Title: Lecture 2: Supersymmetric Quantum Mechanics and All That

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Collection: QFT for Mathematicians

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SQM Lecture 2

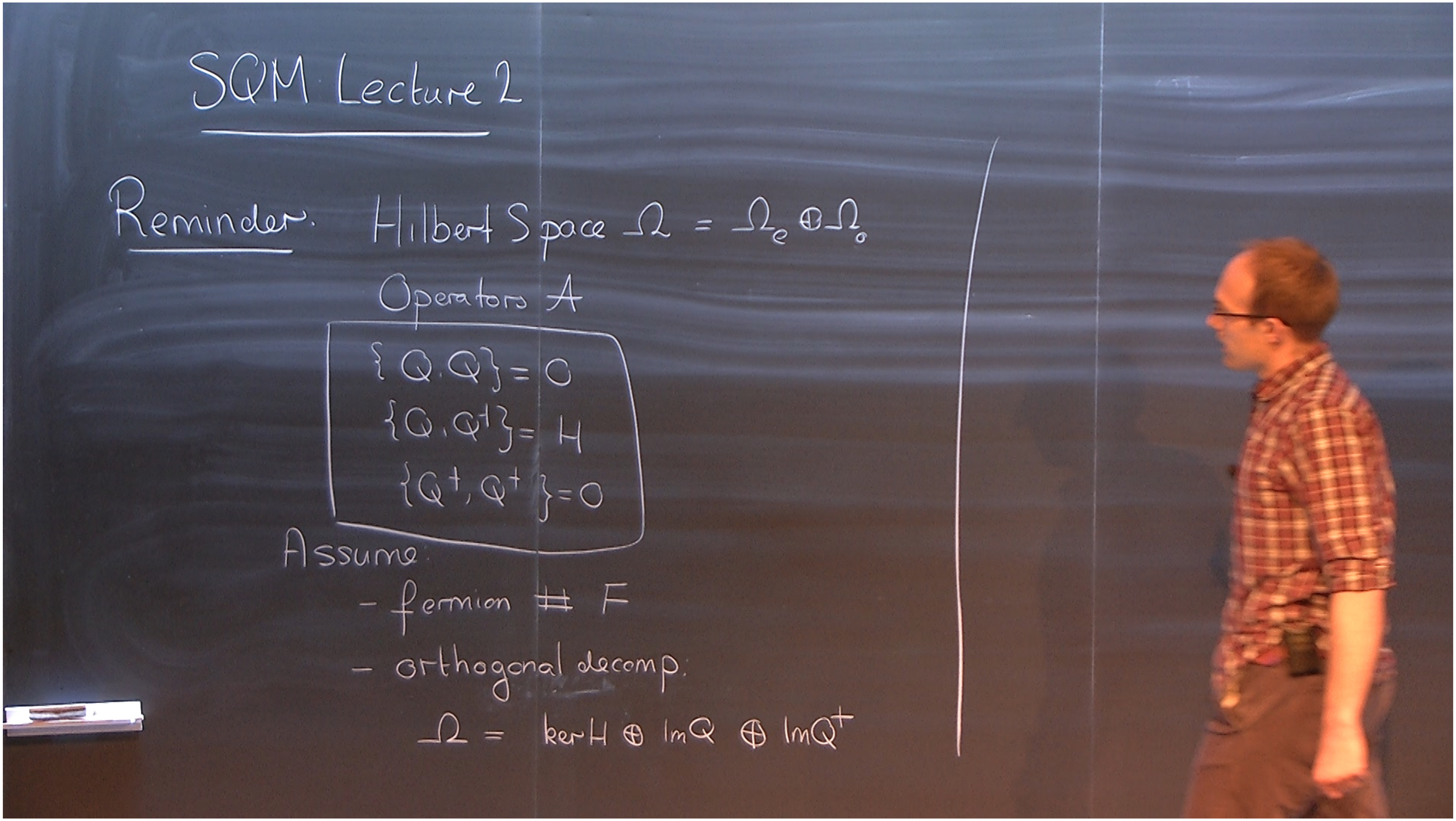
Reminder. Hilbert Space Ω

Operators A

$\{Q, Q\} = 0$

$\{Q, Q^\dagger\} = H$

$\{Q^\dagger, Q^\dagger\} = 0$


SQM Lecture 2
Reminder: Hilbert Space $\Omega = \Omega_e \oplus \Omega_o$
Operators A
$\{Q, Q\} = 0$
$\{Q, Q^\dagger\} = H$
$\{Q^\dagger, Q^\dagger\} = 0$
Assume:
- fermion # F
- orthogonal decomp.
$\Omega = \ker H \oplus \operatorname{Im} Q \oplus \operatorname{Im} Q^\dagger$

2.1) Operators Revisited

Fermion #: F

- $\mathcal{A}$ is $\mathbb{Z}$ graded:

$$\mathcal{A} = \bigoplus_{j\in\mathbb{Z}} \mathcal{A}^j, \quad \mathcal{A}^j = \{ A\in\mathcal{A} \mid [F,A] = jA \}$$

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- Q degree 1:

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$$\left(\quad [A, B] = AB - (-1)^{F(A)F(B)} BA \right)$$

- Compatible w/ product:

$$[Q, AB] = [Q, A]B + (-1)^{F(A)} A[Q, B].$$

...ted

$\longrightarrow (\mathcal{A}^{\bullet}, [Q,\cdot\,])$ is a DG algebra.

...ed:

$\dots\mathcal{A}^{j}, \quad \mathcal{A}^{j} = \{ A \in \mathcal{A} \mid [F, A] = jA \}$

