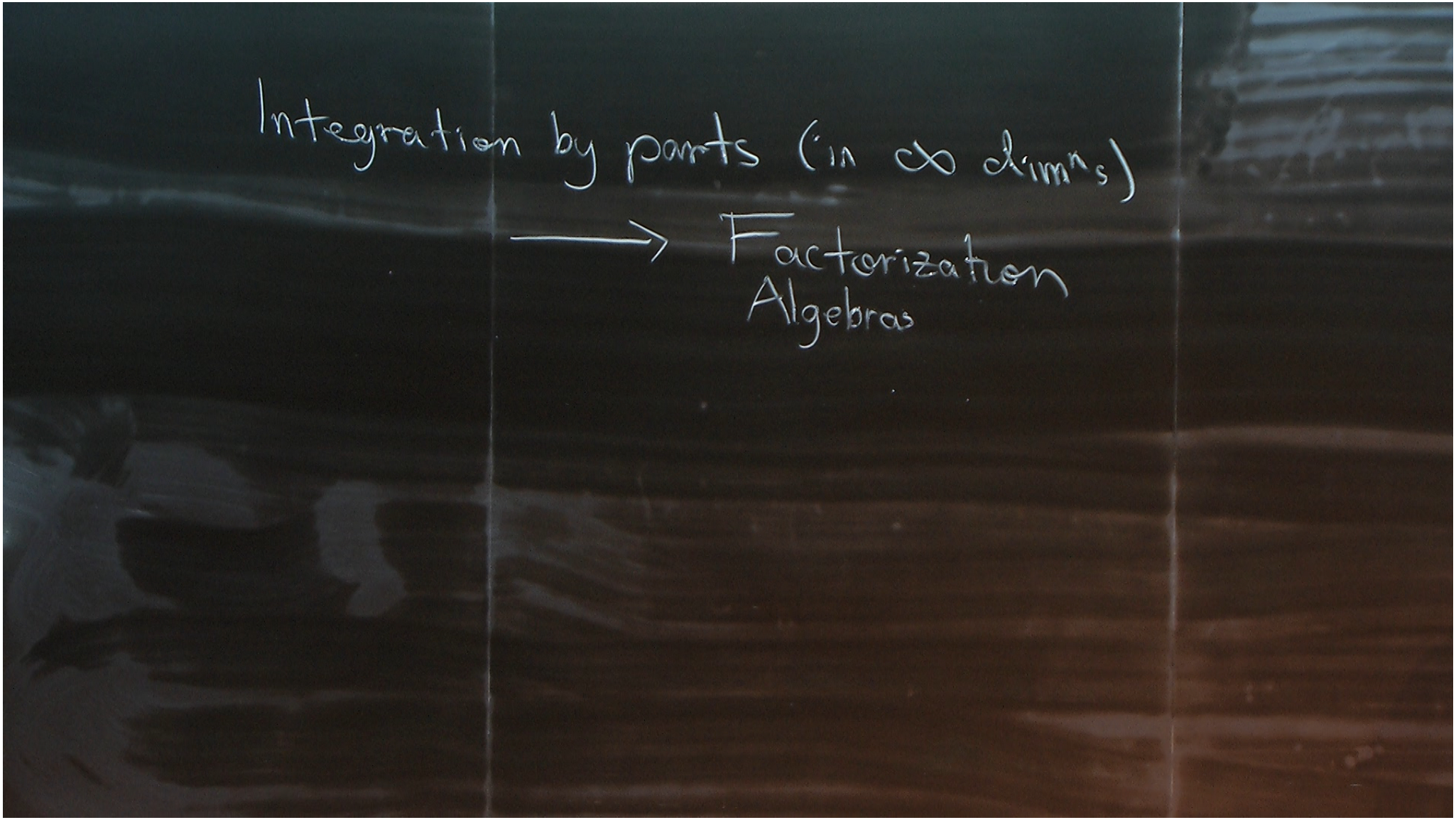
Title: Lecture 2: Factorization Algebras and the General Structure of QFT

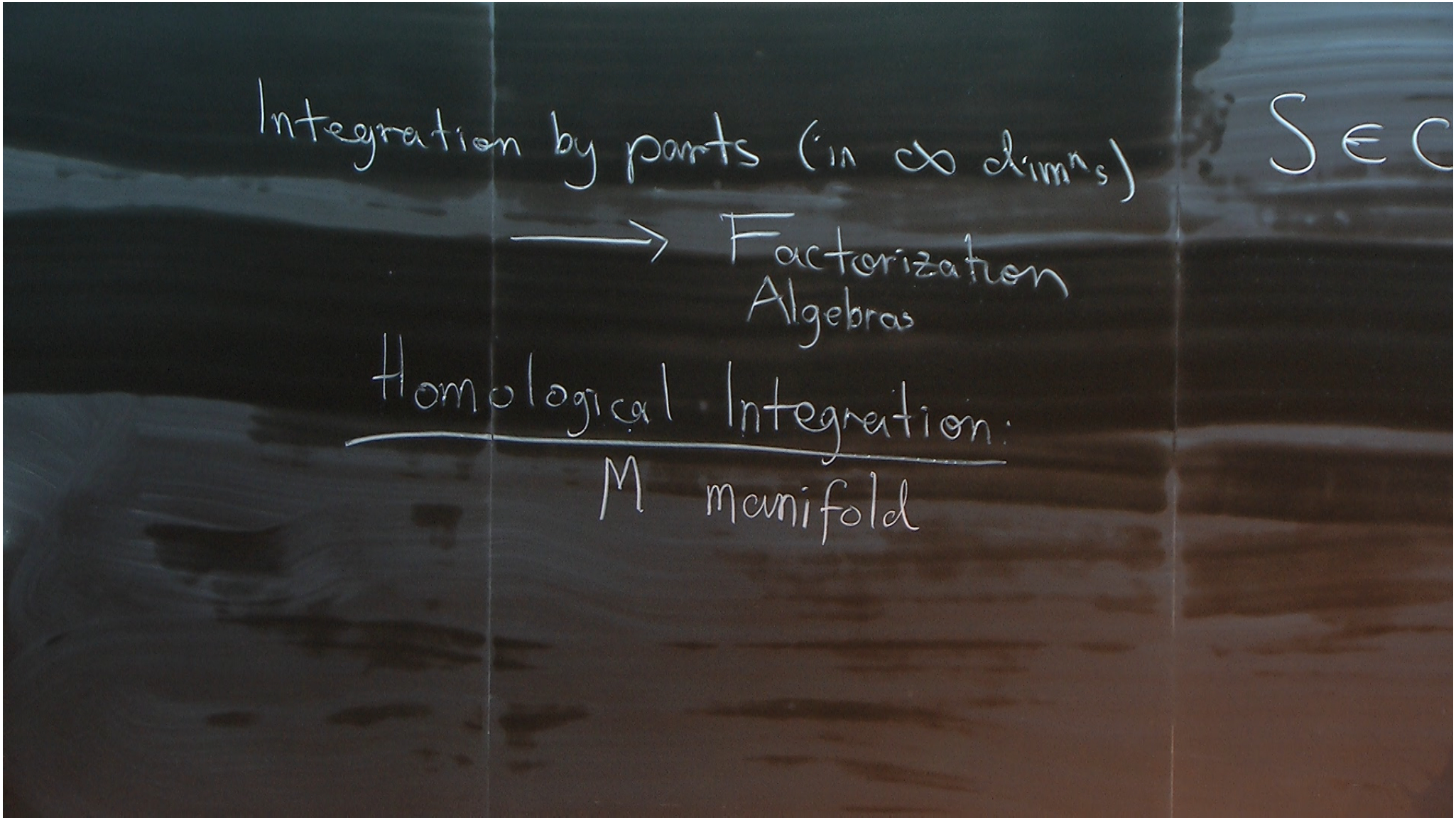
Speakers: Kevin Costello

Collection: QFT for Mathematicians

Date: June 18, 2019 - 11:00 AM

URL: http://pirsa.org/19060007


Integration by parts (in ∞ dim^ns)
⟶ Factorization
Algebras


Integration by parts (in ∞ dim^ns)
Sec
⟶ Factorization Algebras
Homological Integration:
M manifold

∞ dim^n s)

orization

ras

ation:

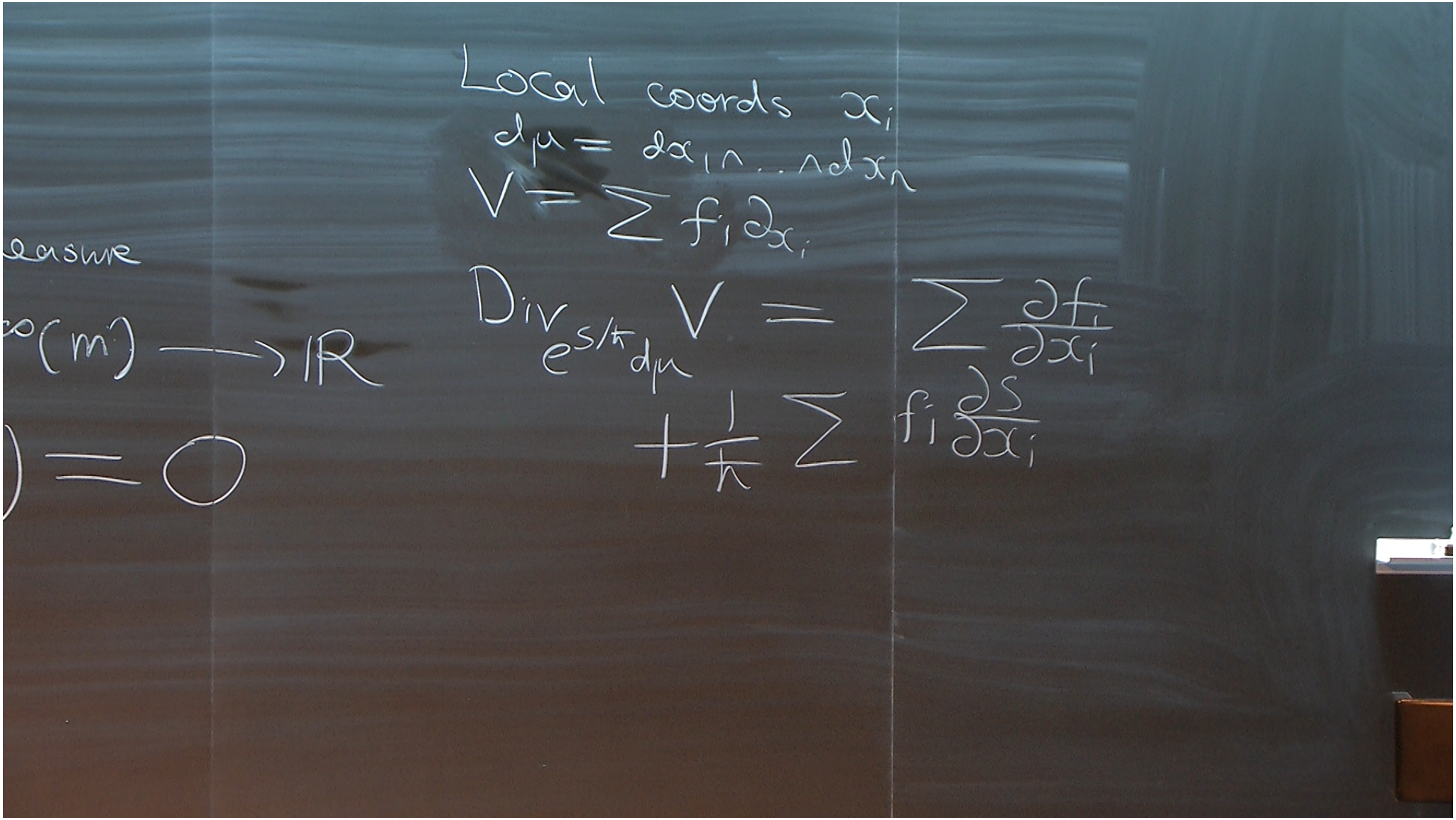
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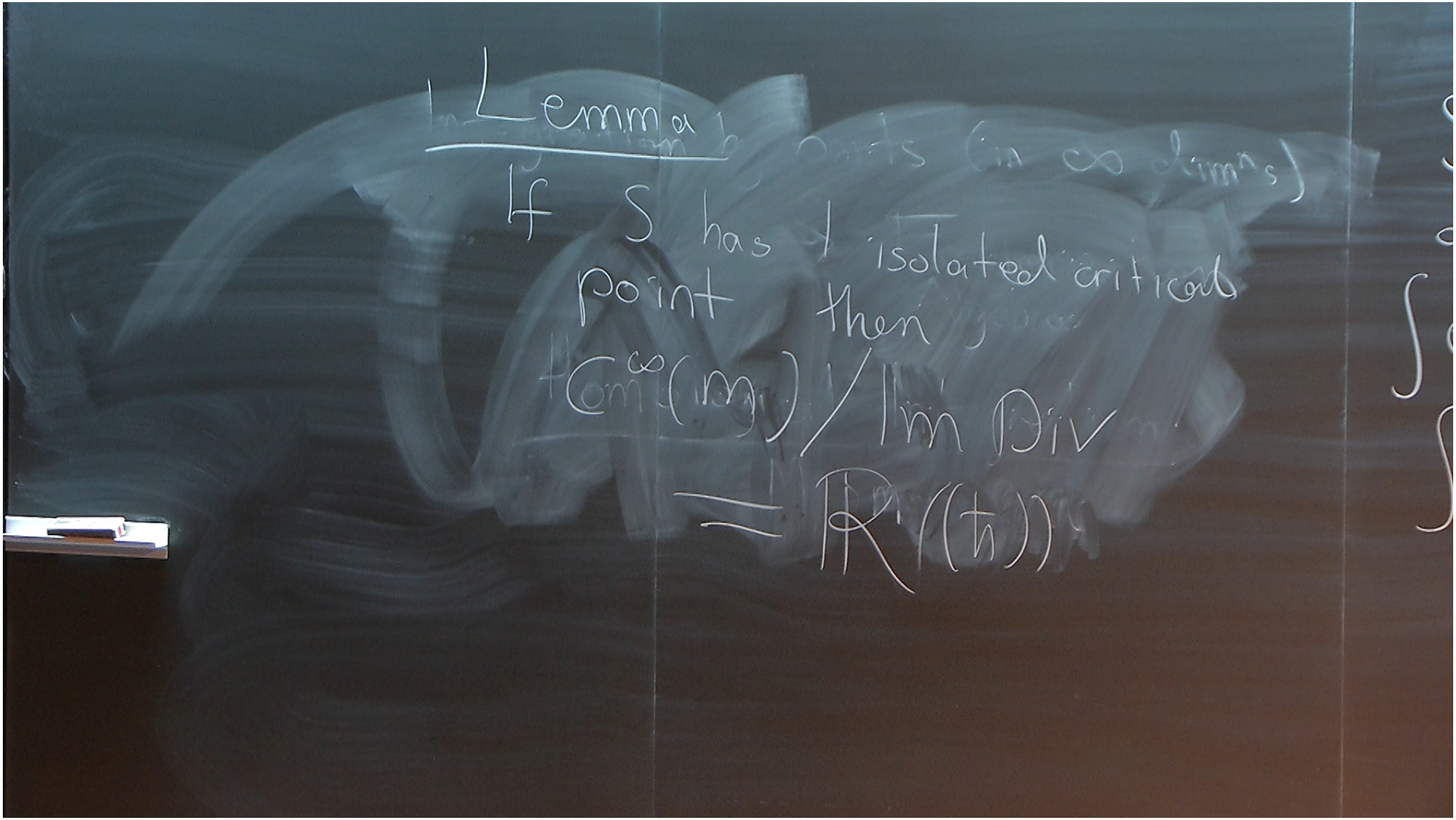
$S \in C^\infty(m)$

