Title: Lecture 1: Supersymmetric Quantum Mechanics and All That

Speakers: Mathew Bullimore

Collection: QFT for Mathematicians

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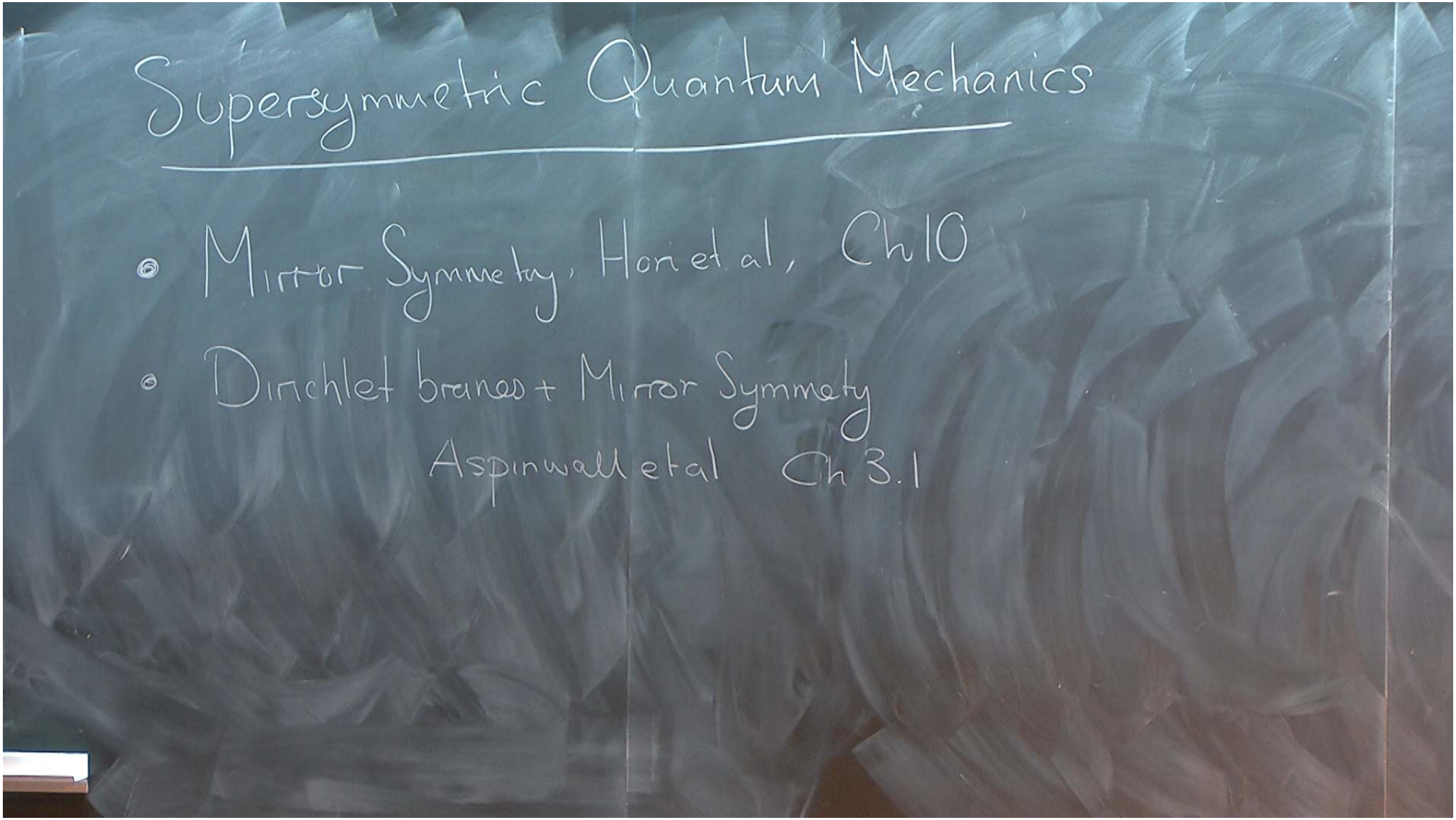
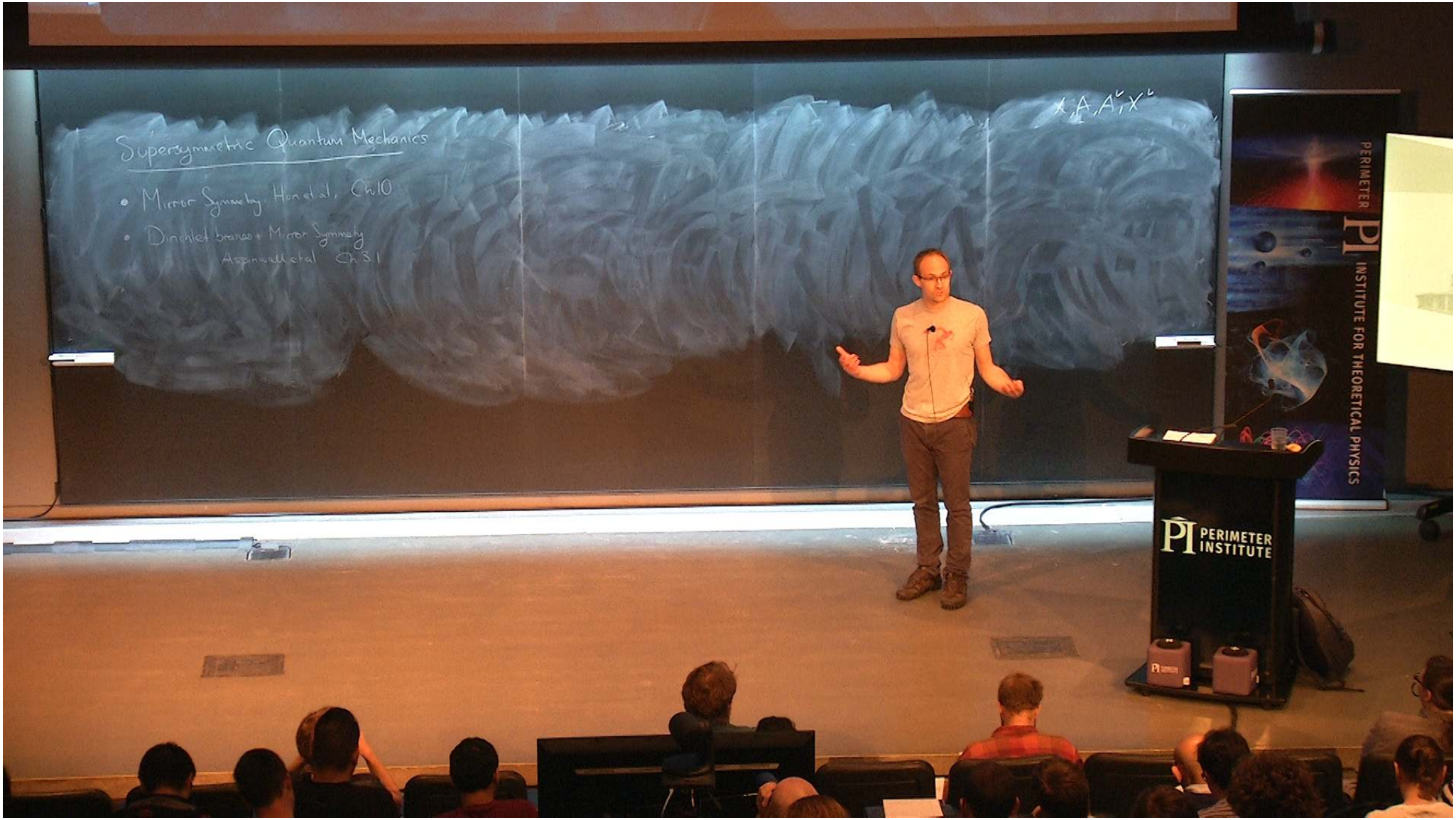

Supersymmetric Quantum Mechanics
Mirror Symmetry, Hori et al, Ch 10
Dirichlet branes + Mirror Symmetry
Aspinwall et al Ch 3.1


Supersymmetric Quantum Mechanics
• Mirror Symmetry, Hori et al, Ch 10
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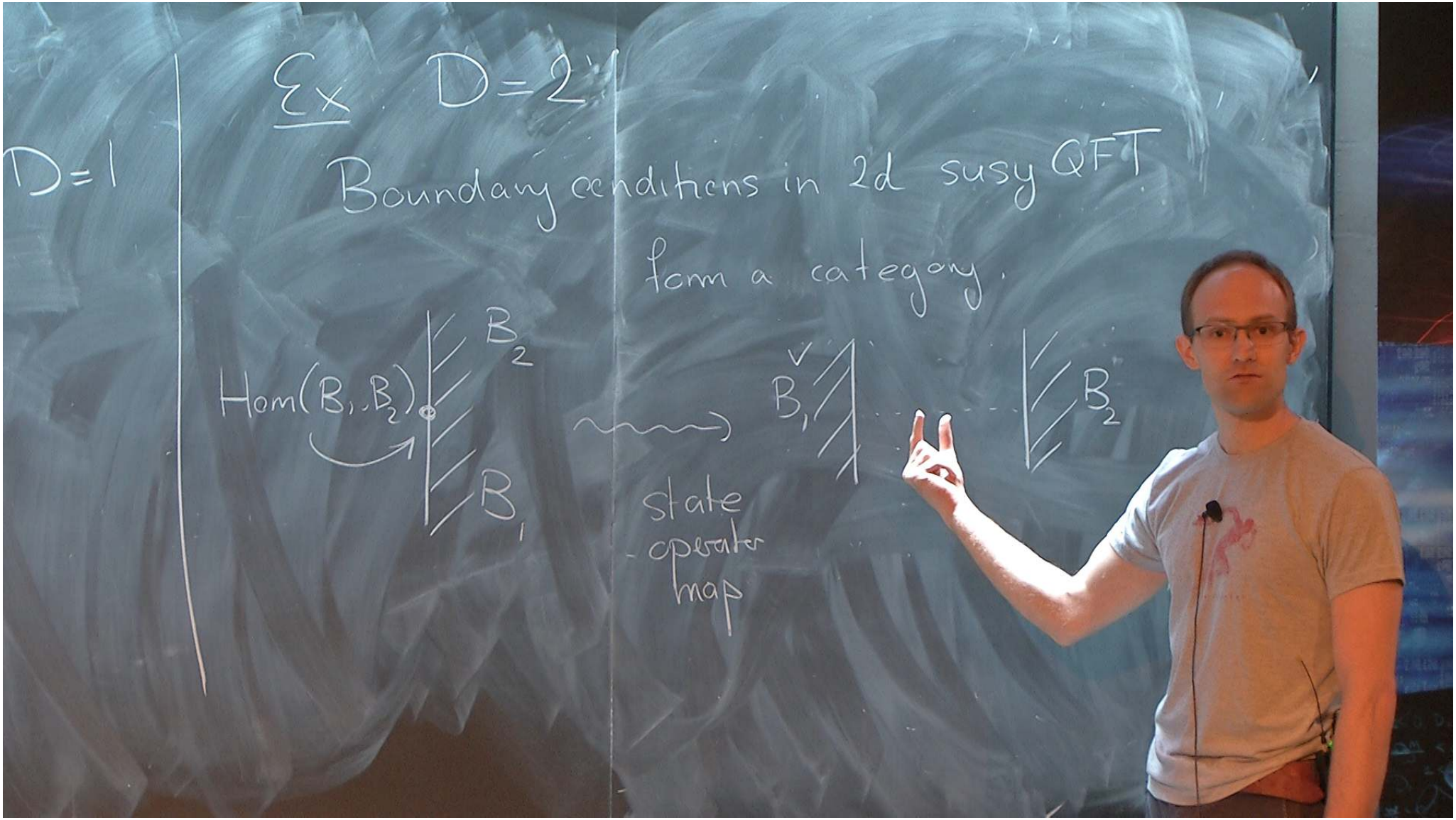
1.1) Motivation

