Title: Lecture 1: Supersymmetric Quantum Mechanics and All That

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Collection: QFT for Mathematicians

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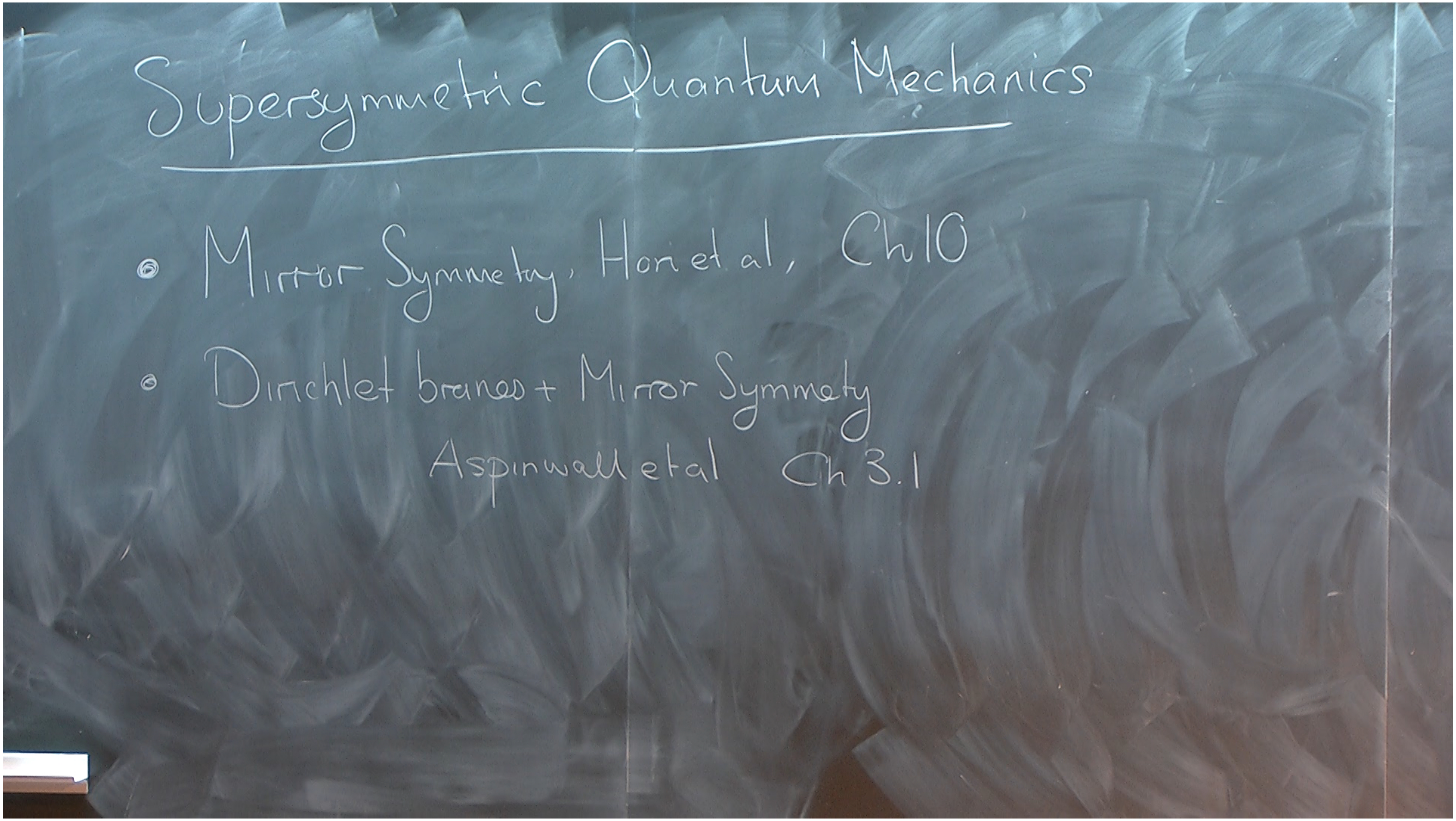
Supersymmetric Quantum Mechanics
Mirror Symmetry, Hori et al, Ch 10
Dirichlet branes + Mirror Symmetry
Aspinwall et al Ch 3.1

Supersymmetric Quantum Mechanics
• Mirror Symmetry, Hori et al, Ch10
• Dirichlet branes + Mirror Symmetry
Aspinwall et al Ch 31
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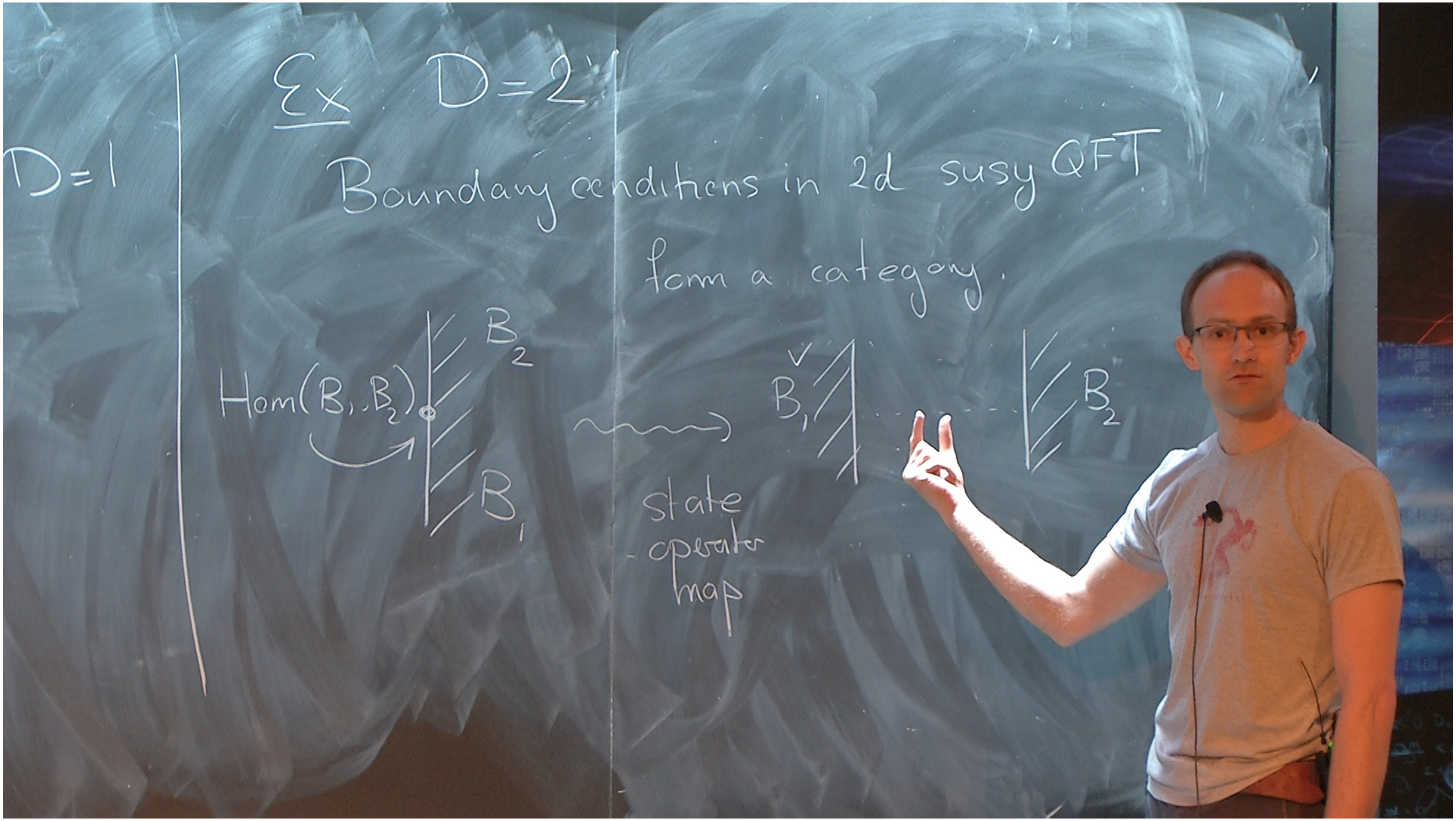
1.1) Motivation