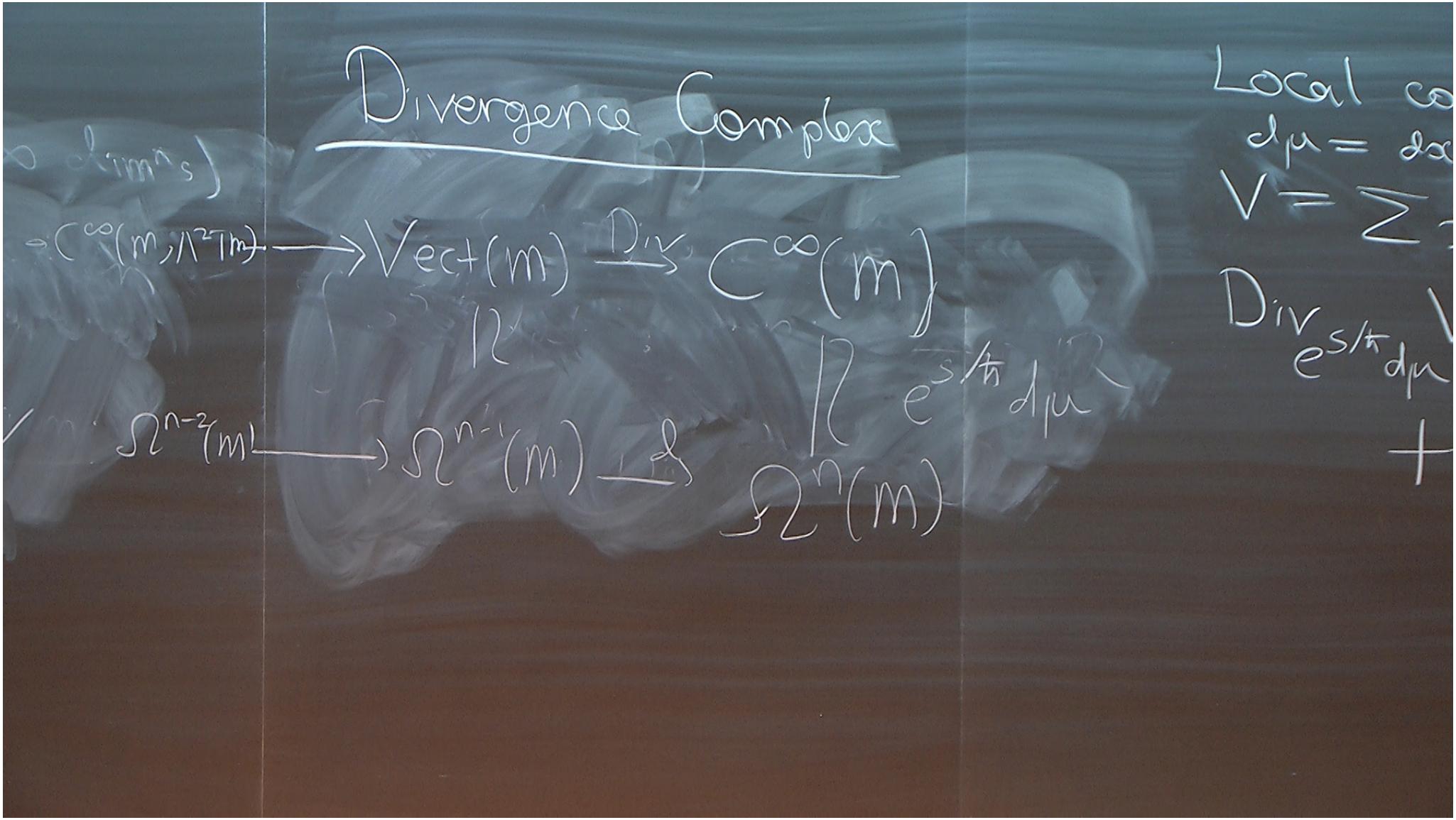
$d\mu \in \Omega^n(m)$ a measure

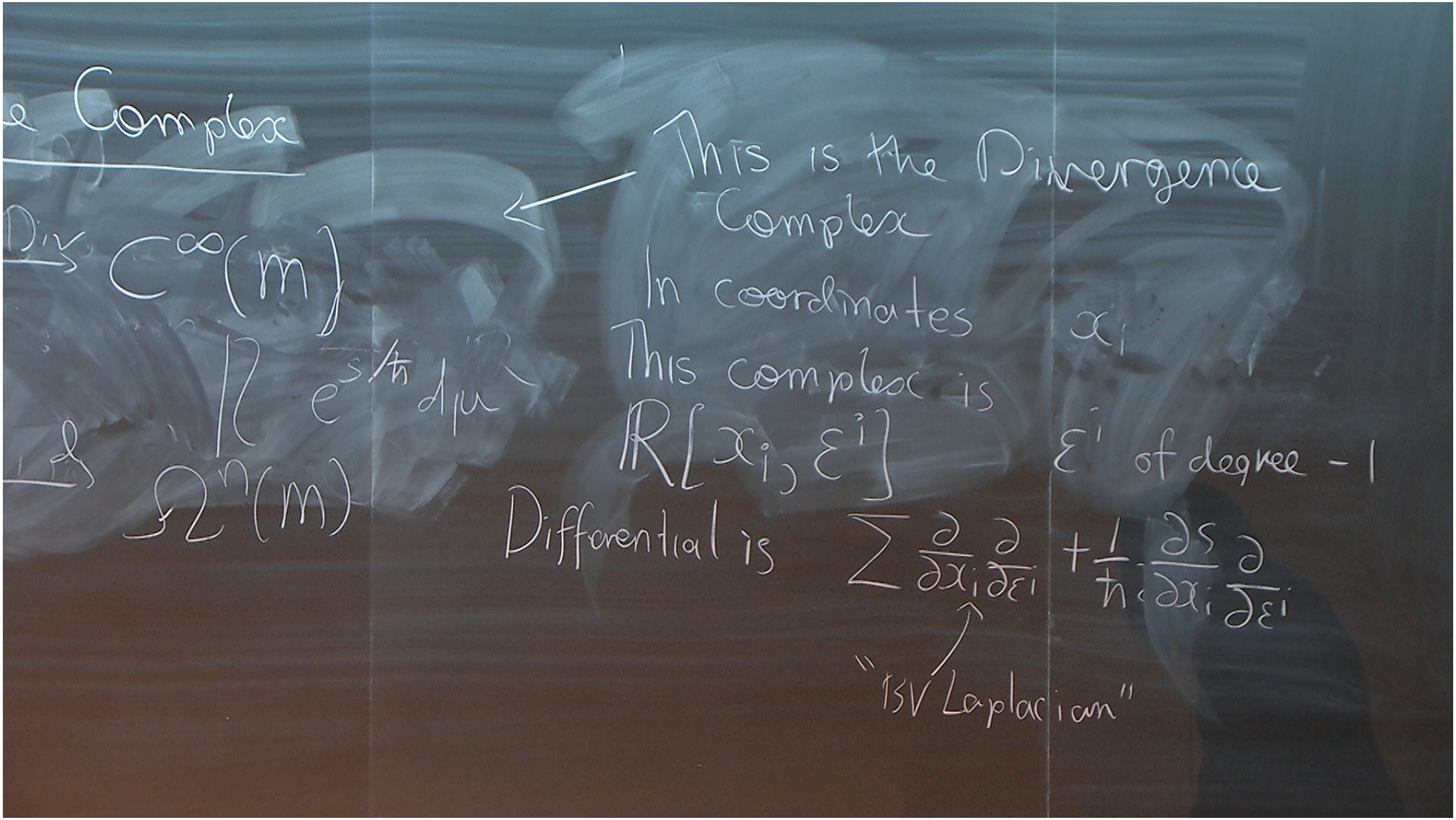
$\int e^{S/\hbar} d\mu(-) : C^\infty(m) \longrightarrow \mathbb{R}$

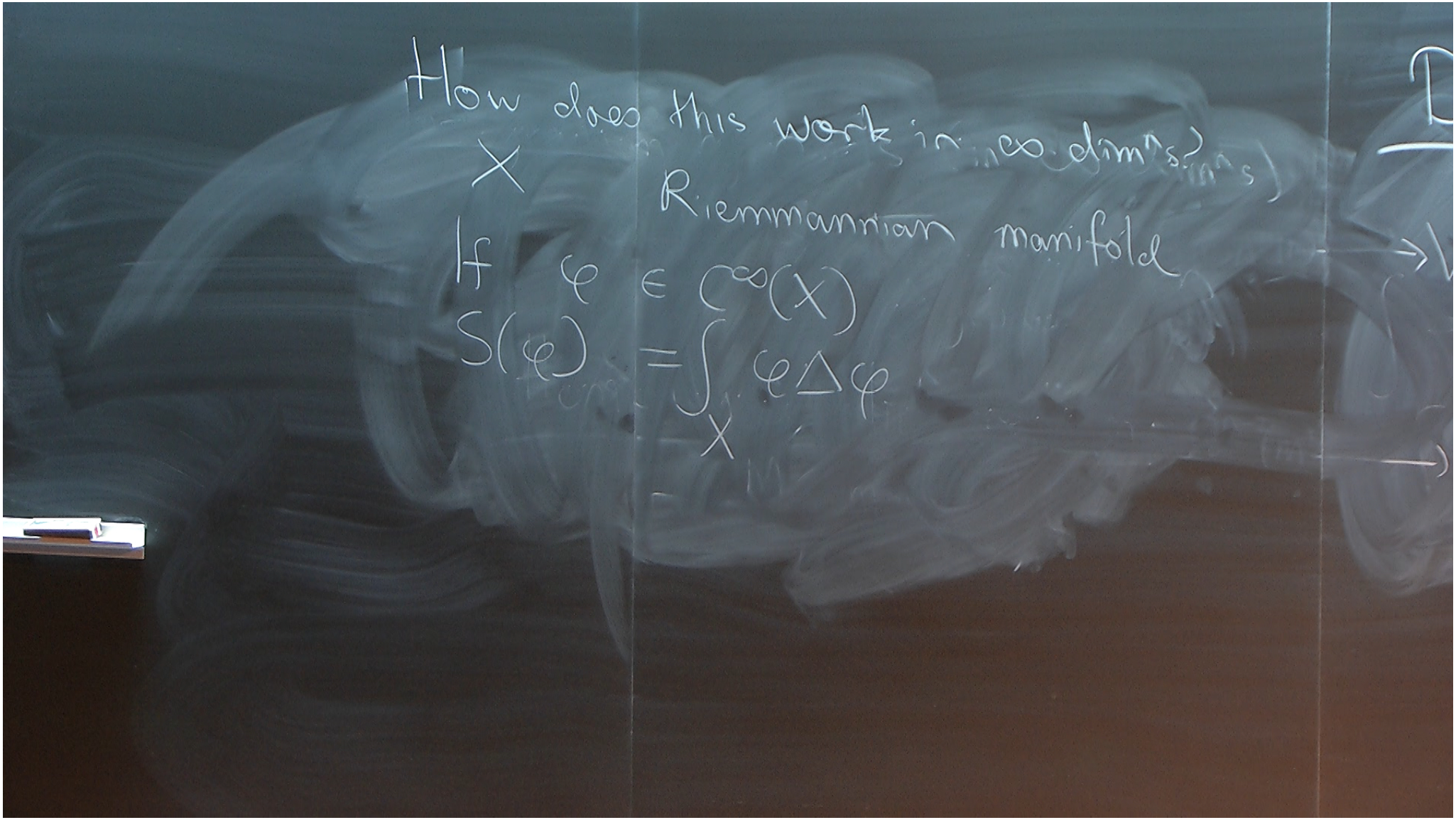
$\int e^{S/\hbar} d\mu(\mathrm{Div}\, V) = 0$


Local coords x_i
$d\mu = dx_1 \wedge \dots \wedge dx_n$
$V = \sum f_i \partial_{x_i}$
$\mathrm{Div}_{e^{S/\hbar} d\mu} V = \sum \frac{\partial f_i}{\partial x_i} + \frac{1}{\hbar} \sum f_i \frac{\partial S}{\partial x_i}$
easure
$\rho(m) \longrightarrow \mathbb{R}$
$) = 0$


Lemma
If S has 1 isolated critical
point then
$\mathcal{O}^\infty(m)$ / Im Div
$\cong \mathbb{R}((t))$


Divergence Complex
$C^\infty(m, \Lambda^2 Tm)$
$\Omega^{n-2}(m)$
$Vect(m)$
$\Omega^{n-1}(m)$
Div
$C^\infty(m)$
$\Omega^n(m)$
$\int e^{s/\hbar} d\mu$
Local
$d\mu =$
$V = \Sigma$
Div
$e^{s/\hbar} d\mu$
+


e Complex
Div
$C^\infty(m)$
$\int e^{S/\hbar} d\mu$
$\Omega^n(m)$
This is the Divergence
Complex
In coordinates x_i
This complex is
$R[x_i, \varepsilon^i]$
ε^i of degree -1
Differential is $\sum \frac{\partial}{\partial x_i}\frac{\partial}{\partial \varepsilon^i} + \frac{1}{\hbar}\frac{\partial S}{\partial x_i}\frac{\partial}{\partial \varepsilon^i}$
"BV Laplacian"


How does this work in ∞ dim's?
X Riemannian manifold
If φ ∈ C∞(X)
S(φ) = ∫X φΔφ

How does this work in ∞ dim^{n}s?

