Title: Lecture 4: Factorization Algebras and the General Structure of QFT

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Collection: QFT for Mathematicians

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Free scalar field

$\varphi \in C^\infty(M)$

$\int \varphi D \varphi$

$\sum \partial_{x_i}^2$

If $U \subseteq M$

$f_i \in C_c^\infty(U)$

$g_i \in C_c^\infty(U)$

$\mathrm{Obs}^q(U) =$ cochain complex spanned by

$\mathcal{O}_{f_1} \cdots \mathcal{O}_{f_n} \mathcal{O}^\vee_{g_1} \cdots \mathcal{O}^\vee_{g_m}$

in degree $-m$

d_{Ob}

$$d_{Obs^q(U)} = \Delta_{BV} + \frac{1}{\hbar} Q$$

Δ_{BV} contracts 1 f with 1 g

$$\Delta_{BV}(\mathcal{O}_f \, \mathcal{O}^\vee_g) = \int_U fg$$

Δ_{BV} = "Divergence wrt the Lebesgue measure"

Q a derivation

$$Q \, \mathcal{O}^*_g = \mathcal{O}_{Dg}$$

Laplacian

Q = Lie derivative of a v. field on

$$S(\varphi) = \int \varphi \Delta \varphi$$

In the interacting case

$$S(\varphi) = \int \varphi D \varphi + \int \varphi^4$$

we need to add a term

$$\{I, -\} \qquad I = \int \varphi^4$$

The extra term is Lie derivative of a v. field on the function

$$\varphi \longmapsto \int \varphi^4$$

φ^4

term

derivative

function

$\to \int \varphi^4$

$$\{I, \mathcal{O}_g^*\} = \int_{x\in M} g(x)\,\varphi(x)^3$$

Problem This function

$$\varphi \longrightarrow \int g\,\varphi^3 \quad \text{is not}$$

in the class of fns we considered.

It's not $\displaystyle\int_{M^3} f_1(x_1) f_2(x_2) f_3(x_3)\,\varphi(x_1)\varphi(x_2)\varphi(x_3)$ $\quad f_i \in C_c^\infty(M)$

$g(x)\,\varphi(x)^3$

$= M$

function

is not

fns we considered.

$x_2)\, f_3(x_3)\, \varphi(x_1)\, \varphi(x_2)\, \varphi(x_3)$

$C_c^\infty(M)$

Instead,

$$\int g\varphi^3 = \int \underbrace{g(x_1)\,\delta_{x_1=x_2=x_3}}\,\varphi(x_1)\,\varphi(x_2)\,\varphi(x_3)$$

$\boxed{\text{This is a distribution}}$

Tentative Sol^n ... case

Enlarge are allowed fns to include things like

$$\varphi \longmapsto \int D(x_1 \ldots x_n)\,\varphi(x_1)\ldots\varphi(x_n)$$

D is a distribution on M^n w. compact support.

Prob

BV

fns

$) \varphi(x_1) \ldots \varphi(x_n)$

istribution on M^n
pact support.

Problem

BV Laplacian ill defined

$f \in \mathcal{D}_c(U)$

$g \in \mathcal{D}_c(U)$

$$\Delta_{BV}(\mathcal{O}_f \, \mathcal{O}^*_g) = \text{“} \int f(x) g(x) \text{”}$$

ill-defined as we can't multiply distributions.

Instead,

$$\int g \varphi^3 = \int g(x, \ldots$$

$\boxed{\text{This}}$

Actual Sol^n

- Use observables which are distributions
- Mollify Δ_{BV} to Δ_ε

where $\Delta_\varepsilon(\mathcal{O}_f\, \mathcal{O}_g^*) = \int K_\varepsilon(x_1, x_2) f(x_1) g(x_2)$

✓ K_ε is smooth. $\varepsilon \to 0$ reproduces Δ_{BV}

$$\Delta_0 - \Delta_\varepsilon = [Q, \partial_{P_0^\varepsilon}]$$

where $P_0^\varepsilon = \int_0^\varepsilon K_t$

$\partial_{P_0^\varepsilon}$ contracts 2 $\mathcal{O}_f$'s by

$$\partial_{P_0^\varepsilon}(\mathcal{O}_{f_1} \mathcal{O}_{f_2}) \longrightarrow \int f_1(x_1) f_2(x_2) P_0^\varepsilon(x_1, x_2)$$

$(x_1, x_2) f(x_1) g(x_2)$

$\to 0$ reproduces Δ_{BV}

$$\Delta_\varepsilon - \Delta_L = [Q, \partial_{P_\varepsilon^L}]$$

The

$Q, \partial_{P_0^\varepsilon}]$

K_t

$\mathcal{O}_f$'s by

$\int f_1(x_1) f_2(x_2) P_0^\varepsilon(x_1, x_2)$

$\partial_{P_\varepsilon^L}]$

The differentials
$\frac{1}{\hbar}Q + \Delta_\varepsilon$ are up to homotopy independent of ε.

