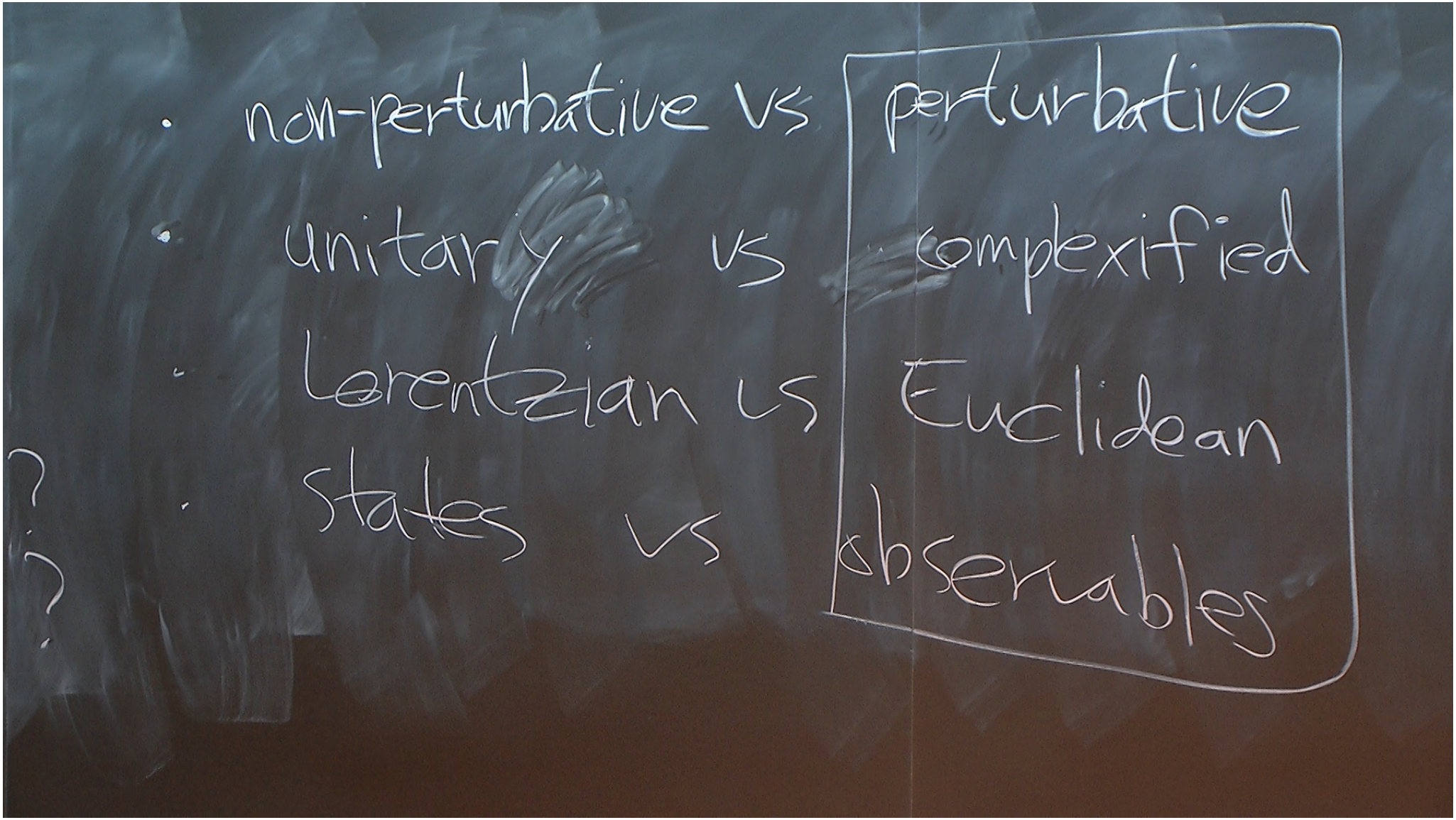
Title: Lecture 1:  Factorization Algebras and the General Structure of QFT

