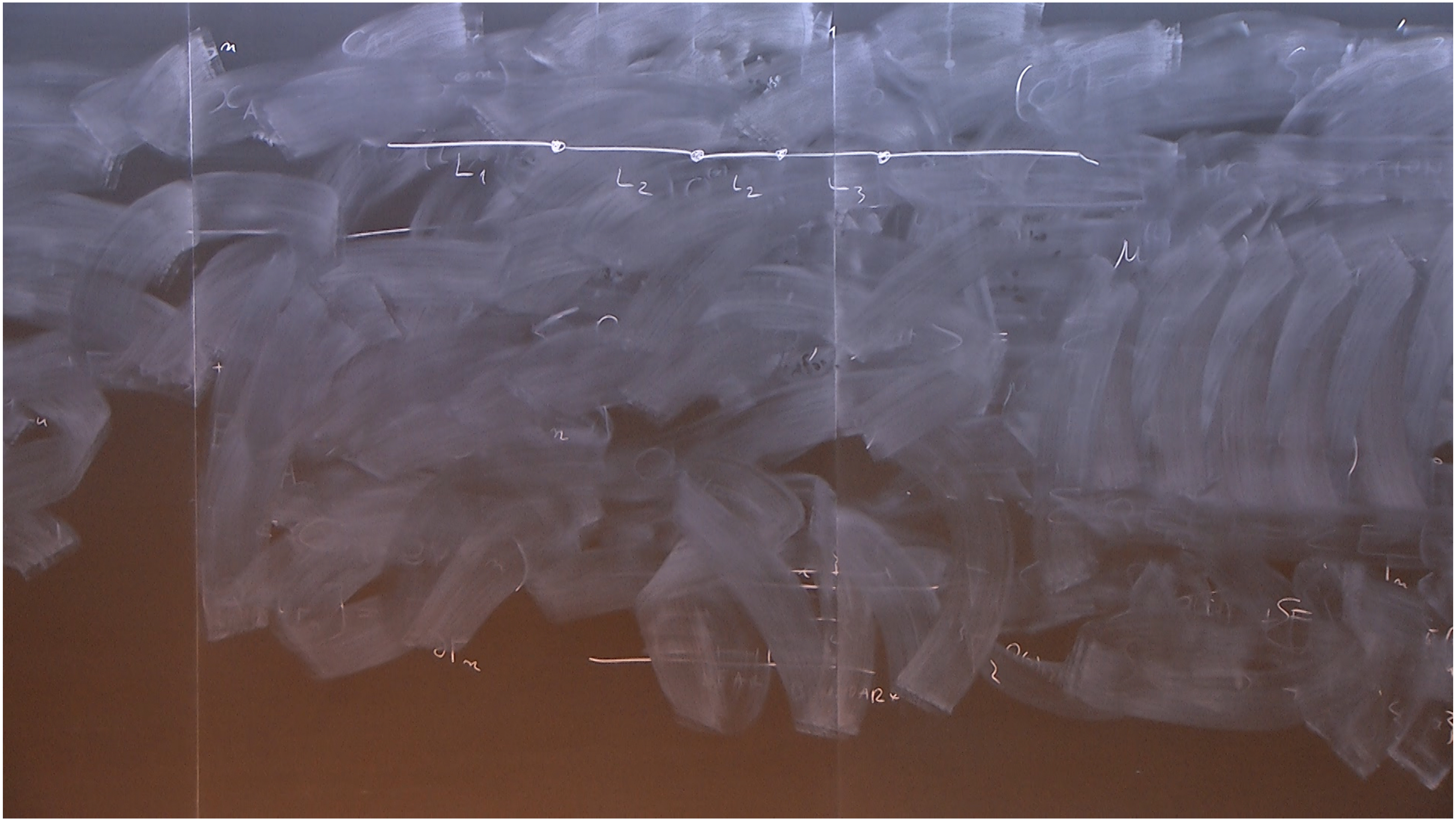
Γ_1