What about add interacting term?

$$\frac{1}{\hbar}Q + \Delta_\varepsilon + \{ I[\varepsilon], - \}_\varepsilon$$

this is up to htpy independent of ε as long as $I[\varepsilon]$ satisfies

"homotopical RG flow"

The differentials

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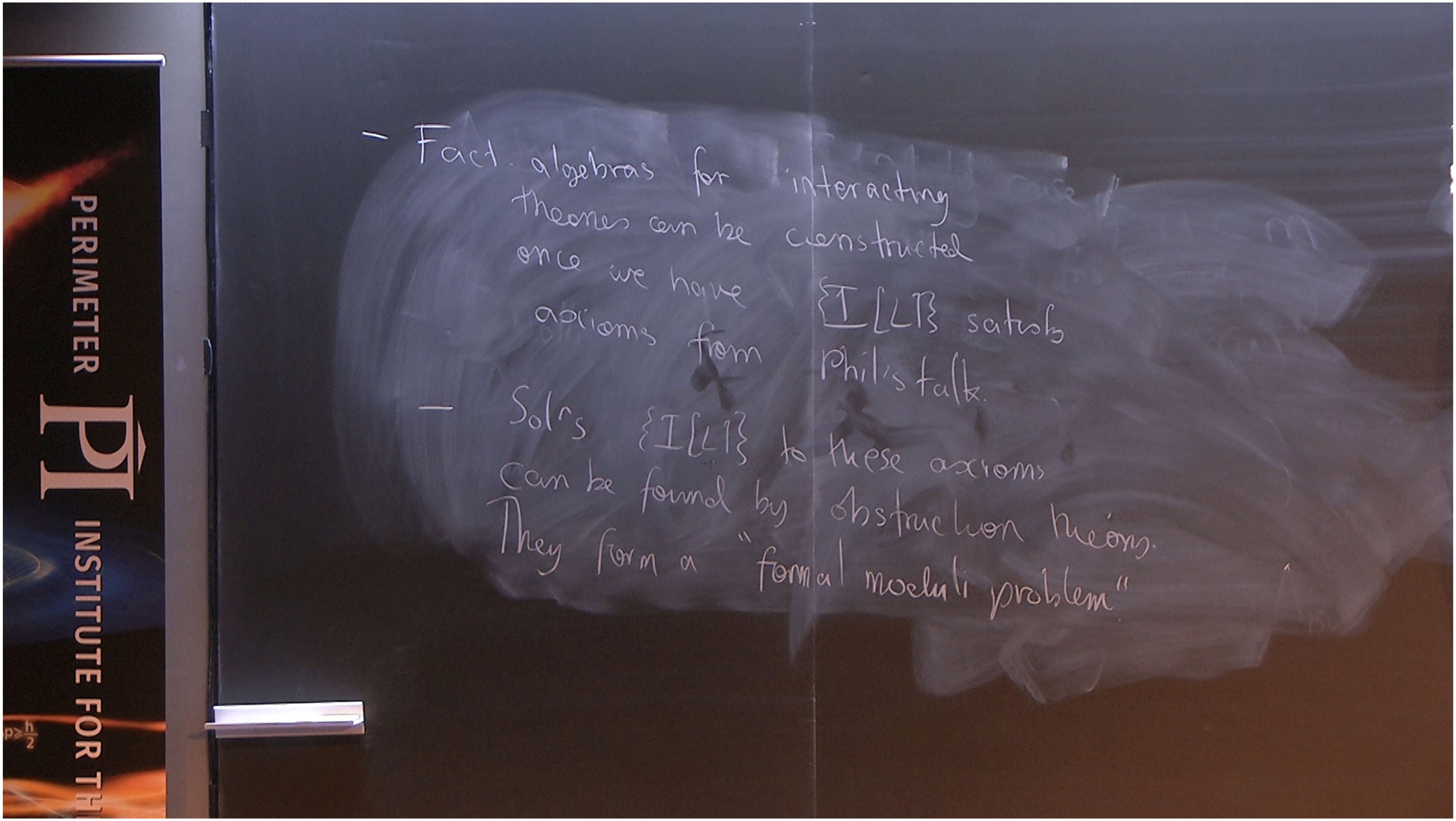
What about add interacting term?

(x_1, x_2)

$$\frac{1}{\hbar}Q + \Delta_\varepsilon + \{I[\varepsilon], -\}_\varepsilon$$

this is up to htpy independent of ε as long as $I[\varepsilon]$ satisfies "homotopical RG flow"

(and $I[\varepsilon]$ must satisfy QME to ensure squares to 0)


PERIMETER
PI
– Fact. algebras for interacting
theories can be constructed
once we have {I[L]} satisfy
axioms from Phil's talk.
– Sol'ns {I[L]} to these axioms
can be found by obstruction theory.
They form a "formal moduli problem"

case

satisfy

k.

axioms

ation theory.

i problem"

In the case of free scalars

Sol'ns $\{Z[L]\}$ to these axioms

$\xleftrightarrow{\text{non-canonical}}$ Lagrangians

$$\sum \hbar^i \int P_i(\varphi, \partial\varphi, \partial^2\varphi, \dots)$$

The

$\frac{1}{\hbar} Q +$

homo

ε

...alars

...e axioms

Renormalizability.

Problem: Seems to be too many theories.

E.g. $\int_{\mathbb{R}^4} \varphi^{10}$ should be a "bad" interaction; why do we include it?

$\mathbb{R}_{>0}$ acts on $\mathbb{R}^n$

If $\mathcal{F}$ is a f. alg. on $\mathbb{R}^n$, $\lambda^* \mathcal{F}$ is also.