SQM = Susy QFT in $D=1$

$\tau \uparrow \qquad M = \mathbb{R}$

Useful for QFT in $\underline{d > 1}$

$$\tau \uparrow \times M_{D-1} \rightsquigarrow \uparrow \tau$$


Ex D=2
D=1
Boundary conditions in 2d susy QFT
form a category.
Hom(B1, B2)
B2
B1
state
operator
map
B1
B2

12) Quantum Mechanics

States: elements (rays) in a complex Hilbert space Ω

$\langle \cdot , \cdot \rangle : \Omega \times \Omega \longrightarrow \mathbb{C}$

Operators: Linear operators $A : \Omega \longrightarrow \Omega$

generate associative algebra $\mathcal{A}$

Measurements:

- measurable quantities are self-adjoint ops A
- Possible outcomes of a measurement are $\mathrm{spec}(A) \subseteq \mathbb{R}$

Time evolution

- euclidean time τ $\uparrow\tau$
- distinguished self-adjoint operator H

• Schrodinger: states evolve

$$\partial_\tau \psi = - H \cdot \psi, \quad \psi \in \Omega$$

• Heisenberg: operators

$$\partial_\tau A = [H, A]$$

$\subseteq \mathbb{R}$

Time evolution

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$$\partial_\tau A = [H, A]$$

Def A is conserved if

$$[H, A] = 0$$

Ex: Particle on S^1

- $\Omega = L^2(S^1, \mathbb{C})$

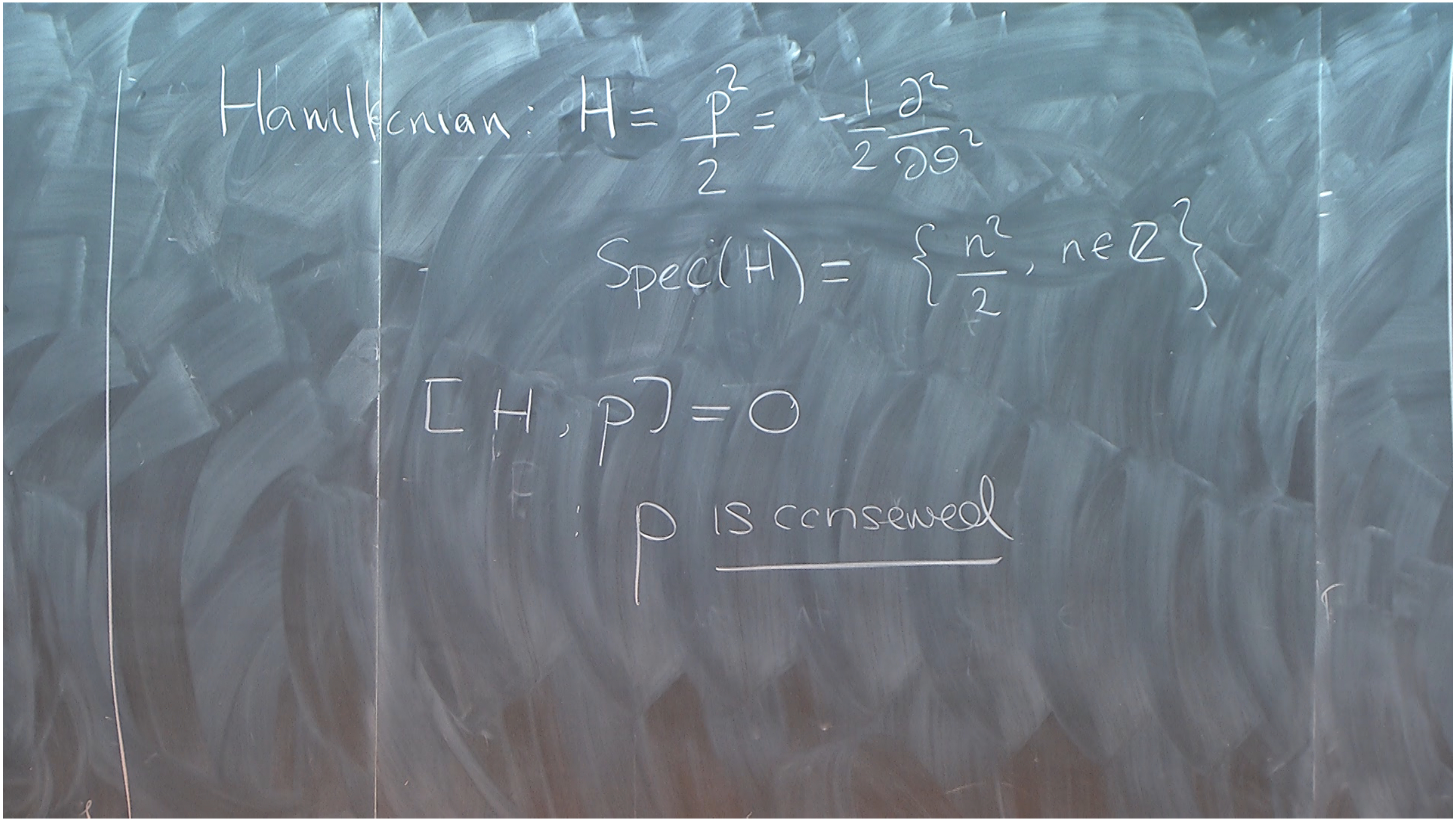
$$\langle f, g \rangle = \int_0^{2\pi} \overline{f(\vartheta)}\, g(\vartheta)\, d\vartheta$$