X Riemmannian manifold

If $\varphi \in C^\infty(X)$ (= replacement for m)

$S(\varphi) = \int_X \varphi \Delta \varphi$

Want to consider $\int_{\varphi \in C^\infty(X)} e^{S(\varphi)/\hbar} \, d\mu$

Divergence

$\to \mathrm{Vect}(m) \xrightarrow{\mathrm{Div}}$

$\mathbb{R}$

$\to \Omega^{n-1}(m) \xrightarrow{d}$

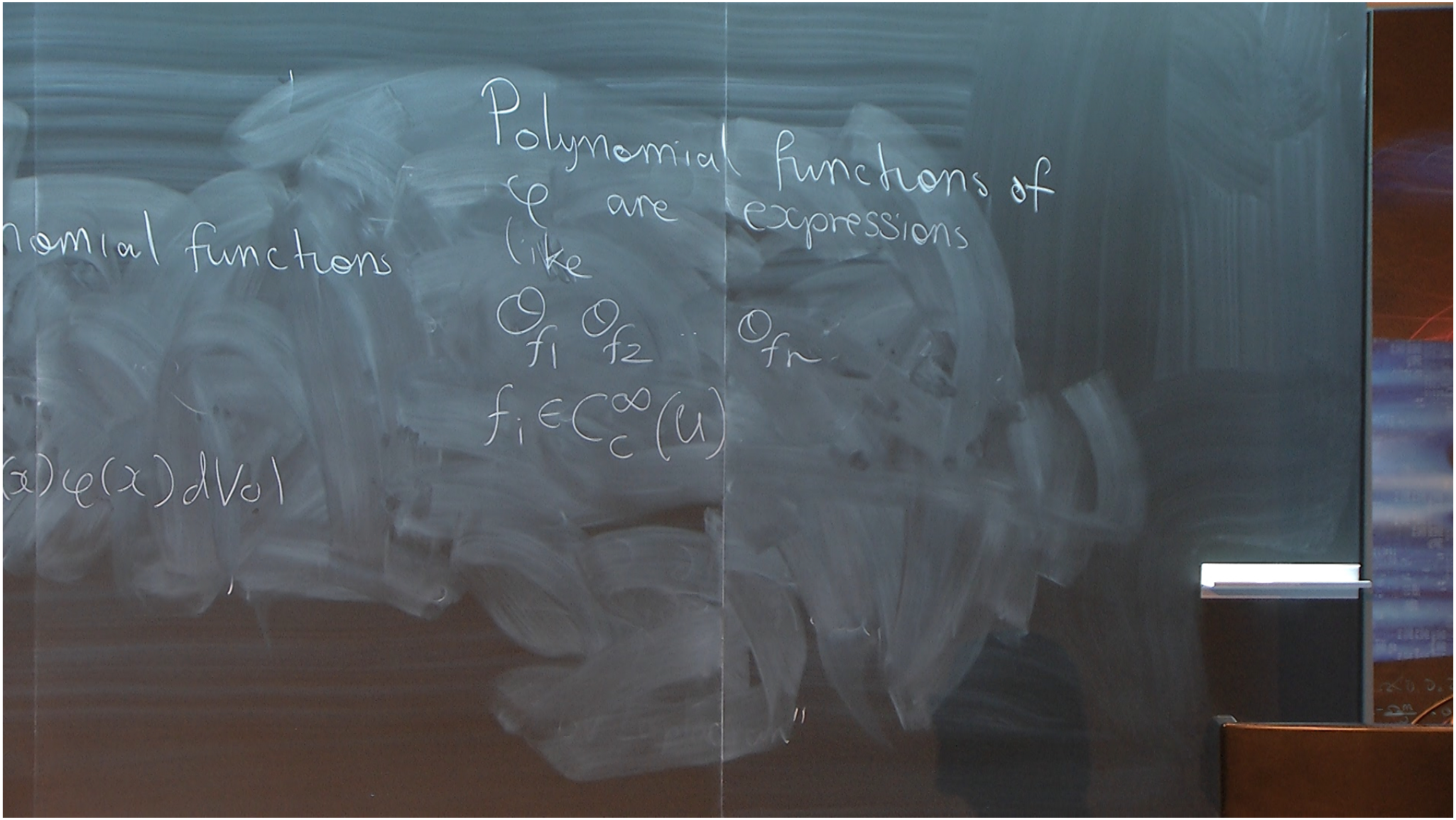
Replace $\mathbb{R}[x_i]$

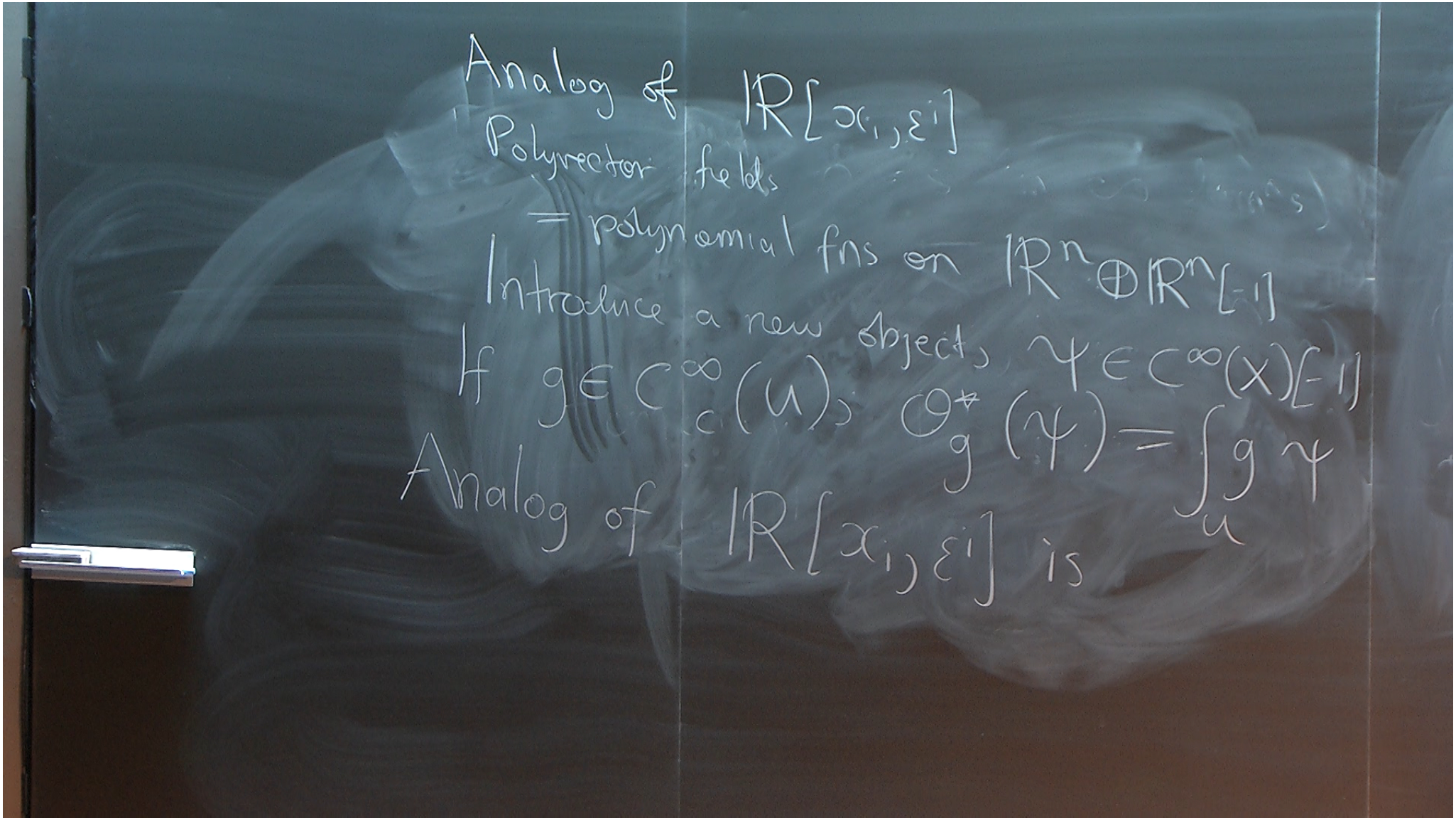
Need to consider polynomial functions of φ