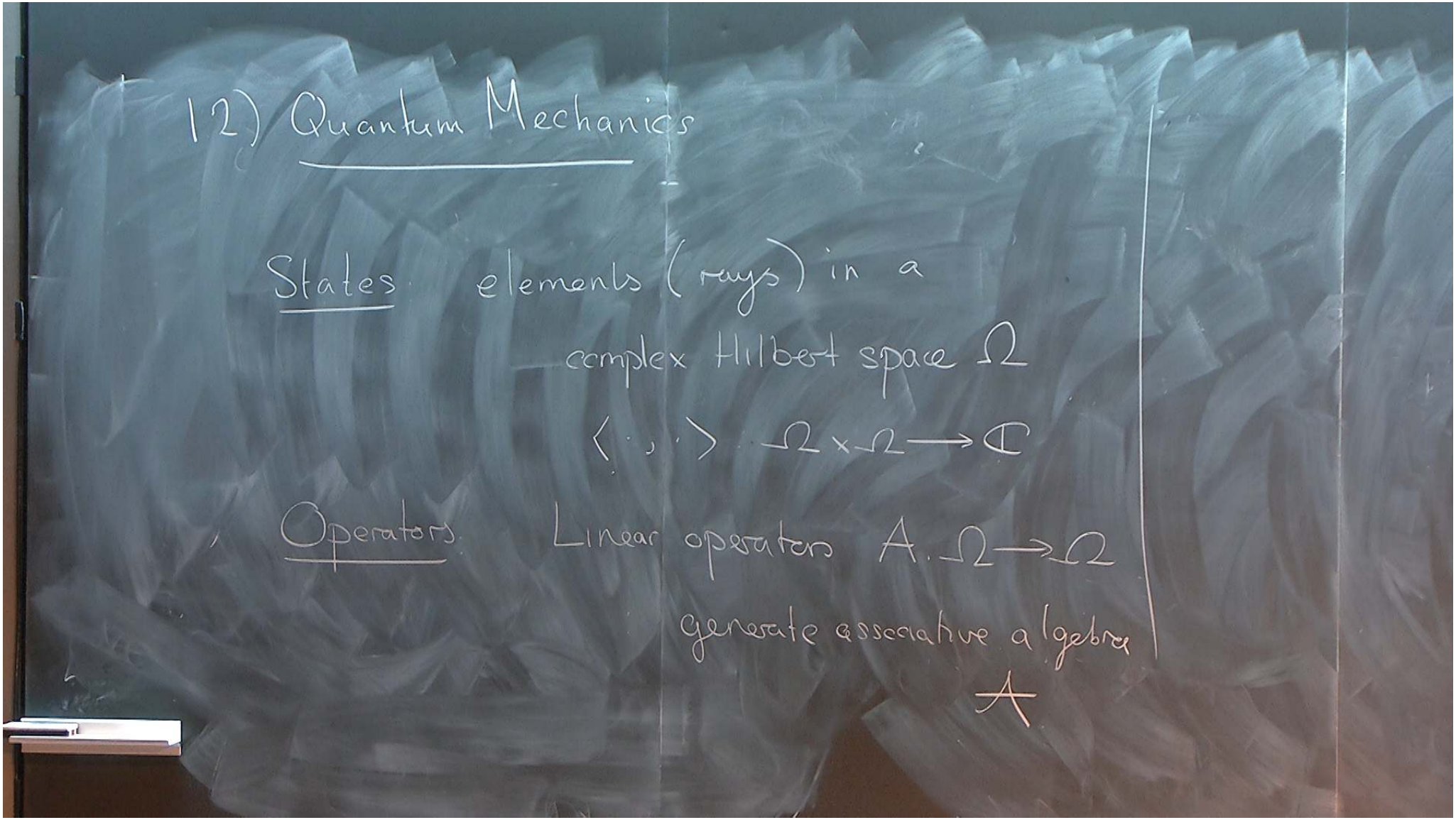
SQM = Susy QFT in D=1

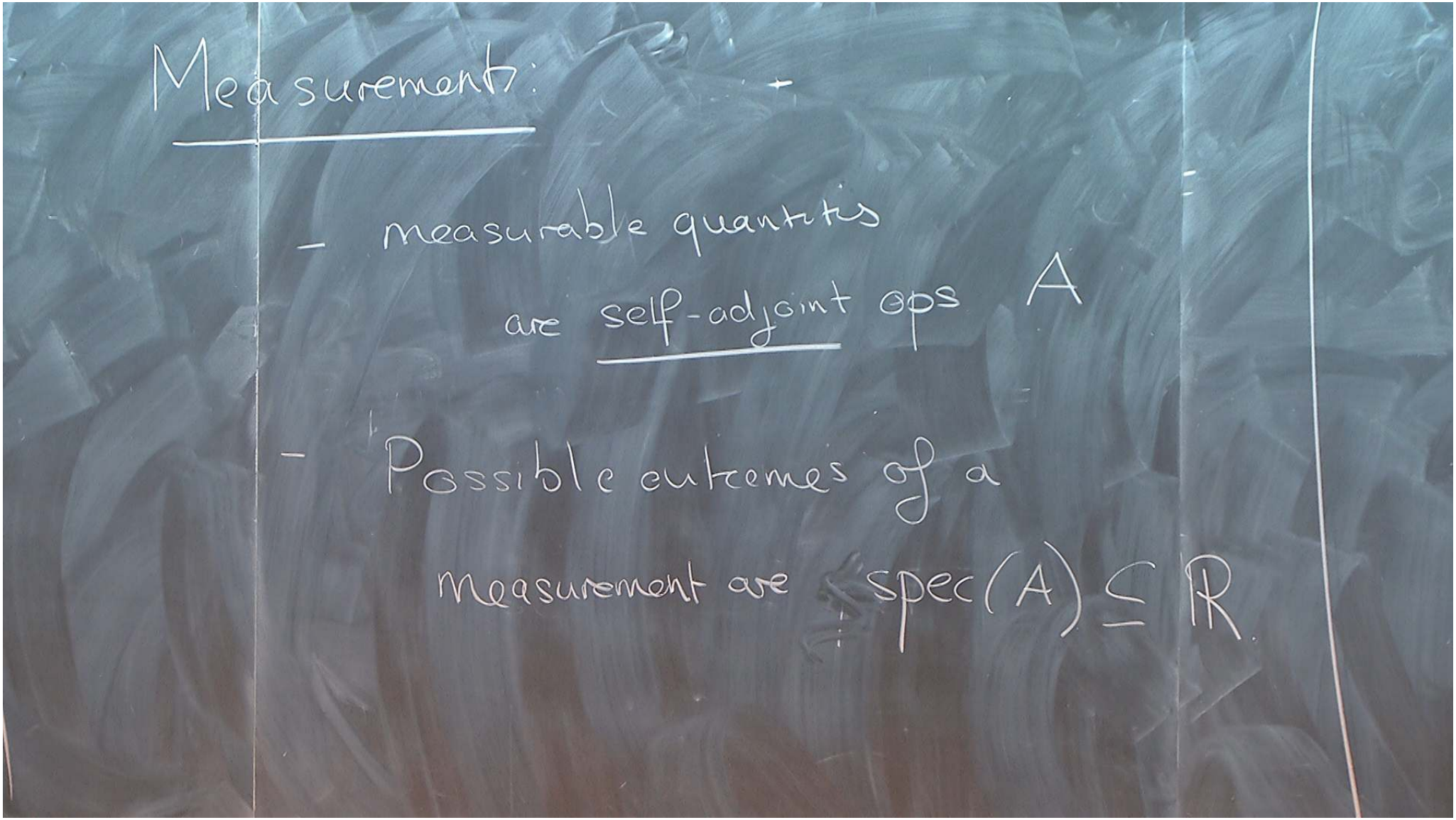
τ | $M = \mathbb{R}$

Useful for QFT in $d > 1$

τ | $\times$ M_{D-1} $\rightsquigarrow$ τ


D=1
Ex D=2
Boundary conditions in 2d susy QFT
form a category.
Hom(B1, B2)
B2
B1
state
operator
map
B1
B2


12) Quantum Mechanics
States elements (rays) in a
complex Hilbert space Ω
⟨ , ⟩ Ω × Ω → ℂ
Operators Linear operators A : Ω → Ω
generate associative algebra
𝒜



Measurements:

- measurable quantities are self-adjoint ops A

- Possible outcomes of a measurement are $\mathrm{spec}(A) \subseteq \mathbb{R}$.