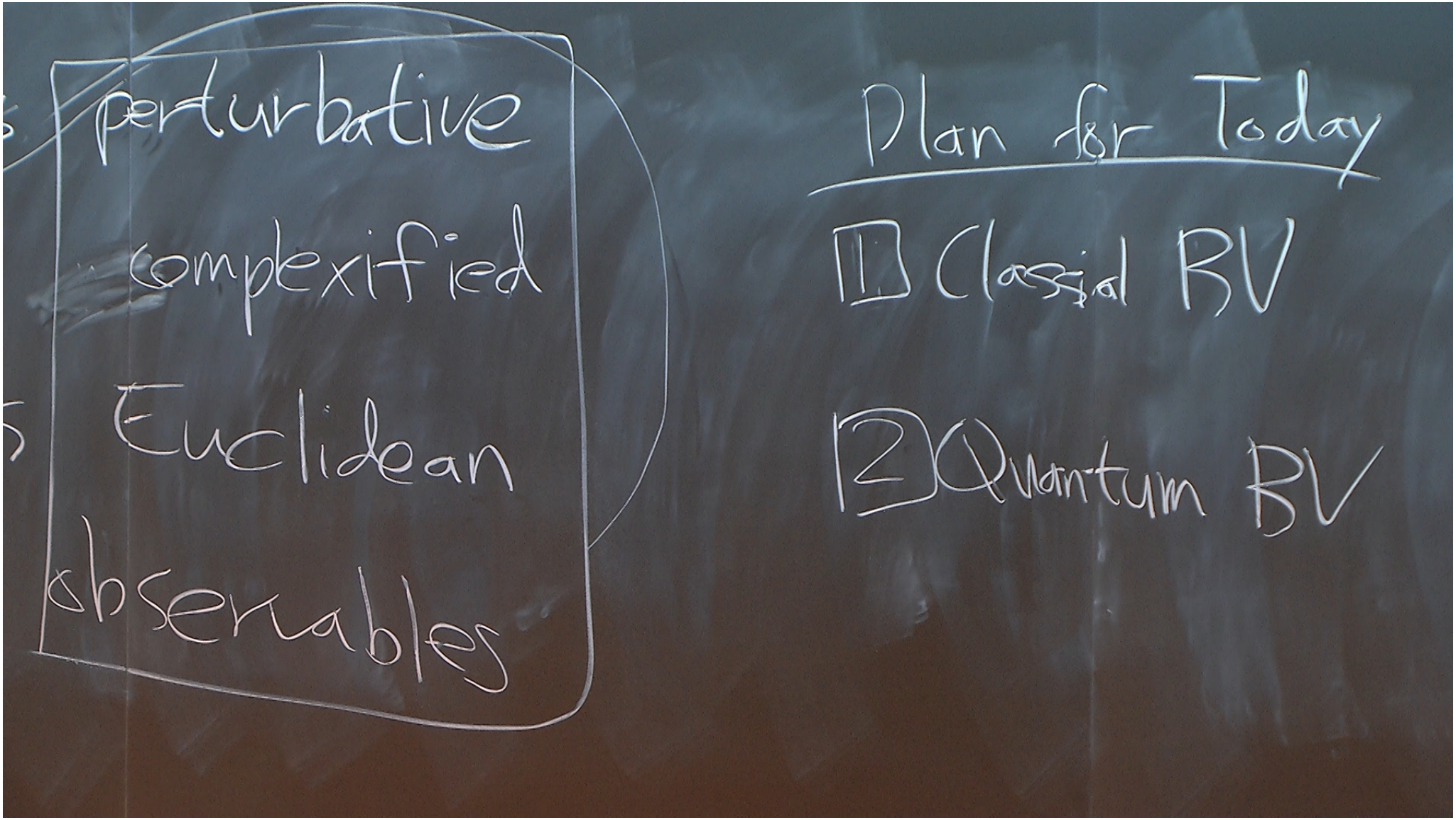
Speakers: Philsang Yoo

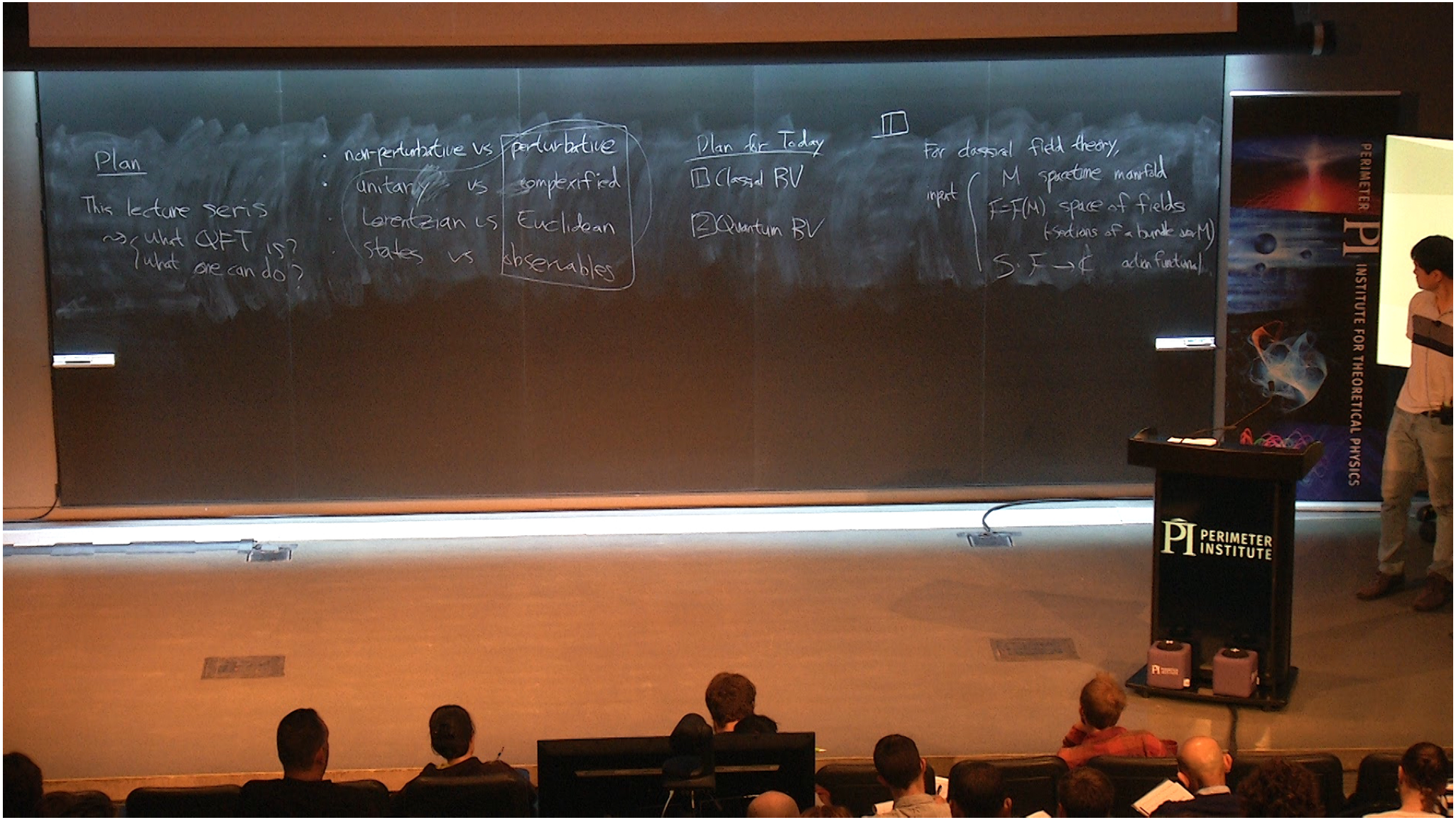
Collection: QFT for Mathematicians

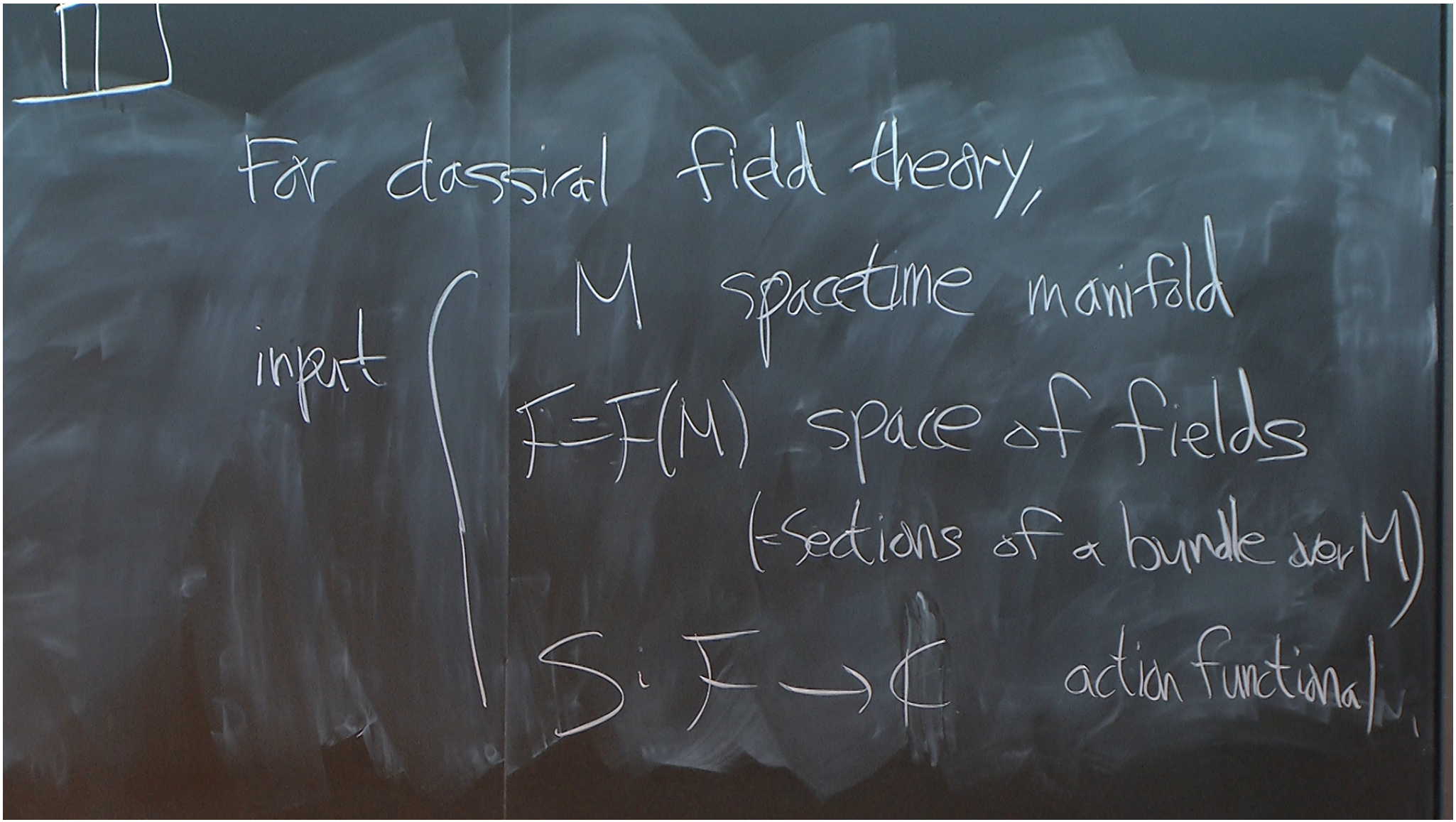
Date: June 17, 2019 - 9:00 AM

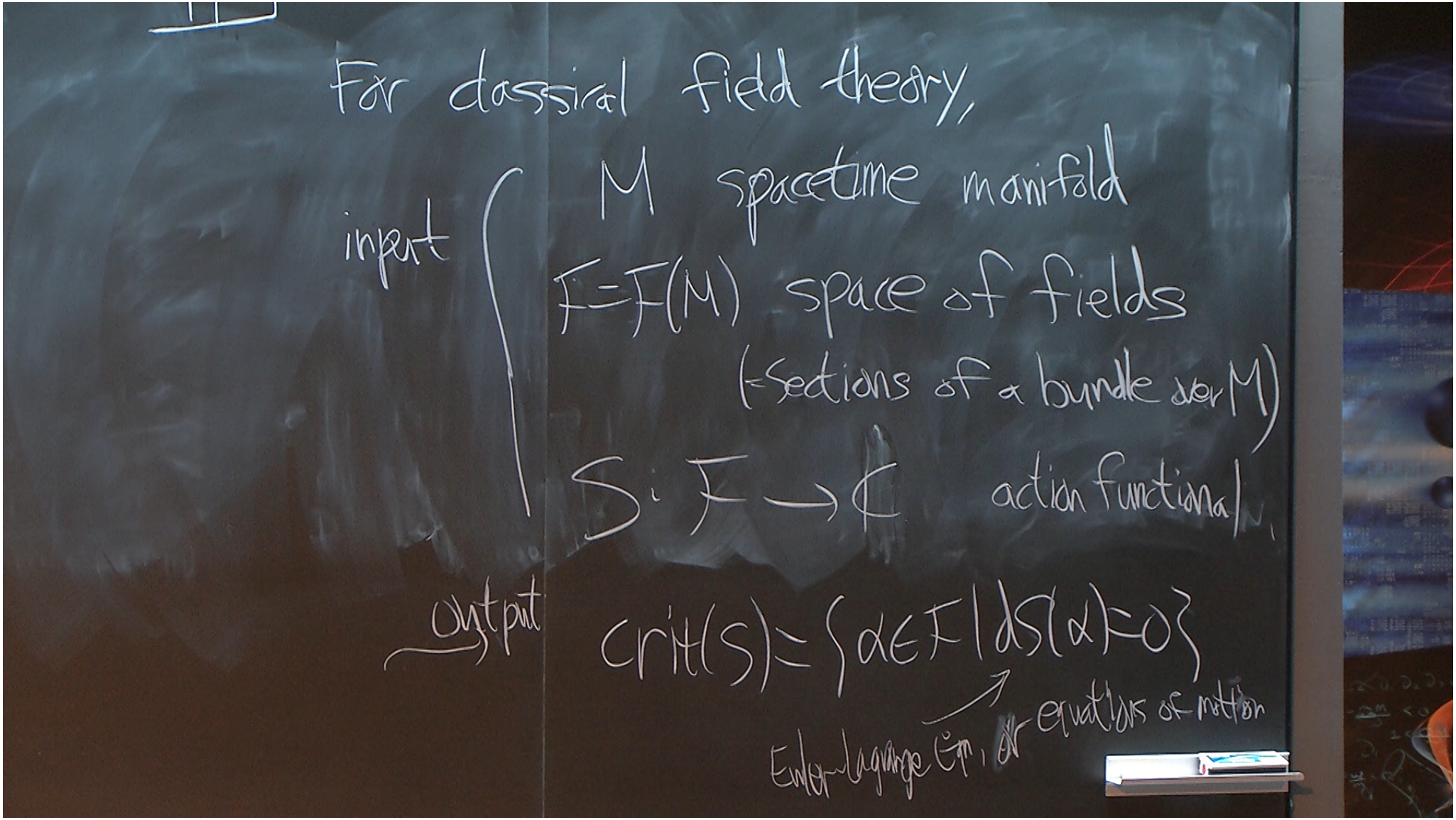
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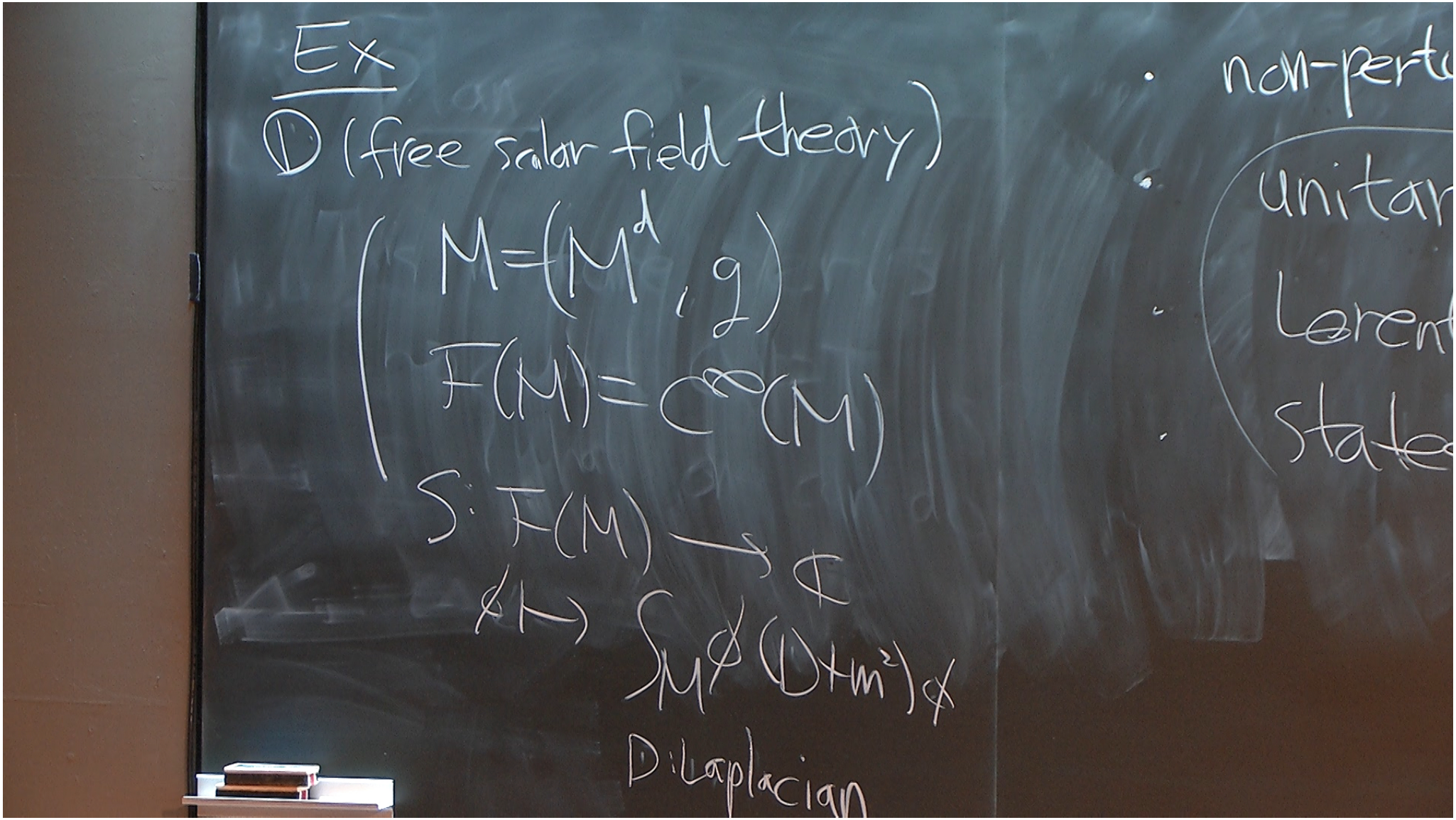

non-perturbative vs perturbative
unitarity vs complexified
Lorentzian vs Euclidean
states vs observables