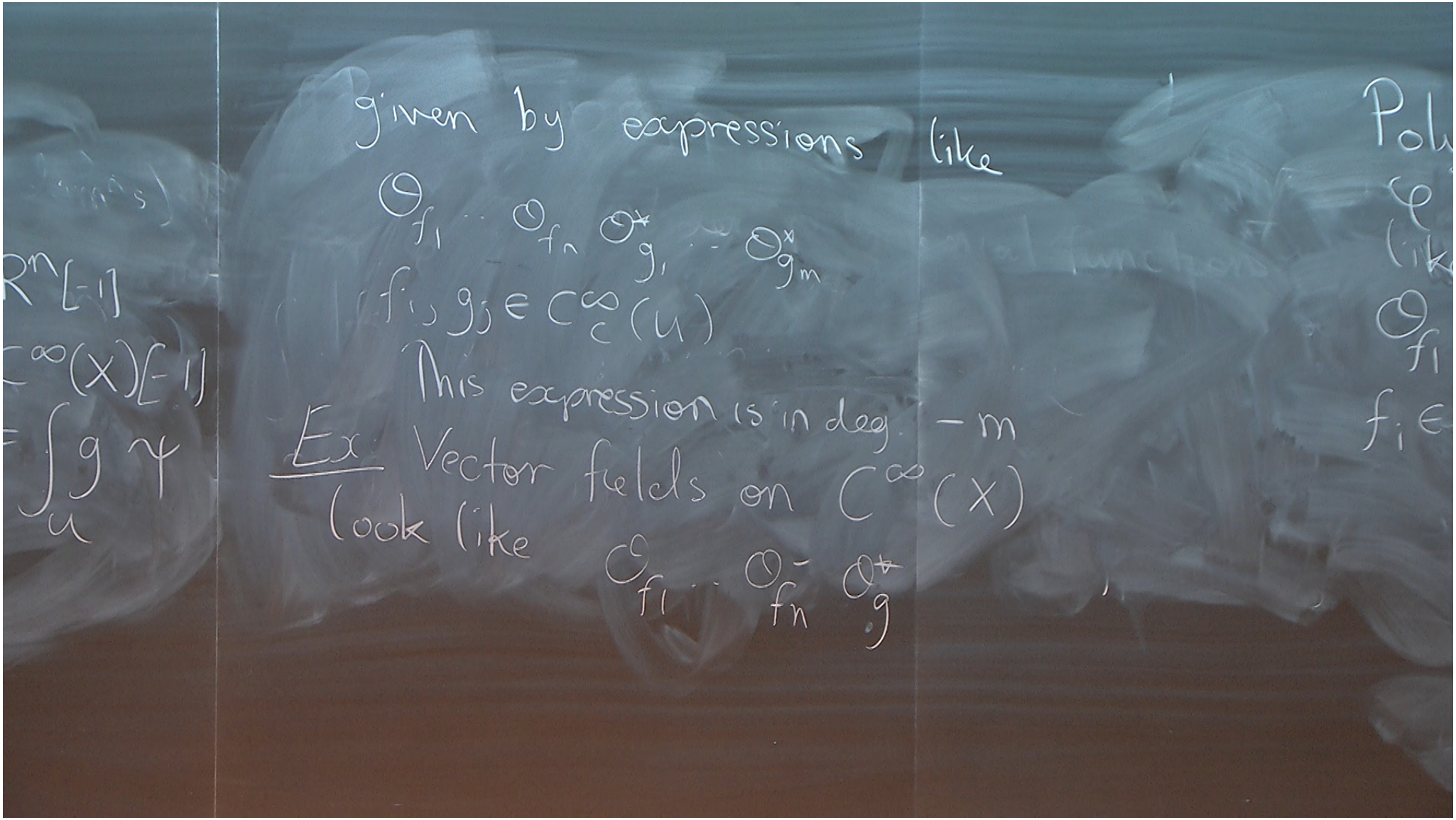
If $U \subseteq X$

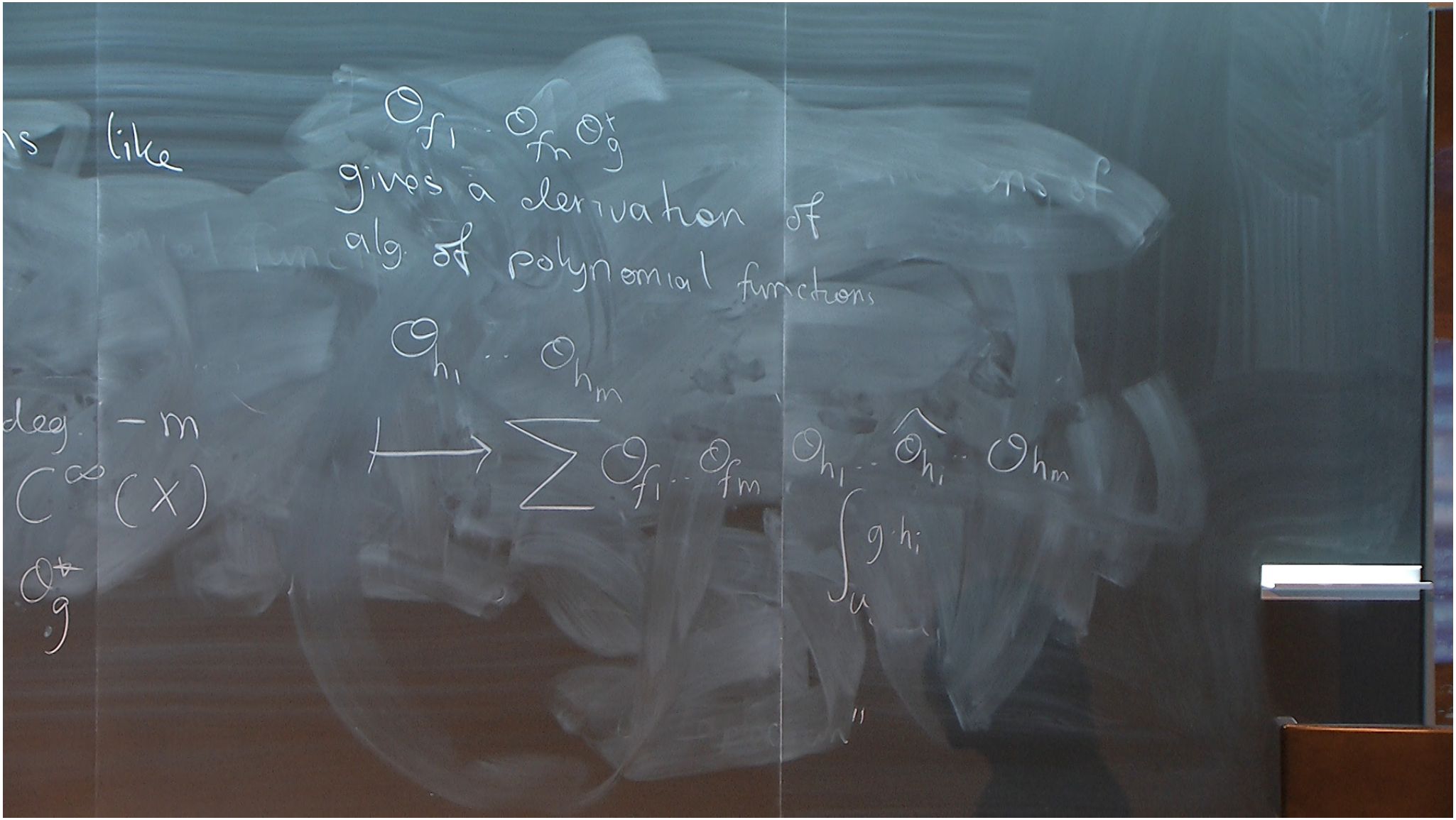
$f \in C_c^\infty(U)$

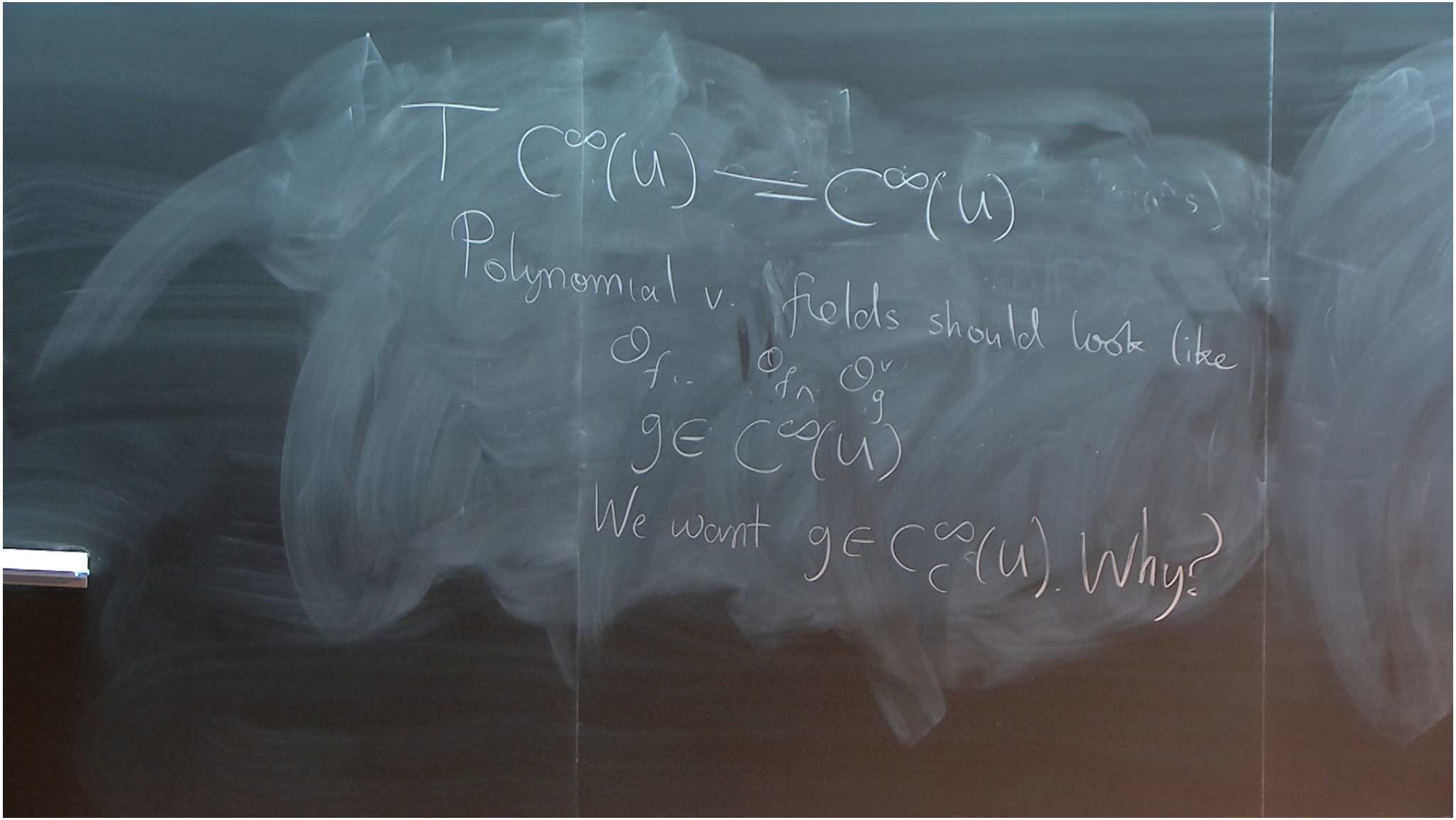
Define $\Theta_f(\varphi) = \int_U f(x)\varphi(x)\,dVol$


nomial functions
(x)φ(x)dVol
Polynomial functions of φ are expressions (like
$\mathcal{O}_{f_1}\ \mathcal{O}_{f_2}\ \cdots\ \mathcal{O}_{f_n}$
$f_i \in C_c^\infty(U)$


Analog of $\mathbb{R}[x_i, \varepsilon^i]$
Polyvector fields
= polynomial fns on $\mathbb{R}^n \oplus \mathbb{R}^n[-1]$
Introduce a new object, $\psi \in C^\infty(X)[-1]$
If $g \in C^\infty_c(U)$, $\Theta^*_g(\psi) = \int_U g\,\psi$
Analog of $\mathbb{R}[x_i, \varepsilon^i]$ is


given by expressions like
$\mathcal{O}_{f_1} \cdots \mathcal{O}_{f_n} \mathcal{O}^*_{g_1} \cdots \mathcal{O}^*_{g_m}$
$f_i, g_j \in C^\infty_c(U)$
This expression is in deg $-m$
Ex Vector fields on $C^\infty(X)$
look like $\mathcal{O}_{f_1} \cdots \mathcal{O}_{f_n} \mathcal{O}^*_g$


like
$\mathcal{O}_{f_1} \cdots \mathcal{O}_{f_n} \mathcal{O}_g^*$
gives a derivation of
alg. of polynomial functions
$\mathcal{O}_{h_1} \cdots \mathcal{O}_{h_m}$
$\longmapsto \sum \mathcal{O}_{f_1} \cdots \mathcal{O}_{f_m} \mathcal{O}_{h_1} \cdots \widehat{\mathcal{O}_{h_i}} \cdots \mathcal{O}_{h_m} \int_U g \cdot h_i$
deg. $-m$
$C^\infty(X)$
$\mathcal{O}_g^*$


$T C^\infty(U) = C^\infty(U)$
Polynomial v. fields should look like
$\partial_{f_1} \dots \partial_{f_n} \partial_g^\nu$
$g \in C^\infty(U)$
We want $g \in C_c^\infty(U)$. Why?

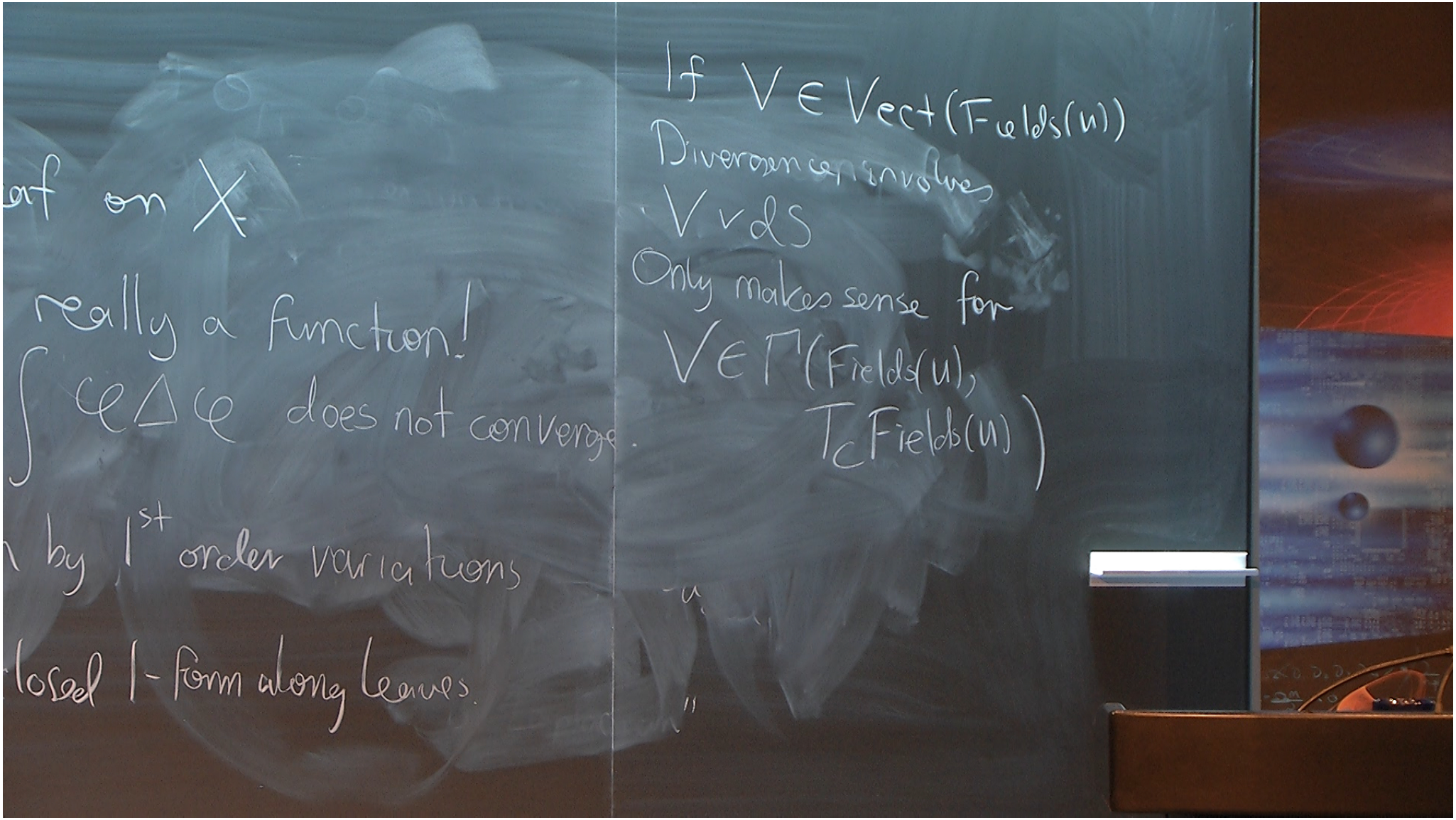
Fields(U) is a sheaf on X.