- momentum operator $\quad p = -i \dfrac{\partial}{\partial \vartheta}$

$$\mathrm{Spec}(p) = \mathbb{Z}$$

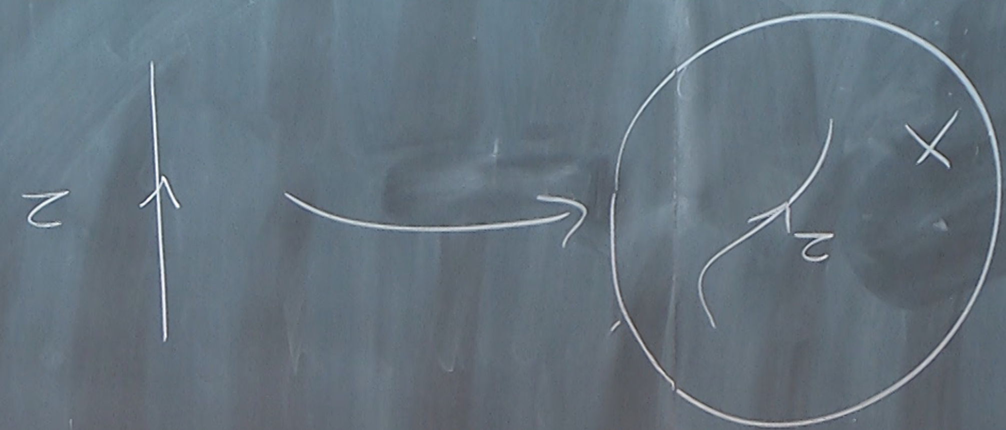
$$\phi_n(\vartheta) = \frac{1}{\sqrt{2\pi}} e^{in\vartheta}, \quad n \in \mathbb{Z}$$

Measure


Hamiltonian: $H = \frac{p^2}{2} = -\frac{1}{2}\frac{\partial^2}{\partial\theta^2}$
$\mathrm{Spec}(H) = \left\{ \frac{n^2}{2},\ n \in \mathbb{Z} \right\}$
$[H, p] = 0$
$\therefore$ p is conserved

Ex: Riemannian σ-model

X = cpt, smooth Riemannian manifold



$$\Omega = L^2(X, \mathbb{C})$$

$$\langle f, g \rangle = \int \bar{f} g \, dVol_X$$

$\mathcal{A}$: differential ops on X

- Smooth function $f : X \to \mathbb{C}$

$$A_f : \psi \longmapsto f\psi$$

commutative subalgebra $\subset \mathcal{A}$

- Vector field V

$$A_V : \psi \longmapsto -iV[\psi] = -iV^i \frac{\partial \psi}{\partial x^i}$$

- Hamiltonian: $H = \Delta$

  ↳ Laplacian

$$\partial_\tau \psi = -\Delta \psi$$

heat eq. on X.

- If V real vecter fied that generate isometry of X

$$[H, A_v] = 0$$

∴ conserved (momentum)

Ex: Ri

X

z

$\Omega =$

$\langle f,$

lacian

at eq.

on X.

at

X

um)

13) Susy QM

Additional structure:

- Hilbert space $\mathbb{Z}_2$-graded

$$\Omega = \Omega_e \oplus \Omega_o$$

- $\exists$ odd operators: $Q, Q^\dagger$

$$\{Q, Q\} = 0$$

$$\{Q, Q^\dagger\} = H$$

$$\{Q^\dagger, Q^\dagger\} = 0$$

Consequence: $[H, Q] = [H, Q^\dagger] = 0$

: $Q, Q^\dagger$ are conserved

"supercharges"

$$\boxed{\begin{aligned} [A, B] &= AB - BA \\ \{A, B\} &= AB + BA \end{aligned}}$$

Outer automorphisms: $O(2) = U(1) \otimes \mathbb{Z}_2$

fermion # / R-charge

	Q	Q^{+}	H
weights	$+1$	-1	0

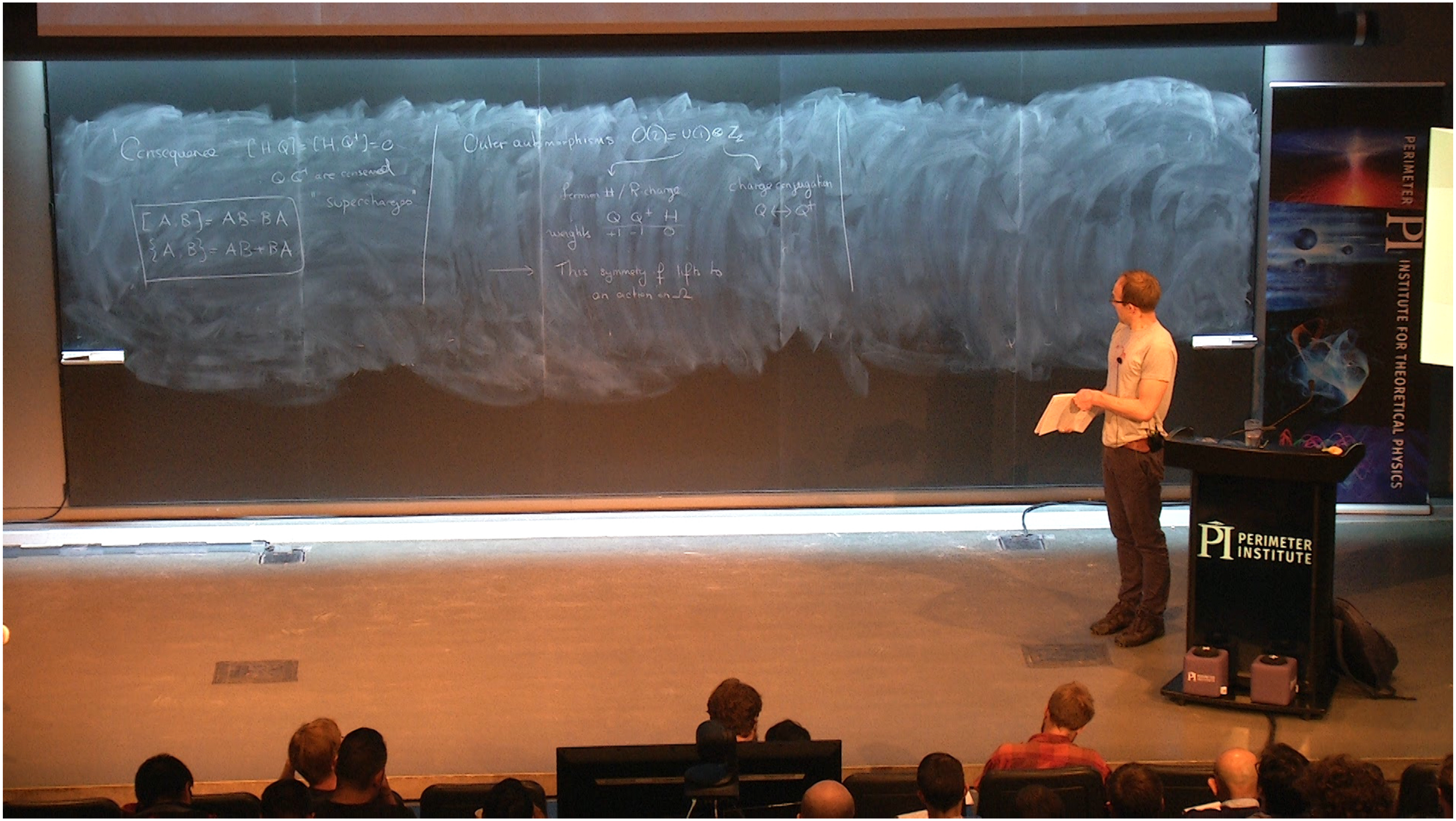
Outer automorphisms: $O(2) = U(1) \otimes \mathbb{Z}_2$