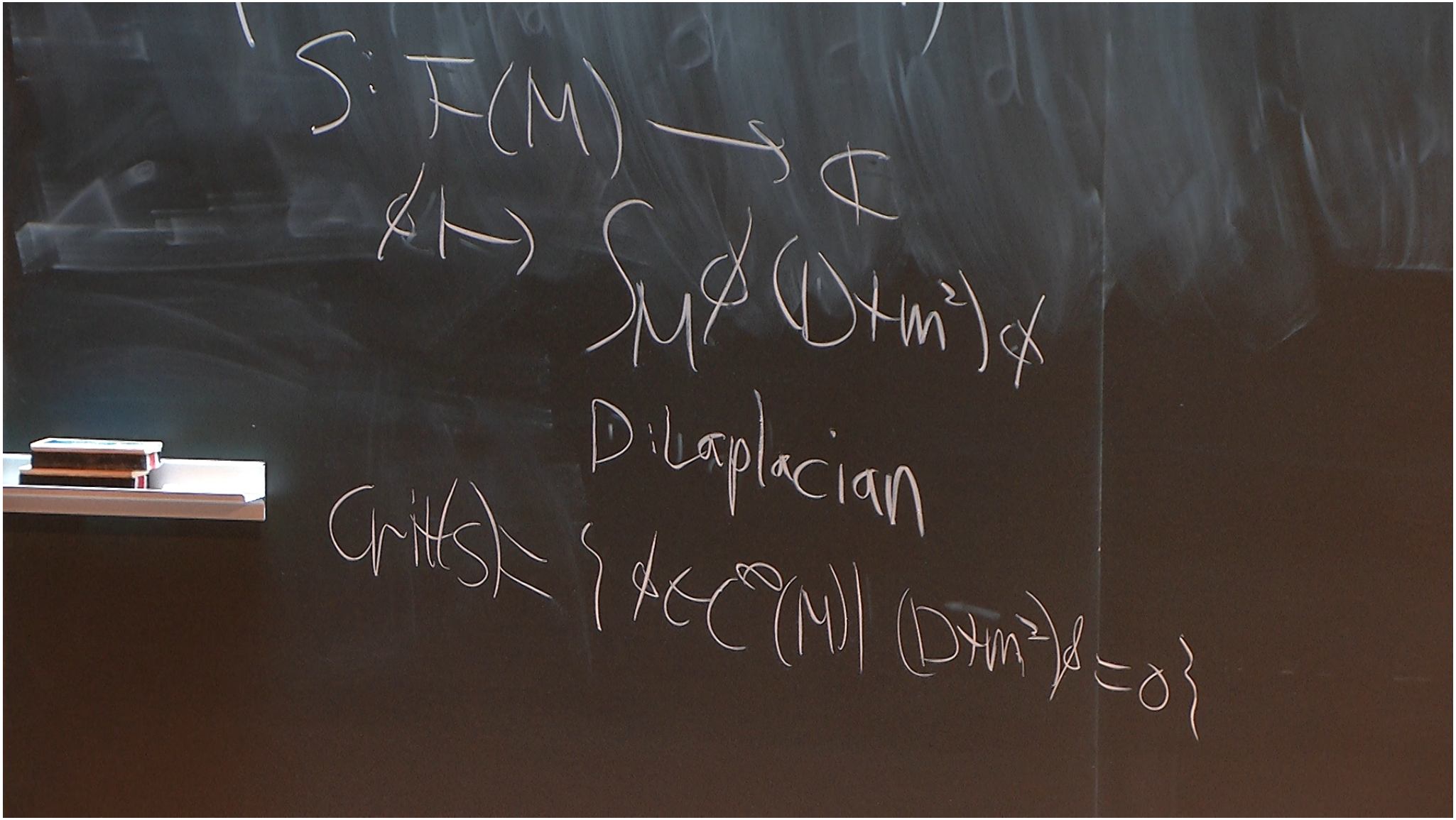

perturbative
complexified
Euclidean
observables
Plan for Today
1 Classical BV
2 Quantum BV

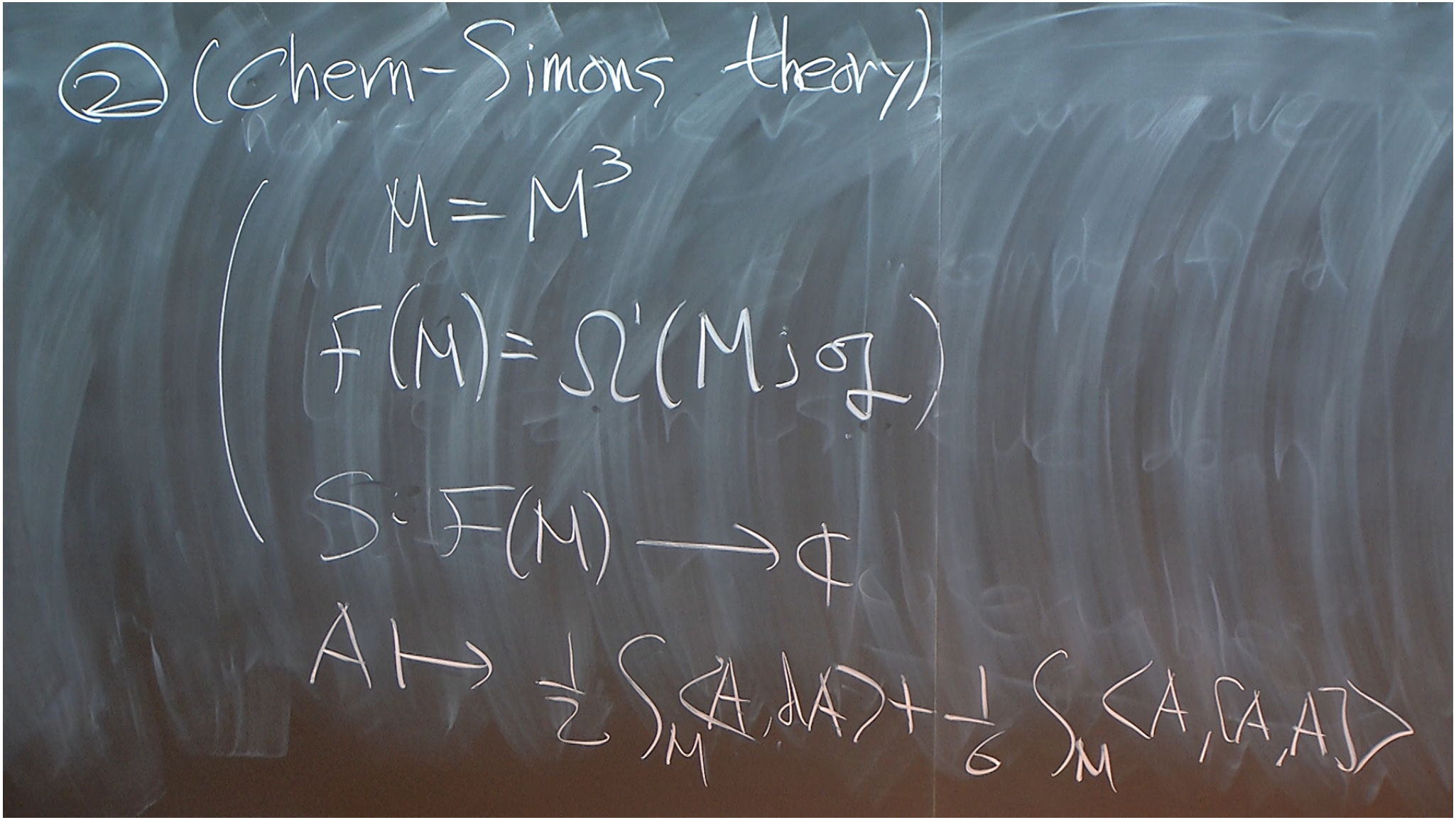

Plan
This lecture series
→ what QFT is?
what one can do?
· non-perturbative vs perturbative
· unitary vs complexified
· Lorentzian vs Euclidean
· states vs observables
Plan for Today
1 Classical BV
2 Quantum BV
For classical field theory,
input
M spacetime manifold
F=F(M) space of fields
(=sections of a bundle over M)
S: F → ℝ action functional
PERIMETER INSTITUTE FOR THEORETICAL PHYSICS
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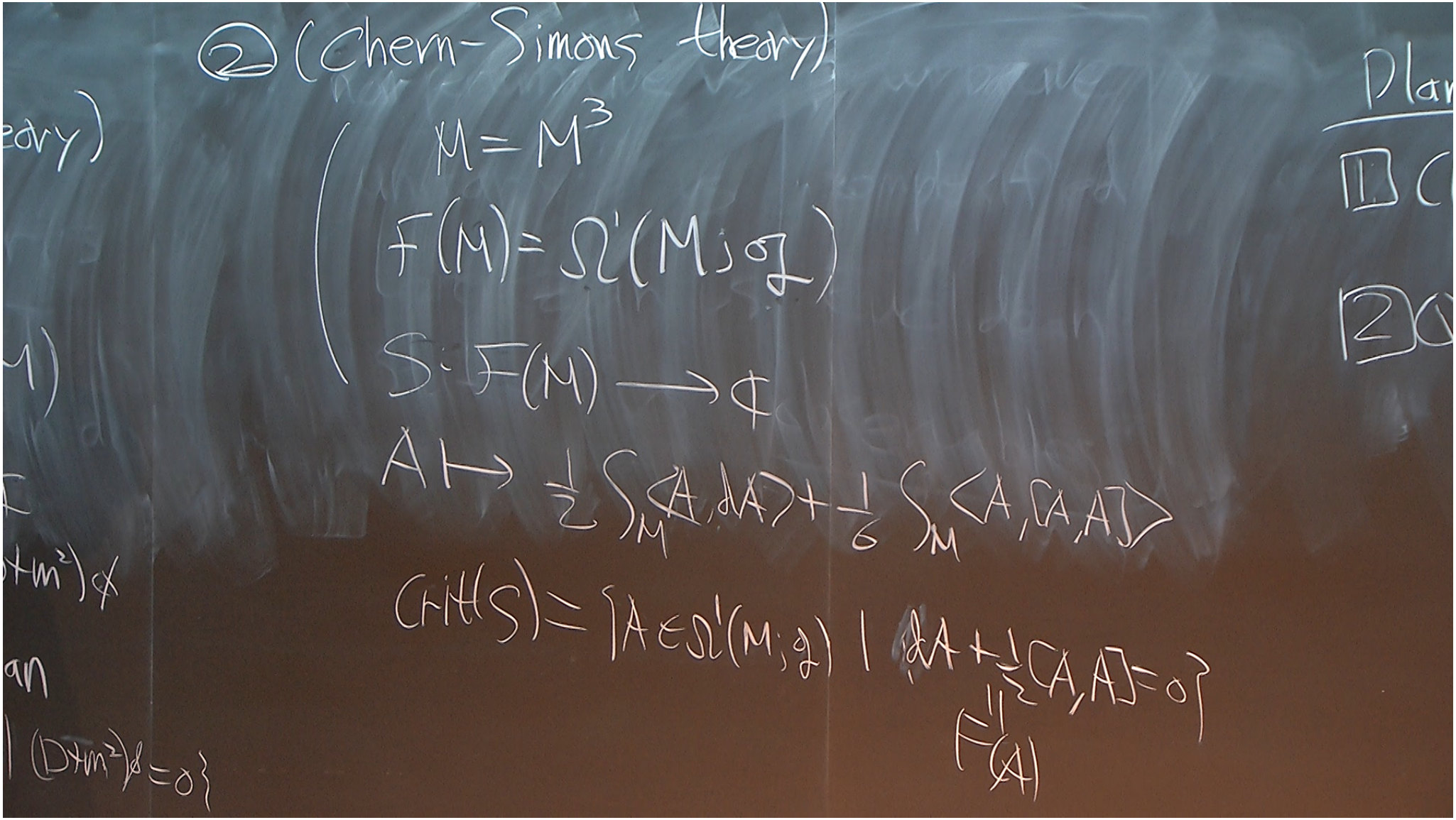