$\mathcal{A}^{j} \xrightarrow{[Q,\,\cdot\,]} \mathcal{A}^{j+1}$

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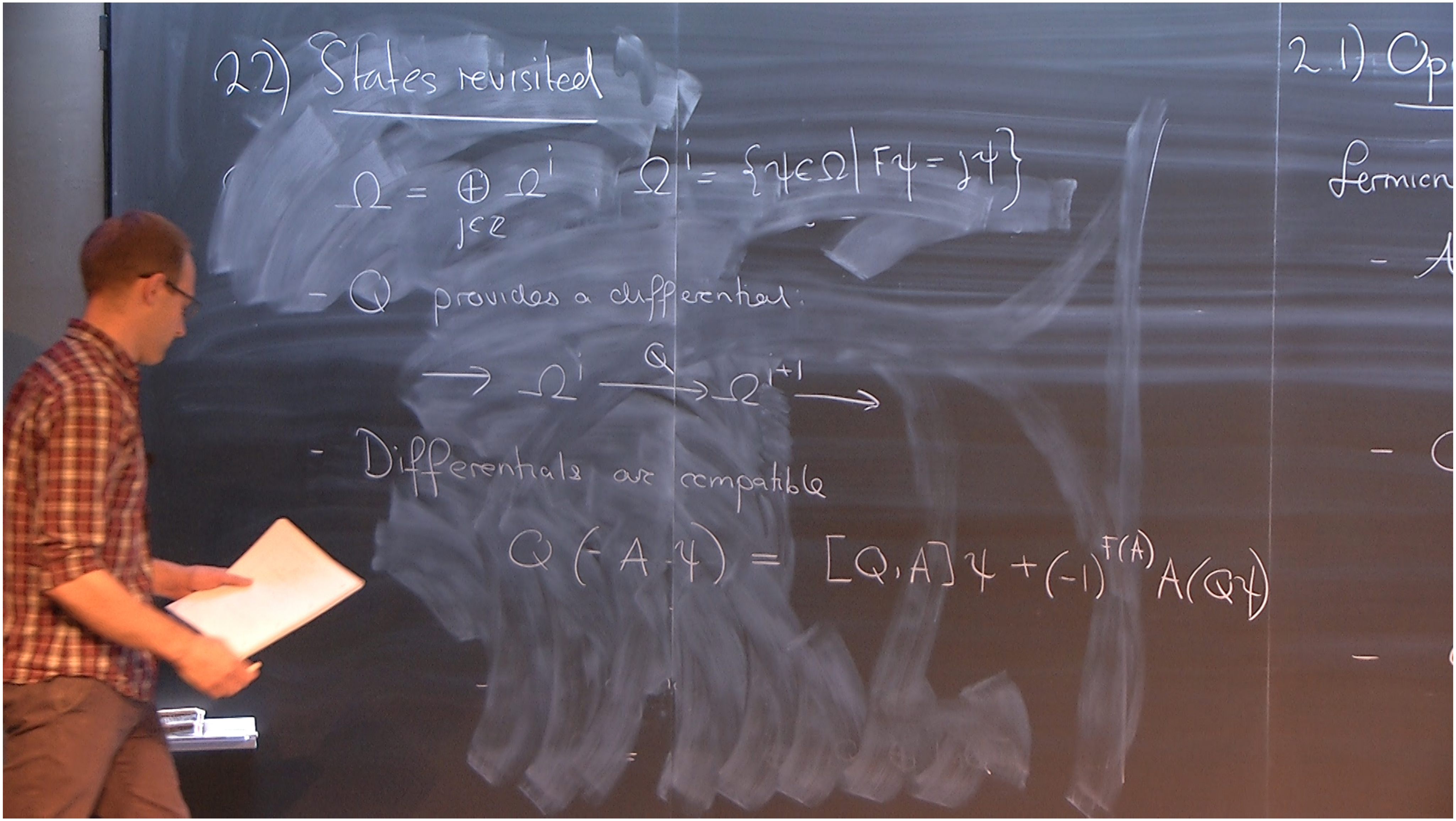
$\dots, AB] = [Q, A]B + (-1)^{F(A)} A[Q, B].$


2.2) States revisited
$\Omega = \bigoplus_{j \in \mathbb{Z}} \Omega^j$, $\Omega^j = \{\psi \in \Omega \mid F\psi = j\psi\}$
- Q provides a differential:
$\longrightarrow \Omega^j \xrightarrow{Q} \Omega^{j+1} \longrightarrow$
- Differentials are compatible
$Q(A\psi) = [Q,A]\psi + (-1)^{F(A)} A(Q\psi)$

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(Ω, Q) DG-module.

2.1) Op

Fermion

2.3) Operators: cohomology of Q

BPS operator: A obeys $[Q, A] = 0$

$\longrightarrow$ Class $[A] \in H^{\cdot}(A) = \mathcal{O}$

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Transfer of structure: $\mathcal{O}^{\bullet}$ inherits a A_∞-structure:

$$\mu_n : \mathcal{O}^{\otimes n} \longrightarrow \mathcal{O}$$

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$\hookrightarrow$ degree $2-n$

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- $\mu_1 = 0$
- μ_2 is inherited from product $A^{\cdot}$

$$[A_1][A_2] = [A_1 A_2]$$

2.3) Operators: cohomology of Q

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- μ_2 is inherited from product $\mathcal{A}$

$$[A_1] \cdot [A_2] = [A_1 A_2]$$

- μ_n, $n \geq 3$ may not be trivial

2.3) Operators: cohomology of Q

BPS operator: A obeys $[Q, A] = 0$

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$$[A_1][A_2] = [A_1 A_2]$$

- μ_n, $n \geq 3$ may not be trivial

Unique up to quasi-isomorphism.

Ex. suppose 3-BPS ops A_1, A_2, A_3

$$[A_1, A_2) = 0$$

$$[A_2, A_3) = 0$$

$$A_1 A_2 = [Q, B_{12}]$$

$$A_2 A_3 = [Q, B_{23}]$$

Define $A_{123} = B_{12} A_3 - (-1)^{F(A_1)} A_1 B_{23}$

$[Q, A_{123}] = 0$ A_{123} also BPS

Time Dependence in Heisenberg picture

$$\partial_z A = [H, A]$$

$$= [Q, [Q^{+}, A]]$$

if A is BPS

$[A]$ is independent of z.

Time Dependence: in Heisenberg picture

$$\partial_\tau A = [H, A]$$

$$= [Q, [Q^\dagger, A]]$$

if A is BPS

$[A]$ is independent of τ.

Finite version. $A \in \mathcal{A}^j$

$\longrightarrow$ descendant: $A^{(1)} = [Q^\dagger, A]\, d\tau$: $\mathcal{A}^{j-1}$ valued 1-form on $\mathbb{R}$: $\uparrow \tau$

$$[Q, A^{(1)}] = d_{\mathbb{R}} A$$

$$A(\tau_2) - A(\tau_1) = \left[Q, \int_{\tau_1}^{\tau_2} A^{(1)}(\tau)\right].$$

Research Idea:

$$\underbrace{A_1(z_1) \cdots\cdots A_n(z_n)}_{\text{BPS ops}}$$

? Construct μ_n by integrating $(n-2)$-form descendent ?