fermion # / R-charge

	Q	$Q^\dagger$	H
weights	$+1$	-1	0

charge conjugation

$Q \longleftrightarrow Q^\dagger$


Consequence [H,Q]=[H,Q†]=0
Q, Q† are conserved
"supercharges"
[A,B]= AB-BA
{A,B}= AB+BA
Outer automorphisms O(2)=U(1)⊗Z2
fermion # / R-charge
charge conjugation
Q ↔ Q†
Q Q† H
weights +1 -1 0
This symmetry lifts to
an action on Ω
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Outer automorphisms: $O(2) = U(1) \otimes \mathbb{Z}_2$

(1) fermion # / R-charge

	Q	Q^+	H
weights	$+1$	-1	0

(2) charge conjugation

$$Q \longleftrightarrow Q^\dagger$$

$\longrightarrow$ This symmetry if lifts to an action on Ω

1. Self-adjoint op F

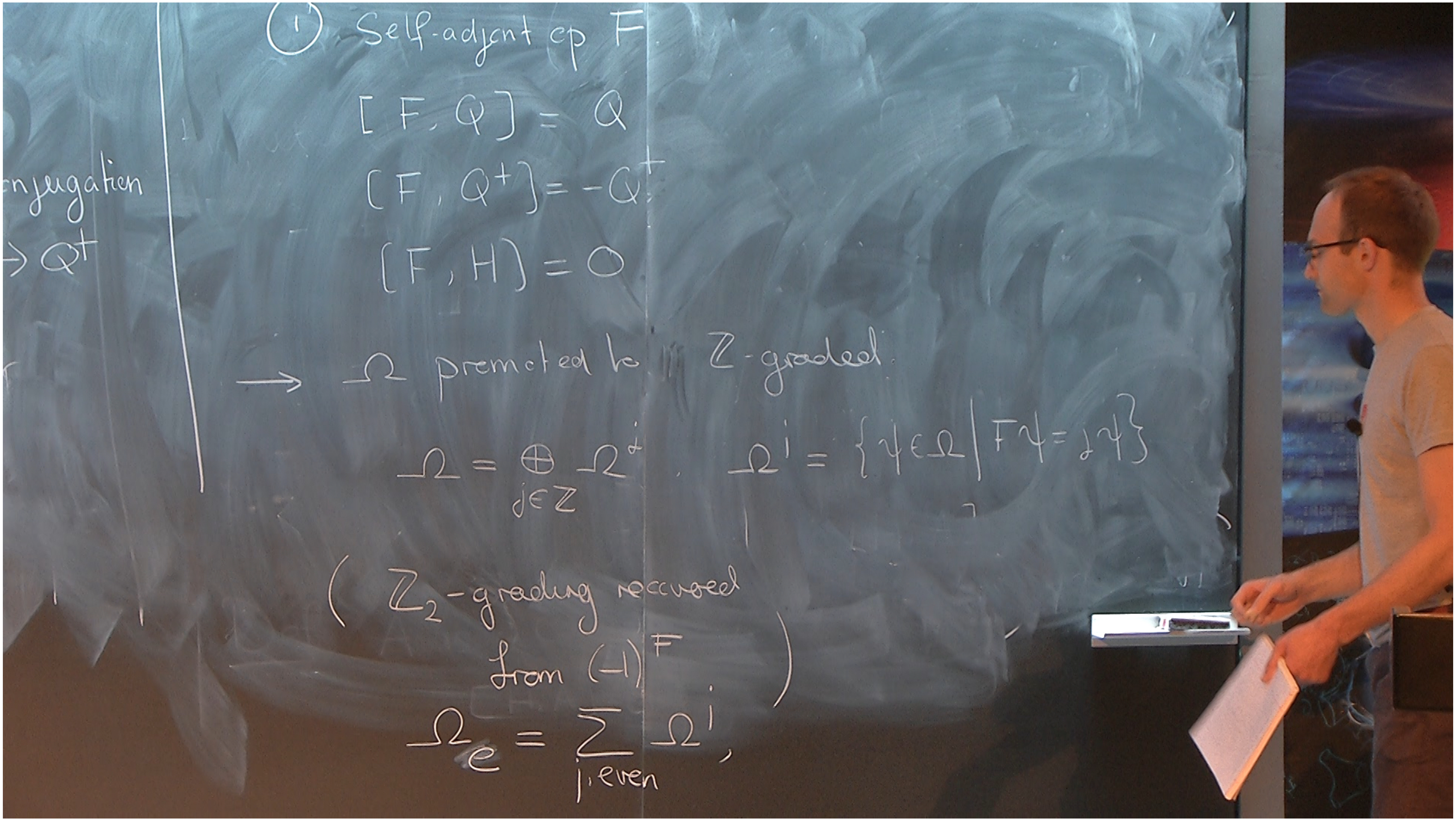
$$[F, Q] = Q$$

$$[F, Q^{\dagger}] = -Q^{\dagger}$$

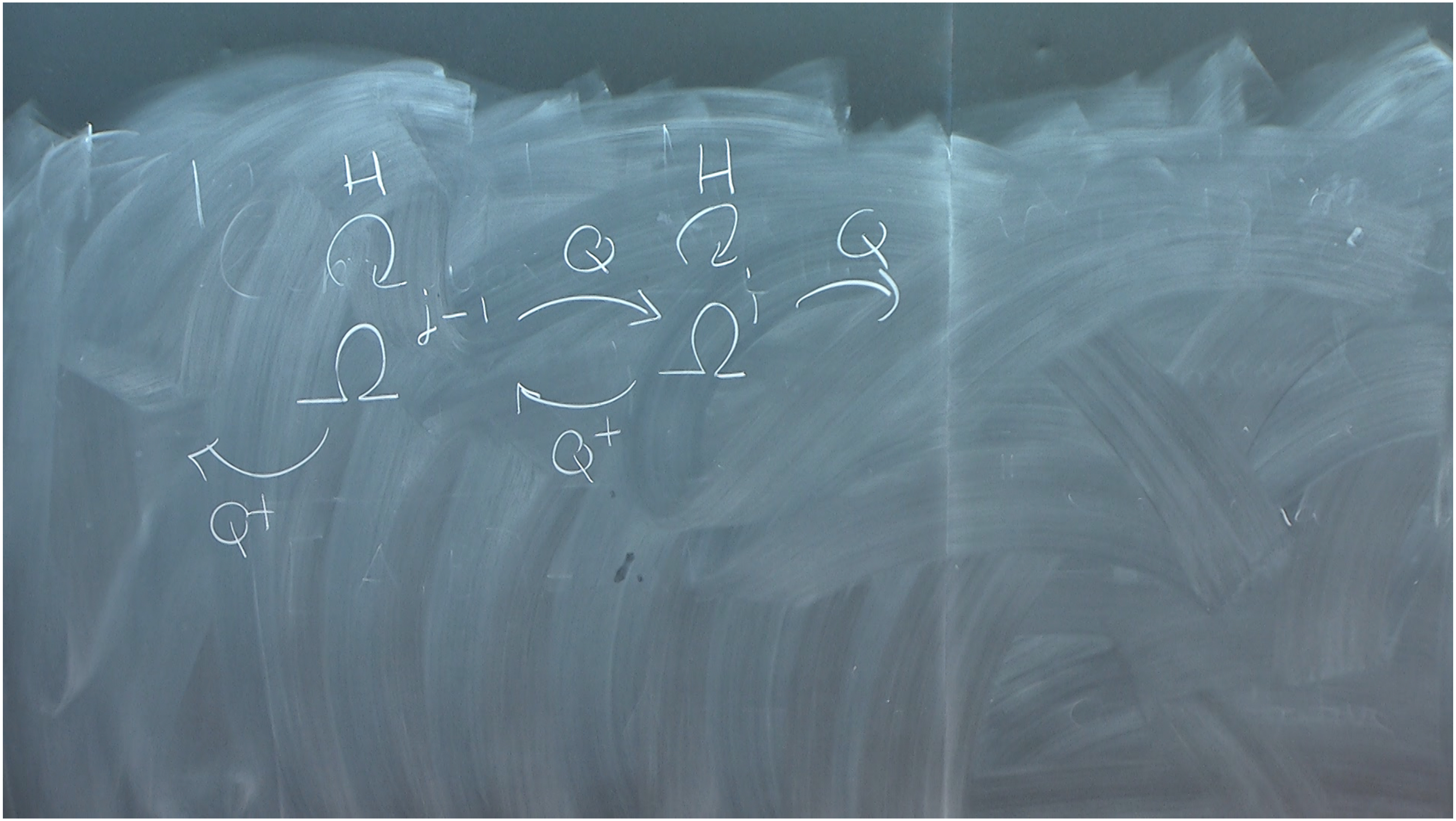
$$[F, H] = 0$$

$\rightarrow$ Ω promoted to $\mathbb{Z}$-graded.