Time evolution

- euclidean time τ $\uparrow \tau$
- distinguished self-adjoint operator H

- Schrodinger: states evolve

$$\partial_\tau \psi = -H \cdot \psi \quad \psi \in \Omega$$

- Heisenberg: operators

$$\partial_\tau A = [H, A]$$

Time evolution

- euclidean time τ $\uparrow \tau$
- distinguished self-adjoint operator H

$\subseteq \mathbb{R}$

- Schrodinger: states evolve

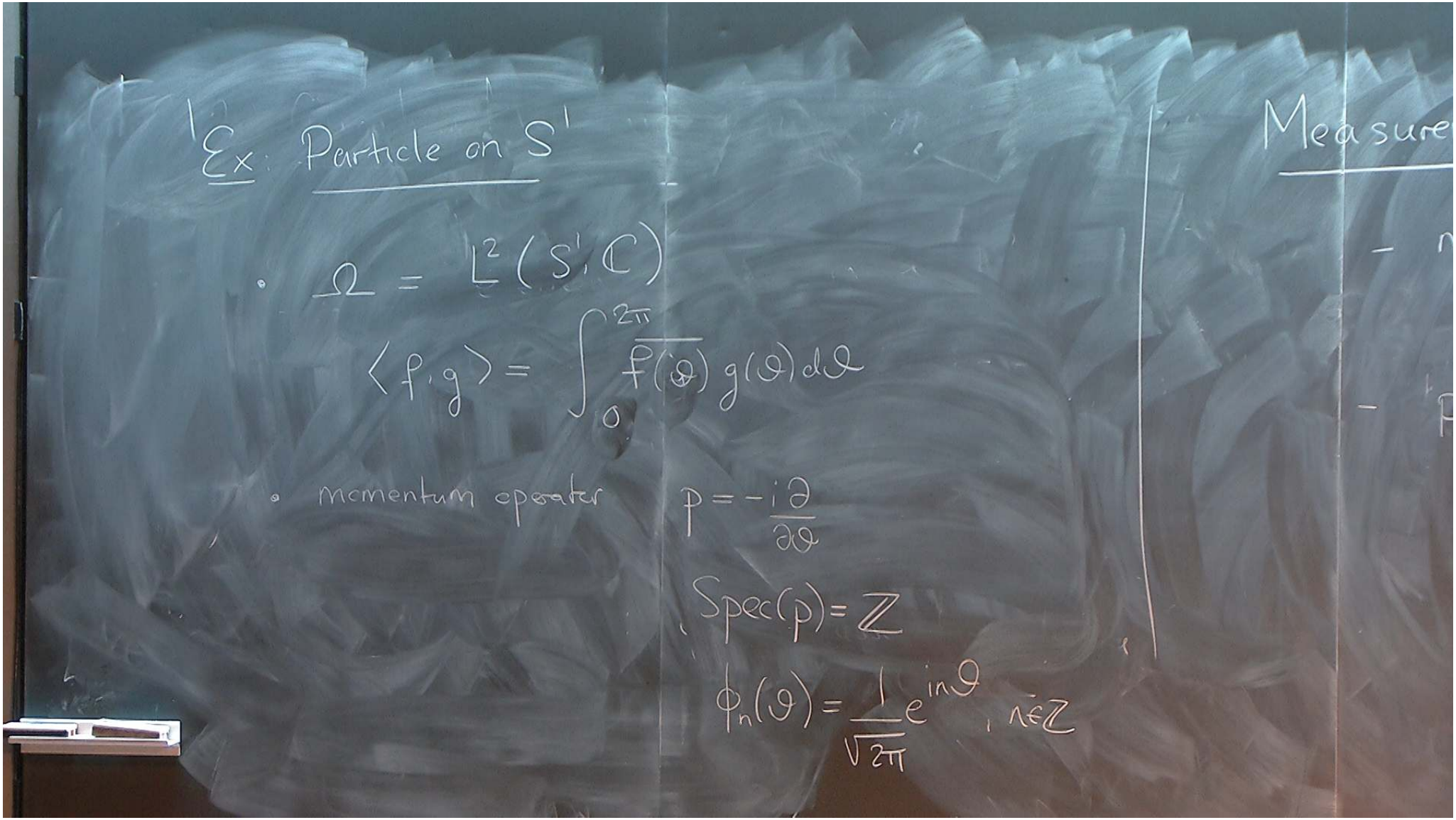
$$\partial_\tau \psi = -H \cdot \psi \quad \psi \in \Omega$$

- Heisenberg: operator

$$\partial_\tau A = [H, A]$$

Def A is conserved if

$$[H, A] = 0$$


Ex: Particle on S^1
$\Omega = L^2(S^1, \mathbb{C})$
$\langle f, g \rangle = \int_0^{2\pi} \overline{f(\theta)}\, g(\theta)\, d\theta$
momentum operator $p = -i\frac{\partial}{\partial\theta}$
$\mathrm{Spec}(p) = \mathbb{Z}$
$\phi_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}$, $n \in \mathbb{Z}$
Measure
- n
- P

Hamiltonian: $H = \frac{p^2}{2} = -\frac{1}{2}\frac{\partial^2}{\partial \theta^2}$

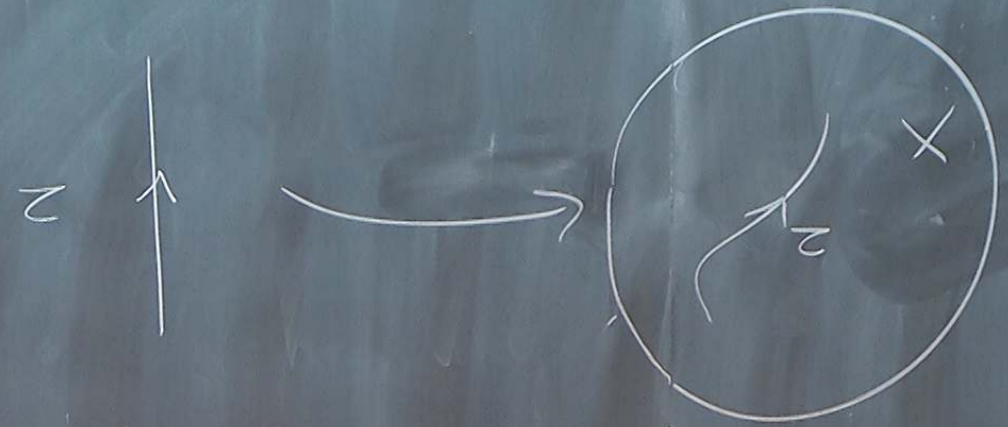
$$Spec(H) = \left\{ \frac{n^2}{2}, n \in \mathbb{Z} \right\}$$

$$[H, p] = 0$$

$\therefore$ p is conserved

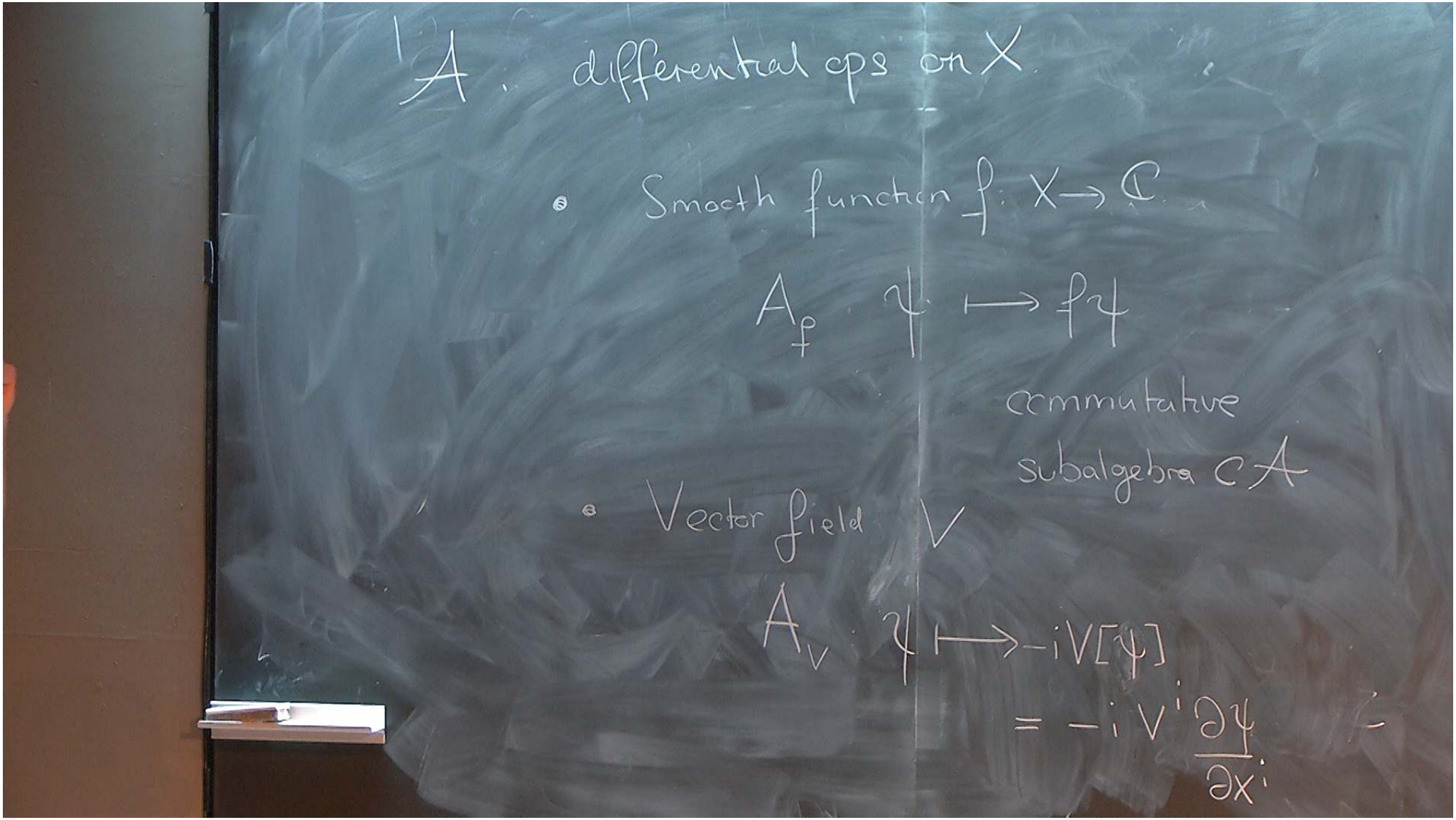
Ex: Riemannian σ-model

X = cpt, smooth Riemannian manifold



$\Omega = L^2(X, \mathbb{C})$

$\langle f, g \rangle = \int_X \bar{f} g \, dVol_X$


$\mathcal{A}$: differential ops on X
• Smooth function $f: X \to \mathbb{C}$
$A_f \quad \psi \longmapsto f\psi$
commutative subalgebra $\subset \mathcal{A}$
• Vector field V
$A_V \quad \psi \longmapsto -iV[\psi]$
$= -i\, V^i \frac{\partial \psi}{\partial x^i}$

- Hamiltonian: $H = \Delta$

↳ Laplacian

$$\partial_t \psi = -\Delta \psi$$

: heat eq. on X.

- If V real vector field that generate isometry of X

$$[H, A_V] = 0$$

: conserved (momentum)

Ex: Ri

X

t

$\Omega =$

$\langle f_i$

13) Susy QM

Additional structure

- Hilbert space $\mathbb{Z}_2$-graded

$$\Omega = \Omega_e \oplus \Omega_o$$

- $\exists$ odd operators $Q, Q^\dagger$

$$\{Q, Q\} = 0$$

$$\{Q, Q^\dagger\} = H$$

$$\{Q^\dagger, Q^\dagger\} = 0$$

Consequence: $[H, Q] = [H, Q^\dagger] = 0$

: $Q, Q^\dagger$ are conserved

"Supercharges"