$\longrightarrow$ Via descent, construct k-forms on

$$\mathrm{Conf}_n(\mathbb{R}) = \{x_1 < \cdots < x_n\}$$

or $\quad$ " $\quad / \mathbb{R}$

or $\quad \{ \text{"} \quad \text{"} \} / \underset{\substack{\uparrow \\ \text{translations}}}{\mathbb{R}} \times \mathbb{R}^+ \rightarrow \text{scalings}$

Time Depe

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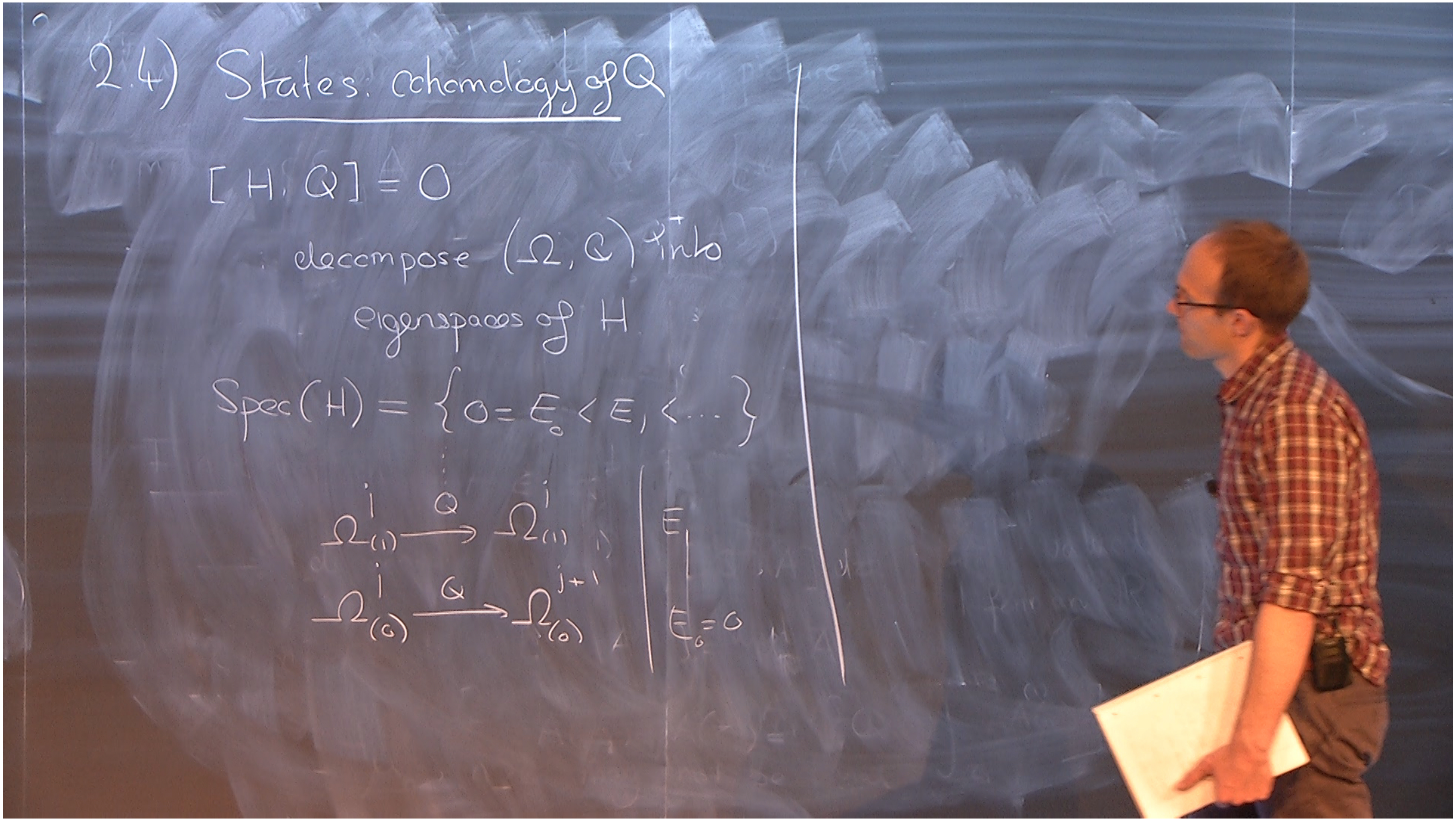
Finite ver

$\longrightarrow$ d


2.4) States: cohomology of Q
$[H, Q] = 0$
decompose (Ω, Q) into
eigenspaces of H
$\mathrm{Spec}(H) = \{0 = E_0 < E_1 < \dots\}$
$\Omega^j_{(1)} \xrightarrow{Q} \Omega^j_{(1)}$ E_1
$\Omega^j_{(0)} \xrightarrow{Q} \Omega^{j+1}_{(0)}$ $E_0 = 0$

2.4) States: cohomology of Q

$[H, Q] = 0$

decompose (Ω, Q) into eigenspaces of H

$\mathrm{Spec}(H) = \{0 = E_0 < E_1 < \dots\}$

$$\Omega^j_{(1)} \xrightarrow{Q} \Omega^j_{(1)}$$

$$\boxed{\Omega^j_{(0)} \xrightarrow{Q} \Omega^{j+1}_{(0)}}$$

E_1

$E_0 = 0$

- Cohomology vanishes for $n > 0$:

$\psi \in \Omega_{(n)}$

$\{Q, Q^+\}\psi = E_n \psi$

if $Q\psi = 0$:

$\psi = Q\left(\frac{Q^+\psi}{E_n}\right)$.

- **Claim:** $H^j(\Omega^\cdot, Q)$

$\simeq \ker H|_{\Omega^j}$

$\simeq \mathcal{H}^j$: susy ground states

Ex) Riemannian X

$$\mathcal{H}^i = \mathrm{Harm}^i(X,\mathbb{C})$$
$$\simeq H^i_{dR}(X,\mathbb{C})$$

Ex) Hermitian (X,E)

$$\mathcal{H}^i = \mathrm{Harm}^{0,i}_{\Delta_{\bar\partial_E}}(X,\mathbb{C})$$
$$\simeq H^{0,i}_{\bar\partial}(X,E)$$