$$\Omega = \bigoplus_{j \in \mathbb{Z}} \Omega^{j}, \quad \Omega^{j} = \{ \psi \in \Omega \mid F\psi = j\psi \}$$



njugation
$\to Q^\dagger$
① Self-adjoint op F
$[F, Q] = Q$
$[F, Q^\dagger] = -Q^\dagger$
$[F, H] = 0$
$\rightarrow$ Ω promoted to $\mathbb{Z}$-graded
$\Omega = \bigoplus_{j \in \mathbb{Z}} \Omega^j$, $\Omega^j = \{\psi \in \Omega \mid F\psi = j\psi\}$
($\mathbb{Z}_2$-grading recovered from $(-1)^F$)
$\Omega_e = \sum_{j\text{ even}} \Omega^j$,


H
Ω
Ω
j−1
Q
H
Ω
Ω
j
Q
Q+
Q+

$$
\begin{tikzcd}
{} & \Omega^{j-1} \arrow[l, bend left, "Q^+"] \arrow[loop above, "H"] \arrow[r, bend left, "Q"] & \Omega^{j} \arrow[l, bend left, "Q^+"] \arrow[loop above, "H"] \arrow[r, bend left, "Q"] & {}
\end{tikzcd}
$$

② unitary operator $C \quad \Omega^{j} \longrightarrow \Omega^{-j}$

$$CQC^{-1} = Q^{+}$$

$$CQ^{+}C^{-1} = Q$$

$\therefore$ Ω premeted to repr of $O(2)$.