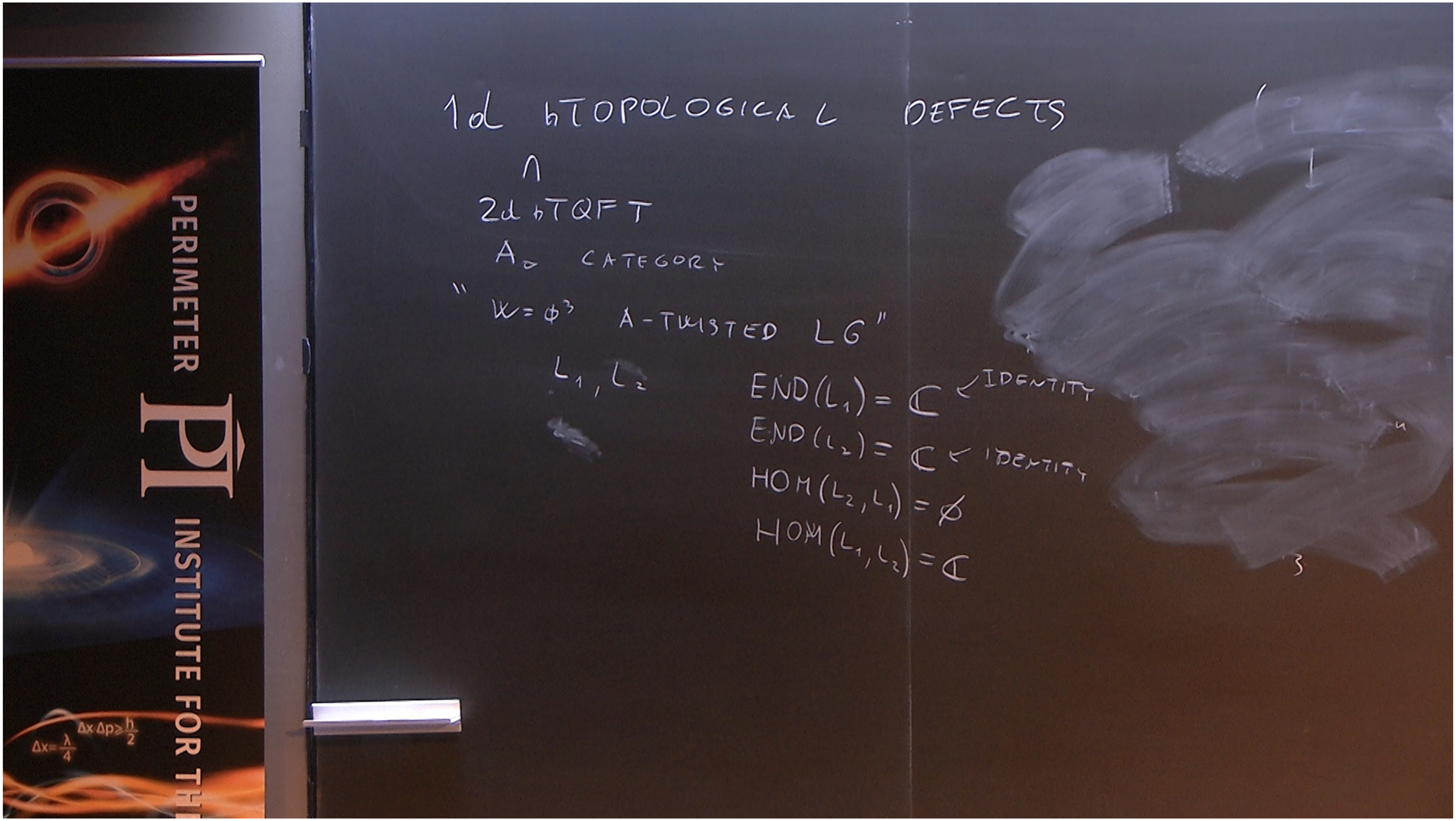
Γ_2

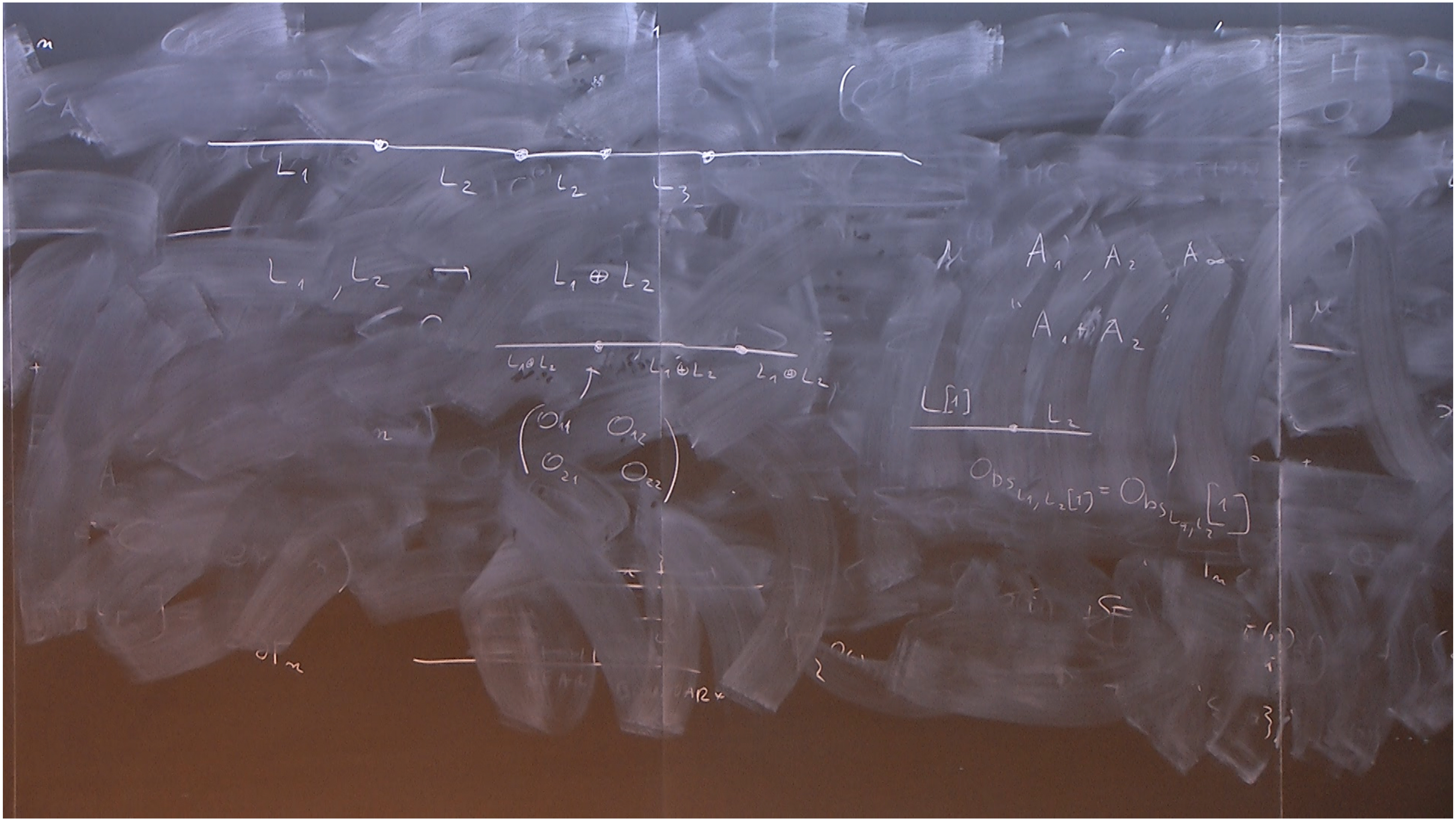
$$+ \sum_n E_{\Gamma_{m-n}}(x, E_{M_n}(\ ), x)$$