$\mathbb{R}_{>0}$ acts on set of f. algebras on $\mathbb{R}^n$

This is called the RG flow.

$\mathcal{F}$ is a fixed point if
$\lambda^* \mathcal{F} \cong \mathcal{F}$
(this is data)

Ex The fact. alg of a free massless scalar field is a fixed point.

$\varphi \in C^\infty(\mathbb{R}^n)$
make $\lambda \in \mathbb{R}_{>0}$
act on φ by
$\varphi(x) \longrightarrow \lambda^{n/2-1}\varphi(\lambda x)$

Then $\int \varphi \Delta \varphi$ is scale invariant.

If λ act on $f \in C^\infty_c(U)$
by $f \to \lambda^{n/2+1} f(\lambda x)$

and on
then
d on
spann
$\mathcal{O}_{f_1 \ldots}$

$^{-1}\varphi(\lambda x)$

is scale

$\in C^\infty_c(U)$

and on $g(x) \longrightarrow \lambda^{n/2-1} g(\lambda x)$

then

d on $\mathrm{Obs}^q(U)$

spanned by

$\mathcal{O}_{f_1} \dots \mathcal{O}_{f_n} \mathcal{O}^\vee_{g_1} \dots \mathcal{O}^\vee_{g_m}$

is scale invariant.

$d = \Delta_{BV} + \frac{1}{\hbar} Q$

commutes w. $\mathbb{R}_{>0}$ action.

and on $g(x) \longrightarrow \lambda^{n/2-1} g(\lambda x)$

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$$d = \Delta_{BV} + \frac{1}{\hbar} Q$$

commutes w. $\mathbb{R}_{>0}$ action.

What about deformations?

$\mathcal{M} = \{$ moduli of translation inv. fact. algebras on $\mathbb{R}^n \}$

$$T_{free}\mathcal{M} = \{ \text{Lagrangians} \}$$

$\mathbb{R}_{>0}$ acts on $T_{free}\mathcal{M}$.

We ce

this ach

What about deformations?

$$\mathcal{M} = \{\text{moduli of translation inv. fact. algebras on } \mathbb{R}^n\}$$

$$T_{\text{free}}\mathcal{M} = \{\text{Lagrangians}\}$$

$\mathbb{R}_{>0}$ acts on $T_{\text{free}}\mathcal{M}$.

We can compute this action (skipping a small subtlety) by asking how a Lagrangian changes $\mathcal{L}(\varphi)$ under

$$\varphi(x) \longrightarrow \lambda^{n/2-1}\varphi(\lambda x)$$

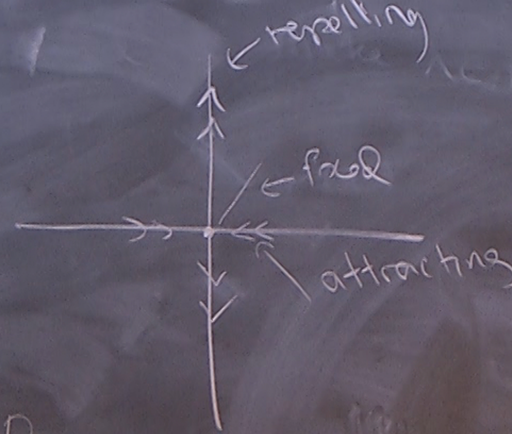
In $\dim^n$ 4:

$$\int_{\mathbb{R}^4} \underbrace{\varphi^n}_{\text{wt } n} \underbrace{dx_1 \cdots dx_4}_{\text{wt } -4}$$