For classical field theory,
input
M spacetime manifold
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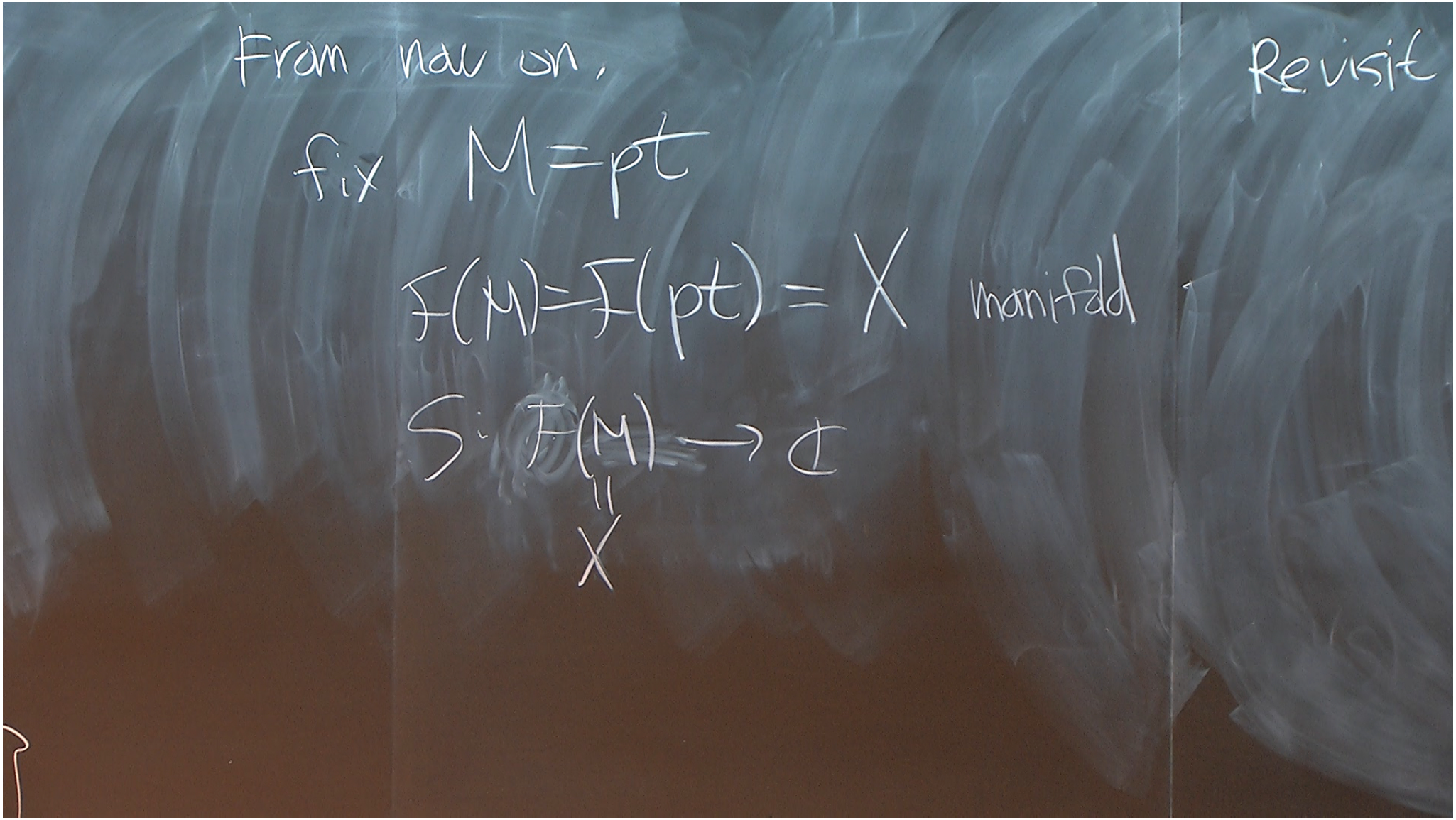

For classical field theory,
input
M spacetime manifold
F=F(M) space of fields
(=sections of a bundle over M)
S: F → ℂ action functional
output
crit(S) = {α ∈ F | dS(α) = 0}
Euler-Lagrange Eqn, or equations of motion

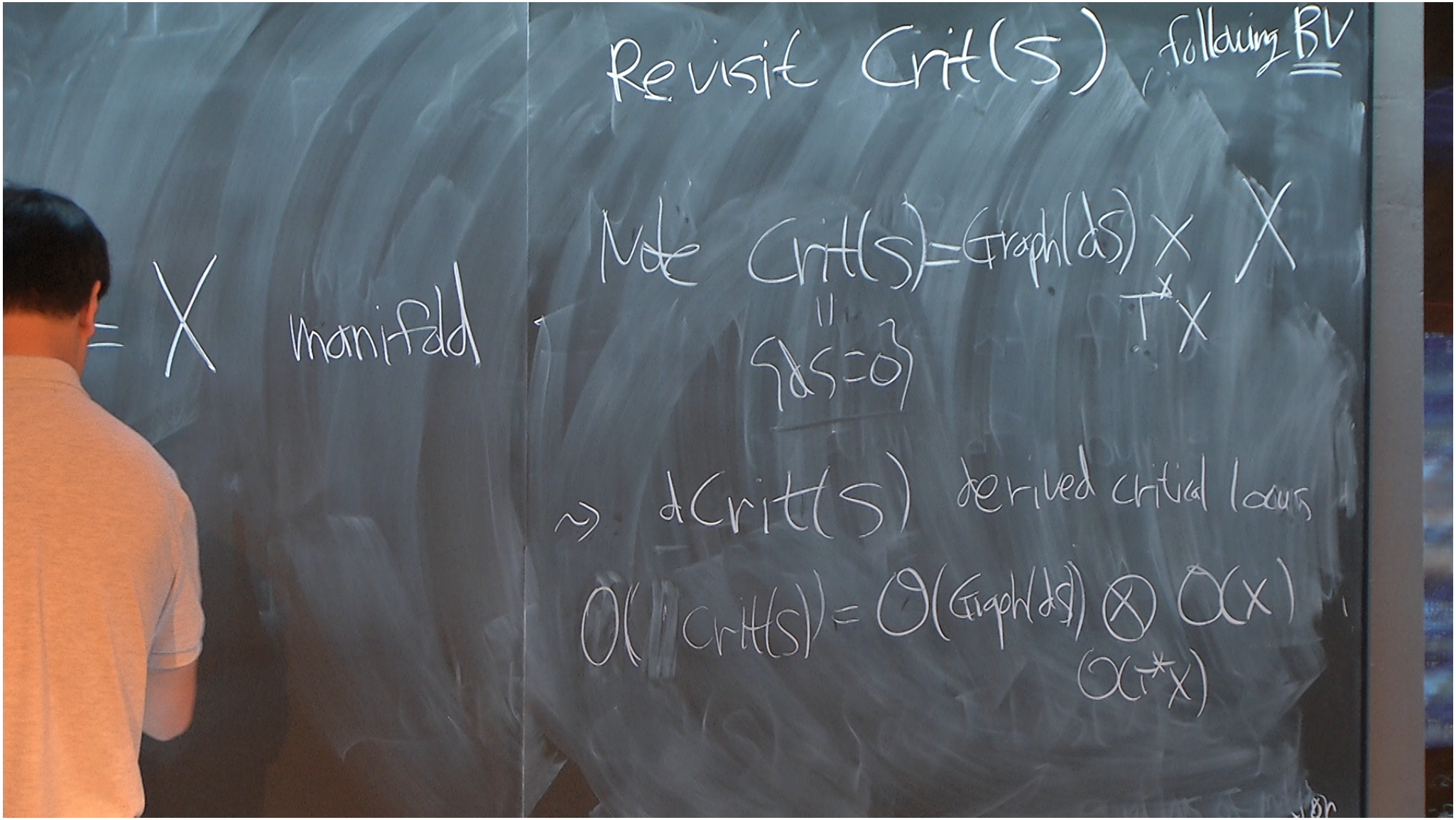

Ex
① (free scalar field theory)
M=(M^d, g)
F(M)=C^∞(M)
S: F(M) → ℂ
φ ↦ ∫_M φ (D+m²) φ
D: Laplacian
: non-pertu
unitar
Lorent
state

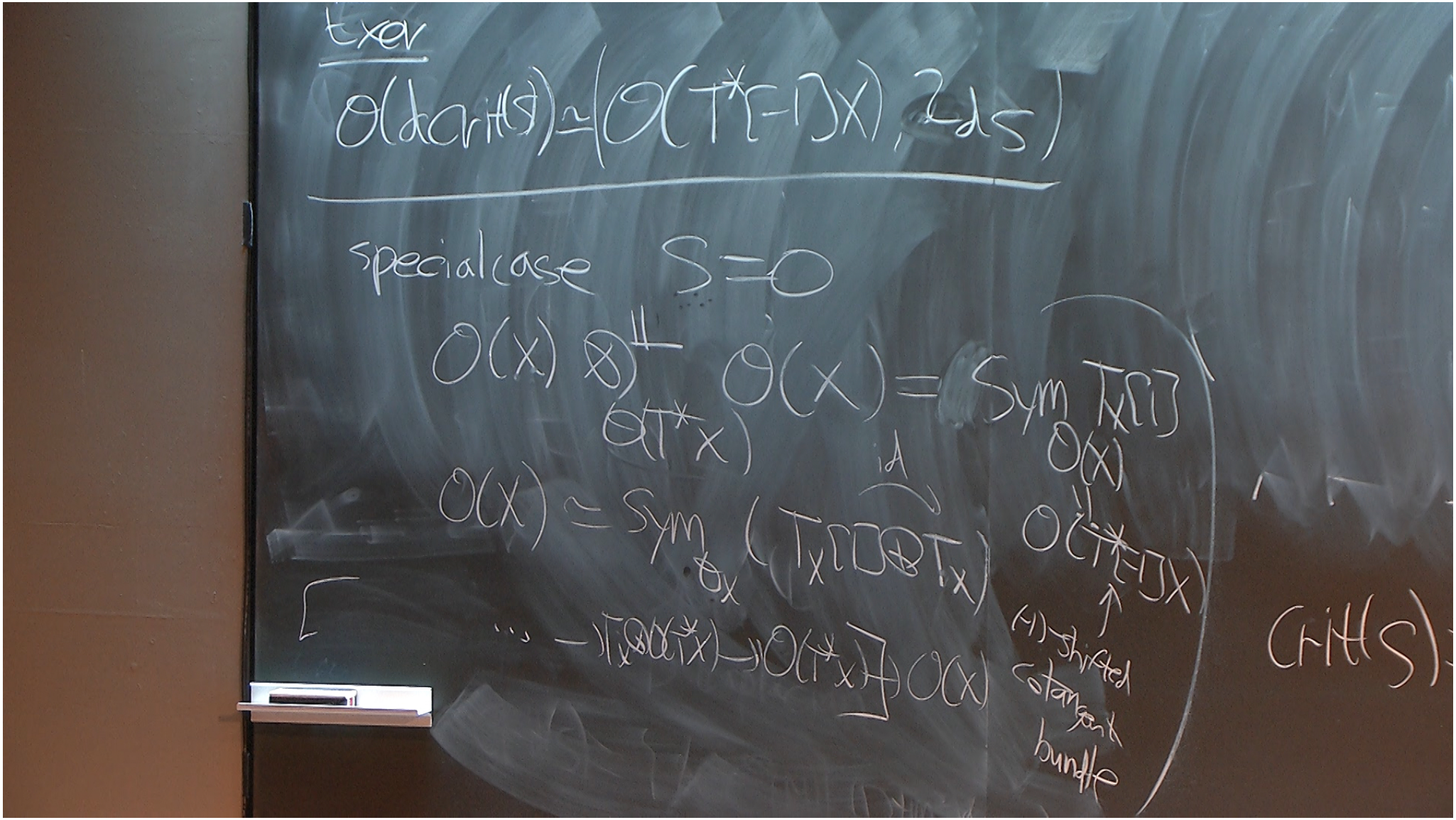

$S: F(M) \longrightarrow \mathbb{C}$
$\phi \longmapsto \int_M \phi (D+m^2)\phi$
D: Laplacian
$Grit(S) = \{\phi \in C^\infty(M) \mid (D+m^2)\phi = 0\}$