NEAR BOUNDARY


1d TOPOLOGICAL DEFECTS
∩
2d TQFT
" W = φ³ A-TWISTED LG"
PERIMETER
INSTITUTE FOR


L_1
L_2
L_2
L_3


1d hTOPOLOGICAL DEFECTS
∩
2d hTQFT
A∞ CATEGORY
" W = φ³ A-TWISTED LG"
L1, L2
END(L1) = ℂ ← IDENTITY
END(L2) = ℂ ← IDENTITY
HOM(L2, L1) = ∅
HOM(L1, L2) = ℂ
PERIMETER
PI
INSTITUTE FOR TH
Δx Δp ≥ ħ/2
Δx = λ/4



L1
L2
L2
L3
L1 , L2 ↦ L1 ⊕ L2
L1⊕L2
L1⊕L2
L1⊕L2
O11 O12
O21 O22
A1 , A2 , A∞
"A1 ⊕ A2"
L[1]
L2
Obs L1,L2[1] = Obs L1,L2 [1]

1d "TOPOLOGICAL" DEFECTS

$\cap$

2d "TQFT"

A_∞ CATEGORY

" $W = \phi^3$ A-TWISTED LG"

L_1, L_2

$\mathrm{END}(L_1) = \mathbb{C}$ ← IDENTITY

$\mathrm{END}(L_2) = \mathbb{C}$ ← IDENTITY

$\mathrm{HOM}(L_2, L_1) = \emptyset$

$\mathrm{HOM}(L_1, L_2) = \mathbb{C}$

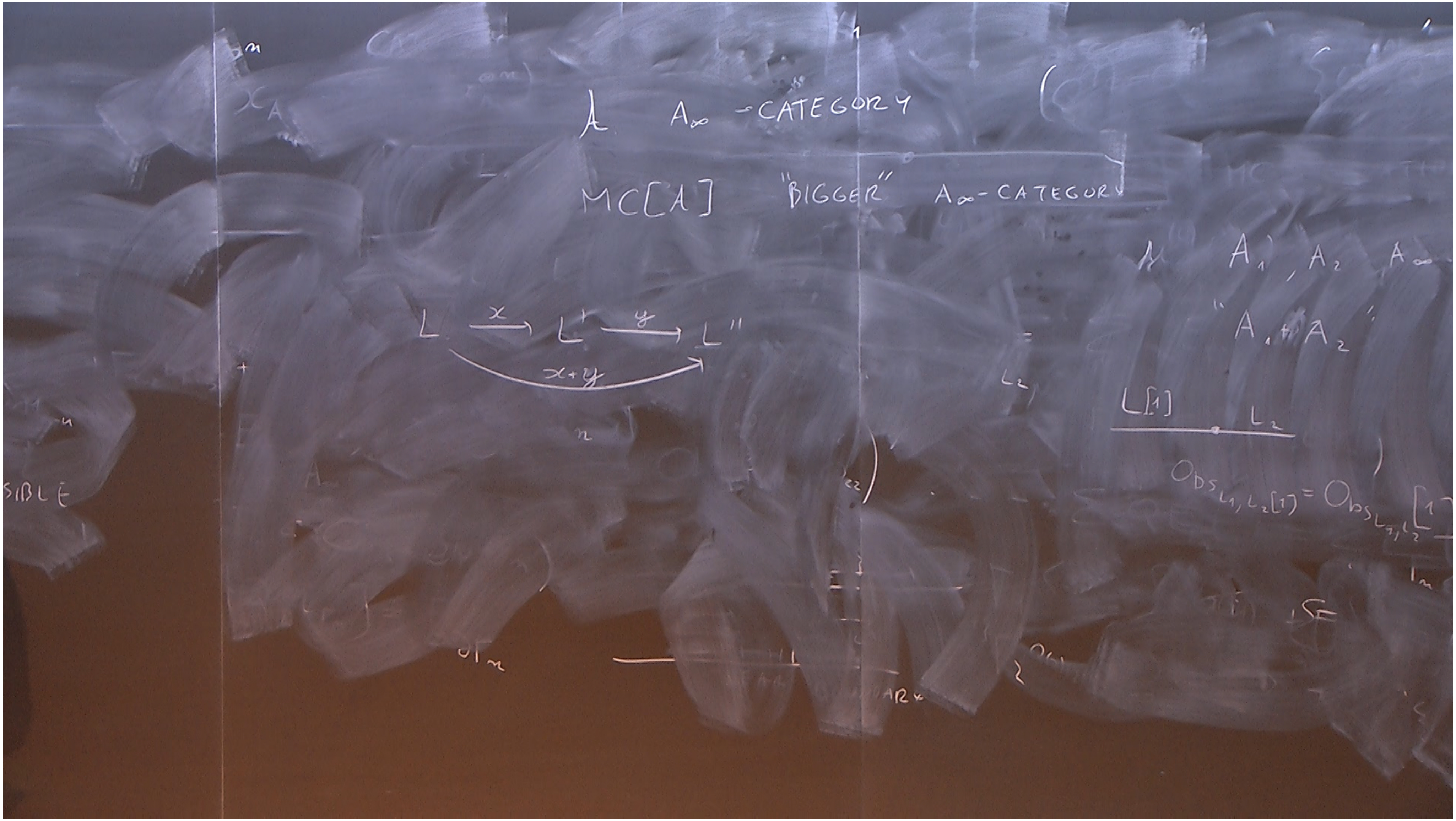
$\tilde{L} = L_1 + L_2[1]$

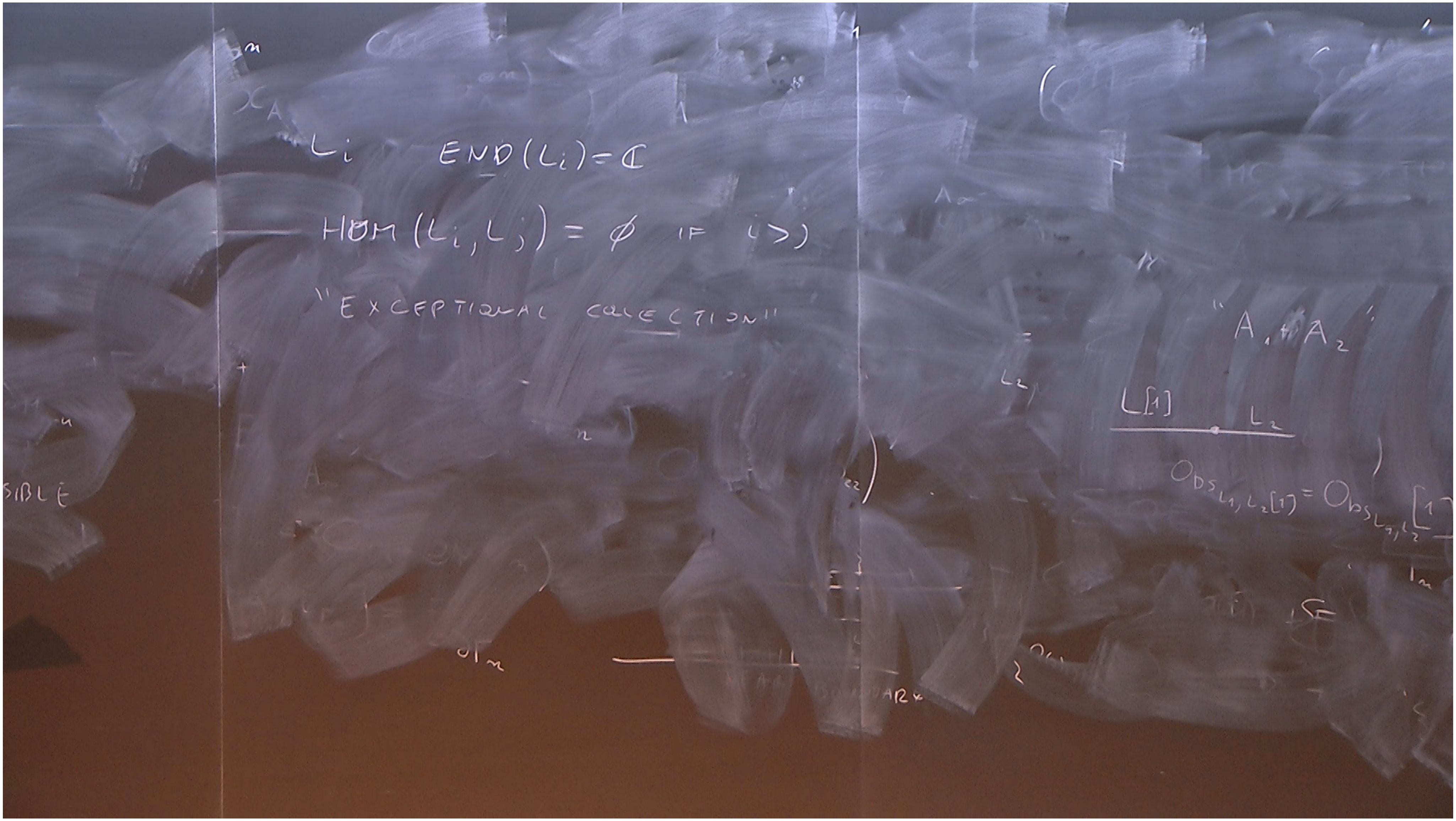
$x \in \mathrm{HOM}(L_1, L_2)[1]$
$\in \mathrm{END}(\tilde{L})$