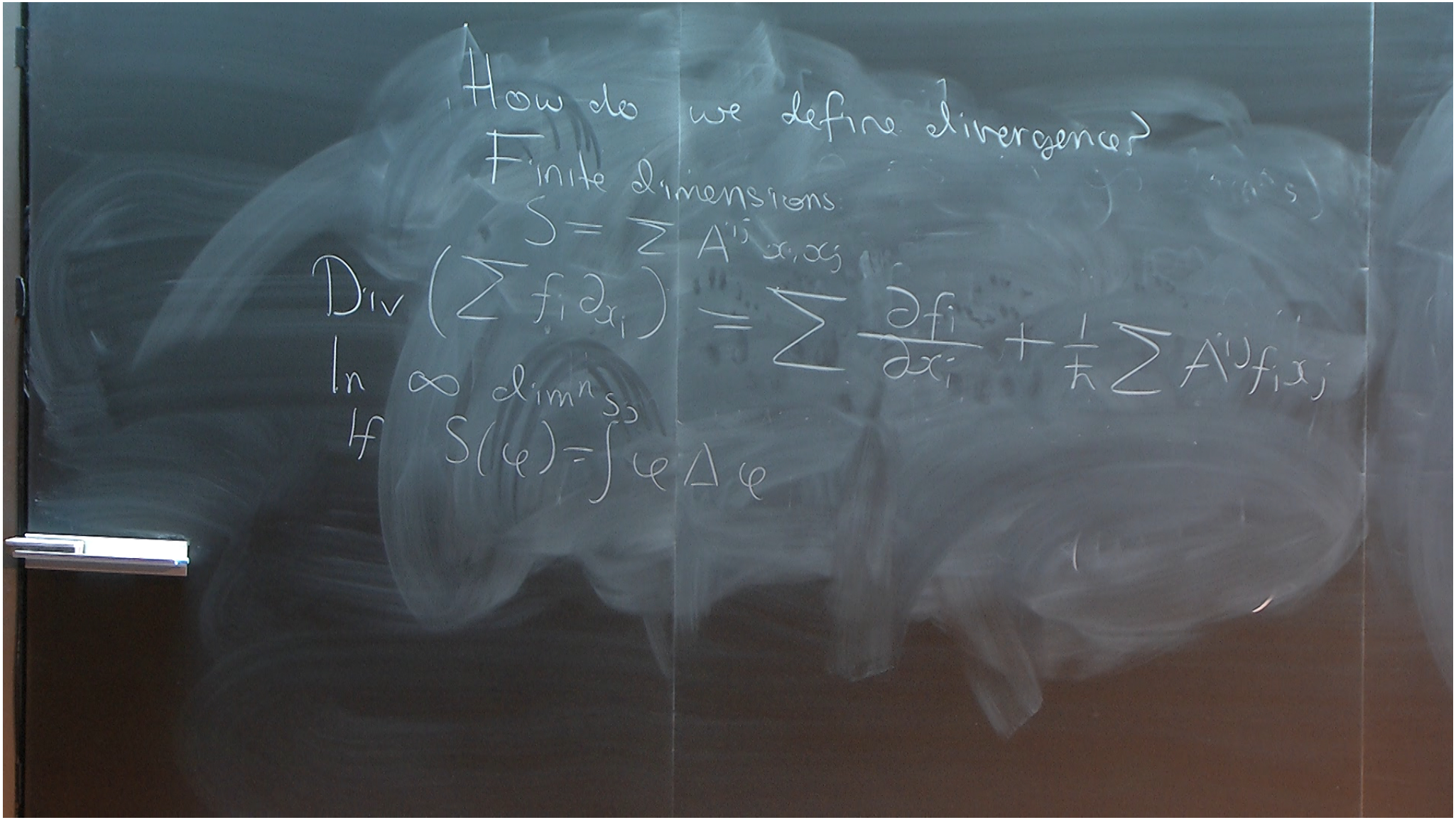
But S is not really a function!

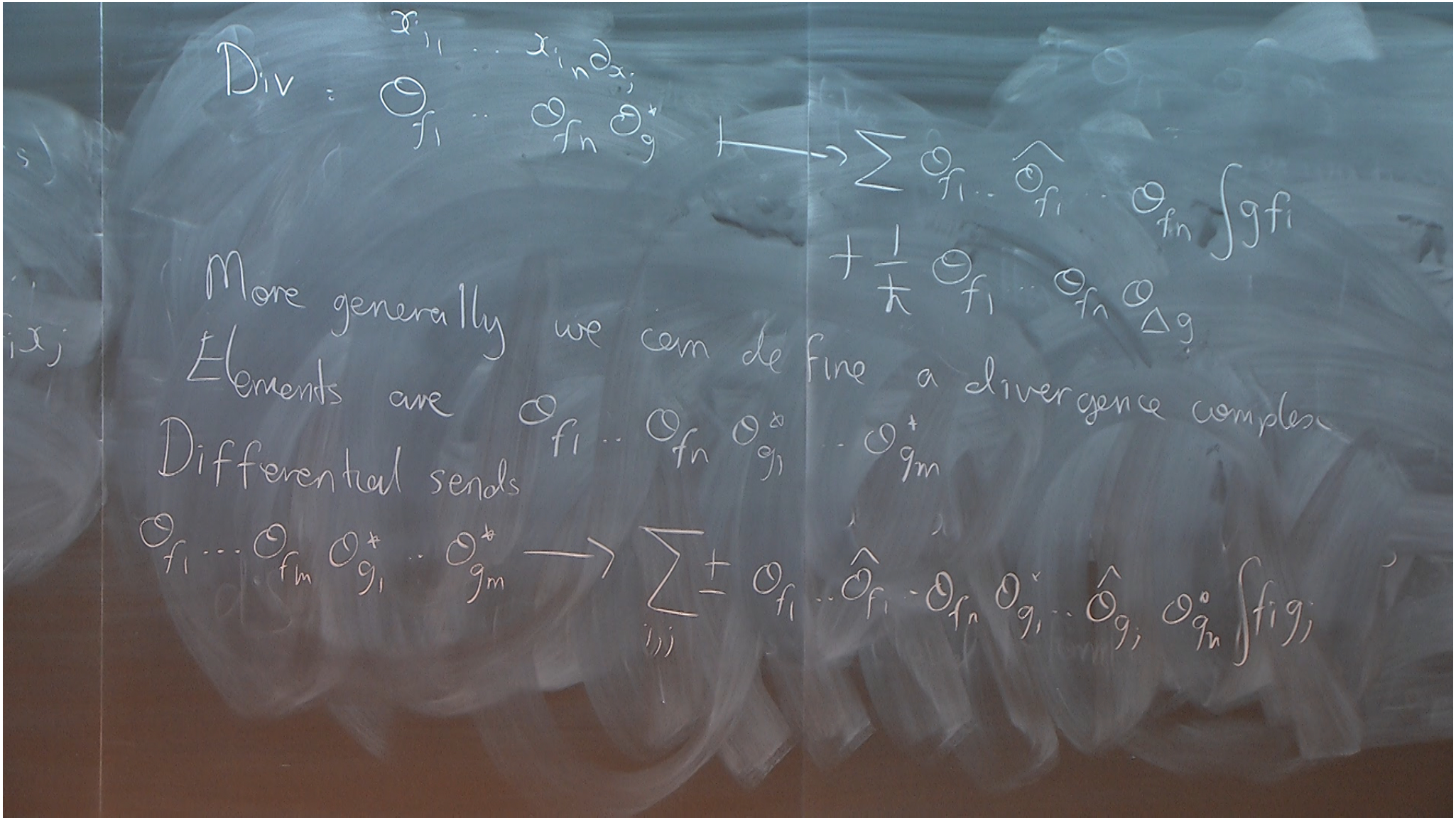
An expression like $\int \varphi \Delta \varphi$ does not converge.

Fields(U) has a foliation by 1st order variations with compact support.

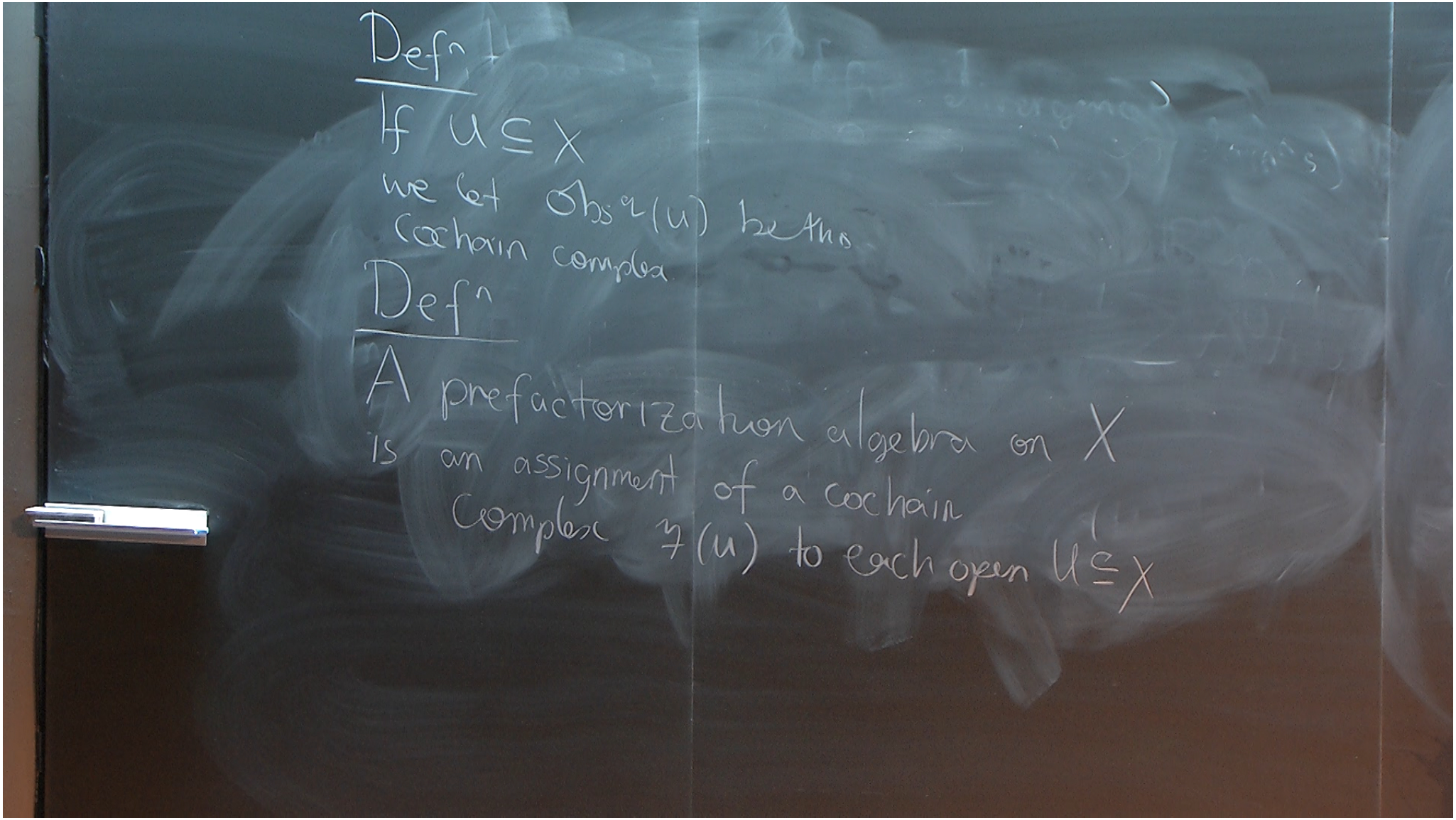
$dS \in T^*_C \text{Fields}(U)$ is a closed 1-form along leaves.


If $V \in \mathrm{Vect}(\mathrm{Fields}(U))$
Divergence involves
$V \lrcorner dS$
Only makes sense for
$V \in \Gamma(\mathrm{Fields}(U), T_C \mathrm{Fields}(U))$
really a function!
does not converge.
by 1st order variations
closed 1-form along leaves.


How do we define divergence?
Finite dimensions
$S = \sum A^{ij} x_i x_j$
$\mathrm{Div}\left(\sum f_i \partial_{x_i}\right) = \sum \frac{\partial f_i}{\partial x_i} + \frac{1}{\hbar} \sum A^{ij} f_i x_j$
In ∞ dimensions
If $S(\varphi) = \int \varphi \Delta \varphi$


Div : $\mathcal{O}_{f_1} \cdots \mathcal{O}_{f_n} \mathcal{O}_g^* \longmapsto \sum \mathcal{O}_{f_1} \cdots \widehat{\mathcal{O}_{f_i}} \cdots \mathcal{O}_{f_n} \int g f_i + \frac{1}{\hbar} \mathcal{O}_{f_1} \cdots \mathcal{O}_{f_n} \mathcal{O}_{\Delta g}$
More generally we can define a divergence complex
Elements are $\mathcal{O}_{f_1} \cdots \mathcal{O}_{f_n} \mathcal{O}_{g_1}^* \cdots \mathcal{O}_{g_m}^*$
Differential sends
$\mathcal{O}_{f_1} \cdots \mathcal{O}_{f_m} \mathcal{O}_{g_1}^* \cdots \mathcal{O}_{g_m}^* \longrightarrow \sum_{i,j} \pm \mathcal{O}_{f_1} \cdots \widehat{\mathcal{O}_{f_i}} \cdots \mathcal{O}_{f_n} \mathcal{O}_{g_1}^* \cdots \widehat{\mathcal{O}_{g_j}} \cdots \mathcal{O}_{g_n}^* \int f_i g_j$