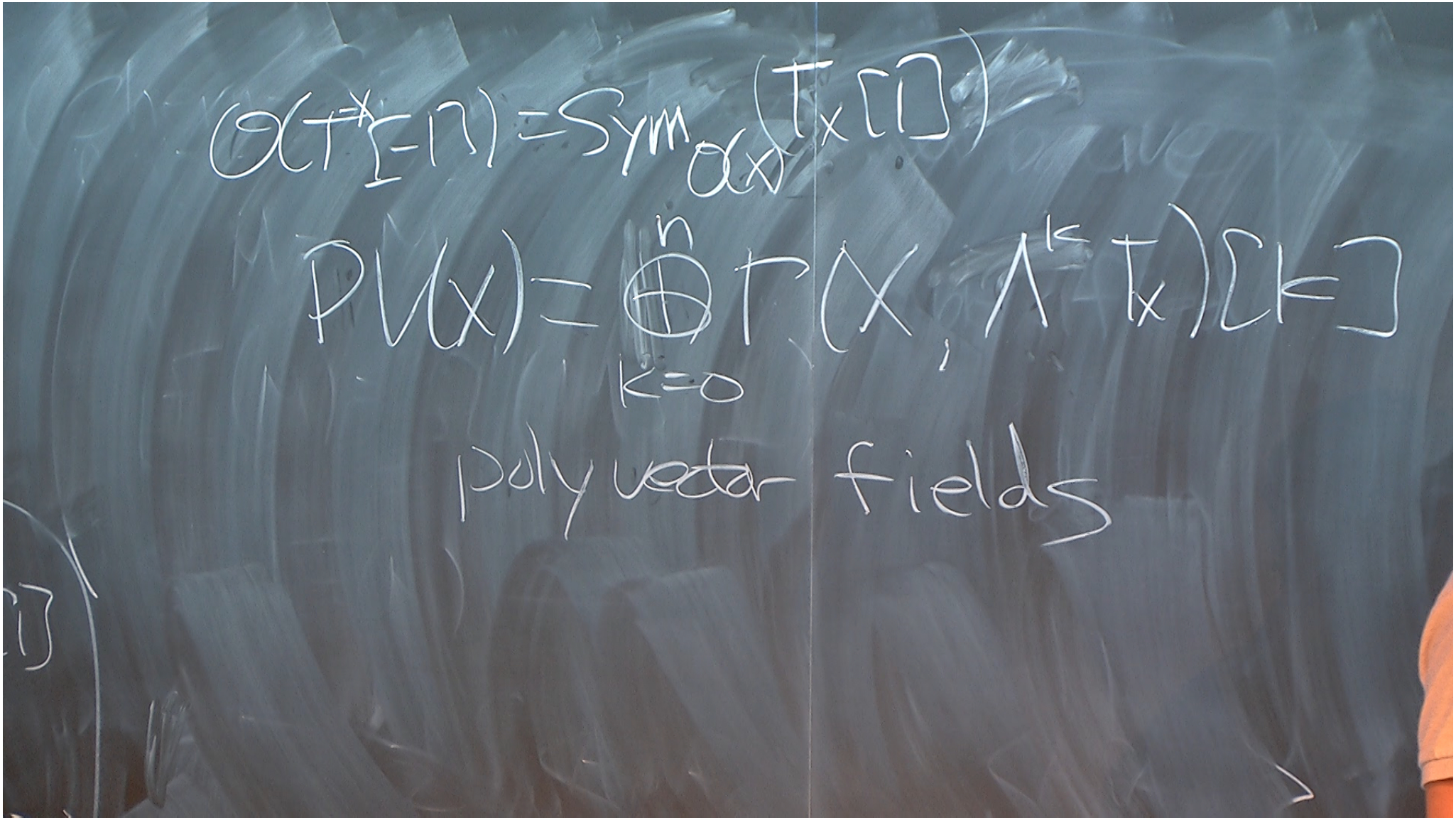

② (Chern-Simons theory)
M = M^3
F(M) = Ω^1(M; g)
S : F(M) ⟶ ℂ
A ↦ 1/2 ∫_M ⟨A, dA⟩ + 1/6 ∫_M ⟨A, [A, A]⟩

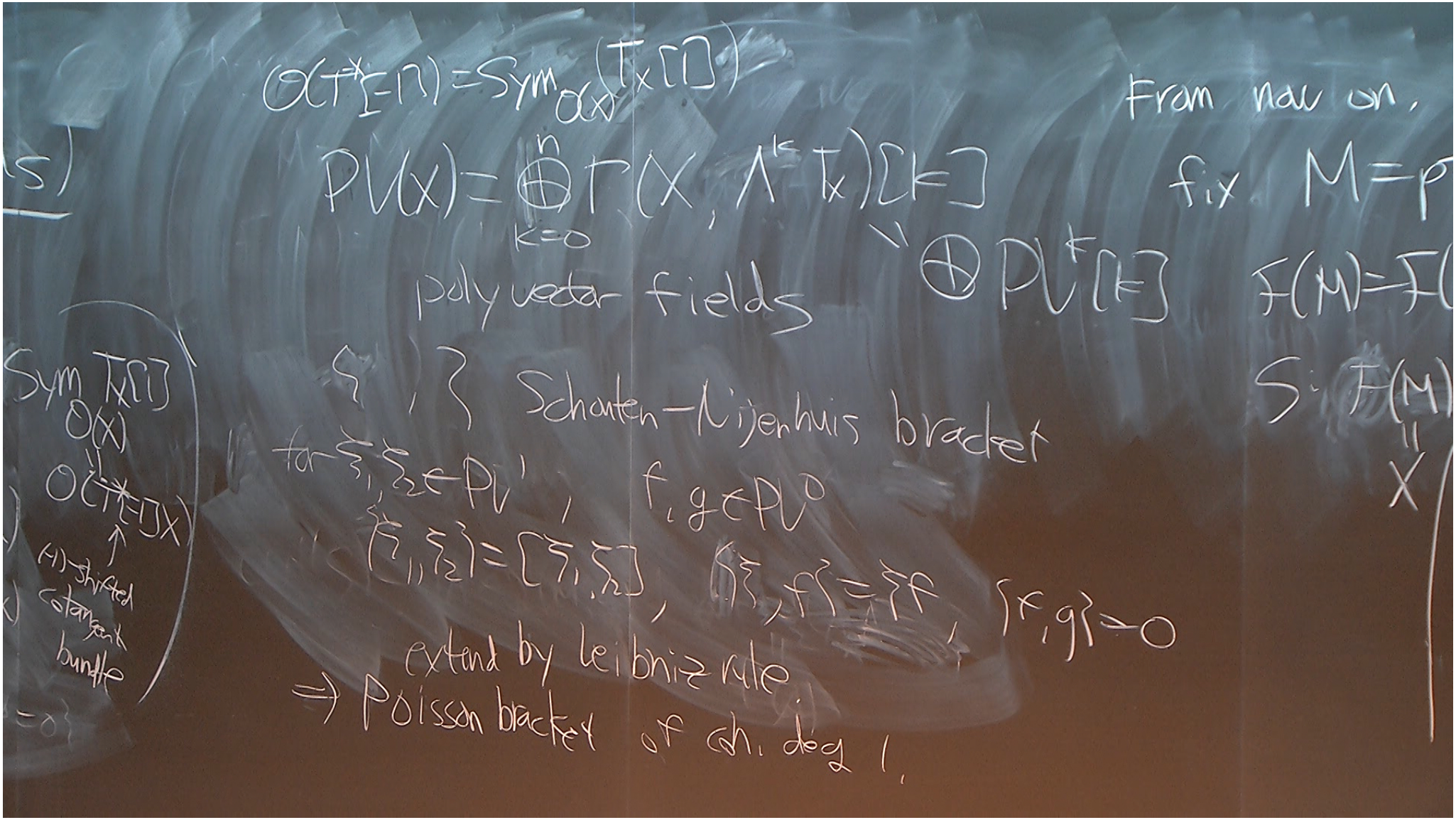

② (Chern-Simons theory)
$M = M^3$
$F(M) = \Omega^1(M;\mathfrak{g})$
$S : F(M) \longrightarrow \mathbb{C}$
$A \mapsto \frac{1}{2}\int_M \langle A, dA\rangle + \frac{1}{6}\int_M \langle A,[A,A]\rangle$
$Crit(S) = \{A \in \Omega^1(M;\mathfrak{g}) \mid dA + \frac{1}{2}[A,A] = 0\}$
$\Gamma(A)$