transforms by λ^{n-4}

If we do some small calculations and consider $SO(4)$ invariant Lagrangians,

the tangent space looks like

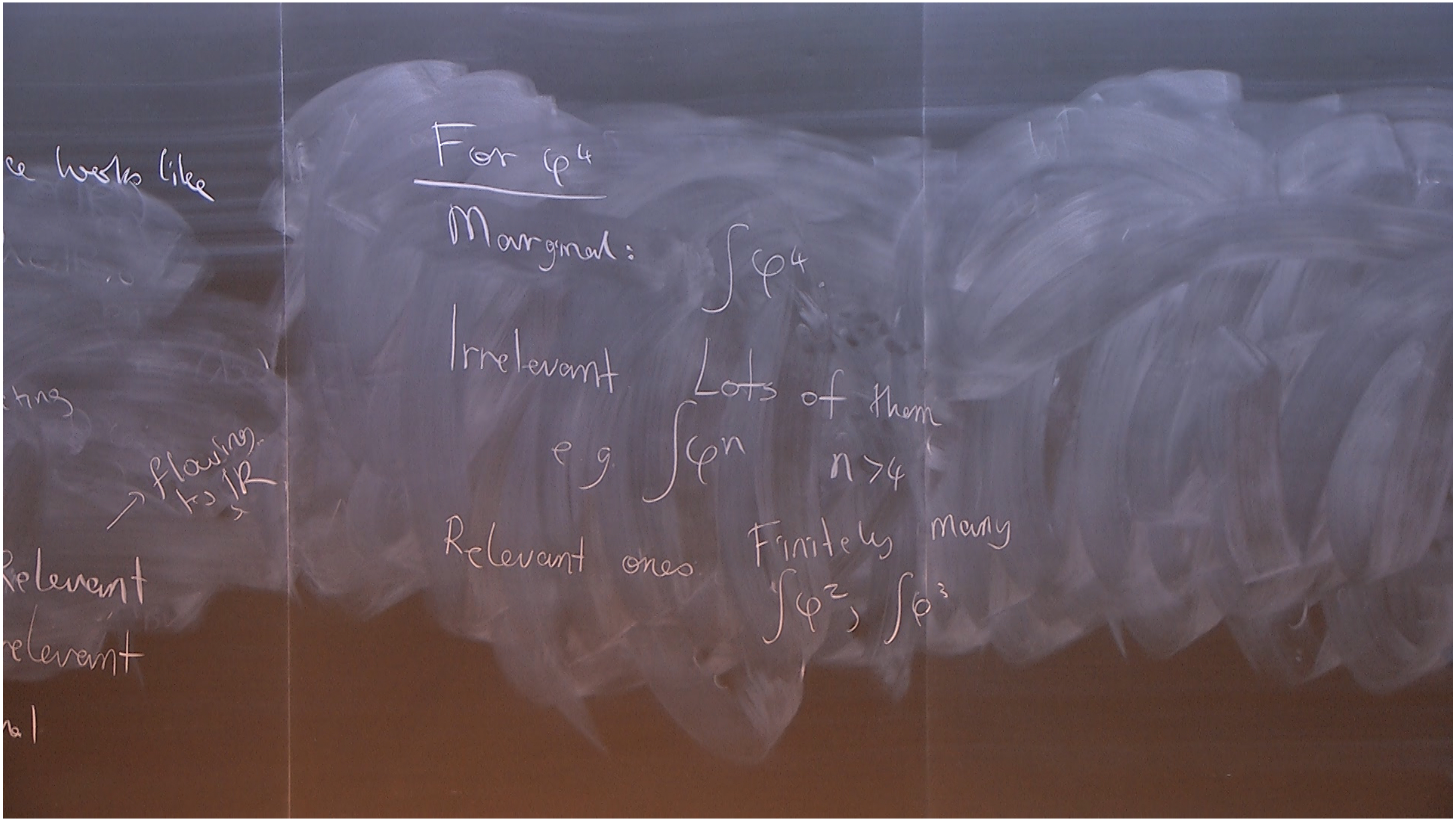


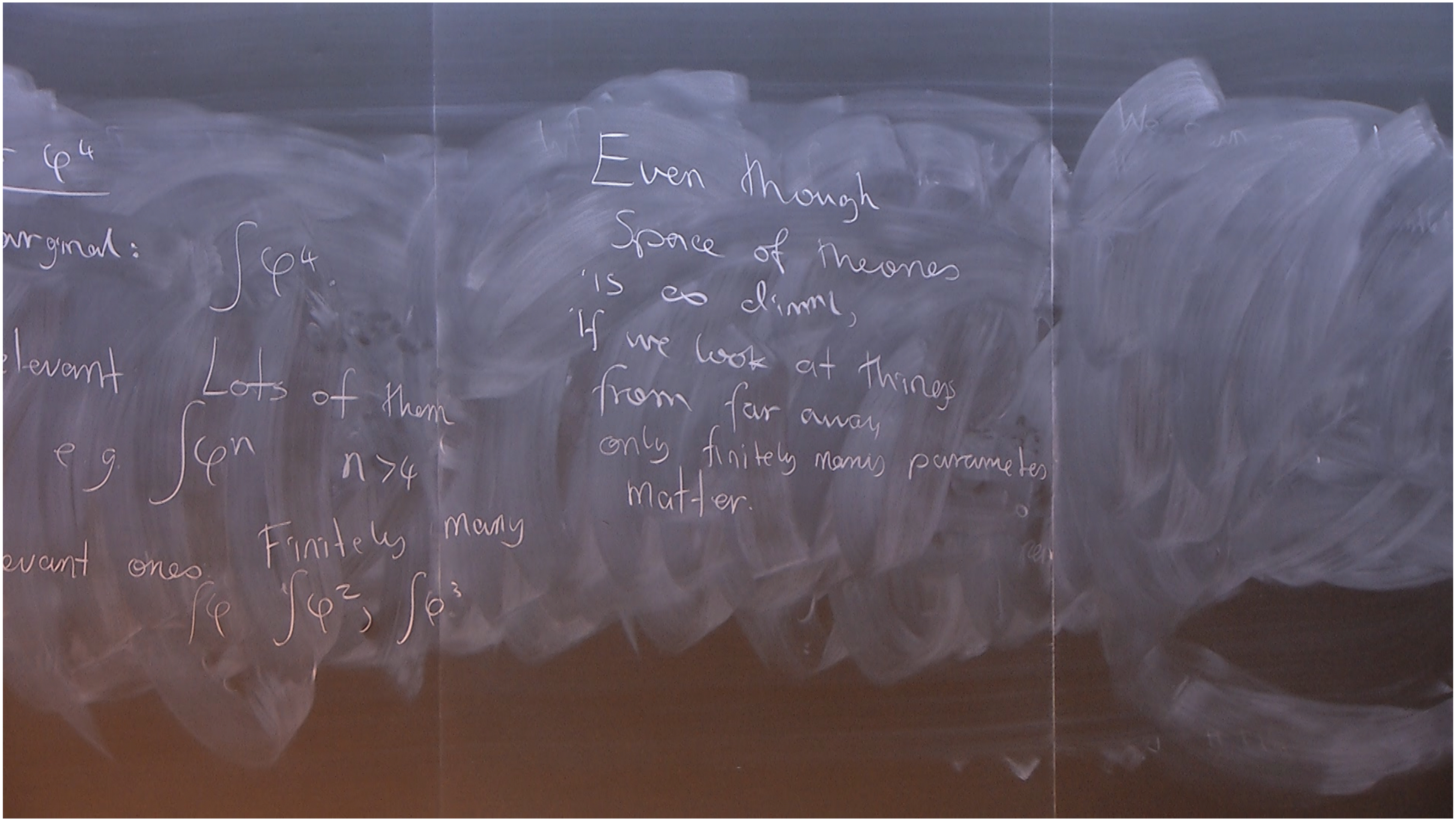
flowing to IR

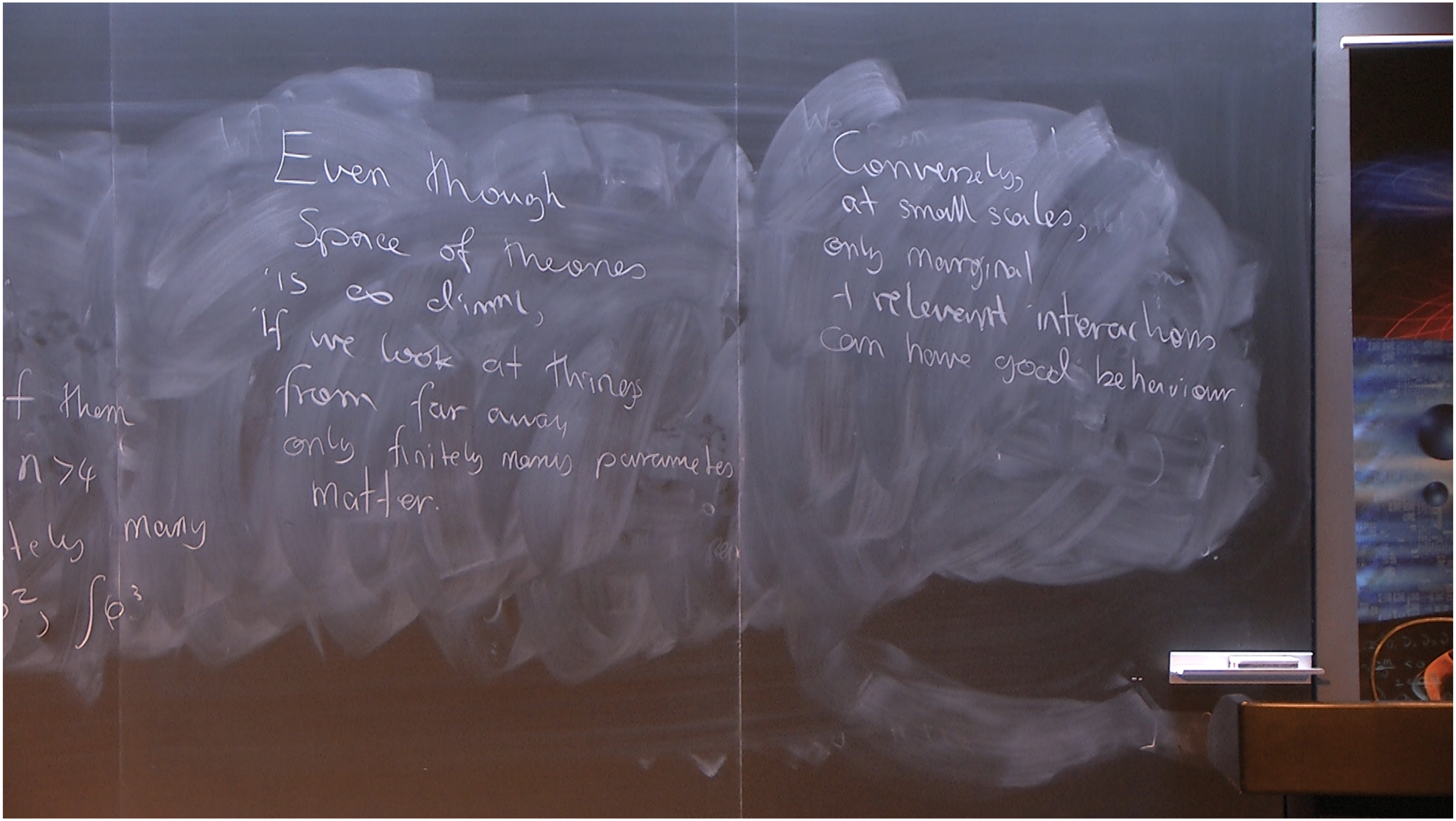
Repelling = Relevant

Attracting = Irrelevant

Fixed = Marginal


ce works like
ting
flowing to IR
Relevant
relevant
For φ^4
Marginal: $\int \varphi^4$.
Irrelevant Lots of them
e.g. $\int \varphi^n$ $n>4$
Relevant ones. Finitely many
$\int \varphi^2$, $\int \varphi^3$