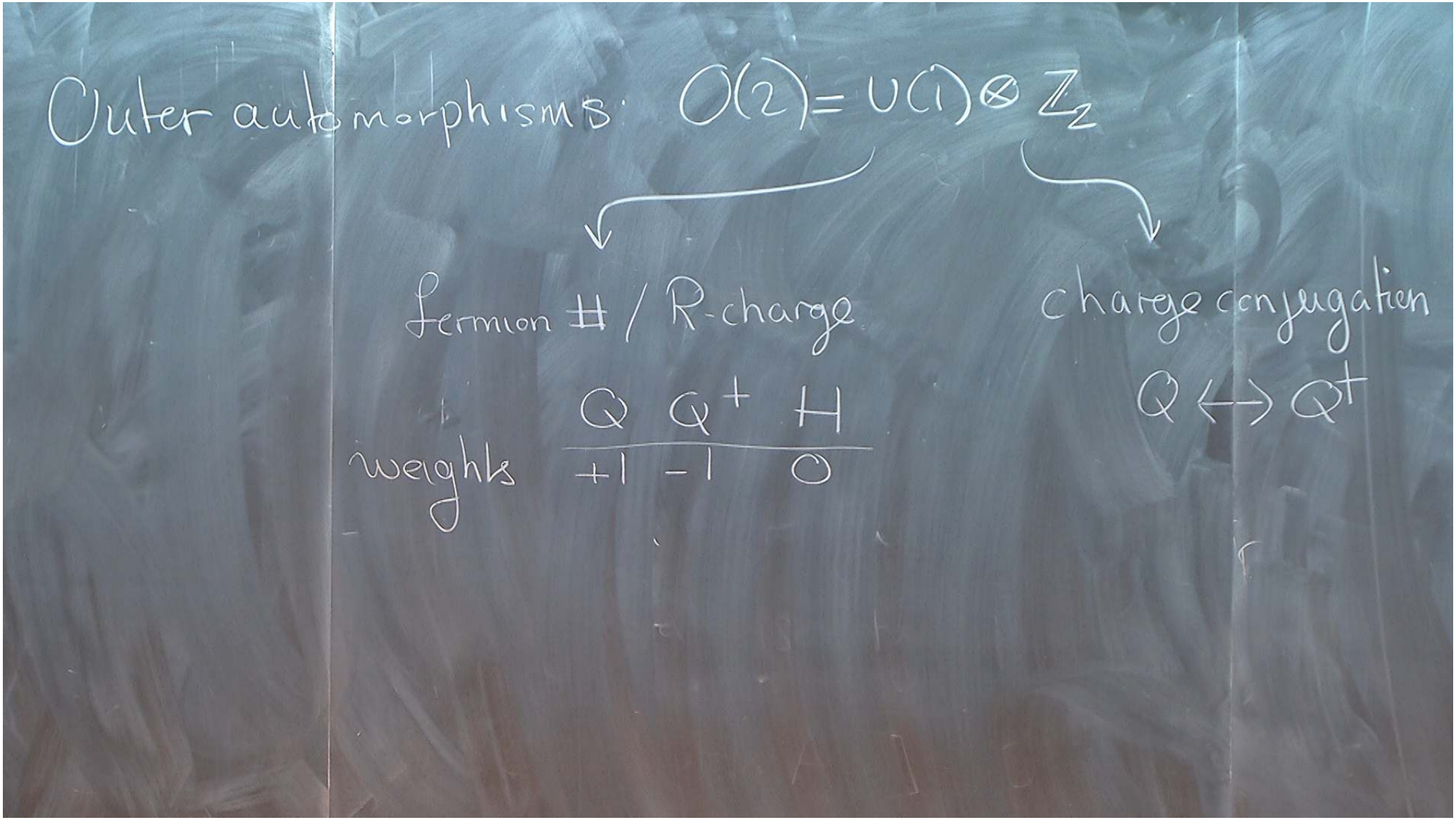
$$[A, B] = AB - BA$$

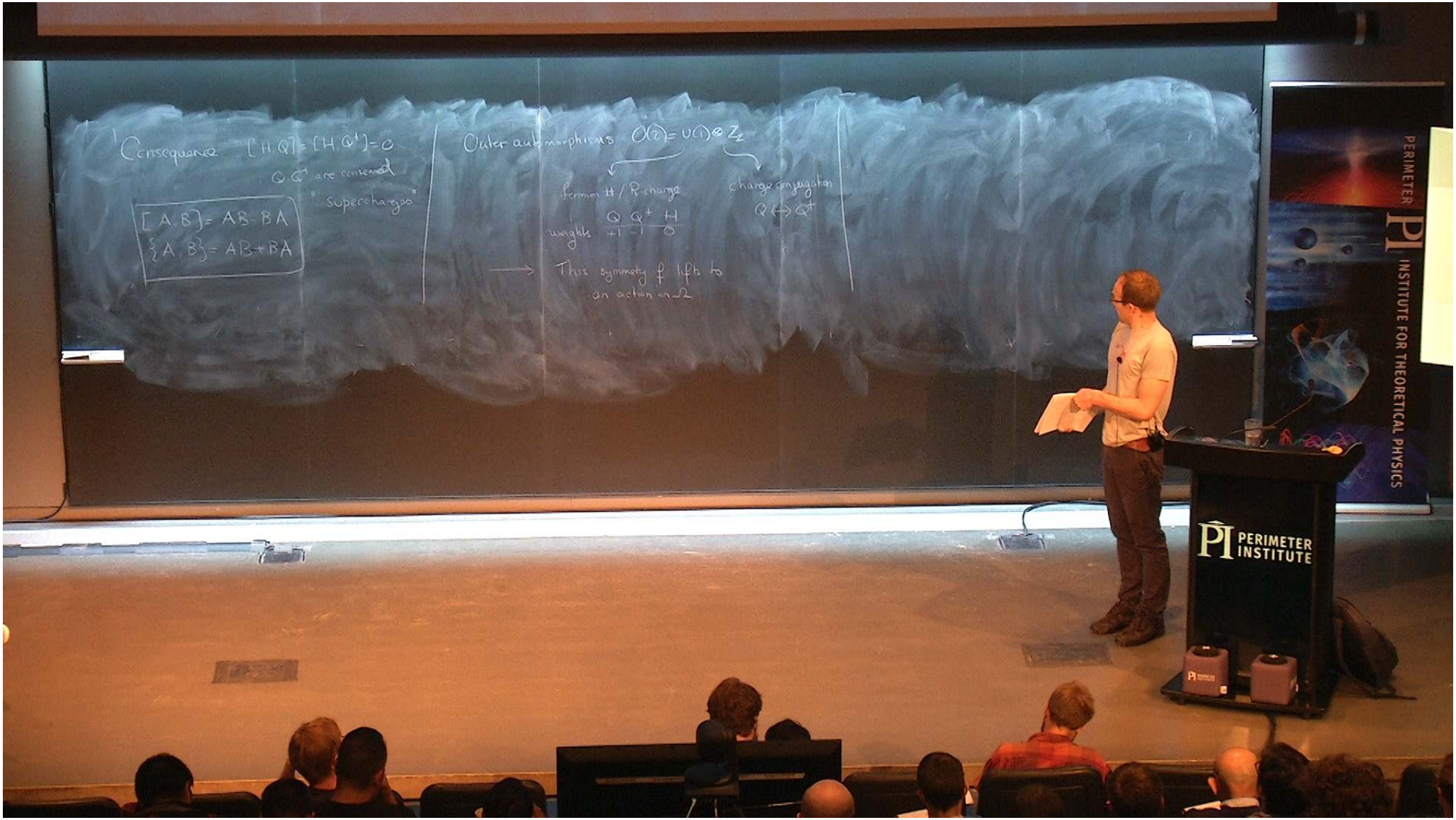
$$\{A, B\} = AB + BA$$

Outer automorphisms: $O(2) = U(1) \otimes Z_2$

fermion # / R-charge

	Q	Q^+	H
weights	$+1$	-1	0


Outer automorphisms: $O(2) = U(1) \otimes Z_2$
fermion # / R-charge
charge conjugation
$Q \leftrightarrow Q^\dagger$
Q Q† H
weights +1 −1 0


Consequence [H,Q] = [H,Q'] = 0
Q, Q' are conserved
"supercharges"
[A,B] = AB - BA
{A,B} = AB + BA
Outer automorphisms O(2) = U(1) ⊗ Z2
fermion # / R-charge
charge conjugation
Q ↔ Q†
weights Q Q† H
+1 -1 0
This symmetry f lifts to an action on Ω
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Outer automorphisms: $O(2) = U(1) \rtimes Z_2$