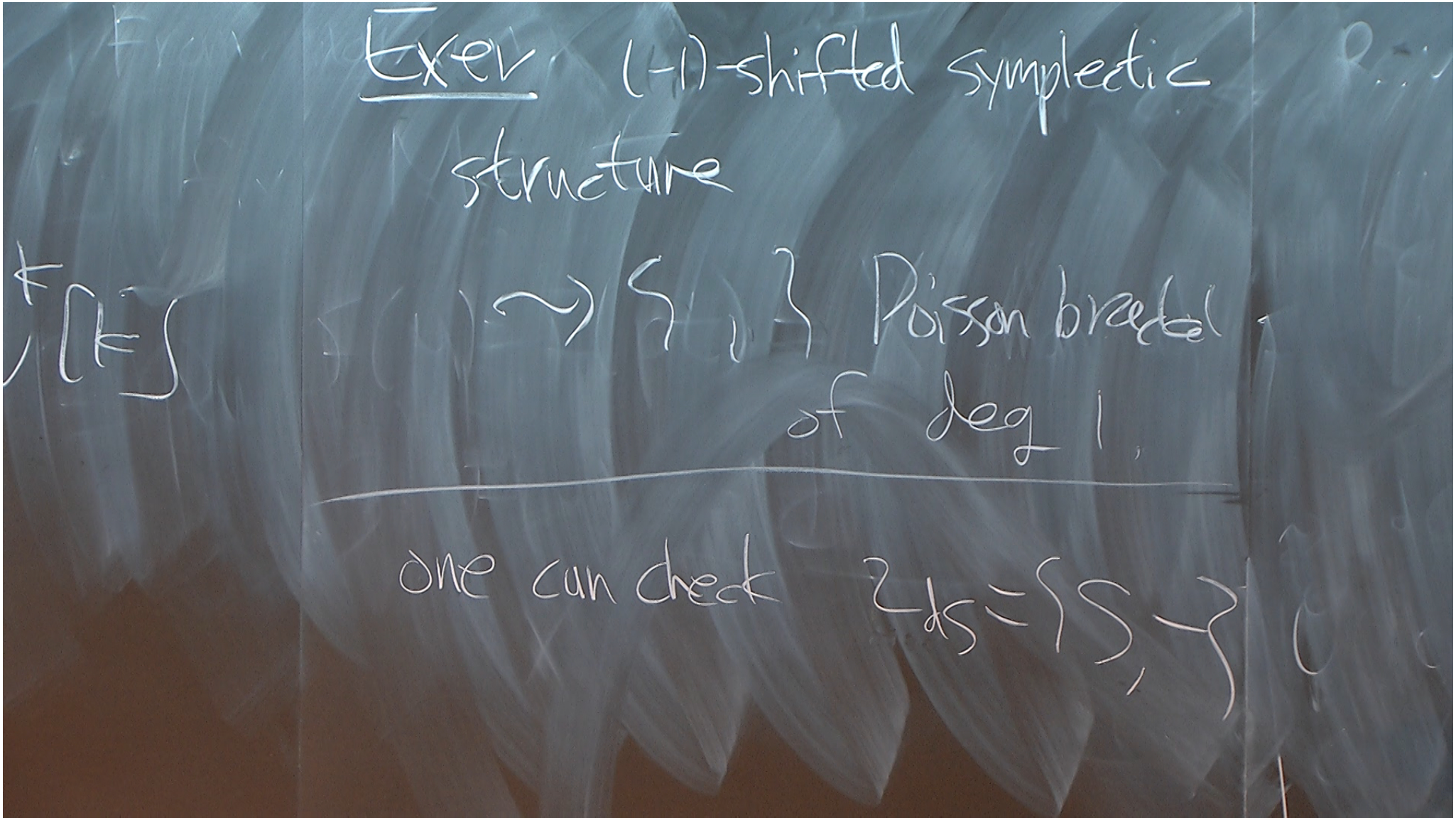

From now on,
Revisit
fix M = pt
F(M) = F(pt) = X
manifold
S: F(M) → ℂ
||
X

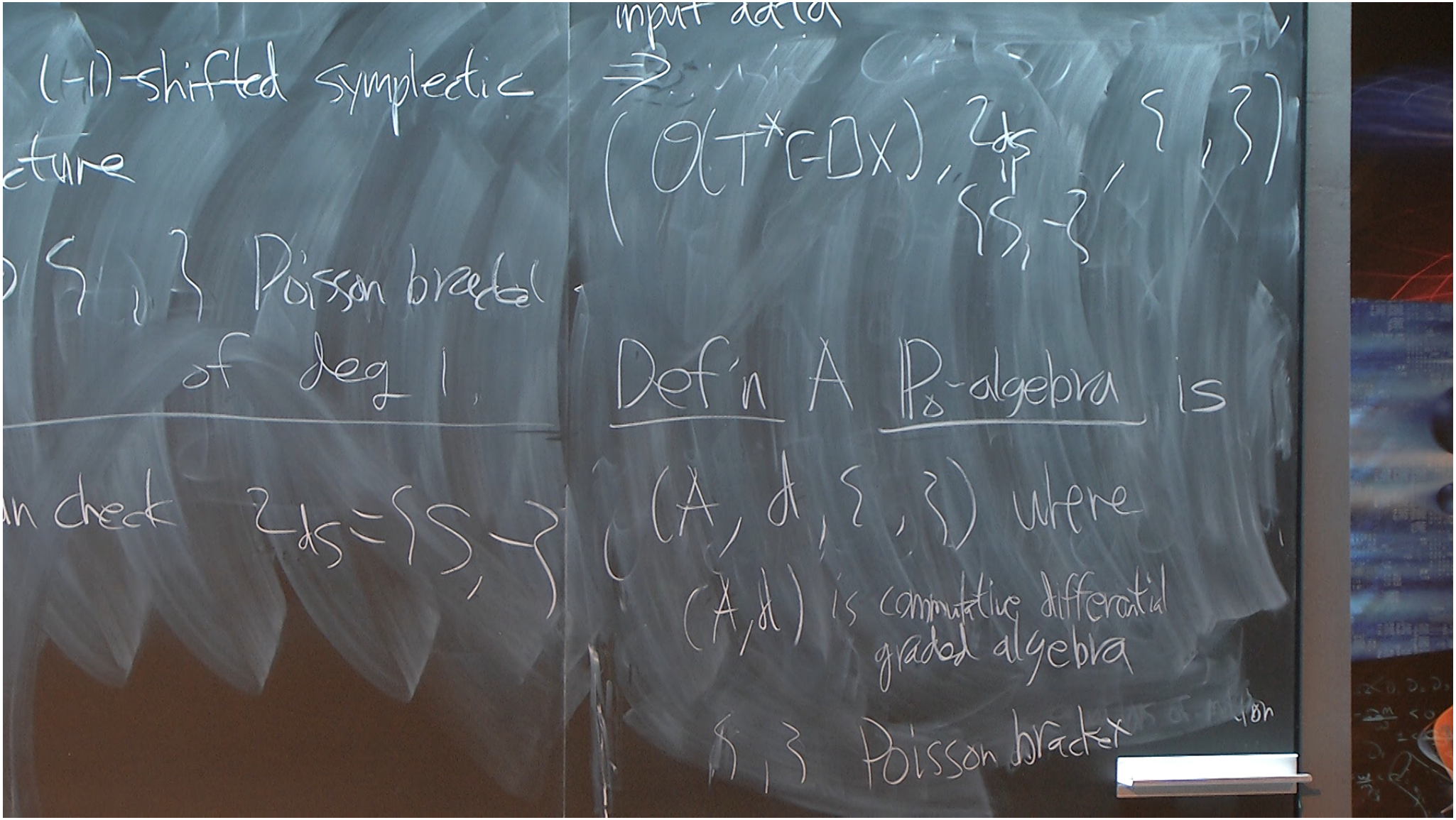

Revisit Crit(S) following BV
= X manifold
Note Crit(S) = Graph(dS) ∩ X
T*X
{dS=0}
≈ dCrit(S) derived critical locus
O(Crit(S)) = O(Graph(dS)) ⊗ O(X)
O(T*X)