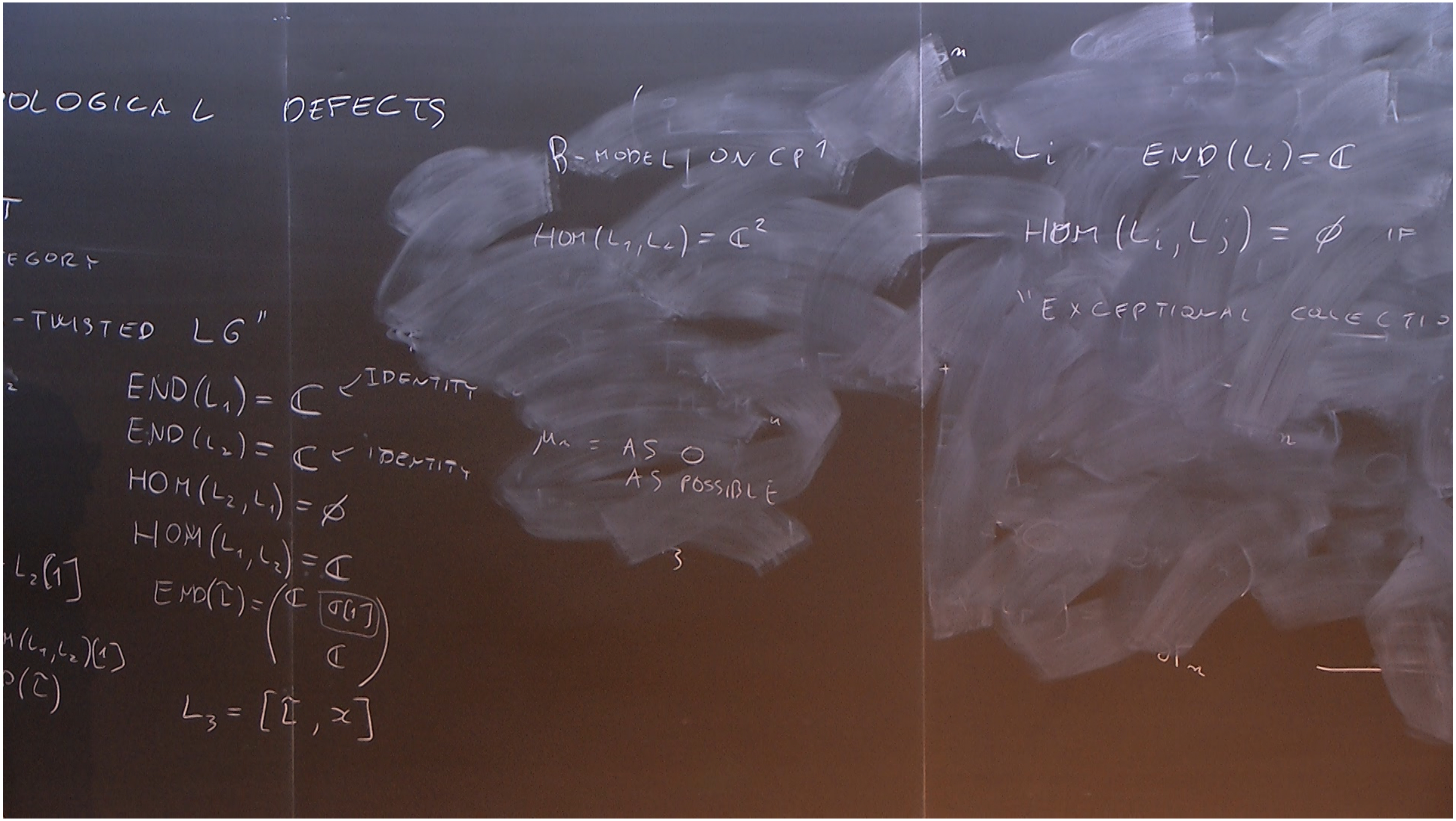
$$\mathrm{END}(\tilde{L}) = \begin{pmatrix} \mathbb{C} & \boxed{\mathbb{C}[1]} \\ & \mathbb{C} \end{pmatrix}$$