Ex: Riemannian σ-model

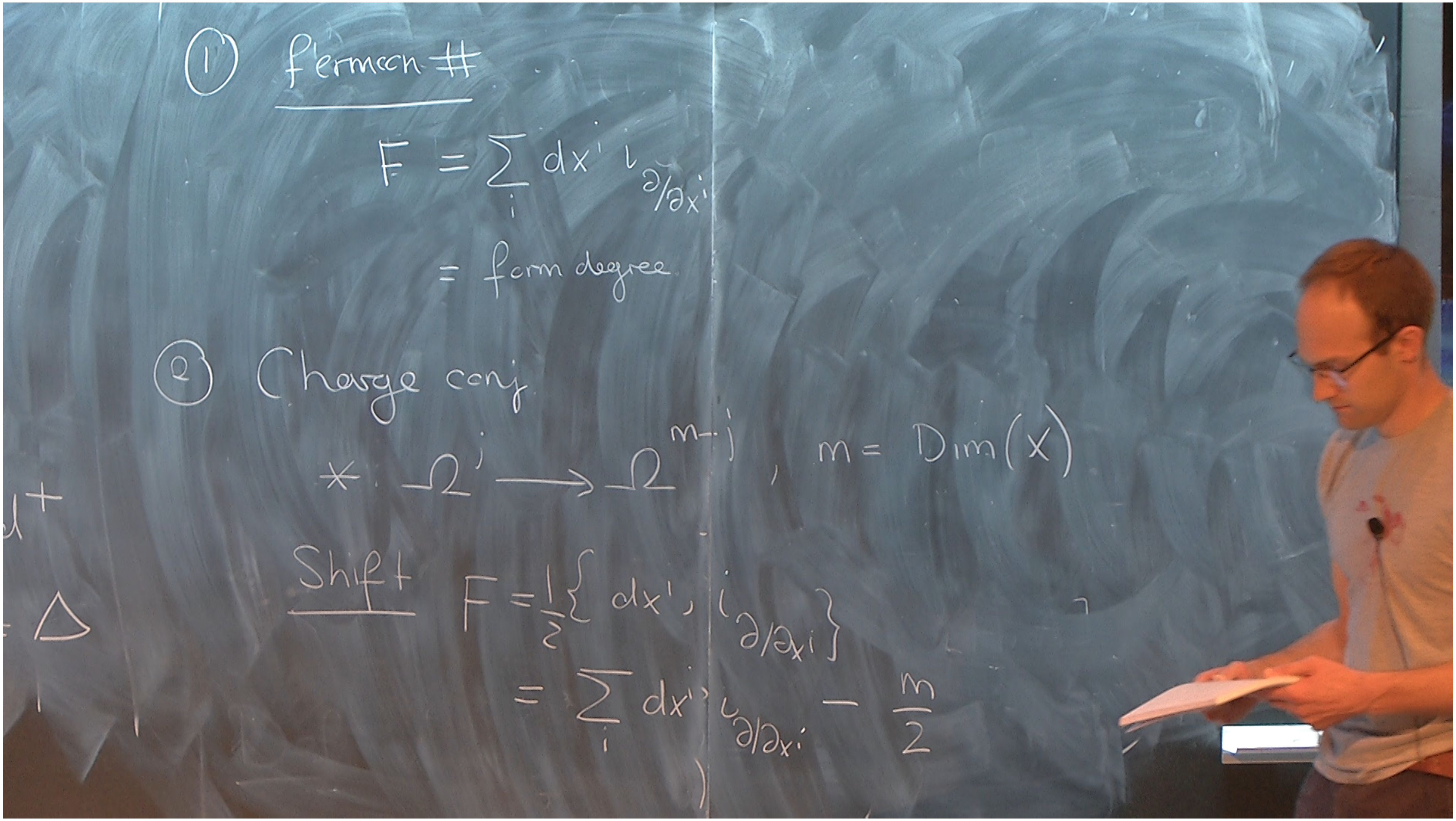
X : cpt smth Riemannian

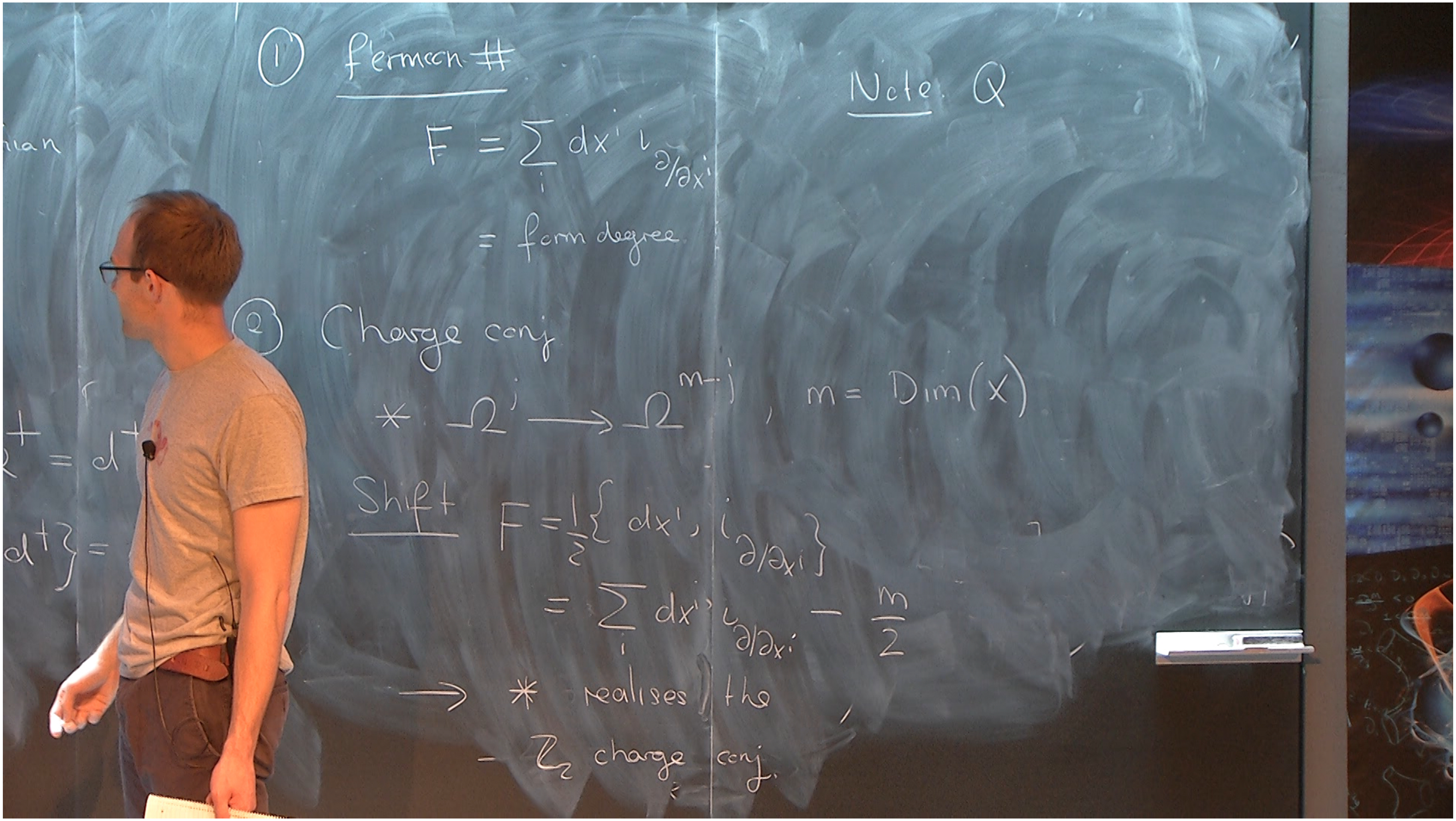
$$\Omega^j = \Omega^j(X, \mathbb{C})$$

$$\langle f, g \rangle = \int_X \bar{f} \wedge * g$$

Supercharges: $Q = d$, $Q^\dagger = d^\dagger$

$$H = \{Q, Q^\dagger\} = \{d, d^\dagger\} = \Delta$$


① Fermion #
$F = \sum_i dx^i \iota_{\partial/\partial x^i}$
= form degree
② Charge conj.
$* : \Omega^i \longrightarrow \Omega^{m-i}$, $m = \mathrm{Dim}(X)$
Shift
$F = \frac{1}{2}\{dx^i, \iota_{\partial/\partial x^i}\}$
$= \sum_i dx^i \iota_{\partial/\partial x^i} - \frac{m}{2}$