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Def^
If $U \subseteq X$
we let $Obs^q(U)$ be the
cochain complex
Def^
A prefactorization algebra on X
is an assignment of a cochain
complex $\mathcal{F}(U)$ to each open $U \subseteq X$

algebra on X

cochain

each open $U \subseteq X$

And, if

$U_1, \ldots, U_n \subseteq V$

U_i are disjoint, we have a cochain map

$$\bigotimes \mathcal{F}(U_i) \longrightarrow \mathcal{F}(V)$$

These satisfy an associativity constraint

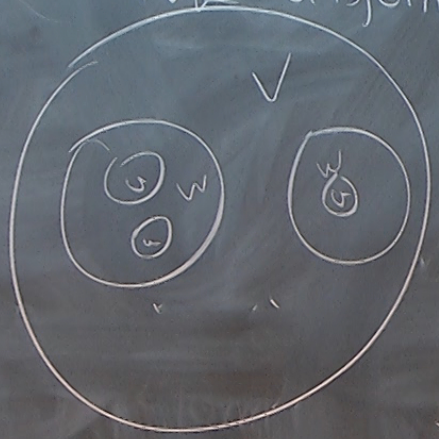
If $W_1 \dots W_n \subseteq V$

$U_{ij} \subseteq W_j$

$i = 1 \dots m_j$

U_{ij} disjoint

W_k disjoint



gativity

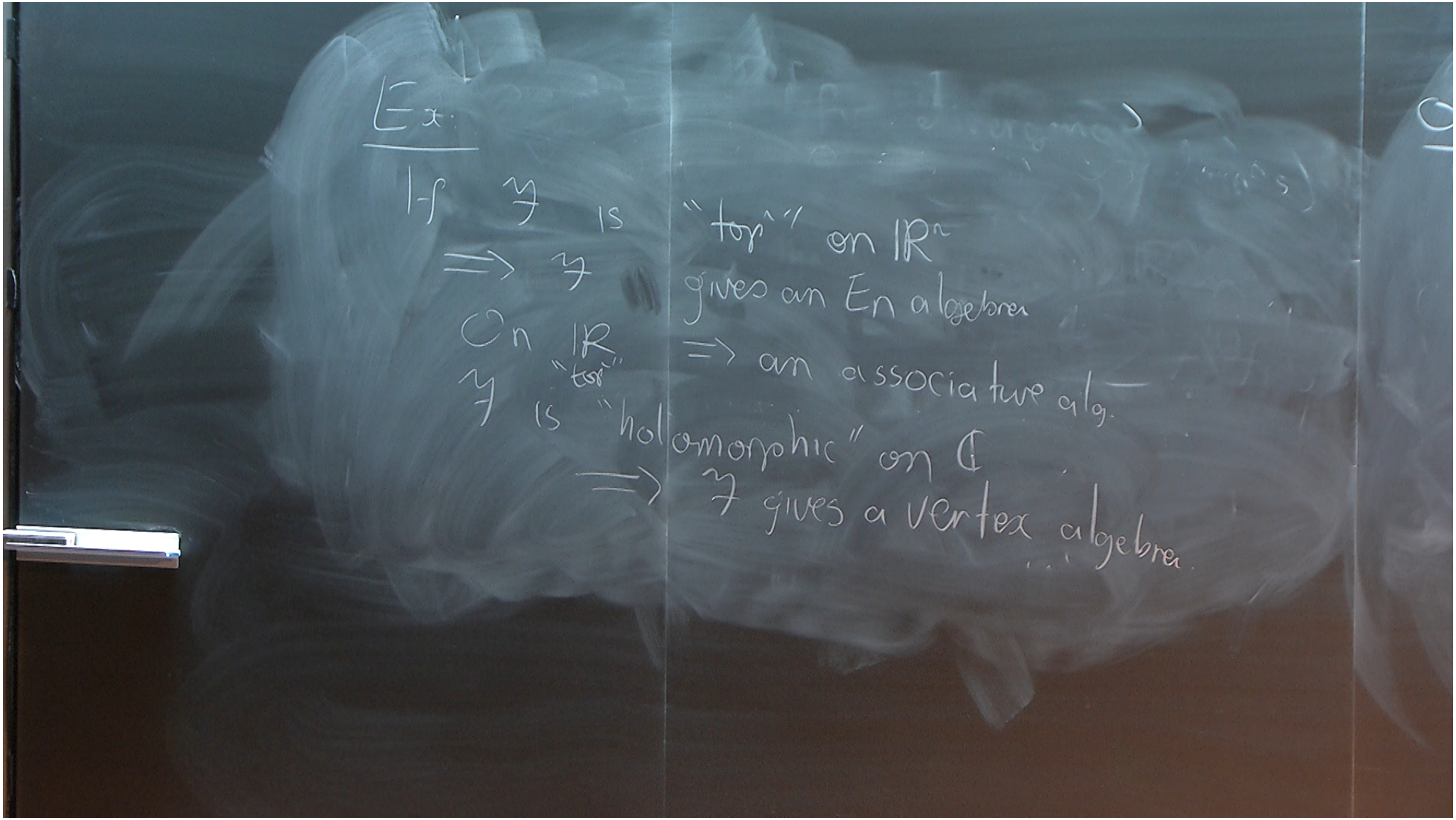
then

$\bigotimes Z(U_{ij}) \longrightarrow Z(V)$

$\downarrow$

$\bigotimes Z(W_k) \nearrow$

commutes

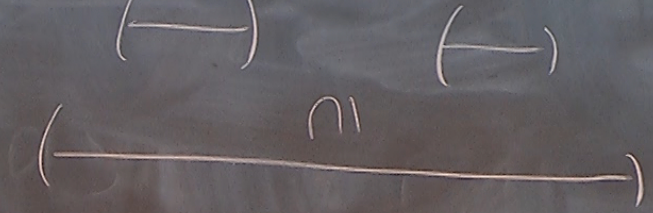

Ex:
If $\mathcal{Y}$ is "top" on $\mathbb{R}^n$
$\Rightarrow \mathcal{Y}$ gives an E_n algebra
On $\mathbb{R}$ "top"
$\Rightarrow$ an associative alg.
$\mathcal{Y}$ is "holomorphic" on $\mathbb{C}$
$\Rightarrow \mathcal{Y}$ gives a vertex algebra.

On $\mathbb{R}$:

If $I \subseteq J$ are intervals
Suppose $\mathcal{F}(I) \longrightarrow \mathcal{F}(J)$
is always a quasi-isomorphism.

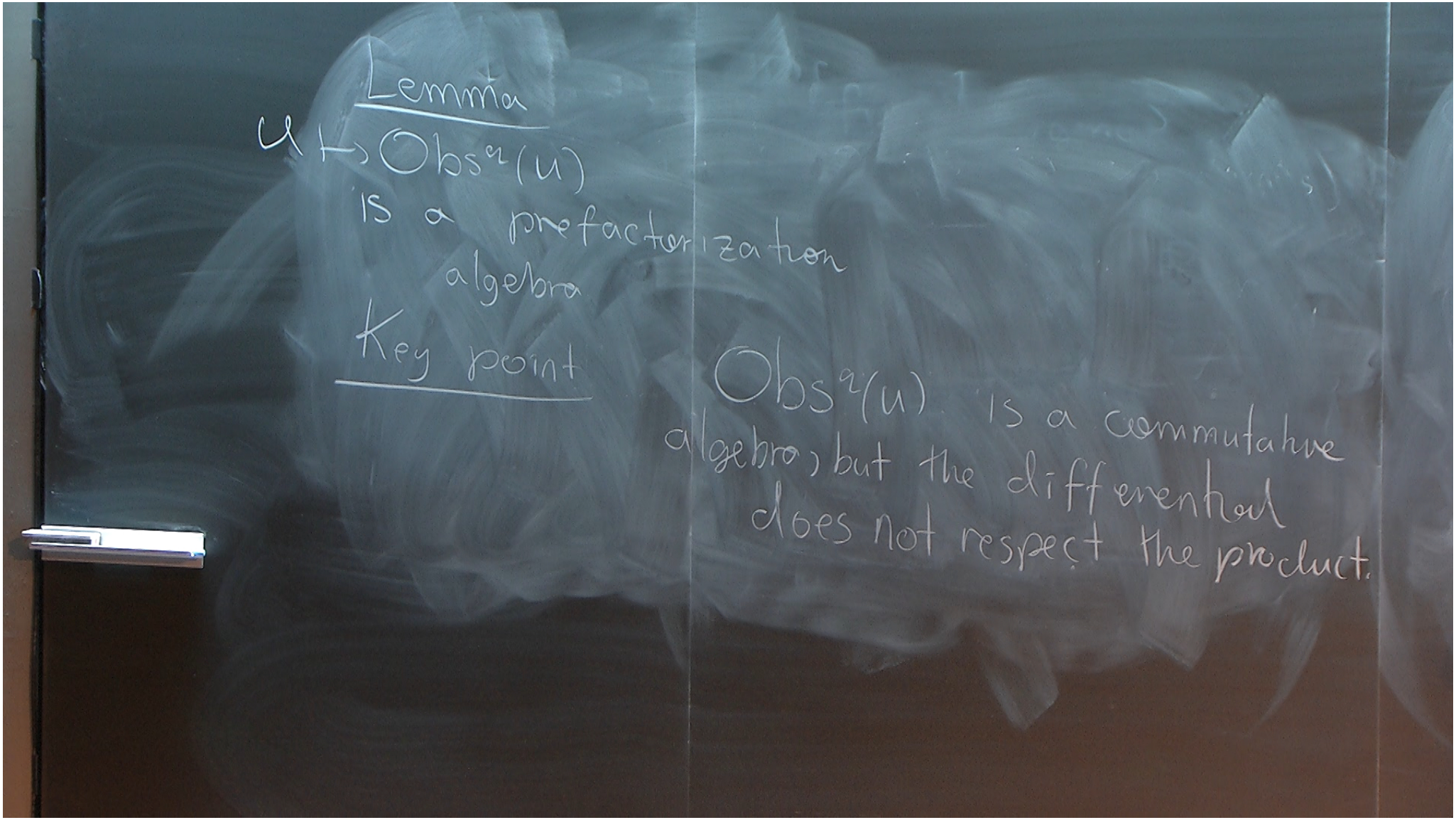
$$H^*(\mathcal{F}((a,b)) \simeq H^*(\mathcal{F}(-\infty,\infty)) = A$$

A is an associative algebra:



$$A \otimes A \longrightarrow A$$

this is the associative product.


Lemma
$U \mapsto Obs^q(U)$
is a prefactorization algebra
Key point
$Obs^q(U)$ is a commutative algebra, but the differential does not respect the product.

zation

$Obs^q(U)$ is a commutative algebra, but the differential does not respect the product.

E.g.

$$\mathcal{O}_f \mathcal{O}_g \longmapsto \frac{1}{i\hbar}\mathcal{O}_{fg} + \left(\int fg\right)\mathbb{1}$$

If $U_1, U_2 \subseteq V$ are disjoint

$\alpha \in Obs^q(U_1) \subseteq Obs^q(V)$

$\beta \in Obs^q(U_2)$

$$d(\alpha\beta) = (d\alpha)\beta \pm \alpha\, d\beta$$

E.g.

$\alpha = \mathcal{O}_f \quad f \in C_c^\infty(U_1)$

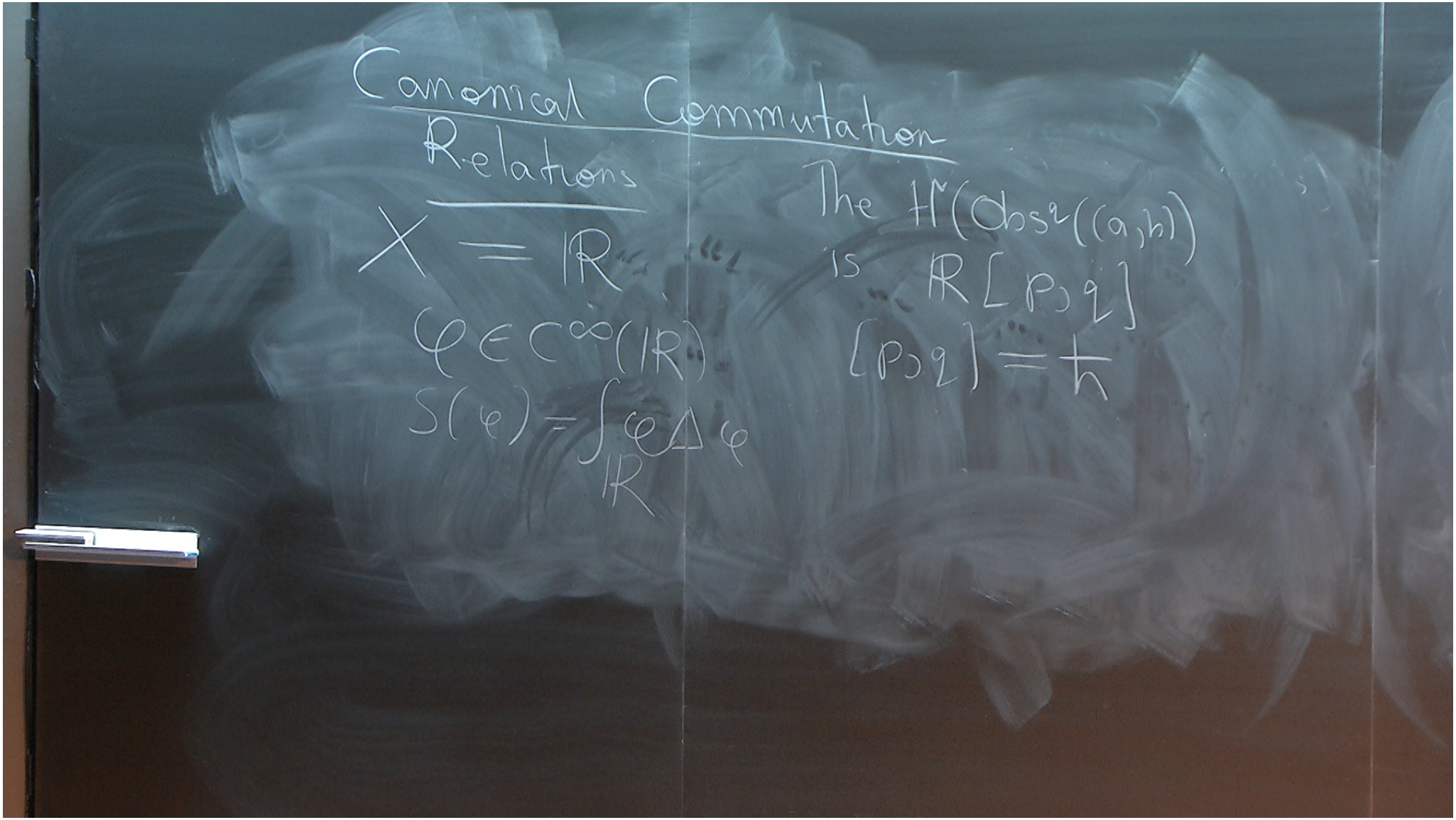
$\beta \in \mathcal{O}_g^* \quad g \in C_c^\infty(U_2)$

$$d(\mathcal{O}_f \mathcal{O}_g^*) = \frac{1}{\hbar}\mathcal{O}_f \mathcal{O}_{\Delta g} + \left(\int f_1 g\right) 1$$

$\frac{1}{\hbar}\mathcal{O}_f \mathcal{O}_{\Delta g}$

$\left(\int f g\right) 1$

$\subseteq Obs^n(V)$


Canonical Commutation Relations
$X = \mathbb{R}$
$\varphi \in C^\infty(\mathbb{R})$
$S(\varphi) = \int_{\mathbb{R}} \varphi \Delta \varphi$
The $\mathcal{H}(Obs^2((a,b))$ is $\mathbb{R}[p,q]$
$[p,q] = \hbar$

nmutation

The $\mathcal{H}(\mathrm{Obs}^2((a,b))$ is $\mathbb{R}[p,q]$

$[p,q] = \hbar$

What are linear observables?