2.4) Sta

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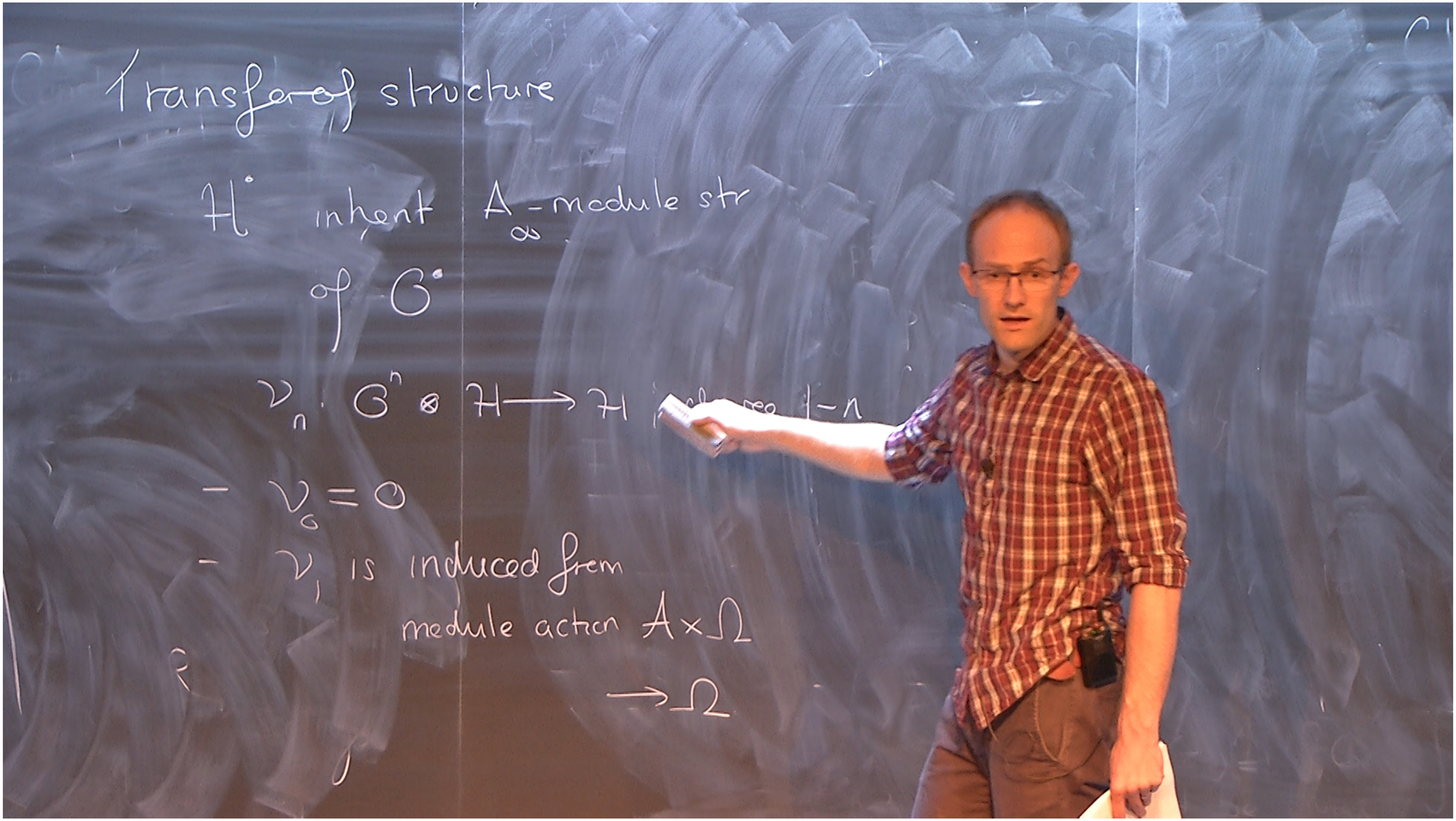
Spec(


Transfer of structure
H° inherit A_∞-module str
of C°
$\nu_n : C^n \otimes H \longrightarrow H$ … $1-n$
$-\ \nu_0 = 0$
$-\ \nu_1$ is induced from
module action $A \times \Omega$
$\longrightarrow \Omega$

2.5) Flavour Symmetry.

G = cpt. connected Lie group

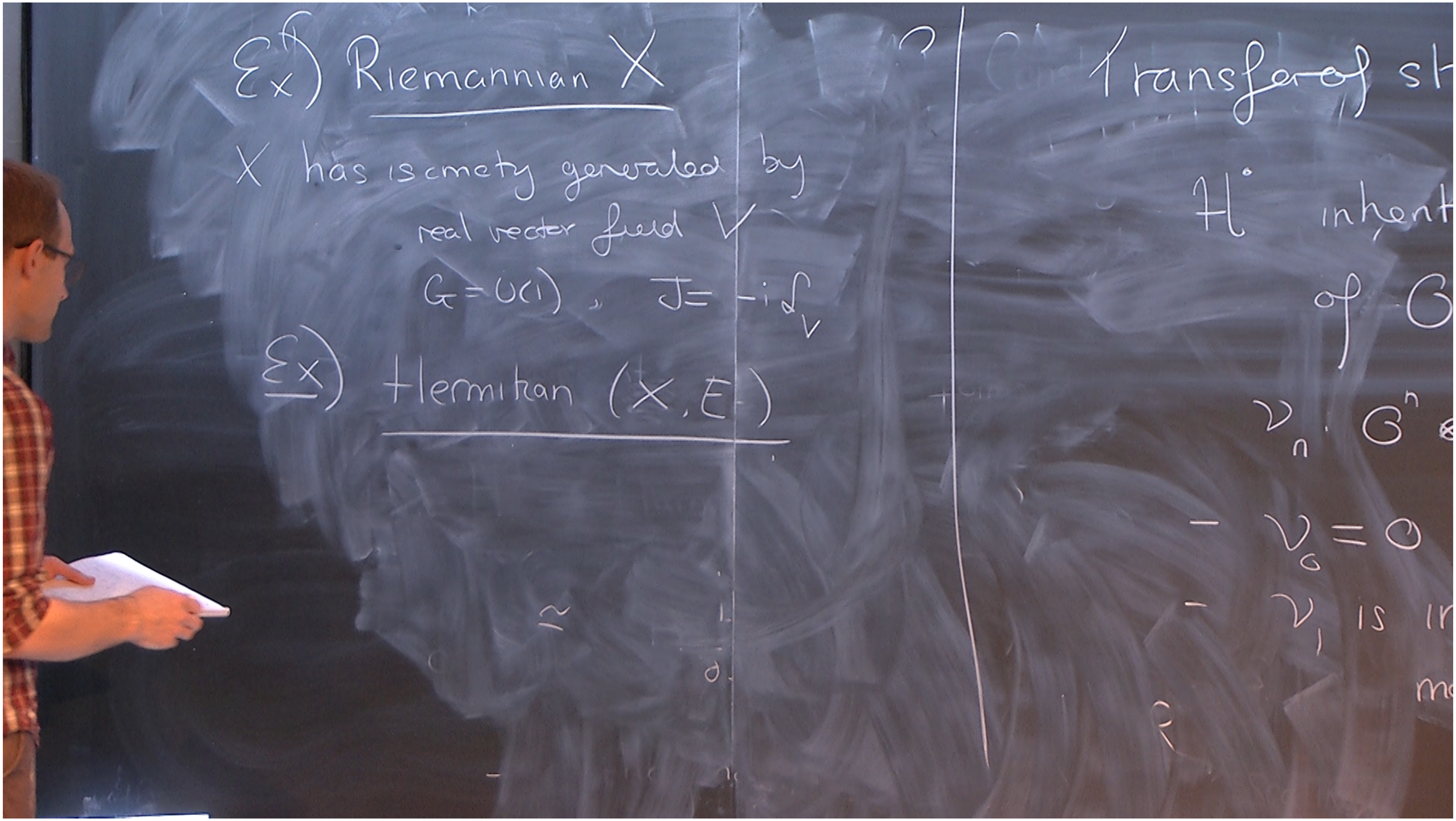
- Ω is unitary repr of G
- commute with $Q, Q^\dagger, H$.