φ^4
original: $\int \varphi^4$.
relevant Lots of them
e.g. $\int \varphi^n$ $n > 4$
relevant ones: Finitely many
$\int \varphi$ $\int \varphi^2$, $\int \varphi^3$
Even though
Space of theories
is ∞ dim'l,
If we look at things
from far away,
only finitely many parameters
matter.


f them
n >4
tely many
Even though
Space of theories
is ∞ dimnl,
If we look at things
from far away
only finitely many parameters
matter.
Conversely,
at small scales,
only marginal
+ relevant interactions
can have good behaviour.

t space looks like
pelling

fixed

attracting

flowing to IR

For 4 dim^n free theory:

Marginal: $\int \varphi^4$

Irrelevant Lots of them

e.g. $\int \varphi^n$ $n>4$

Relevant ones: Finitely many

$\int \varphi$, $\int \varphi^2$, $\int \varphi^3$

Even though

Space of the

is ∞ dim^l,

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matter.

Corrections to the flow

Consider
$$\int \varphi \Delta \varphi + c \int \varphi^4$$

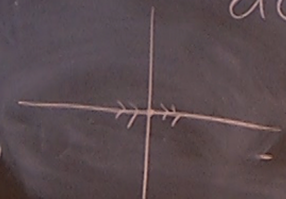
We claimed that to leading order in c, this is scale invariant.

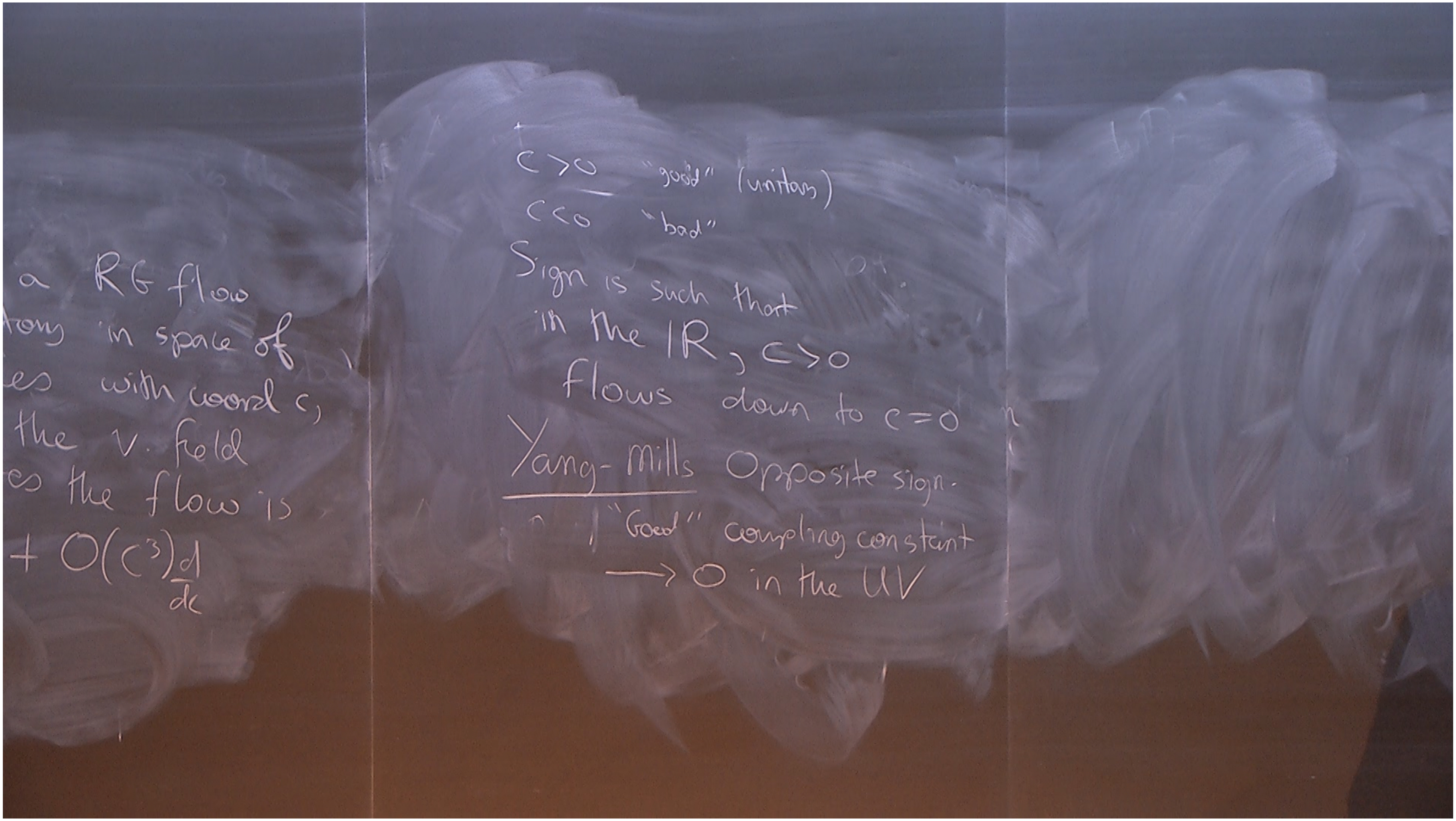
Does this extend beyond leading order?