① fermion # / R-charge

	Q	$Q^\dagger$	H
weights	$+1$	-1	0

② charge conjugation

$Q \leftrightarrow Q^\dagger$

⟶ This symmetry if lifts to an action on Ω

(1) Self-adjoint op F

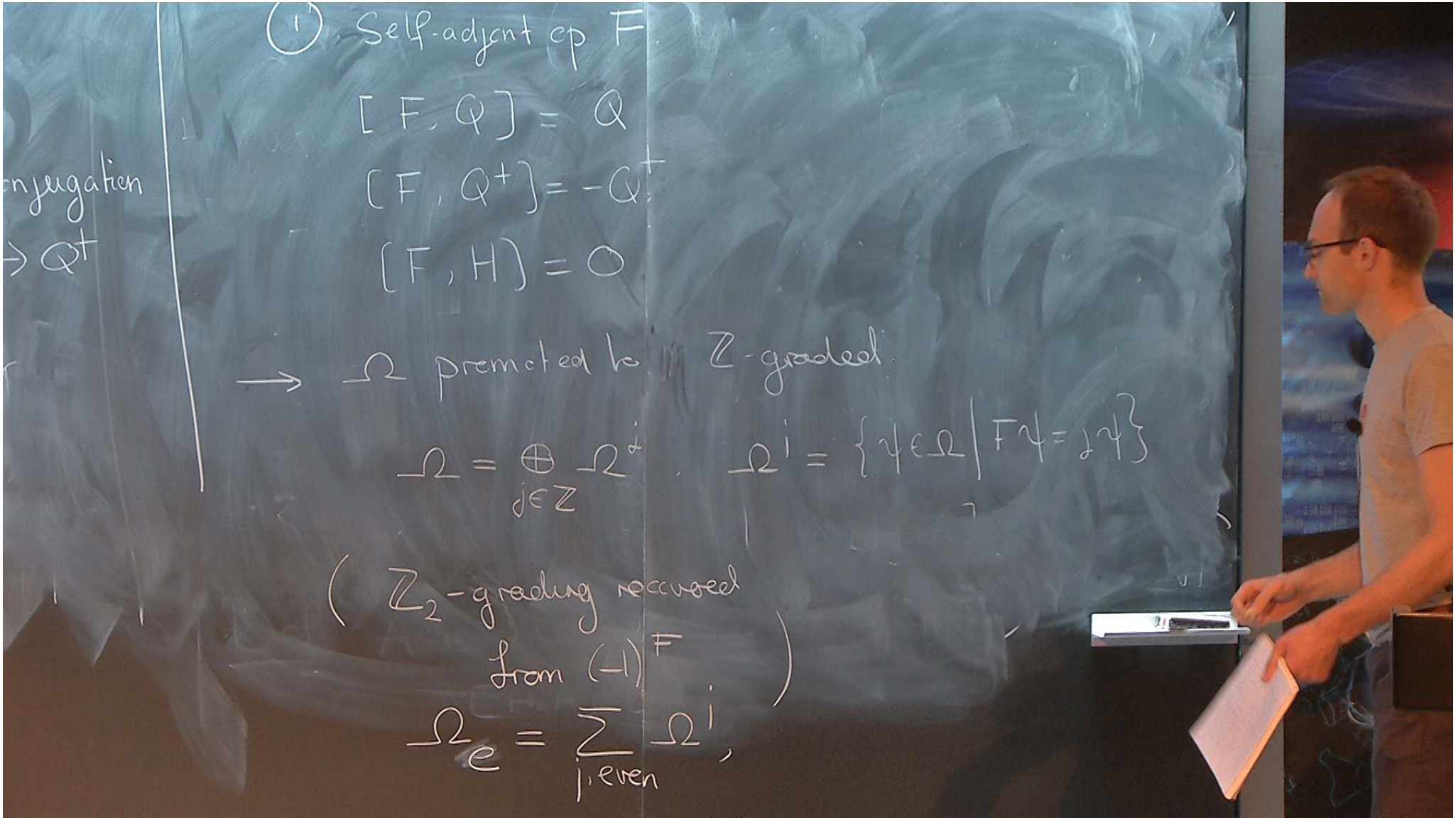
$[F, Q] = Q$

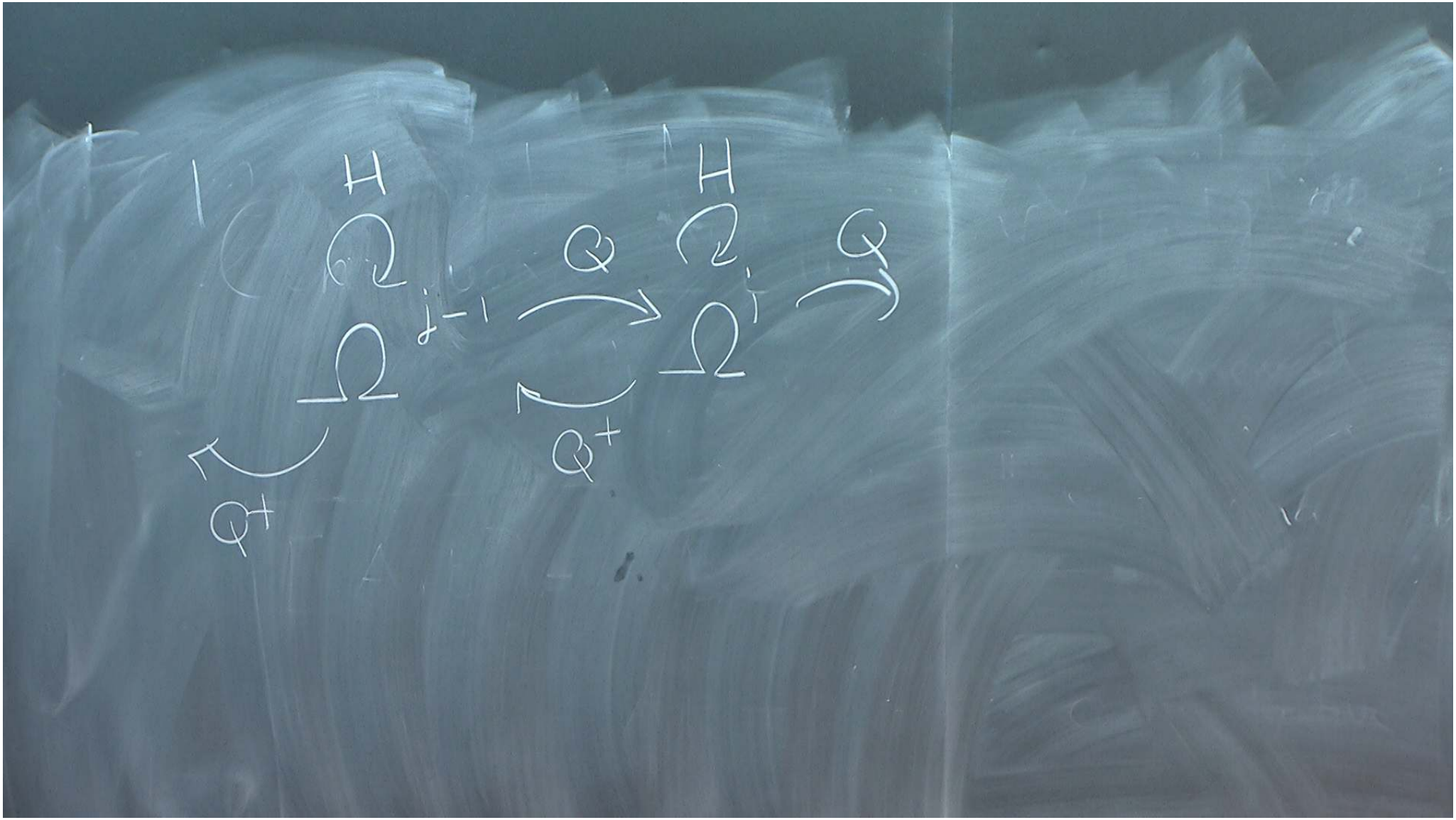
$[F, Q^\dagger] = -Q^\dagger$

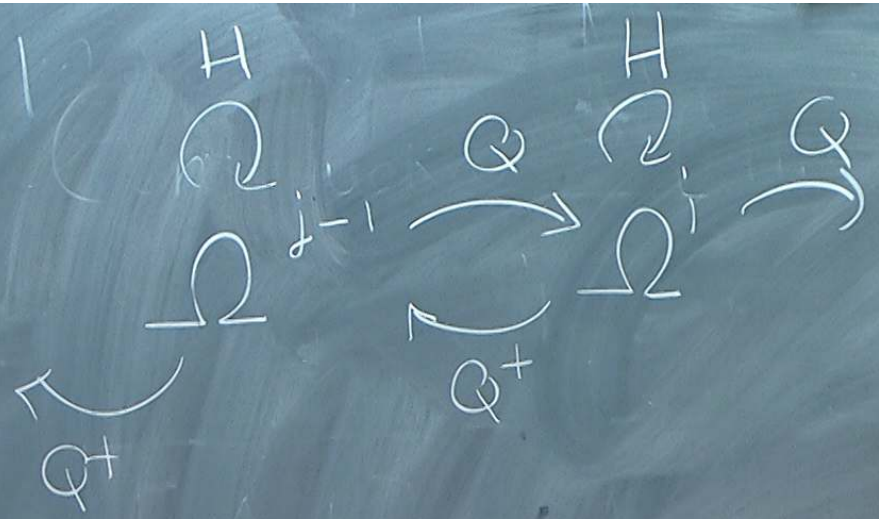
$[F, H] = 0$

$\longrightarrow$ Ω promoted to $\mathbb{Z}$-graded.

$\Omega = \bigoplus_{j \in \mathbb{Z}} \Omega^j$ $\quad$ $\Omega^j = \{\psi \in \Omega \mid F\psi = j\psi\}$


① Self-adjoint op F
[F, Q] = Q
[F, Q†] = −Q†
[F, H] = 0
→ Ω promoted to ℤ-graded
$\Omega = \bigoplus_{j\in\mathbb{Z}} \Omega^j$
$\Omega^j = \{\psi \in \Omega \mid F\psi = j\psi\}$
($\mathbb{Z}_2$-grading recovered from $(-1)^F$)
$\Omega_e = \sum_{j\,\text{even}} \Omega^j$,