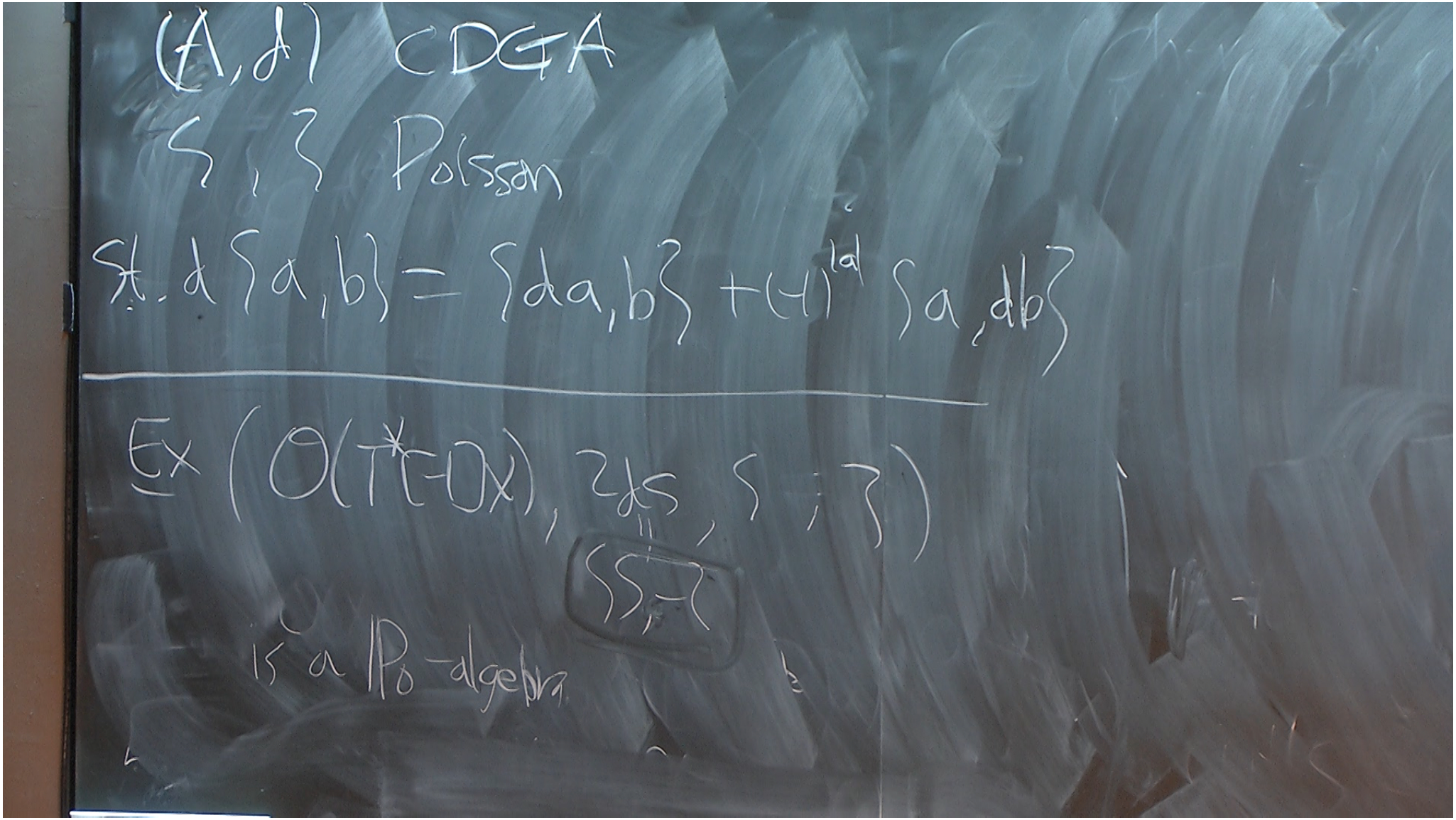

exer
special case S=0
(-1)-shifted cotangent bundle

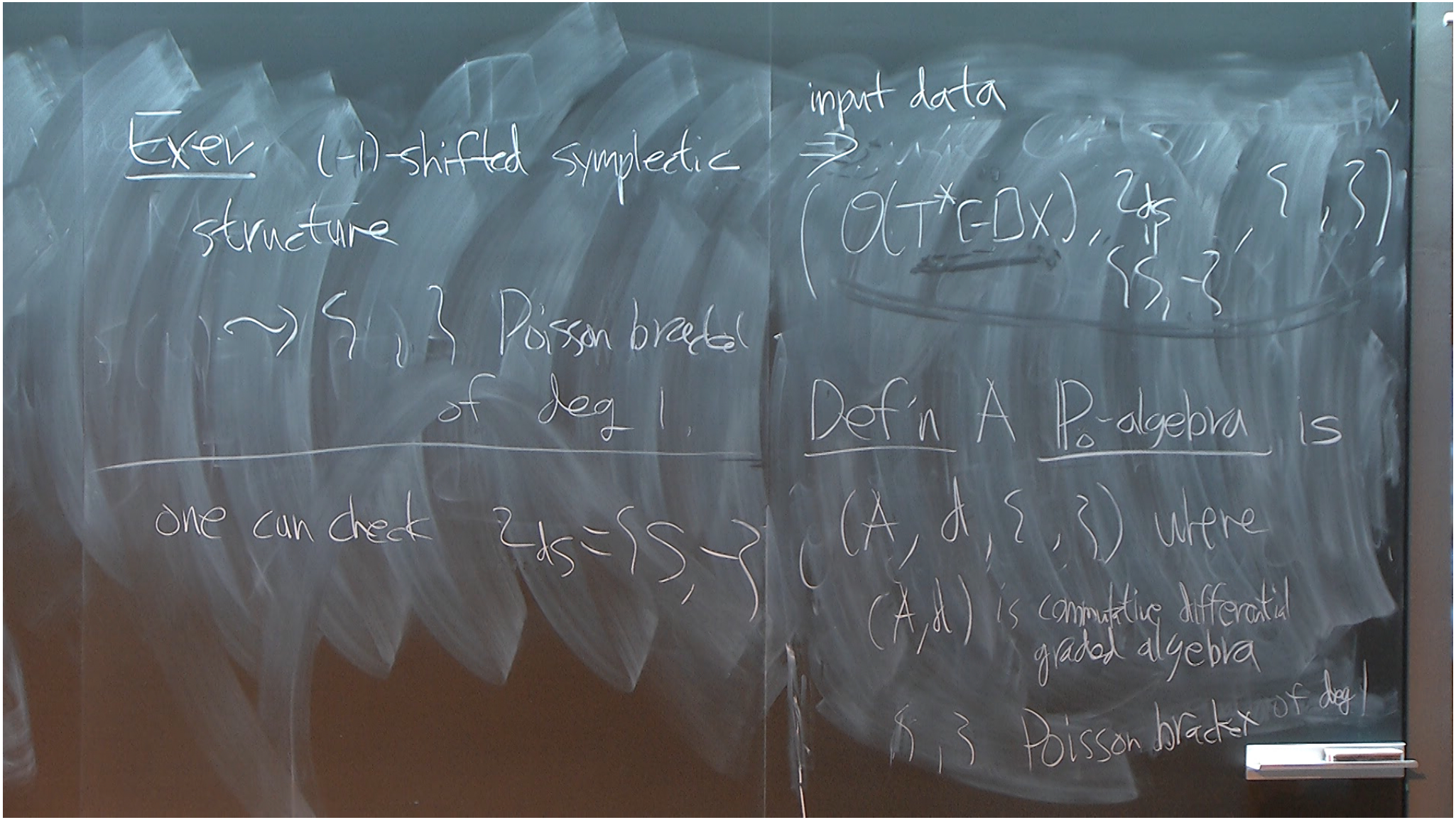

$\mathcal{O}(T^*\Sigma[-1]) = \mathrm{Sym}_{\mathcal{O}(X)}(T_X[1])$
$PV(X) = \bigoplus_{k=0}^{n} \Gamma(X, \Lambda^k T_X)[k]$
polyvector fields

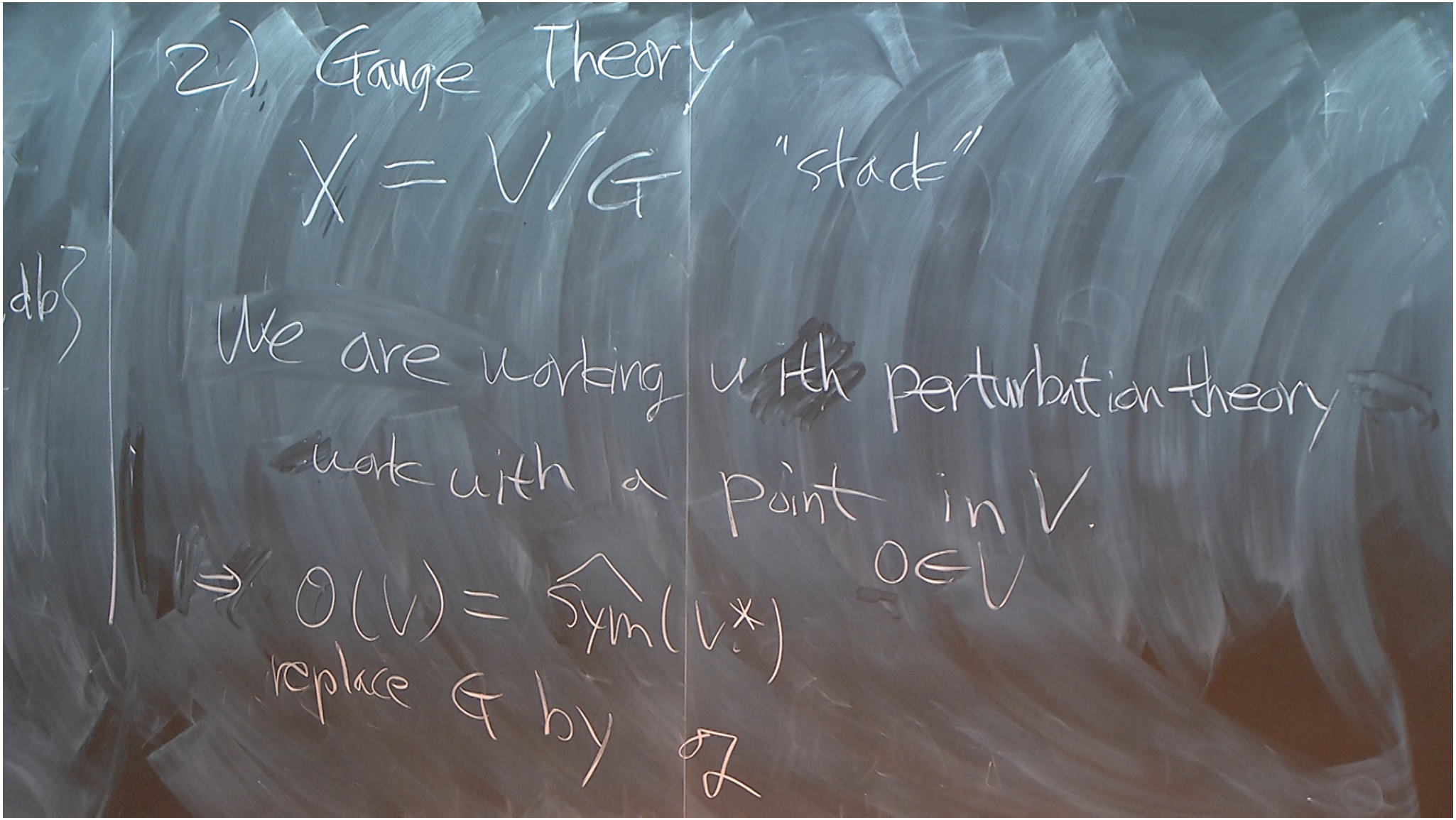

From now on,
fix M=
polyvector fields
Schouten–Nijenhuis bracket
extend by Leibniz rule
Poisson bracket of ch. deg 1.
(-1)-shifted cotangent bundle