No:

There is a RG flow trajectory in space of theories with coord c, where the v. field generates the flow is

$$c^2 \frac{d}{dc} + O(c^3)\frac{d}{dc}$$




a RG flow
tory in space of
es with coord c,
the v. field
es the flow is
$+ O(c^3)\frac{d}{dc}$
$c>0$ "good" (unitary)
$c<0$ "bad"
Sign is such that
in the IR, $c>0$
flows down to $c=0$
Yang-Mills Opposite sign.
"Good" coupling constant
$\longrightarrow 0$ in the UV

"good" (unitary)

"bad"

such that

$\mathbb{R}$, $c>0$

ws down to $c=0$

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"ood" coupling constant

$\to 0$ in the UV

How to compute

- Understand leading order corrections to the f alg as we deform away from a free theory
- Compute with the $I[L]$'s

$c = 0$

sign.

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dV

How to compute

- Understand leading order corrections to the f alg as we deform away from a free theory.
- Compute with the $I[L]$'s.

Notation

R_λ = how $\lambda \in \mathbb{R}_{>0}$ acts on $\varphi \in C^\infty(\mathbb{R}^4)$

$I[L](\varphi)$

define

$$R_\lambda(I[L])(\varphi) = I[\lambda^2 L](R_\lambda \varphi)$$

$$K_L = e^{-\|x_1 - x_2\|^2/L} L^{-n/2}$$

Check:

If $I[L]$ satisfies the axioms so does $R_\lambda(I)[L]$

Computation

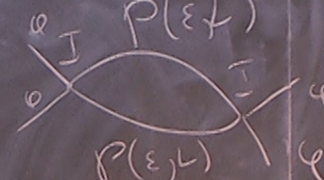
Start with $\int \varphi D\varphi + c\int \varphi^4$

Build $I[L](c)$

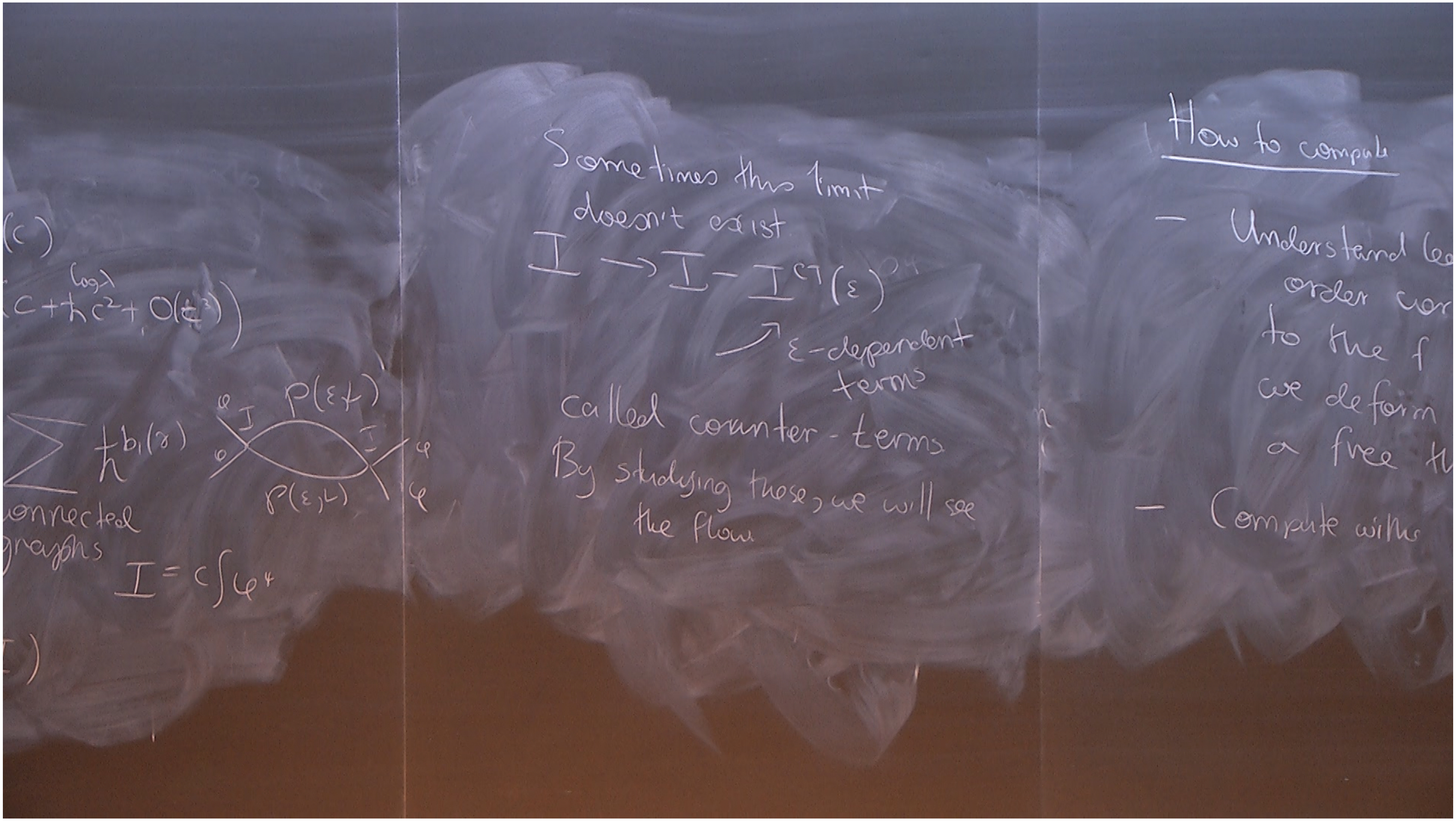
$$R_\lambda(I)[L](c) = I[L]\left(c + \hbar \log\lambda \, c^2 + O(c^3)\right)$$