H
H
Q
Q
Q+
Q+

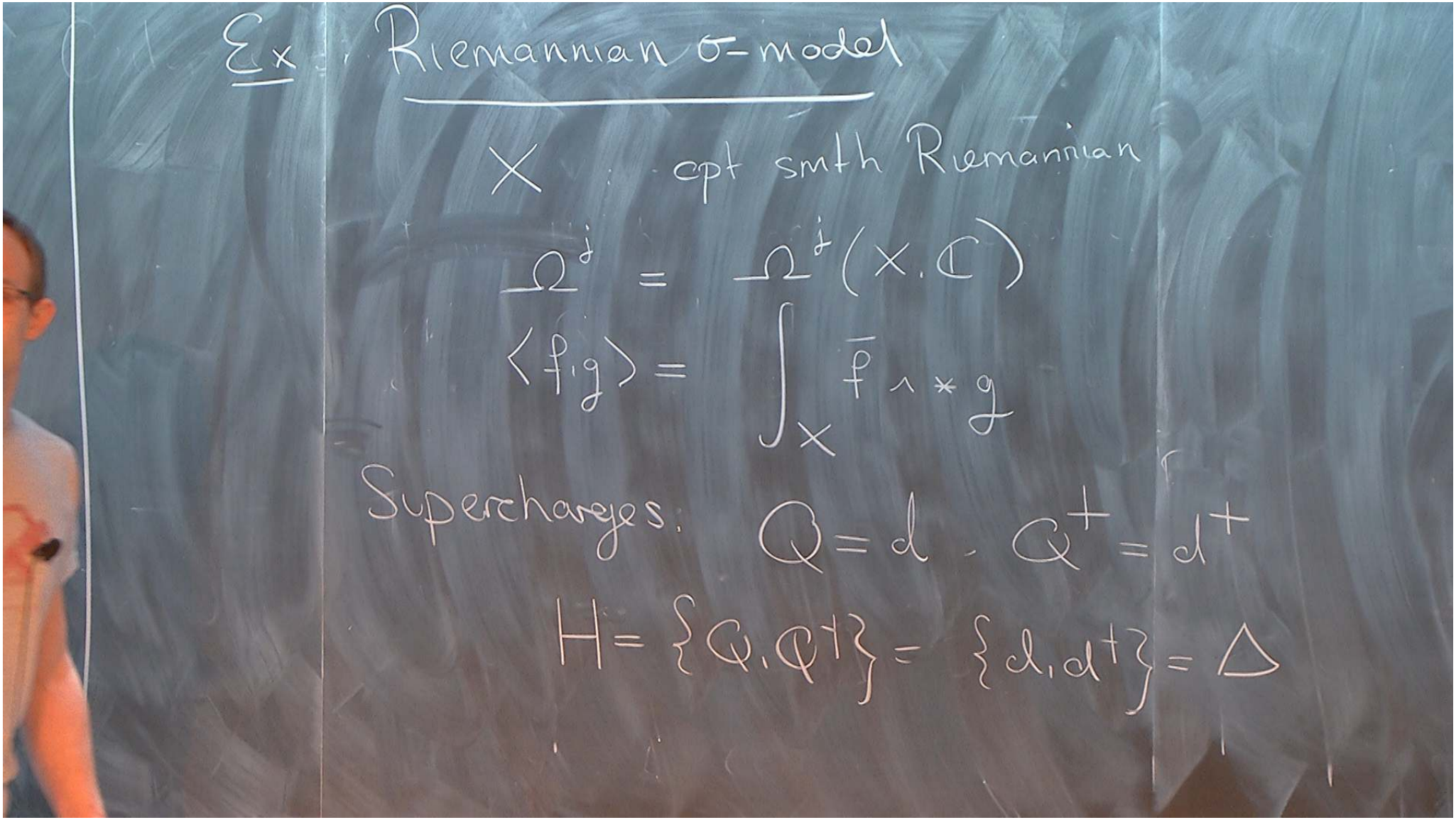


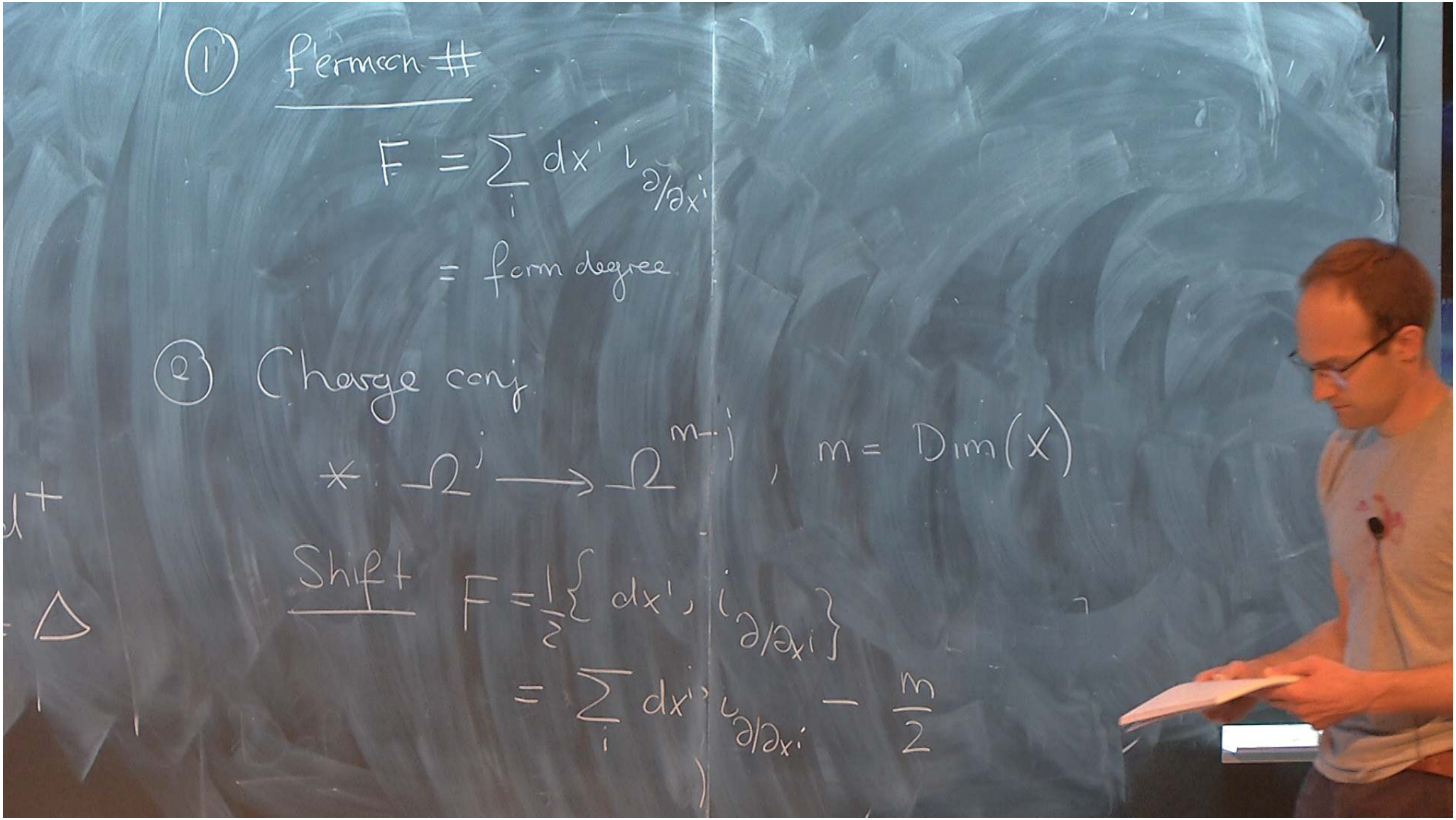
(2) unitary operator C $\Omega^i \to \Omega^{-i}$

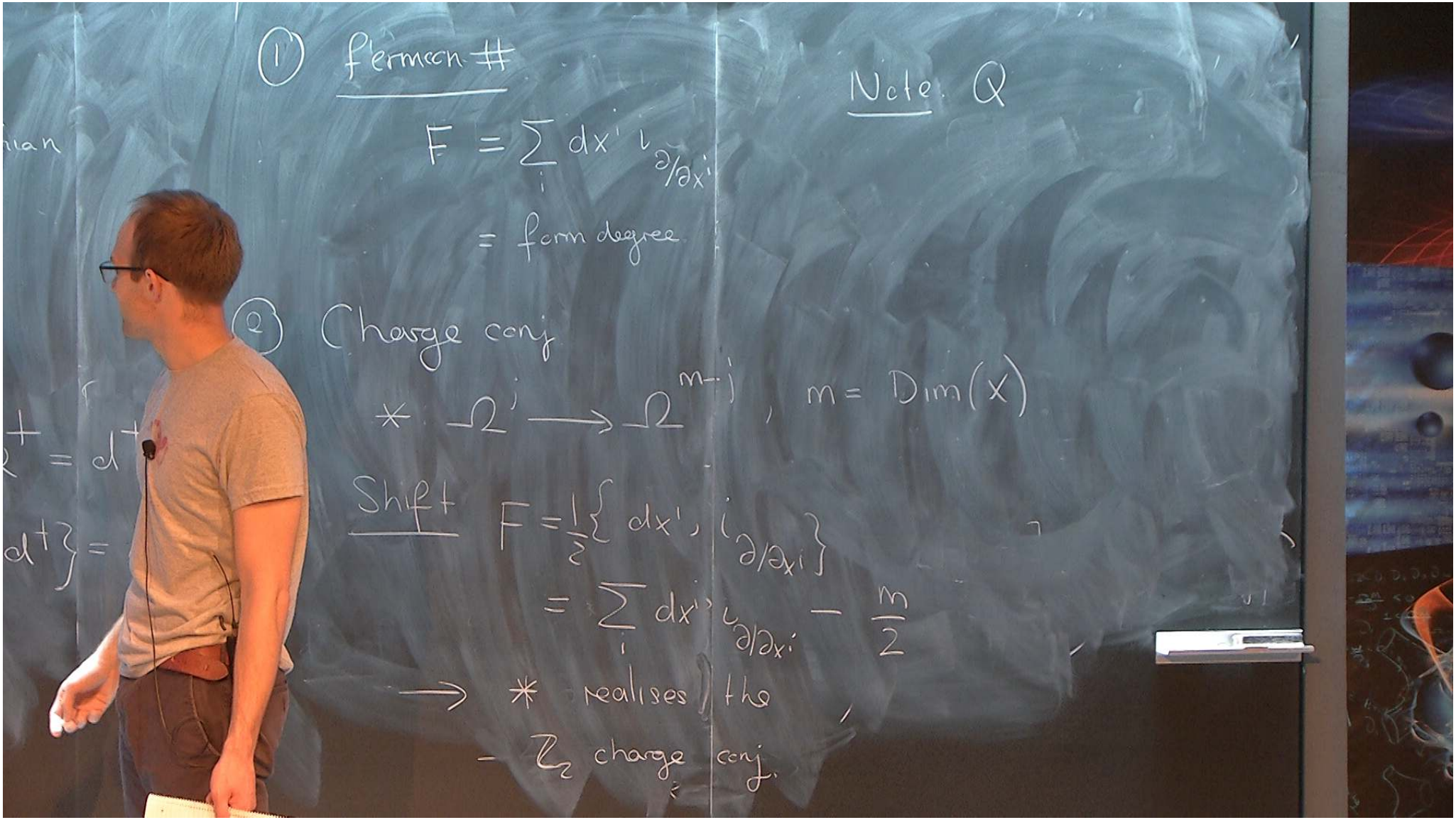
$$C Q C^{-1} = Q^+$$

$$C Q^+ C^{-1} = Q$$

Ω promoted to repr of $O(2)$.


Ex: Riemannian σ-model
X : cpt smth Riemannian
$\Omega^i = \Omega^i(X, \mathbb{C})$
$\langle f, g \rangle = \int_X \bar{f} \wedge * g$
Superchareges: $Q = d$. $Q^+ = d^+$
$H = \{Q, Q^+\} = \{d, d^+\} = \Delta$


(1) fermion #
$F = \sum_i dx^i \, \iota_{\partial/\partial x^i}$
= form degree
(2) Charge conj
$* : \Omega^j \longrightarrow \Omega^{m-j}$, $m = \mathrm{Dim}(X)$
Shift $F = \frac{1}{2}\{ dx^i, \iota_{\partial/\partial x^i} \}$
$= \sum_i dx^i \, \iota_{\partial/\partial x^i} - \frac{m}{2}$


① fermion #
F = Σ dx^i ∧ ∂/∂x^i
= form degree
Note. Q
② Charge conj
* : Ω^i ⟶ Ω^{m-i} , m = Dim(X)
Shift F = 1/2 { dx^i , ∂/∂x^i }
= Σ dx^i ∧ ∂/∂x^i − m/2
⟶ * realises the
− ℤ_2 charge conj.

Ex Hermitian σ-model

X : hermitian manifold

E : hol^c vector bundle

+ hermitian metric

$$\Omega^d = \Omega^{0,j}(X, E)$$

$$\langle f, g \rangle = \int_X \bar{f} \wedge * g$$

$Q = \bar{\partial}_E \; ; \quad Q^+ = \bar{\partial}_E^+$

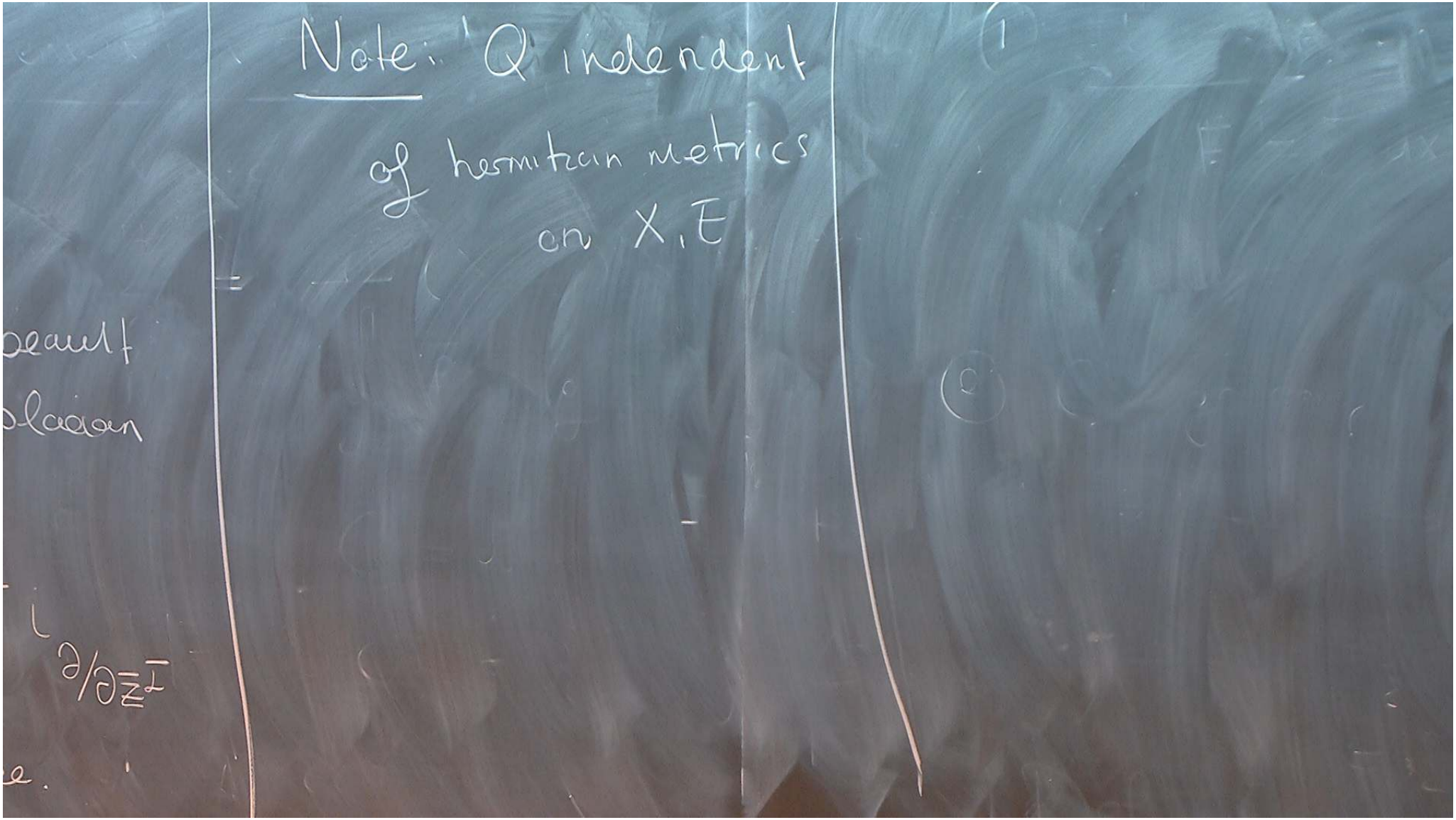
$H = \{\bar{\partial}_E \; \bar{\partial}_E^+\}$