$f \in C_c^\infty((a,b))$

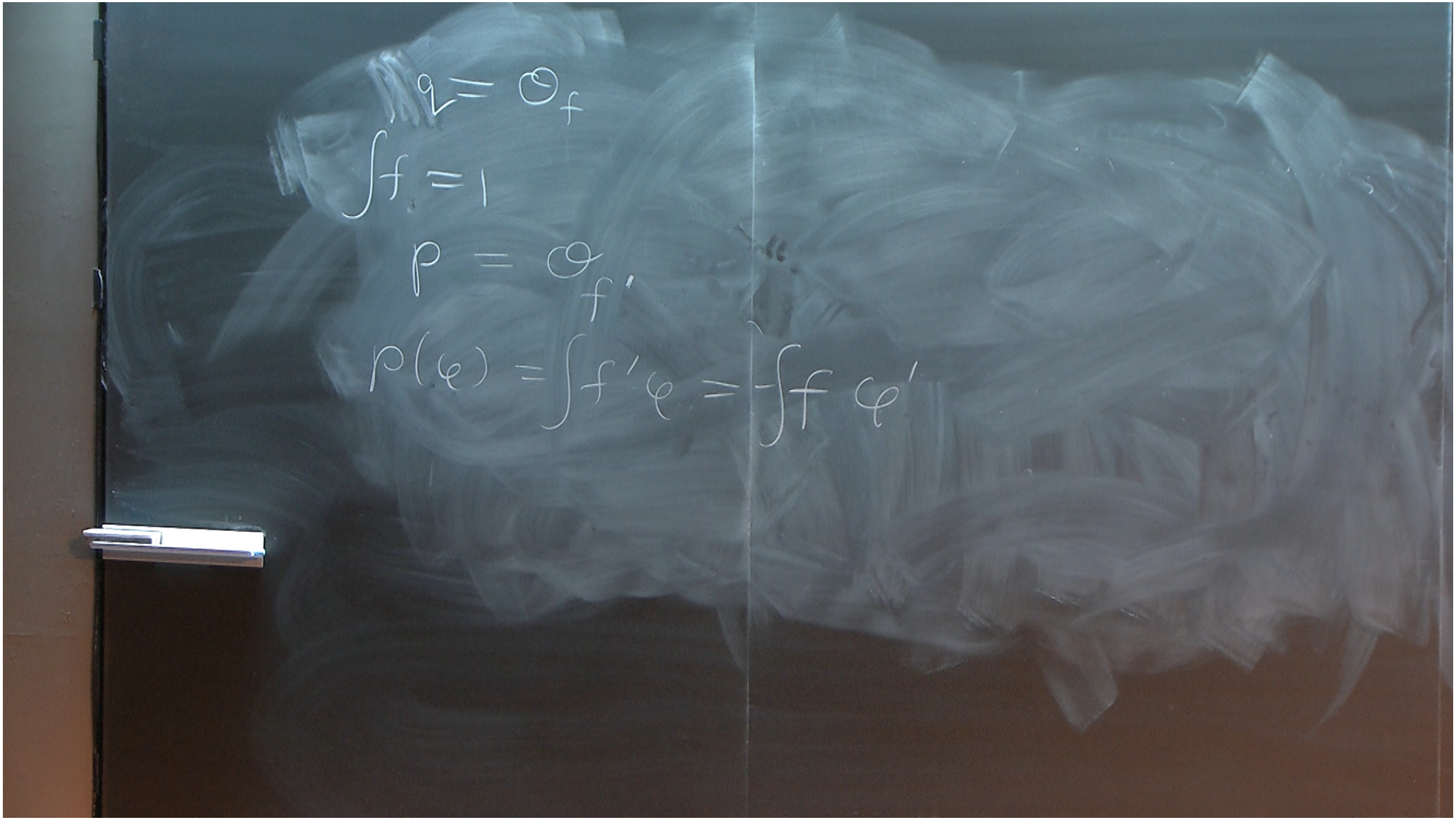
$\mathcal{O}_f$

$[\mathcal{O}_{f_1}] = [\mathcal{O}_{f_2}]$

if $\exists$ g, where

$\frac{\partial^2}{\partial x^2} g = f_1 - f_2$

As, $d\mathcal{O}_g^* = \mathcal{O}_{\Delta g}$


$q = \mathcal{O}_f$
$\int f = 1$
$p = \mathcal{O}_{f'}$
$p(\varphi) = \int f' \varphi = -\int f \varphi'$

To make life easier:

Represent

$$q = \Theta_{\delta_{x=0}}$$

$$p = \Theta_{\delta'_{x=0}}$$

How do we compute $[p, q]$?

$$\Theta_{\delta_{x=0}} \Theta_{\delta'_{x=\varepsilon}} - \Theta_{\delta_{x=0}} \Theta_{\delta'_{x=-\varepsilon}}$$



C_c^∞ ((

Is 2 d

Spanned

f $\int$

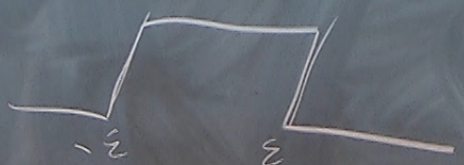
where $\int$

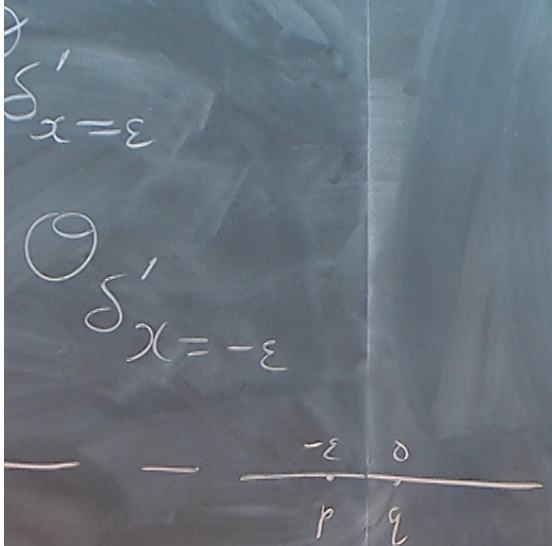
And, $\frac{\partial}{\partial}$

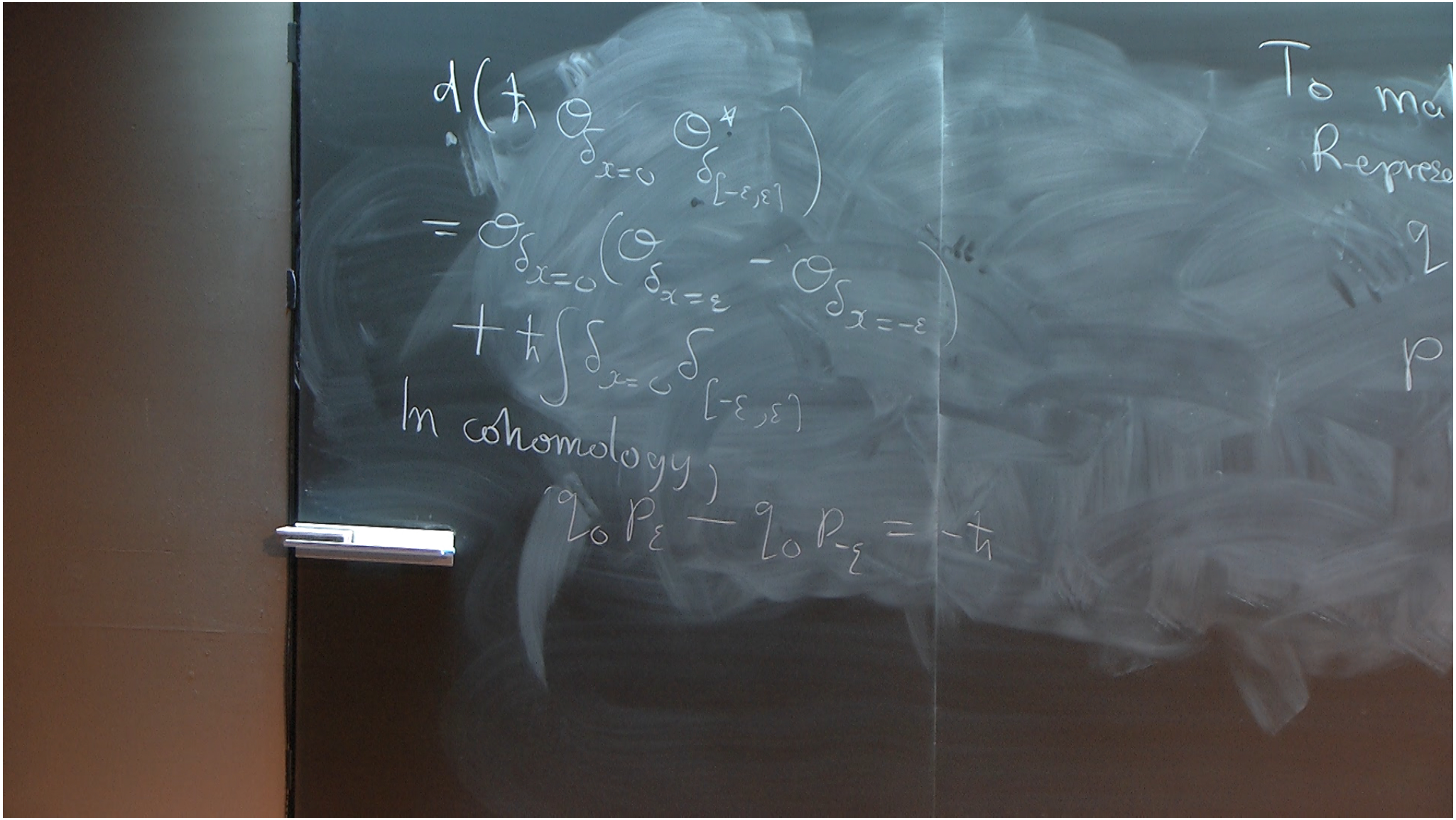
we compute

Notice that

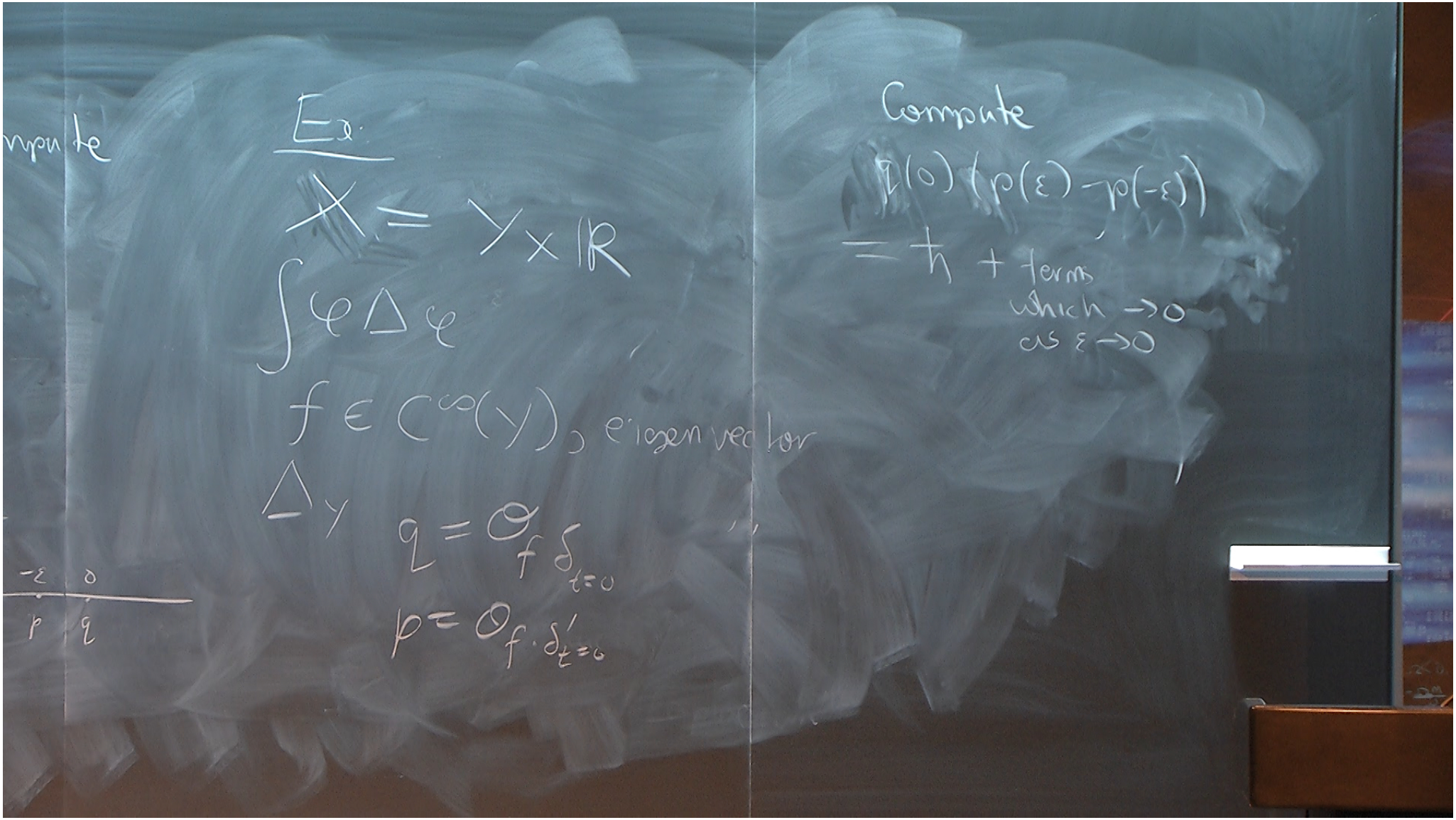
$$\frac{d}{dx}\left(\theta * \delta_{[-\varepsilon,\varepsilon]}\right) = \theta * \delta'_{[-\varepsilon,\varepsilon]} = \theta * \delta'_{x=\varepsilon} - \theta * \delta'_{x=-\varepsilon}$$