Infinitessimal action:

- Self-adjoint $J_a \in A^0$, $a = 1, \dots, \dim G$
- $[J_a, J_b] = i f_{ab}{}^{c} J_c$
- $[J_a, Q] = [J_a, Q^\dagger] = 0$.

$\longrightarrow$ $\mathfrak{g}$ act on (Ω, Q) by cochain maps.

(Ω, Q) DG-module for $\mathfrak{g}$.


Ex) Riemannian X
X has isometry generated by
real vector field V
G = U(1), J = -i £_V
Ex) Hermitian (X, E)

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X has isometry generated by
real vector field V

$G = U(1)$, $J = -i\mathcal{L}_V$

Ex) Hermitian (X, E)

" "

that lifts to an equiv
action on E preserving
hermitian metric.

$\nu_n : \mathcal{O}^n$

$- \ \nu_0 = \mathcal{O}$

$- \ \nu_1$ is

ent

2.6) Flavour Action on Cohomology

$[Q, J] = 0$

$\longrightarrow$ cohomology class $[J] \in \mathcal{O}^{\circ}$

Two cases:

① $[J] = 0$, $J = [Q, I]$

$\longrightarrow$ $\mathcal{H}^{\circ}$ = trivial module

$J \cdot \psi = [Q, J]\psi$

$= Q(I\psi)$

② $[J] \neq 0$

$\mathcal{H}^{\circ}$ may be non-trivial

2.7) Flavour S

G = cpt.

– Ω

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Infinitessimal actio

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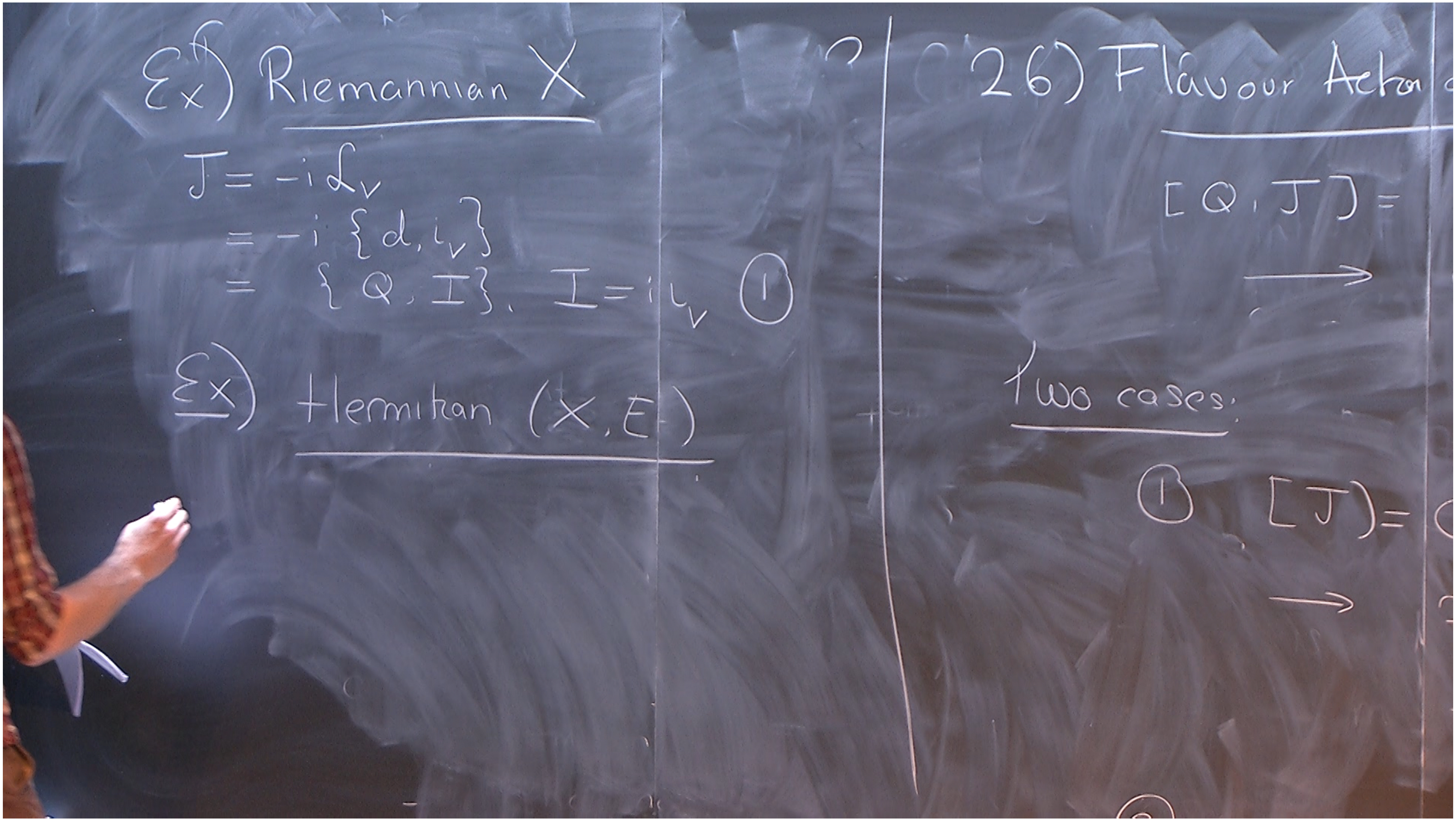
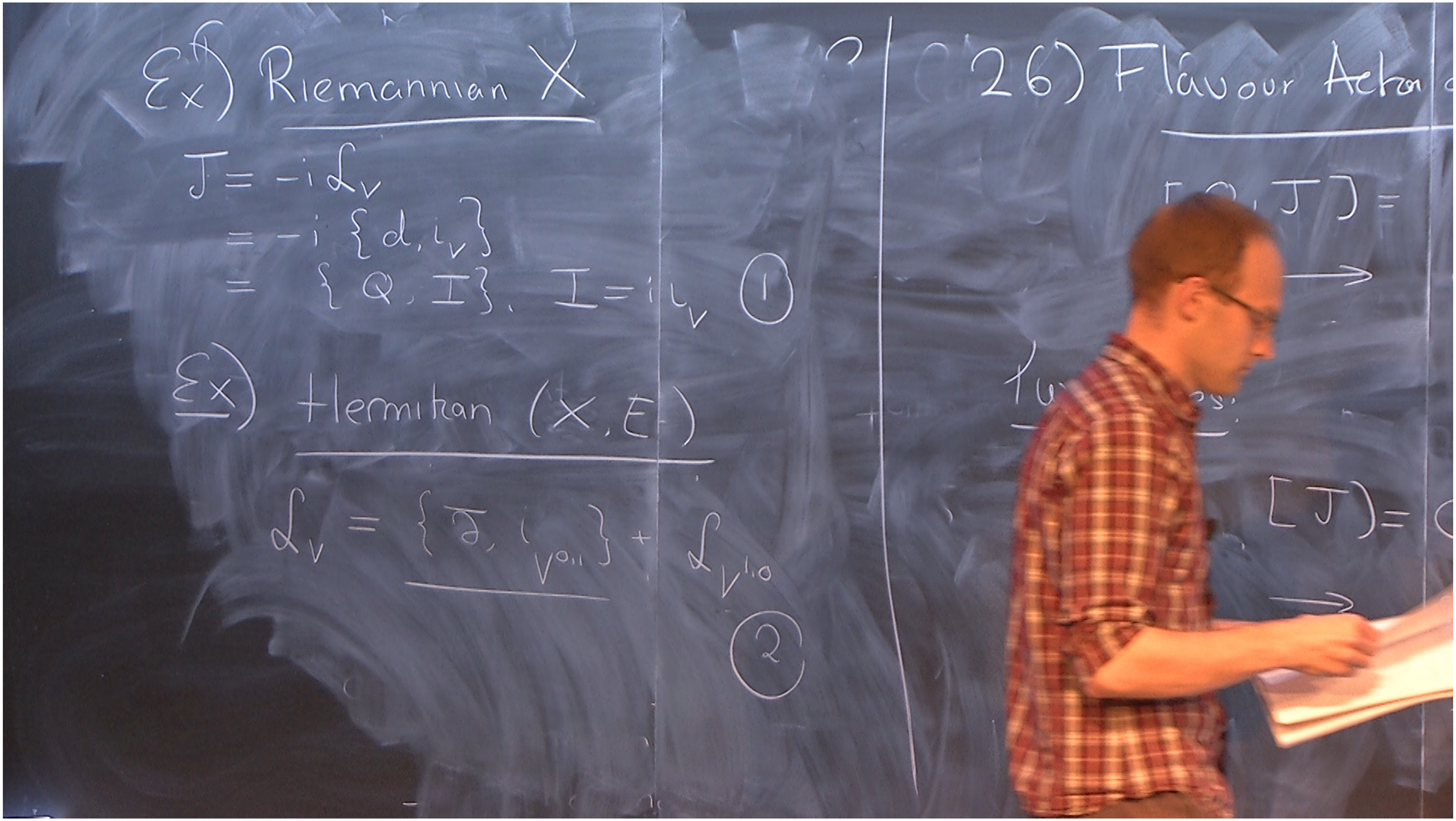