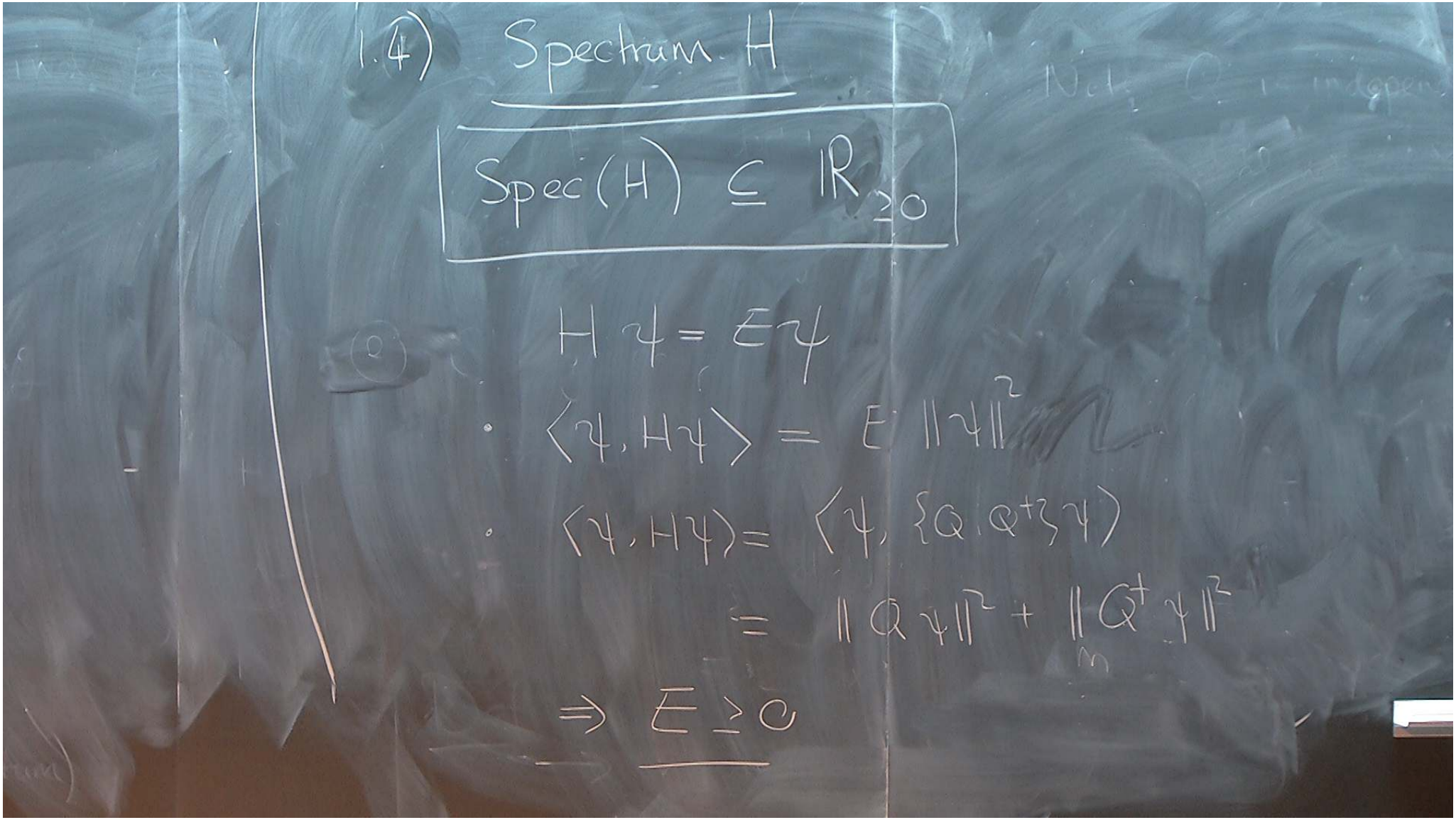
$= \frac{1}{2} \Delta_{\bar{\partial}_E}$ Dolbeault Laplacian

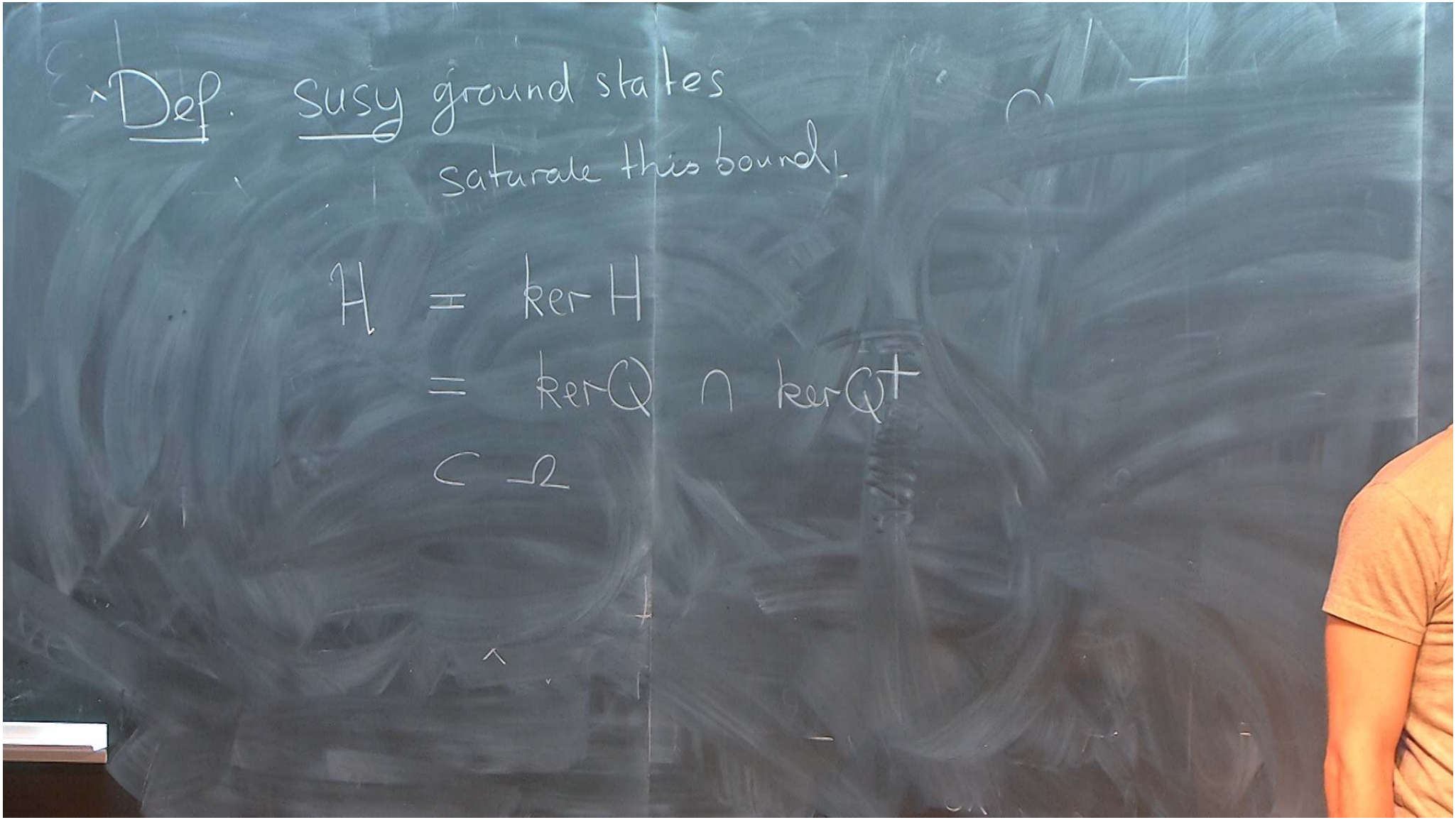
① fermion #

$F = \sum_i d\bar{z}^{\bar{i}} \, \iota_{\partial/\partial \bar{z}^{\bar{i}}}$

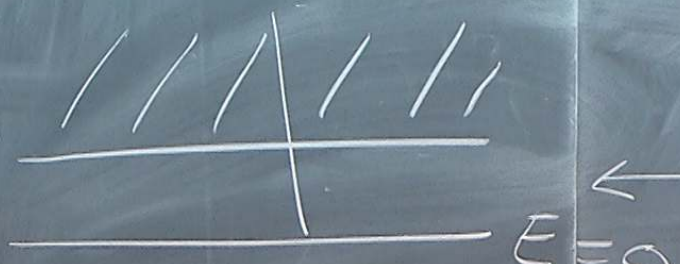
= form degree.


Note: 'Q' independent
of hermitian metrics
on X, E


1.4) Spectrum H
Spec(H) ⊆ ℝ≥0
H ψ = E ψ
⟨ψ, Hψ⟩ = E ‖ψ‖²
⟨ψ, Hψ⟩ = ⟨ψ, {Q, Q†} ψ⟩
= ‖Q ψ‖² + ‖Q† ψ‖²
⇒ E ≥ 0