Naively,

$$I[L] = \lim_{\varepsilon \to 0} \sum_{\substack{\text{connected} \\ \text{graphs}}} \hbar^{b_1(\gamma)} \, W(P_\varepsilon^L, I)$$



$I = c\int \varphi^4$


Sometimes this limit doesn't exist
$I \longrightarrow I - I^{CT}(\varepsilon)$
ε-dependent terms
called counter-terms
By studying these, we will see the flow
How to compute
— Understand lea order cor to the f we deform a free th
— Compute with
$\sum \hbar^{b_1(\gamma)}$
connected graphs
$I = c\int \varphi^4$
$P(\varepsilon, L)$

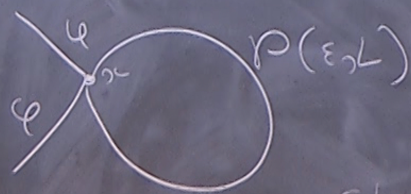
...pendent
...rms

...erms

will see

Tree diagrams: ✓

1 Loop:

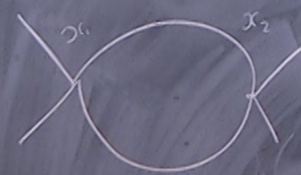


$$= \int_{x\in\mathbb{R}^4} \varphi(x)^2 \int_{t=\varepsilon}^{L} t^{-2} \cancel{e^{-\|x_1-x_2\|^2/t}}$$

$$= \int_{x\in\mathbb{R}^4} \varphi(x)^2 \left(\frac{1}{\varepsilon} - \frac{1}{L}\right)$$

1st Counter term

$$\hbar \frac{1}{\varepsilon} \int \varphi(x)^2$$



$$\int_{x_1,x_2\in\mathbb{R}^4} \varphi(x_1)^2 \varphi(x_2)^2 P_\varepsilon^L(x_1,x_2)^2$$

This is

$$\int_{t_1,t_2=\varepsilon}^{L}\int_{x_1,x_2}\varphi(x_1)^2\varphi(x_2)^2\, t_1^{-2}t_2^{-2}\, e^{\left(-\frac{1}{t_1}-\frac{1}{t_2}\right)\|x_1-x_2\|^2}$$

$\int$ over x_1-x_2 we get

$$\int_{t_1,t_2=\varepsilon}^{L}\int_{x}\varphi(x)^4\, t_1^{-2}t_2^{-2}\left(\frac{t_1t_2}{t_1+t_2}\right)^2 dt_1\,dt_2 \sim \log\varepsilon\int\varphi^4$$

$\int \varphi^4$

When we scale everything

$$I[L] = \lim_{\varepsilon \to 0} W\left(P_\varepsilon^L, I - I^{CT}(\varepsilon)\right)$$

$$R_\lambda I[L] = \lim_{\varepsilon \to 0} W\left(P_{\lambda^2 \varepsilon}^{\lambda^2 L}, I - R_\lambda I^{CT}(\lambda^2 \varepsilon)\right)$$

$\varepsilon^{-1} \int \varphi^2$ If we scale everything this is fixed

$$\log \varepsilon \int \varphi^4 \longrightarrow \log \varepsilon \int \varphi^4 + 2 \log \lambda \int \varphi^4$$

Tree diagrams: ✓

$$\cancel{\int_{t=\varepsilon}^{L} t^{-2} e^{-\|x_1 - x_2\|^2/t}}$$

$$= \int_{x \in \mathbb{R}^4} \varphi(x)^2 \left(\frac{1}{\varepsilon} - \frac{1}{L}\right)$$

we scale everything

$\lim_{\varepsilon \to 0} W(P_\varepsilon^L, I - I^{CT}(\varepsilon))$

$\lim_{\varepsilon \to 0} W(P^{\lambda^2 L}_{\lambda^2 \varepsilon}, I - R_\lambda I^{CT}(\lambda^2 \varepsilon))$

If we scale everything this is fixed

$\to \log \varepsilon \int \varphi^4 + 2 \log \lambda \int \varphi^4$

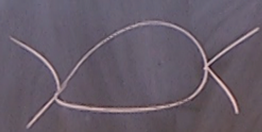
Conclude:

If we scale,

$I^{CT} \to I^{CT} + \hbar \log \lambda \int \varphi^4$

This means

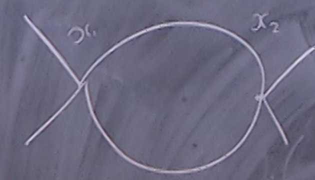
$I \to I + \hbar \log \lambda \int \varphi^4$



C^{2-d}_{KC}

1st Counter term

$\hbar \frac{1}{\varepsilon} \int \varphi(x)^2$



$\int_{x_1, x_2 \in \mathbb{R}^4} \varphi(x_1)^2 \varphi(x_2)^2 P_\varepsilon^L(x_1, x_2)^2$