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$= \sum_i dx^i \iota_{\partial/\partial x^i} - \frac{m}{2}$
$\longrightarrow$ * realises the
$\mathbb{Z}_2$ charge conj.
Note: Q

Ex | Hermitian σ-model

X : hermitian manifold

E : hol^c vector bundle

+ hermitian metric

$$\Omega^j = \Omega^{0,j}(X,E)$$

$$\langle f, g \rangle = \int_X \bar{f} \wedge * g$$

d, cpt

le

trie

$$Q = \bar{\partial}_E \, ; \quad Q^\dagger = \bar{\partial}_E^\dagger$$

$$H = \{ \bar{\partial}_E , \bar{\partial}_E^\dagger \}$$

$$= \frac{1}{2} \Delta_{\bar{\partial}_E}$$

Dolbeault Laplacian

(1) <u>fermion #</u>

$$F = \sum_{\bar{i}} d\bar{z}^{\bar{i}} \, \iota_{\partial/\partial \bar{z}^{\bar{i}}}$$

$$= \text{form degree.}$$

Note: Q independent

of hermitian metrics

on X, E

...eault

...laaon

$\partial/\partial \bar{z}^{\bar{i}}$

...e.

1.4) Spectrum H

$$\boxed{\mathrm{Spec}(H) \subseteq \mathbb{R}_{\geq 0}}$$

$$H\psi = E\psi$$

- $\langle \psi, H\psi \rangle = E \, \|\psi\|^2$
- $\langle \psi, H\psi \rangle = \langle \psi, \{Q, Q^\dagger\}\psi \rangle$

$$= \|Q\psi\|^2 + \|Q^\dagger \psi\|^2$$

$$\Rightarrow \underline{E \geq 0}$$

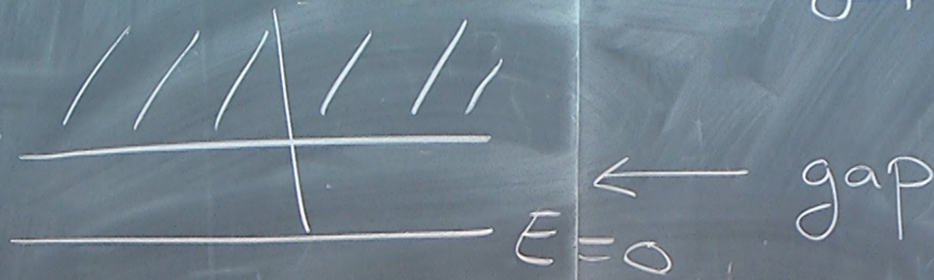
Def. susy ground states saturate this bound.