Ex) Riemannian X
$J = -i\mathcal{L}_v$
$= -i\{d, i_v\}$
$= \{Q, I\}, \quad I = i\,i_v$ ①
Ex) Hermitian (X, E)
26) Flavour Action
$[Q, J] =$
Two cases:
① $[J] =$



Ex) Riemannian X
J = -i L_v
= -i {d, i_v}
= {Q, I}, I = i i_v
Ex) Hermitian (X, E)
L_v = {∂̄, i_{v^{0,1}}} + L_{v^{1,0}}
26) Flavour
[J] =

eg. $X = \mathbb{CP}^1$, $E = \mathcal{O}(n)$

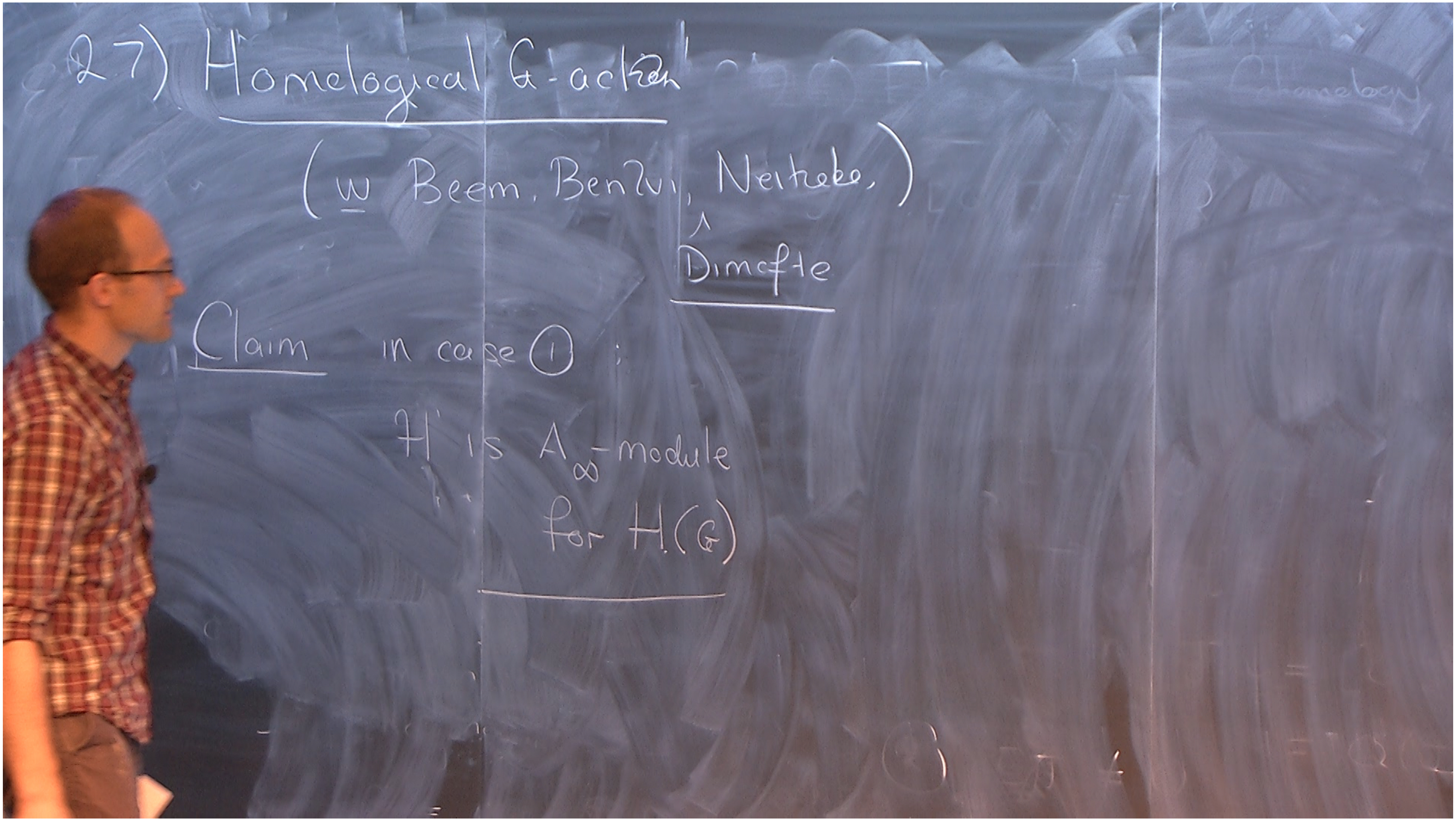
$\circlearrowleft$ $\qquad\qquad n \geq 0$