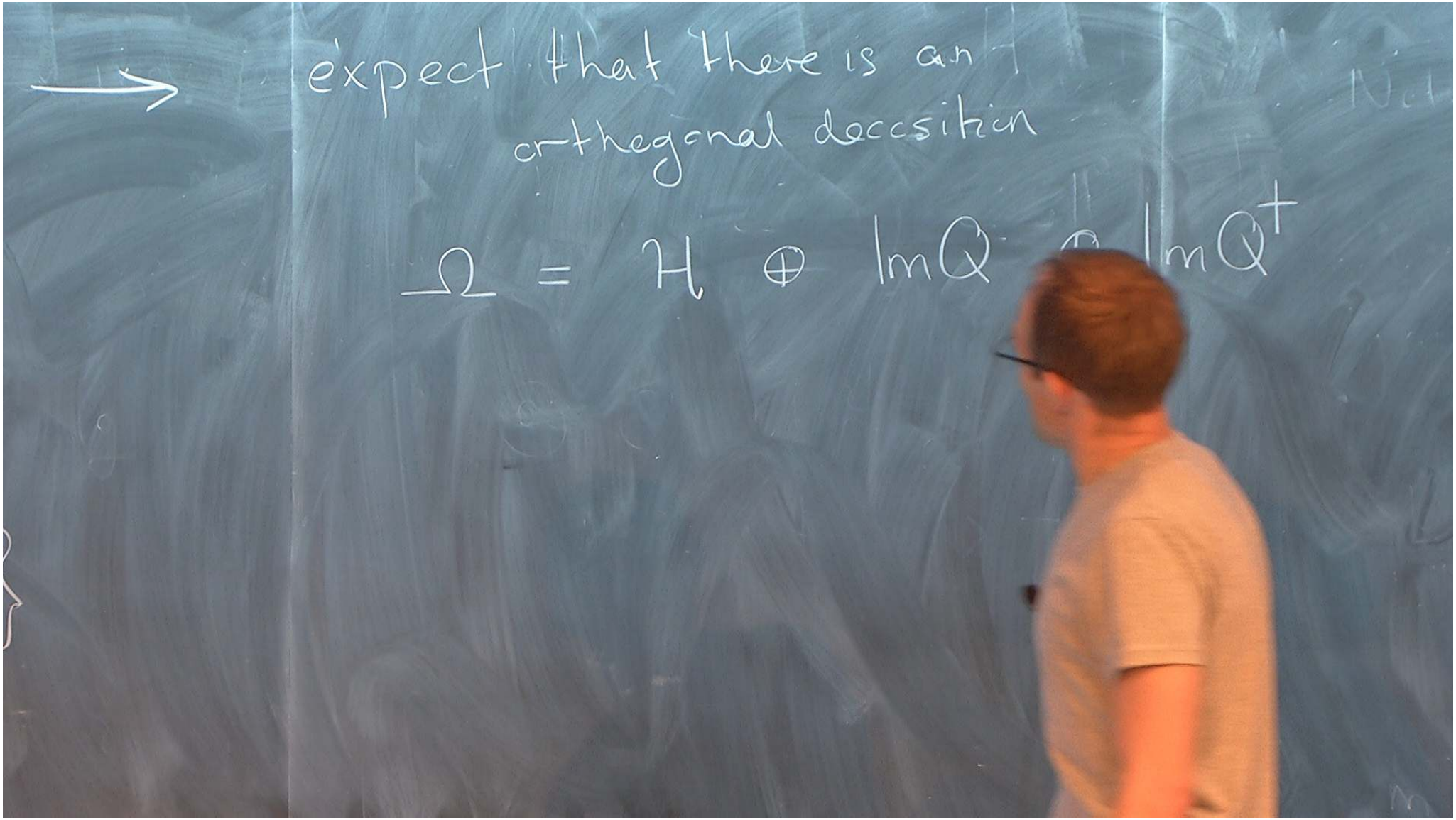

Def. susy ground states
saturate this bound!
$\mathcal{H} = \ker H$
$= \ker Q \cap \ker Q^\dagger$
$\subset \mathcal{Q}$

Typically assume spectrum is gapped



eg: discrete spectrum

$$\text{spec}H = \{0 = E_0 < E_1 < E_2 < \dots\}$$


expect that there is an
orthogonal decomposition
$\Omega = \mathcal{H} \oplus \mathrm{Im}Q \oplus \mathrm{Im}Q^{\dagger}$

→ expect that there is an orthogonal decomposition

$$\Omega = \mathcal{H} \oplus \mathrm{Im}\,Q \oplus \mathrm{Im}\,Q^\dagger$$

Two steps:

1) Hard $\Omega = \underset{\mathcal{H}}{\underset{\parallel}{\ker H}} \oplus \mathrm{Im}\,H$

2) Easy:

$$\psi \in \mathrm{Im}\,H, \quad \psi = H\omega$$
$$= \{Q, Q^\dagger\}\,\omega$$
$$= \underbrace{Q(Q^\dagger\omega)}_{\mathrm{Im}\,Q} + \underbrace{Q^\dagger(Q\omega)}_{\mathrm{Im}\,Q^\dagger}$$

Riemannian σ-model

$$\mathcal{H}^i = \ker \Delta|_{\Omega^i}$$

$$\cong \mathrm{Harm}^i(X, \mathbb{C})$$

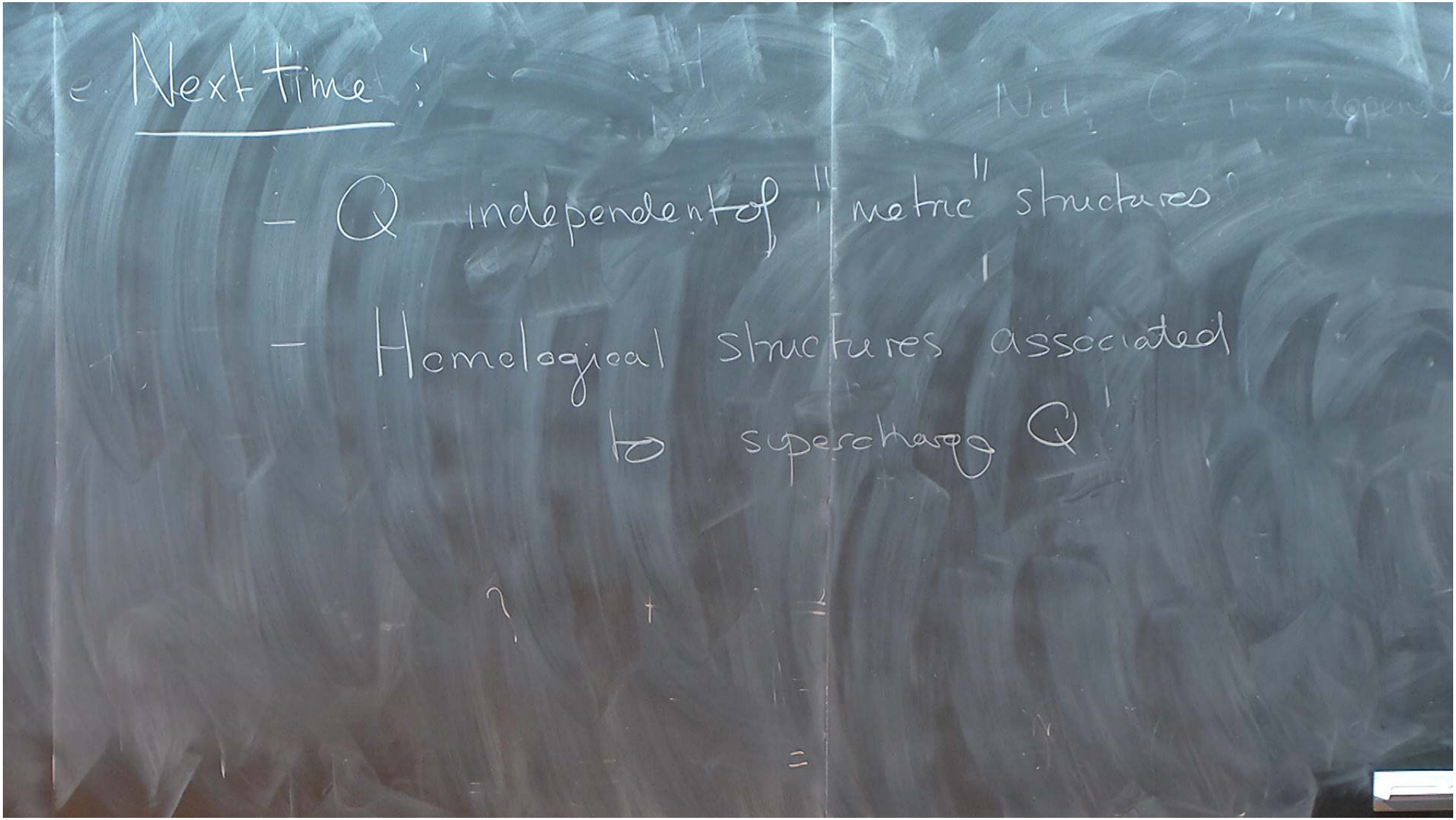
$$\Omega = \mathcal{H} \oplus \mathrm{Im}\, d \oplus \mathrm{Im}\, d^+$$

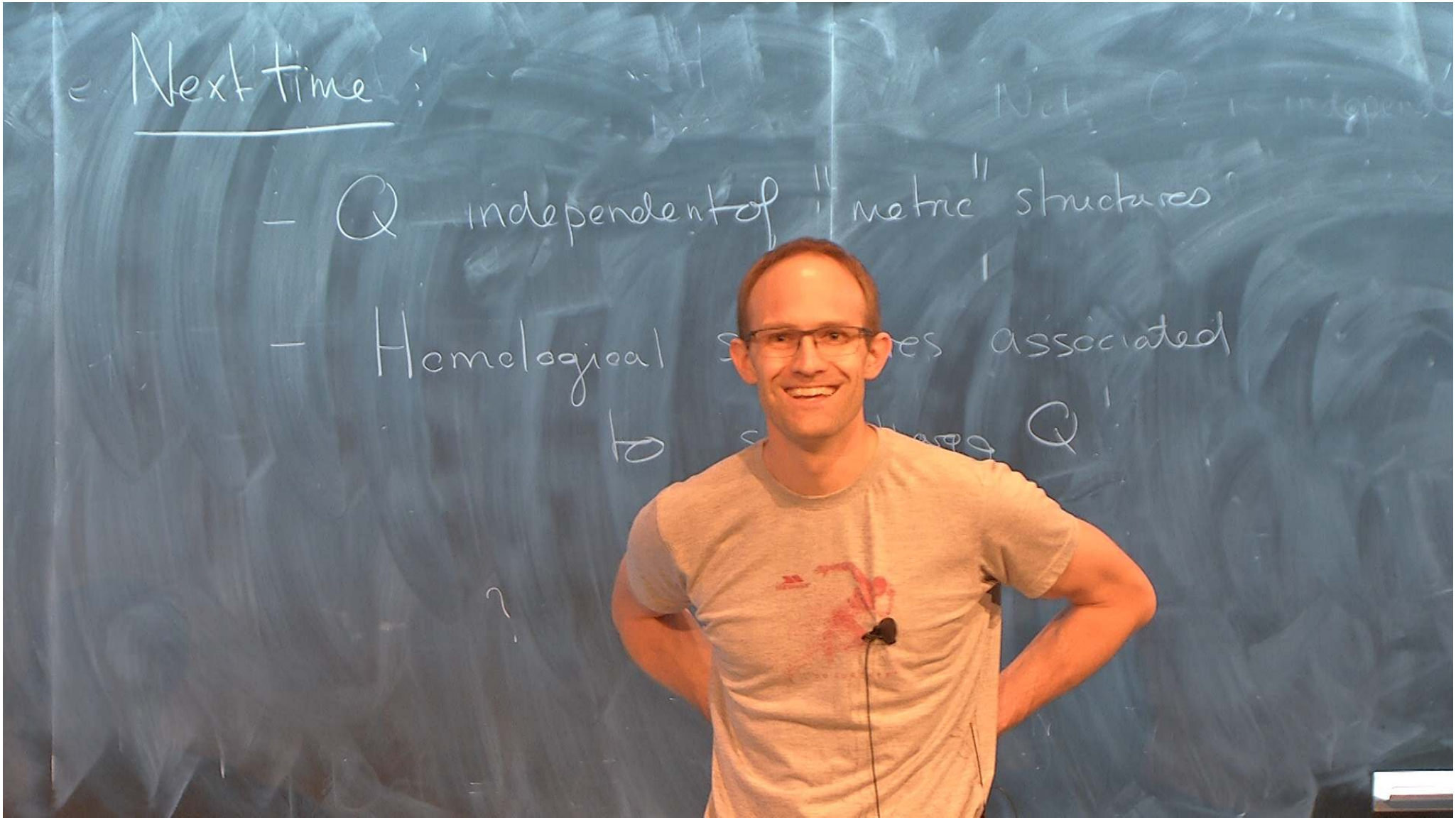
: Hodge decomposition

Ex Hermitian σ-model

$$\mathcal{H}^i = \mathrm{Harm}^{0,i}_{\Delta_{\bar\partial_E}}(X, \mathbb{C})$$

$$\Omega^i = \mathcal{H}^i \oplus \mathrm{Im}\, \bar\partial_E \oplus \mathrm{Im}\, \bar\partial_E^+$$


Next time:
- Q independent of "metric" structures
- Homological structures associated to supercharge Q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