$$\mathcal{H} = \ker H$$

$$= \ker Q \cap \ker Q^\dagger$$

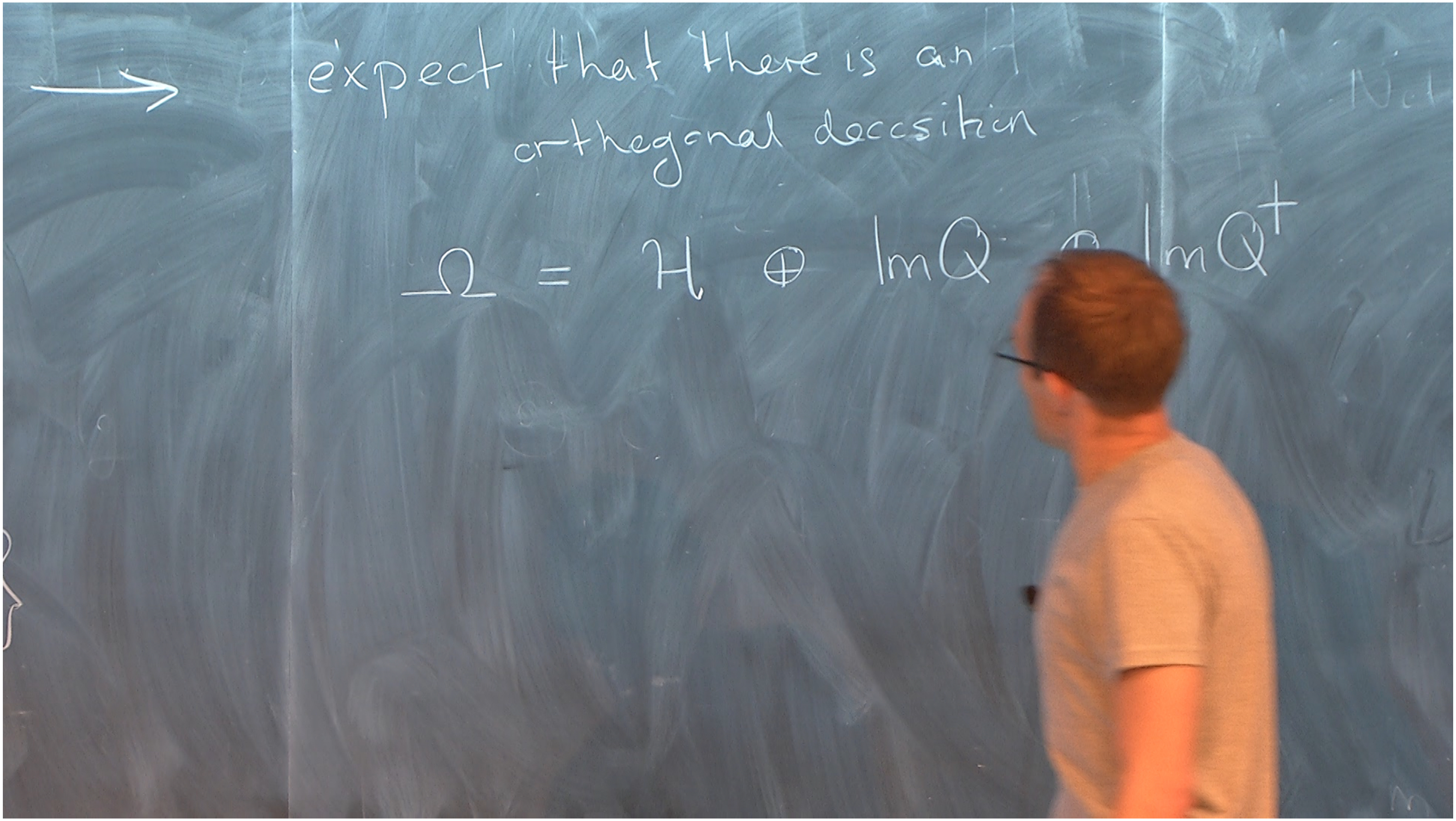
$$\subset \Omega$$

Typically assume spectrum is gapped



eg. discrete spectrum

$$\mathrm{spec}\, H = \{ 0 = E_0 < E_1 < E_2 < \dots \}$$


expect that there is an
orthogonal decomposition
$\Omega = \mathcal{H} \oplus \mathrm{Im}Q \oplus \mathrm{Im}Q^\dagger$

→ expect that there is an orthogonal decosition

$$\Omega = \mathcal{H} \oplus \mathrm{Im}\,Q \oplus \mathrm{Im}\,Q^\dagger$$

Two steps: 1) Hard: $\Omega = \ker H \oplus \mathrm{Im}\,H$, $\quad \ker H = \mathcal{H}$

2) Easy:

$$\psi \in \mathrm{Im}\,H, \quad \psi = H\omega = \{Q, Q^\dagger\}\omega = \underbrace{Q(Q^\dagger\omega)}_{\mathrm{Im}\,Q} + \underbrace{Q^\dagger(Q\omega)}_{\mathrm{Im}\,Q^\dagger}$$

Riemannian σ-model

$$\mathcal{H}^i = \ker \Delta\big|_{\Omega^i}$$

$$\cong \mathrm{Harm}^i(X,\mathbb{C})$$

$$\Omega = \mathcal{H} \oplus \mathrm{Im}\, d \oplus \mathrm{Im}\, d^\dagger$$

Hodge decomposition

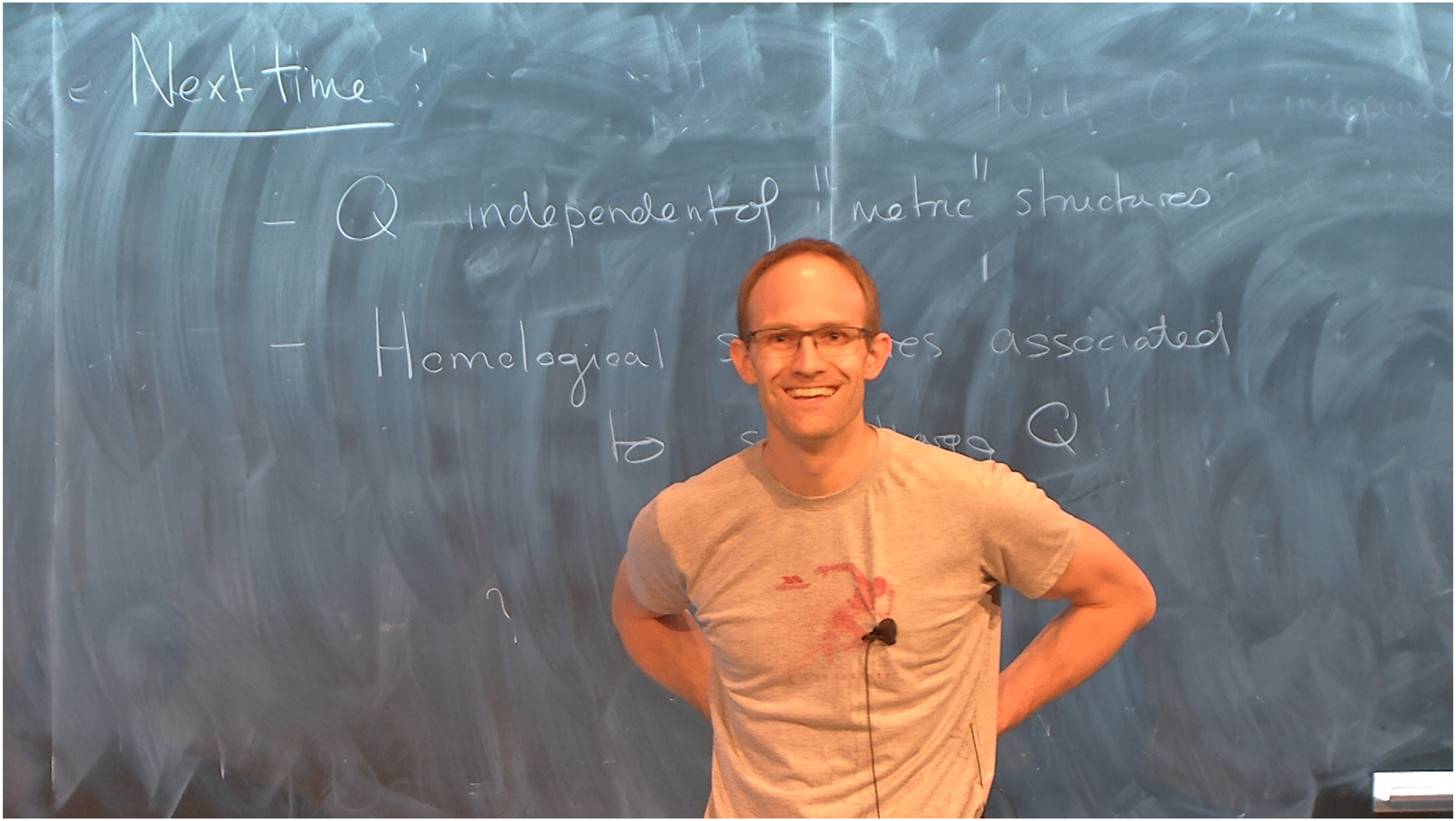
Ex Hermitian σ-model

$$\mathcal{H}^i = \mathrm{Harm}^{0,i}_{\Delta_{\bar\partial_E}}(X,\mathbb{C})$$

$$\Omega^i = \mathcal{H}^i \oplus \mathrm{Im}\,\bar\partial_E \oplus \mathrm{Im}\,\bar\partial_E^\dagger$$

Next time:

- Q independent of "metric" structures
- Homological structures associated to supercharge Q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