$\mathfrak{g} = su(2)$

$$\mathcal{H}^j = H^{0,j}_{\bar\partial}(\mathbb{CP}^1, \mathcal{O}(n))$$

Borel-Weil-Bott

$=$ $(n+1)$-dimensional repr of $su(2)$


2.7) Homological G-action
(w Beem, BenZvi, Neitzke)
Dimofte
Claim in case 1 :
H' is A∞-module
for H.(G)

Focus $G = U(1)$

<u>Idea</u>: descent on G

— $\psi \in \Omega^{j}$, $Q\psi = 0$

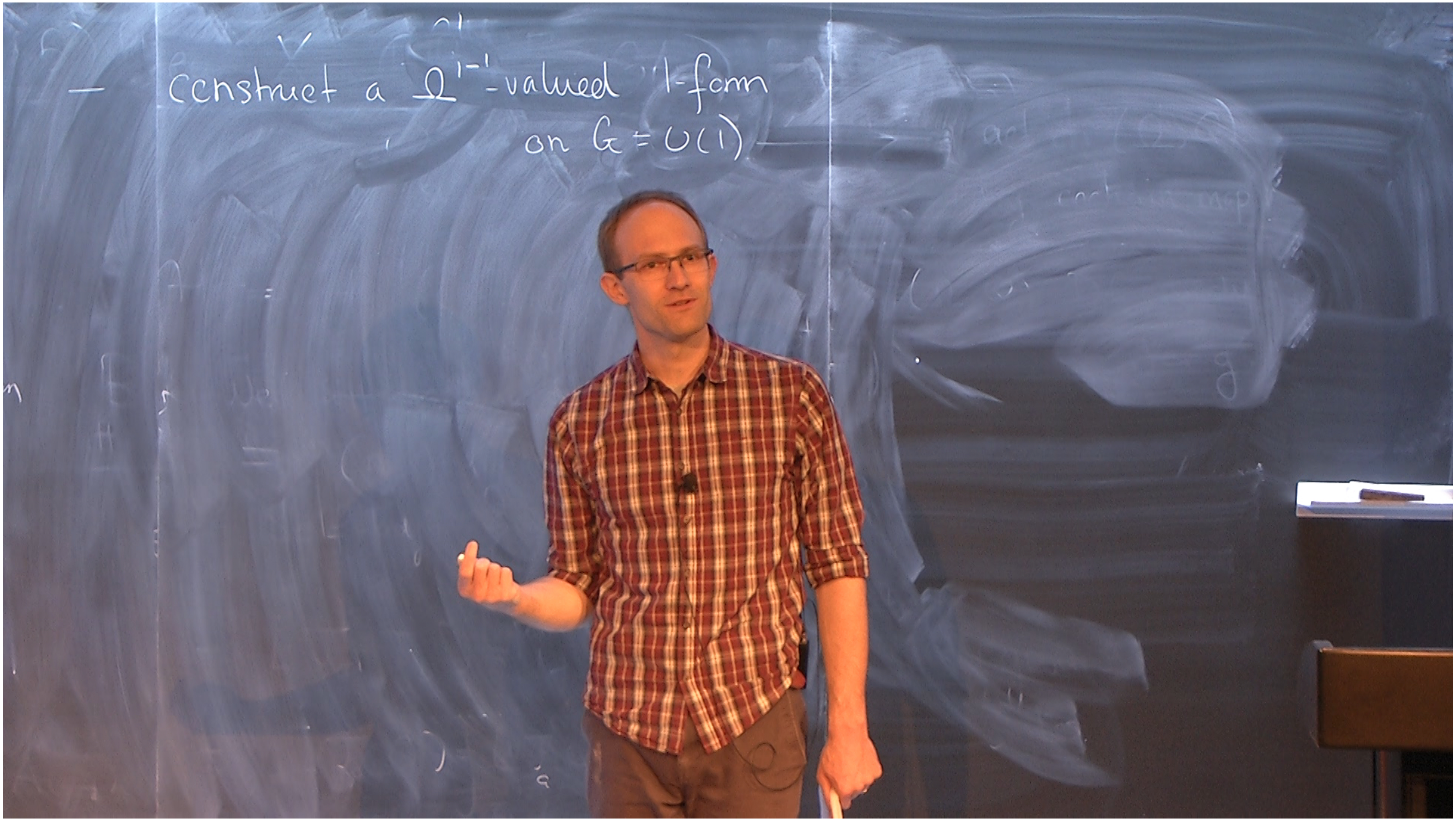
— p

Focus $G = U(1)$

Idea: descent on G

- $\psi \in \Omega^j$, $Q\psi = 0$
- promote to Ω^j-valued function on $G = U(1)$

$$\psi^{(0)} := e^{i\vartheta J}\psi$$


— construct a Ω^{1-1}-valued 1-form
on G = U(1)

— construct a $\Omega^{|-|}$-valued 1-form on $G = U(1)$

$$\boxed{J = \{Q, I\}}$$

$$\longrightarrow \quad \psi^{(1)} = \left(I\, \psi^{(0)}\right) d\theta$$

$$Q\psi^{(1)} = d_{U(1)}\psi^{(0)}$$

— $\gamma \in C_1(G)$

$$\gamma \cdot \psi = \int_\gamma \psi^{(1)}$$

— construct a Ω^{j-1}-valued 1-form on $G = U(1)$

$$\boxed{J = \{Q, I\}}\,, \quad I \text{ degree } -1$$

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$$Q\psi^{(1)} = d_{U(1)}\psi^{(0)}$$