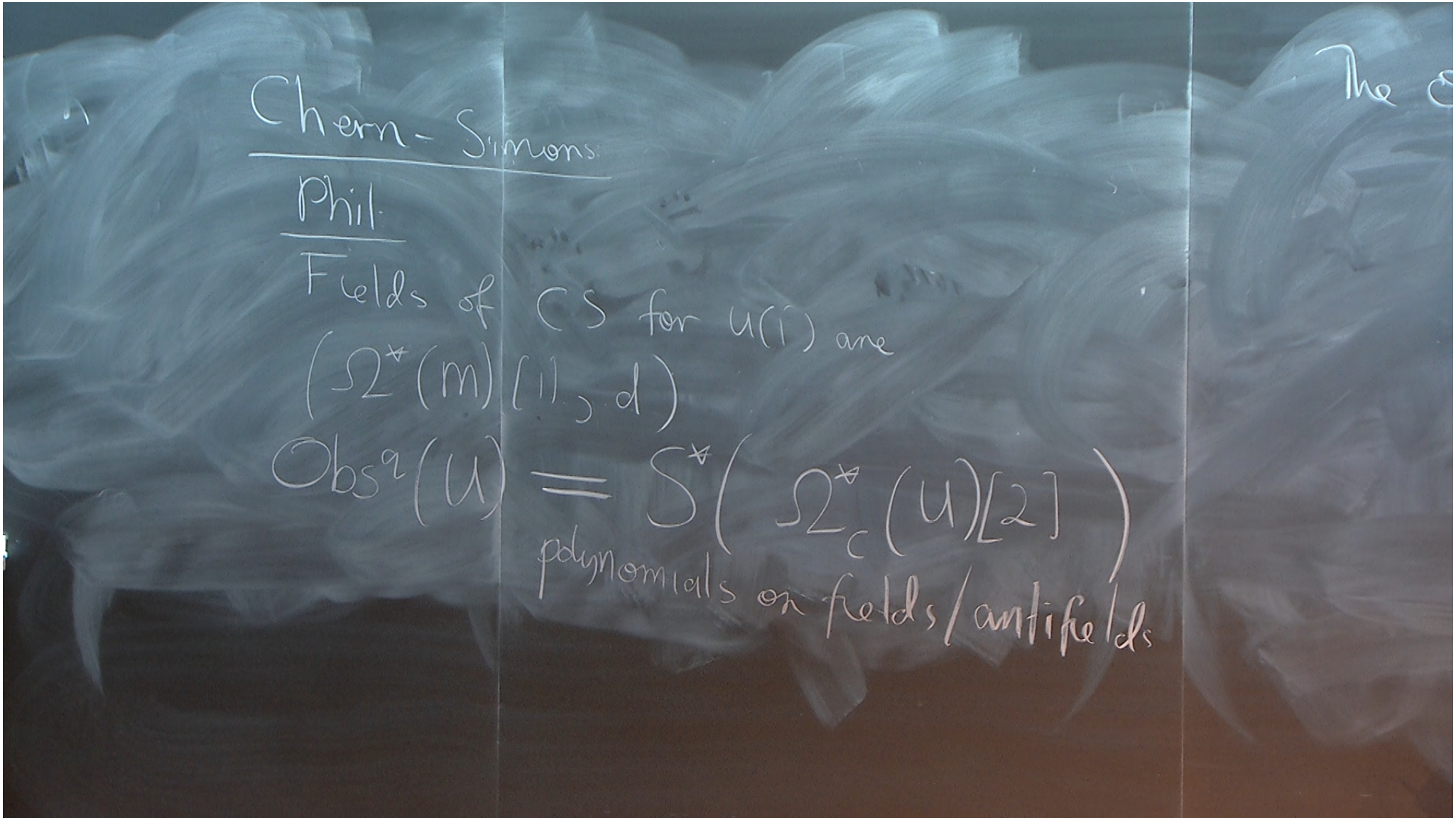





d(ħ
= ħ
In cohomology,


homotopy


mpute
Ex.
$X = Y \times \mathbb{R}$
$\int \varphi \Delta \varphi$
$f \in C^\infty(Y)$, eigenvector
Δ_Y
$q = \mathcal{O}_{f \cdot \delta_{t=0}}$
$p = \mathcal{O}_{f \cdot \delta'_{t=0}}$
$-\varepsilon$ 0
p q
Compute
$\Omega(0)(p(\varepsilon) - p(-\varepsilon))$
$= \hbar$ + terms which $\to 0$ as $\varepsilon \to 0$


Chern-Simons
Phil.
Fields of CS for U(1) are
$(\Omega^*(m)[1], d)$
$Obs^q(U) = S^*(\Omega^*_c(U)[2])$
polynomials on fields/antifields

ane

$\mathcal{O}_c(U)[2]$)

fields/antifields

The $\mathcal{O}^*_g$ fns are already built in.

If $f_i \in \Omega^*_c(U)[2]$

$$d(\mathcal{O}_{f_1} \cdots \mathcal{O}_{f_n})$$

$$= \sum \pm \mathcal{O}_{f_1} \cdots \mathcal{O}_{df_i} \cdots \mathcal{O}_{f_n}$$

$$+ \sum \pm \mathcal{O}_{f_1} \cdots \widehat{\mathcal{O}_{f_i}} \cdots \widehat{\mathcal{O}_{f_j}} \cdots \mathcal{O}_{f_n} \int f_i f_j \, \Delta_y$$

Ex.

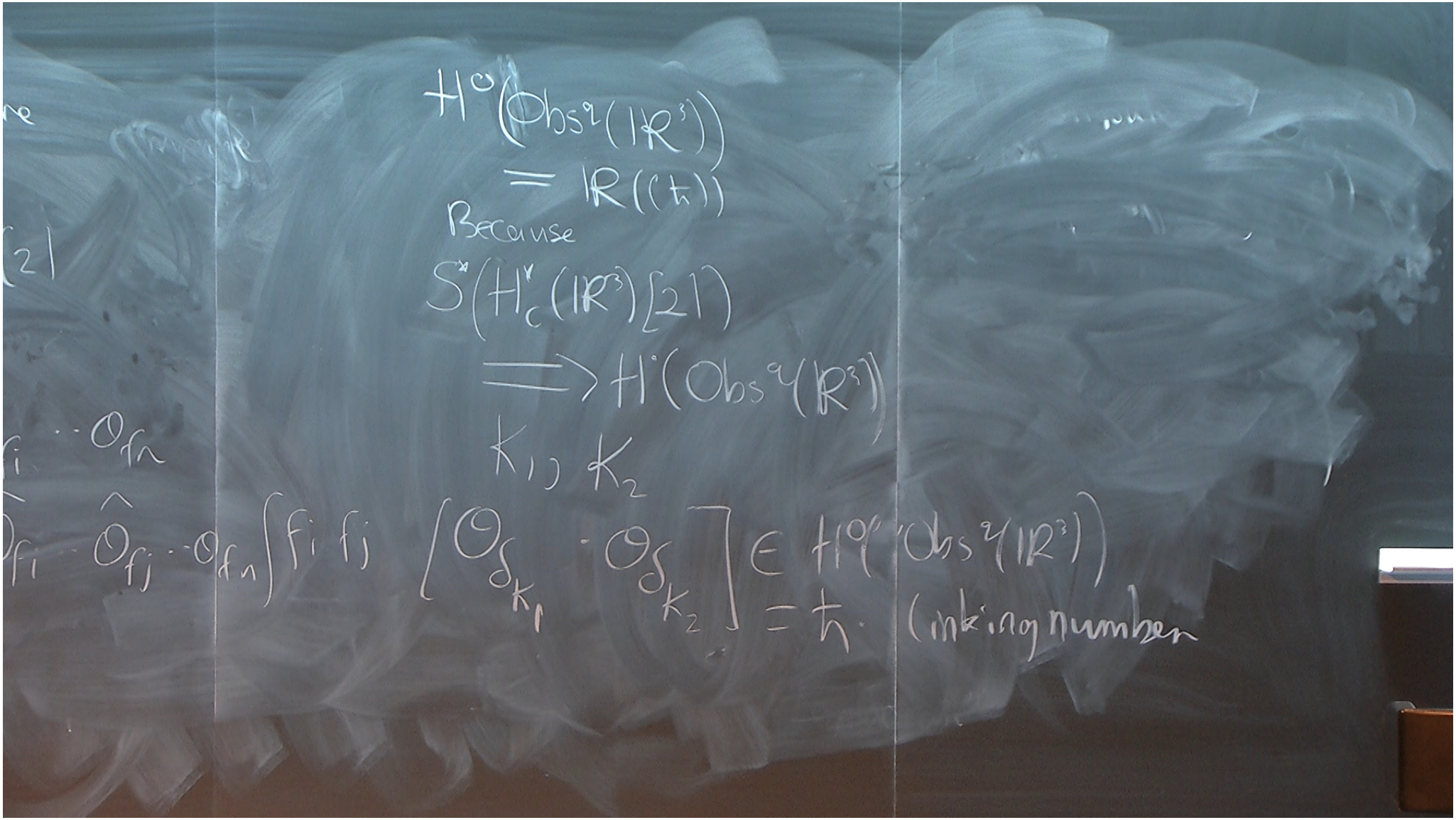
$\chi =$

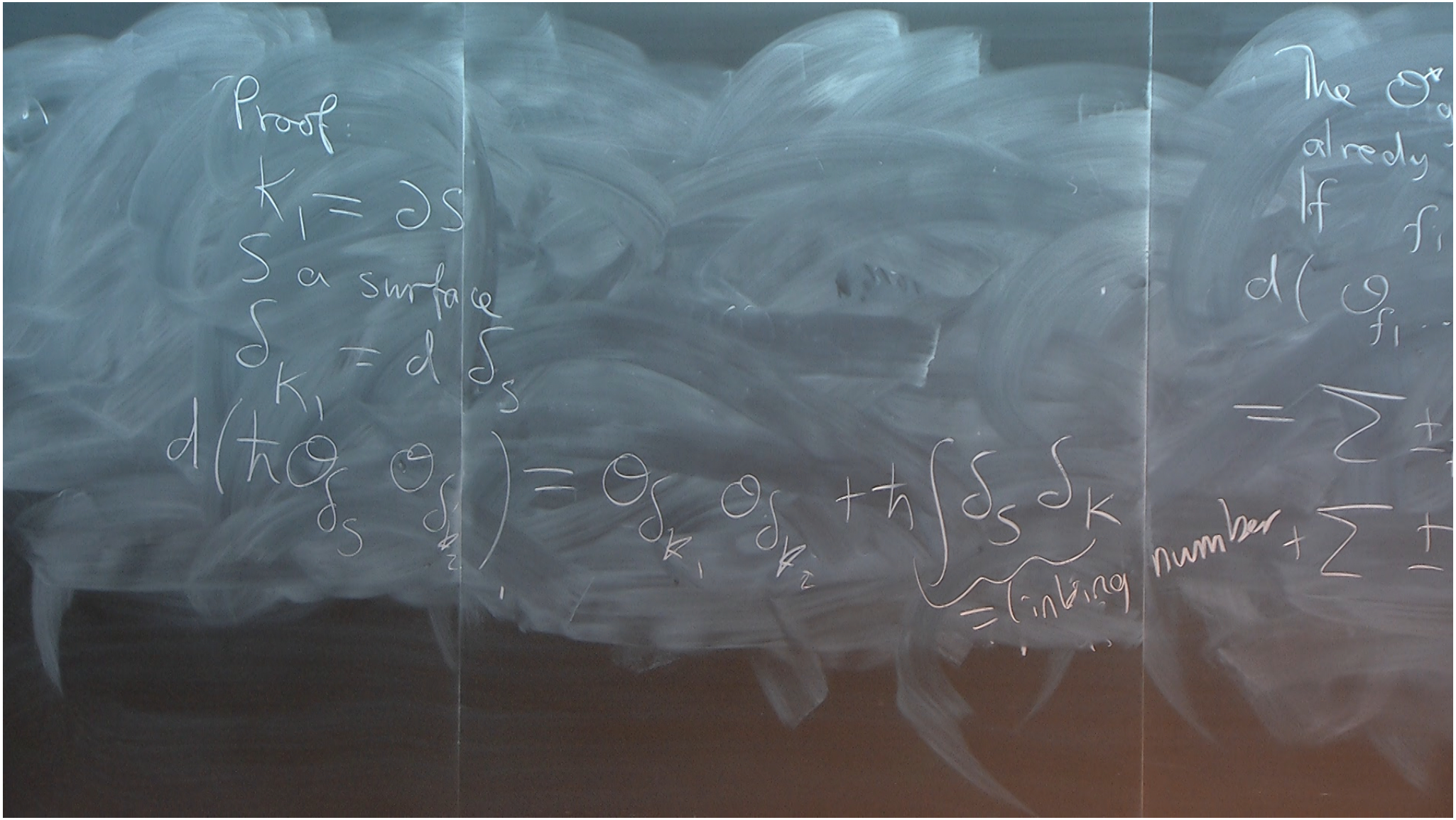
$\int \varphi \Delta \varphi$

$f \in C^$

$Q =$

$P =$


$H^0(Obs^q(\mathbb{R}^3)) = \mathbb{R}((\hbar))$
Because
$S^*(H^\bullet_c(\mathbb{R}^3)[2])$
$\Longrightarrow H^\bullet(Obs^q(\mathbb{R}^3))$
k_1, k_2
$\mathcal{O}_{f_1} \cdots \mathcal{O}_{f_n}$
$\mathcal{O}_{f_i} \cdots \hat{\mathcal{O}}_{f_j} \cdots \mathcal{O}_{f_n} \int f_i f_j \left[\mathcal{O}_{g_{k_1}}, \mathcal{O}_{g_{k_2}}\right] \in H^{\bullet}(Obs^q(\mathbb{R}^3))$
$= \hbar \cdot$ (linking number


Proof:
$K_1 = \partial S$
S a surface
$\delta_{K_1} = d\delta_S$
$d(\hbar \mathcal{O}_{\delta_S} \mathcal{O}_{\delta_{K_2}}) = \mathcal{O}_{\delta_{K_1}} \mathcal{O}_{\delta_{K_2}} + \hbar \int \delta_S \delta_K$
(linking number)
The $\mathcal{O}^*$
already
If
$d(\mathcal{O}_{f_1}$
$= \sum \pm$
$+ \sum \pm$