$L_3 = [\tilde{L}, x]$

μ_n = AS 0
AS POSSIBLE


A A_∞-CATEGORY
MC[A] "BIGGER" A_∞-CATEGORY
L x L' y L''
x+y
A A_1, A_2 A_∞
"A_1 + A_2"
L[1] L_2
$Obs_{L_1,L_2[1]} = Obs_{L_1,L_2}[1]$


L_i END$(L_i) = \mathbb{C}$
HOM$(L_i, L_j) = \emptyset$ IF $i > j$
"EXCEPTIONAL COLLECTION"
"$A_1 \oplus A_2$"
$L[1]$ L_2
$Obs_{L_1, L_2[1]} = Obs_{L_1, L_2}[1]$


OLOGICAL DEFECTS
EGORY
-TWISTED LG"
END(L1) = C ← IDENTITY
END(L2) = C ← IDENTITY
HOM(L2, L1) = ∅
HOM(L1, L2) = C
L2[1]
END(T) = (C σ[1] ; C)
H(L1, L2)[1]
L3 = [T, x]
B-MODEL ON CP1
HOM(L1, L2) = C2
μn = AS O
AS POSSIBLE
Li END(Li) = C
HOM(Li, Lj) = ∅ IF
"EXCEPTIONAL COLLECTIO

1d TOPOLOGICAL DEFECTS

$\cap$

2d TQFT

A_∞ CATEGORY

" $W = \phi^3$ A-TWISTED LG "

L_1, L_2

$END(L_1) = \mathbb{C}$ ← IDENTITY

$END(L_2) = \mathbb{C}$ ← IDENTITY

$HOM(L_2, L_1) = \emptyset$

$HOM(L_1, L_2) = \mathbb{C}$

$\tilde{L} = L_1 + L_2[1]$

$END(\tilde{L}) = \begin{pmatrix} \mathbb{C} & \boxed{\sigma[1]} \\ & \mathbb{C} \end{pmatrix}$

$x \in HOM(L_1, L_2)[1]$

$\in END(\tilde{L})$

$L_3 = [\tilde{L}, x]$

B-MODEL ON CP^1

$HOM(L_1, L_2) = \mathbb{C}^2$

?

$HOM(L_1, L_2) = \mathbb{C}^3$

μ_n = AS 0

AS POSSIBLE

3

x_A

L_i END

$HOM(L_i, L$

"EXCEPTIO