— $\gamma \in C_1(G)$

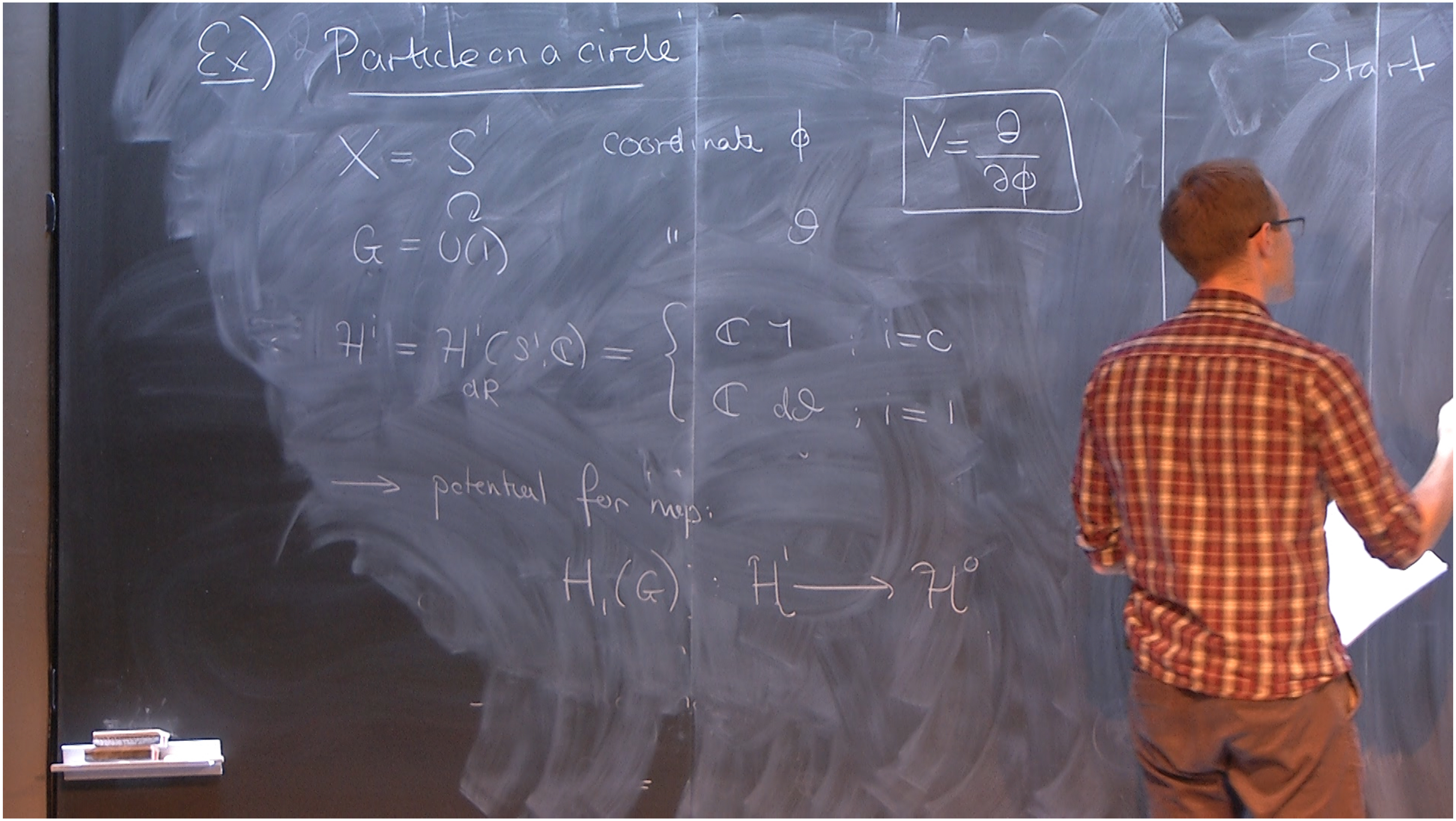
$$\gamma \cdot \psi = \int_\gamma \psi^{(1)}$$

$$Q \int_\gamma \psi^{(1)} = \int_\gamma d_{U(1)}\psi^{(0)} = \int_{\partial\gamma} \psi^{(0)}$$

$$= 0 \text{ if } \partial\gamma = 0$$

— descends to map

$$H_1(G) : \mathcal{H}^j \longrightarrow \mathcal{H}^{j-1}$$



Ex) Particle on a circle
$X = S^1$ coordinate ϕ
$G = U(1)$ " ϑ
$V = \frac{\partial}{\partial \phi}$
$\mathcal{H}^i = H^i_{dR}(S^1, \mathbb{C}) = \begin{cases} \mathbb{C} & ; i=0 \\ \mathbb{C}\, d\vartheta & ; i=1 \end{cases}$
→ potential for map:
$H_1(G) : \mathcal{H}^1 \longrightarrow \mathcal{H}^0$
Start

$$\boxed{V = \frac{\partial}{\partial \phi}}$$

Start $\psi = d\phi$ — Construct a Ω

$$\psi^{(0)} = e^{i\theta T} d\phi$$
$$= e^{\theta \cdot \mathcal{L}_V} d\phi$$
$$= d\phi$$

$$\psi^{(1)} = (I \cdot \psi^{(0)})\, d\theta$$
$$= (\iota_V \psi^{(0)})\, d\theta$$
$$= d\theta$$

$\boxed{J = \{Q}$

$\longrightarrow \psi^{(1)}$

$Q\psi$

$\gamma \in C_1(G)$

$\gamma \cdot \psi = \int_\gamma$

$Q \int_\gamma \psi^{(1)} =$

$$\boxed{V = \frac{\partial}{\partial \phi}}$$

$= c$

$= 1$

$\mathcal{H}^0$

Start $\psi = d\phi$ — Construct a Ω

$$\psi^{(0)} = e^{i\vartheta T} d\phi$$

$$= e^{\vartheta \cdot \mathcal{L}_V} d\phi$$

$$= d\phi$$

$$\psi^{(1)} = (I \cdot \psi^{(0)})\, d\vartheta$$

$$= (\iota_V \psi^{(0)})\, d\vartheta$$

$$= d\vartheta$$

$$\gamma \cdot \psi = \frac{1}{2\pi}\int_0^{2\pi} d\vartheta = 1$$

non-vanishing!

$$\boxed{J = \{Q}$$

$$\longrightarrow \psi^{(1)}$$

$Q\psi$

$\gamma \in C_1(G)$

$$\gamma \cdot \psi = \int_\gamma$$

$$Q \int_\gamma \psi^{(1)} =$$

Problem: $X = S^3$

$G = U(\ )$

Show that $\exists$ non-trivial

A_∞ operation

$$\nu_2(\gamma, \gamma, \omega_3) = 1$$

nishing'.