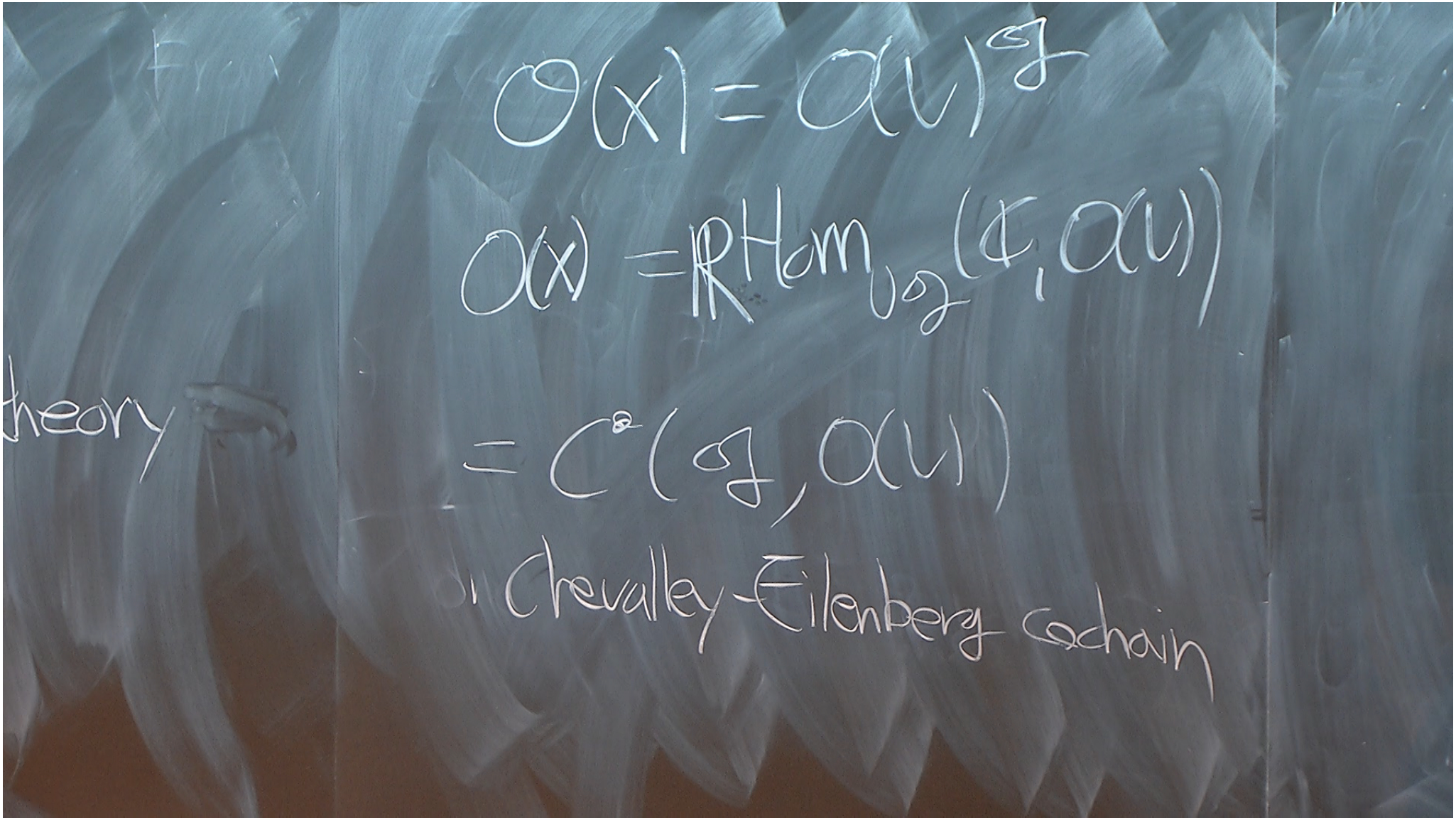

Exer (-1)-shifted symplectic
structure
Poisson bracket
of deg 1
one can check

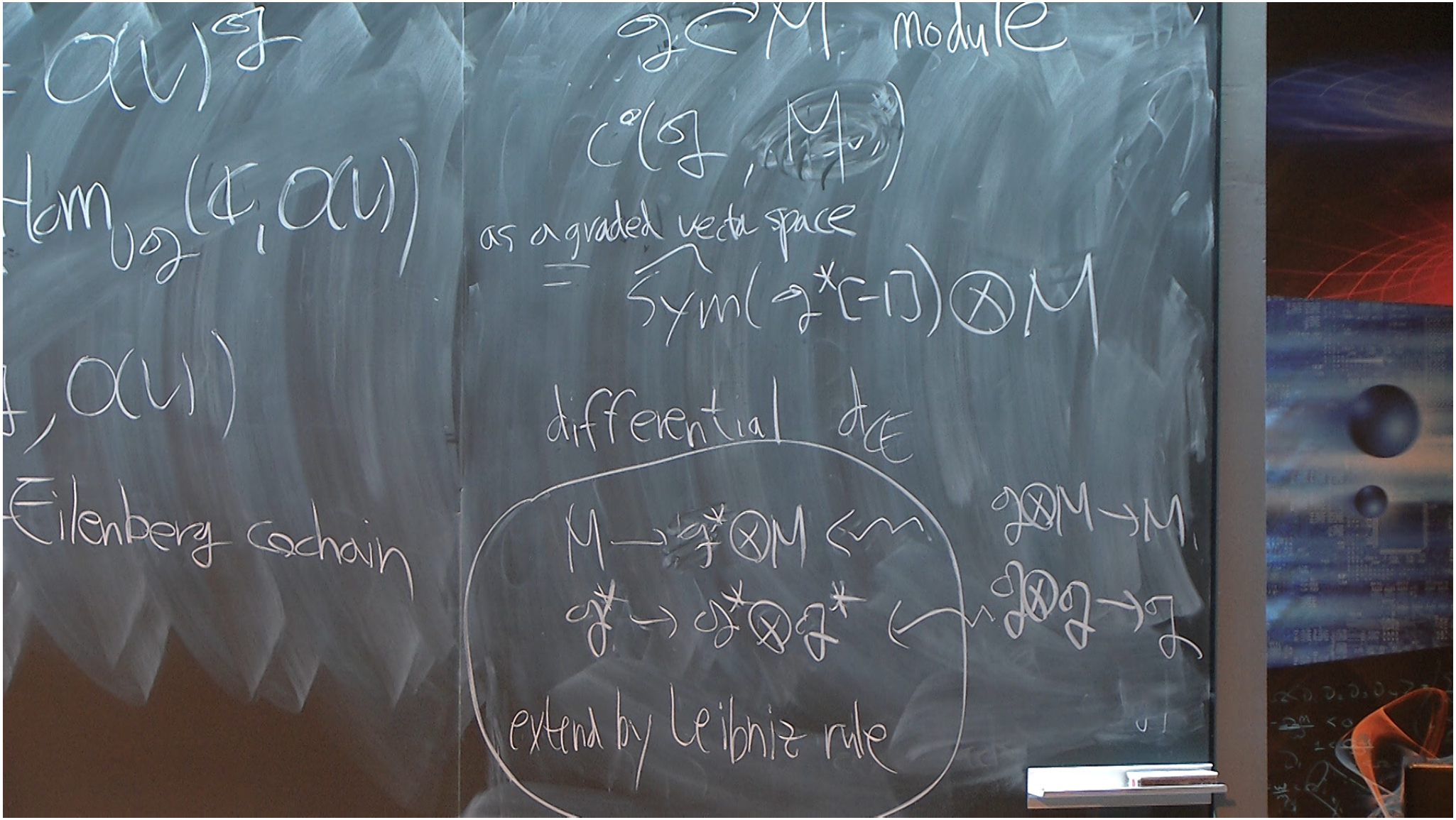

(-1)-shifted symplectic
Poisson bracket
of deg 1.
Def'n A $P_0$-algebra is
$(A, d, \{,\}, \cdot)$ where
$(A, d)$ is commutative differential graded algebra
$\{,\}$ Poisson bracket

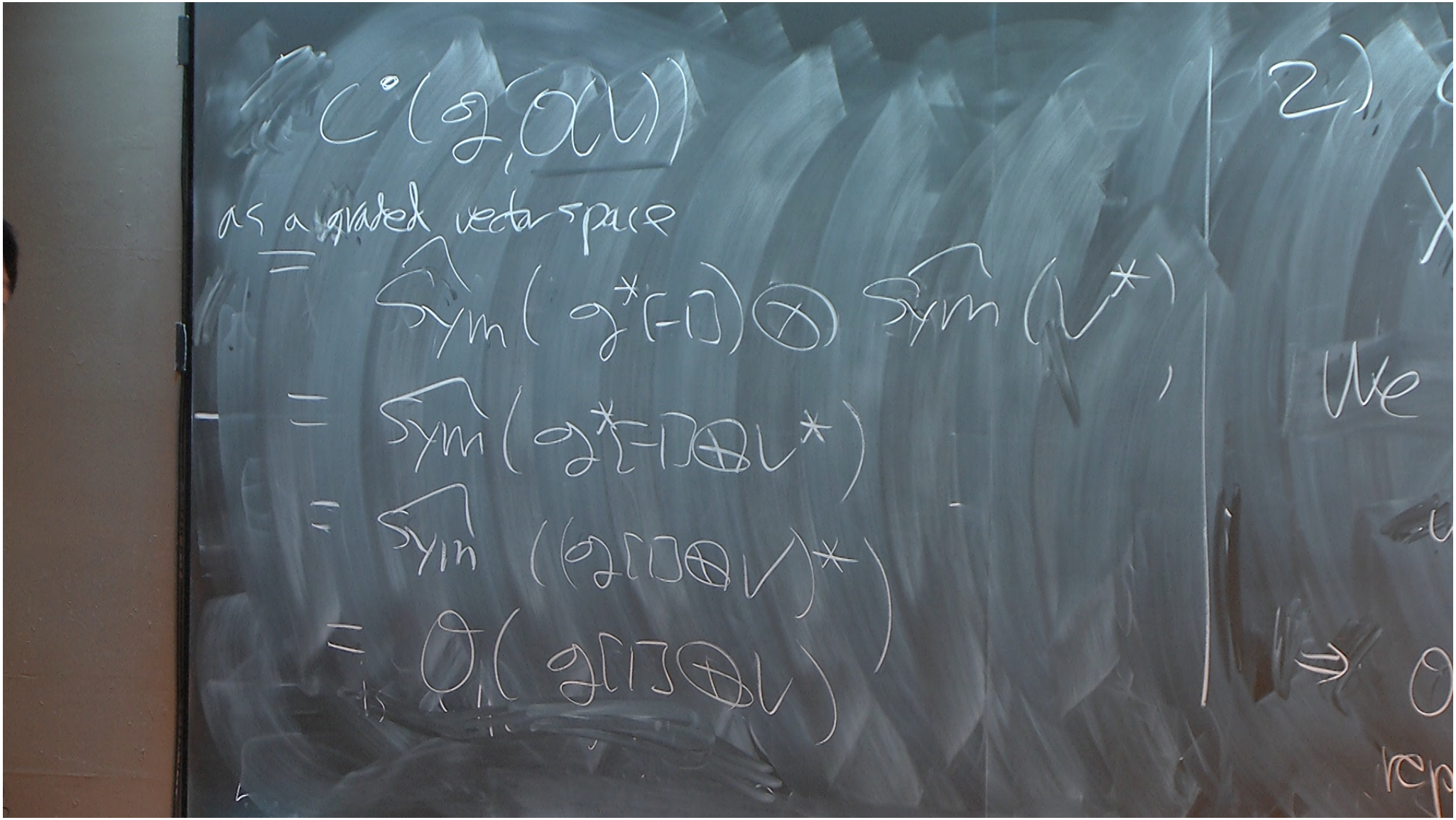

(A, d) CDGA
{ , } Poisson
s.t. d{a,b} = {da,b} + (-1)^|a| {a,db}
Ex (O(T*(-0X)), 2ys, { ; })
is a P0-algebra

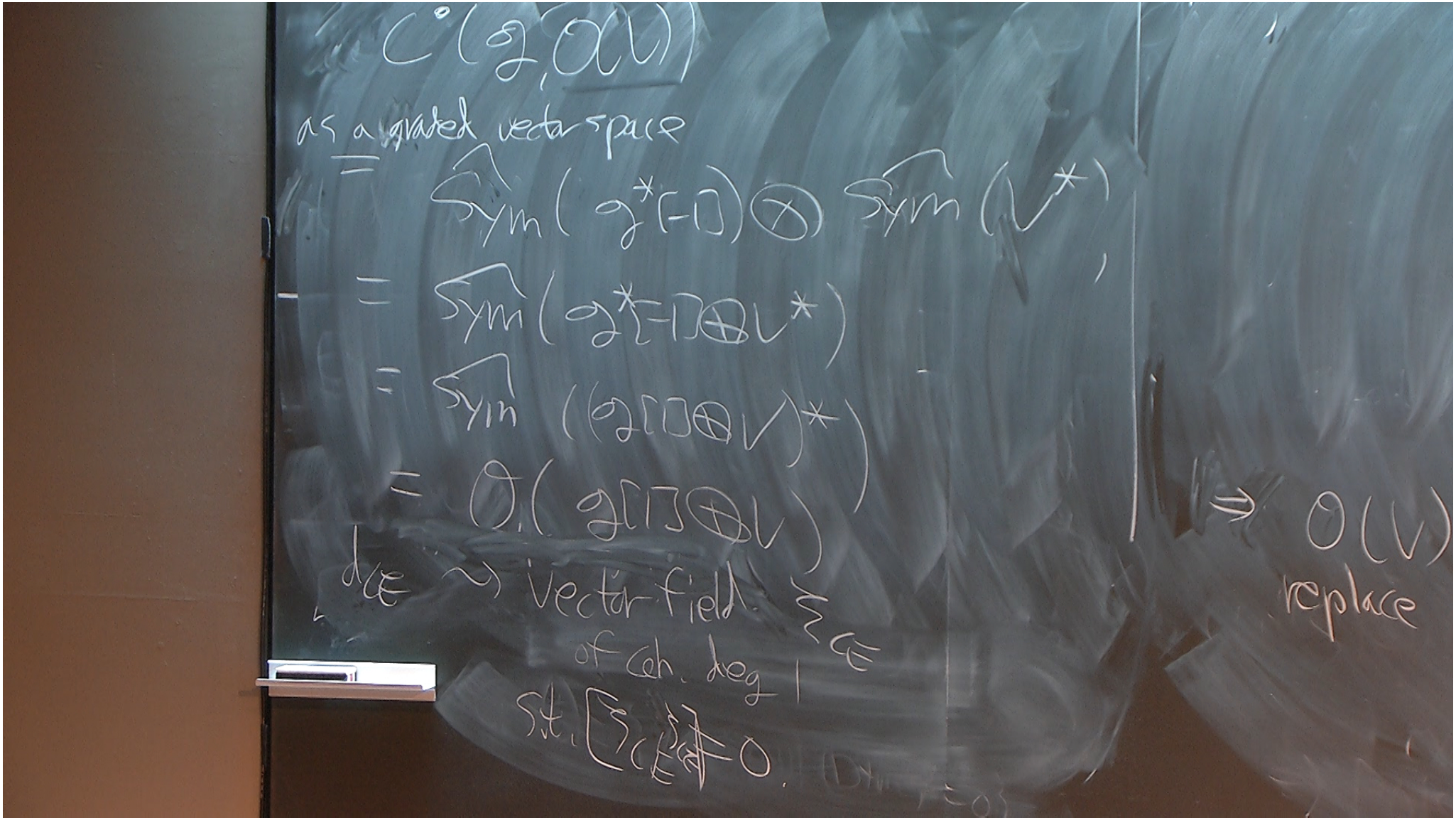

Exer. (-1)-shifted symplectic structure
$\rightsquigarrow \{\,,\,\}$ Poisson bracket of deg 1.
one can check $2d_S = \{S,S\}$
input data
$\Rightarrow$
$(\mathcal{O}(T^*[-1]X), 2d_S, \{\,,\,\})$
$\{S,S\}=0$
Def'n A $P_0$-algebra is
$(A, d, \{\,,\,\})$ where
$(A,d)$ is commutative differential graded algebra
$\{\,,\,\}$ Poisson bracket of deg 1

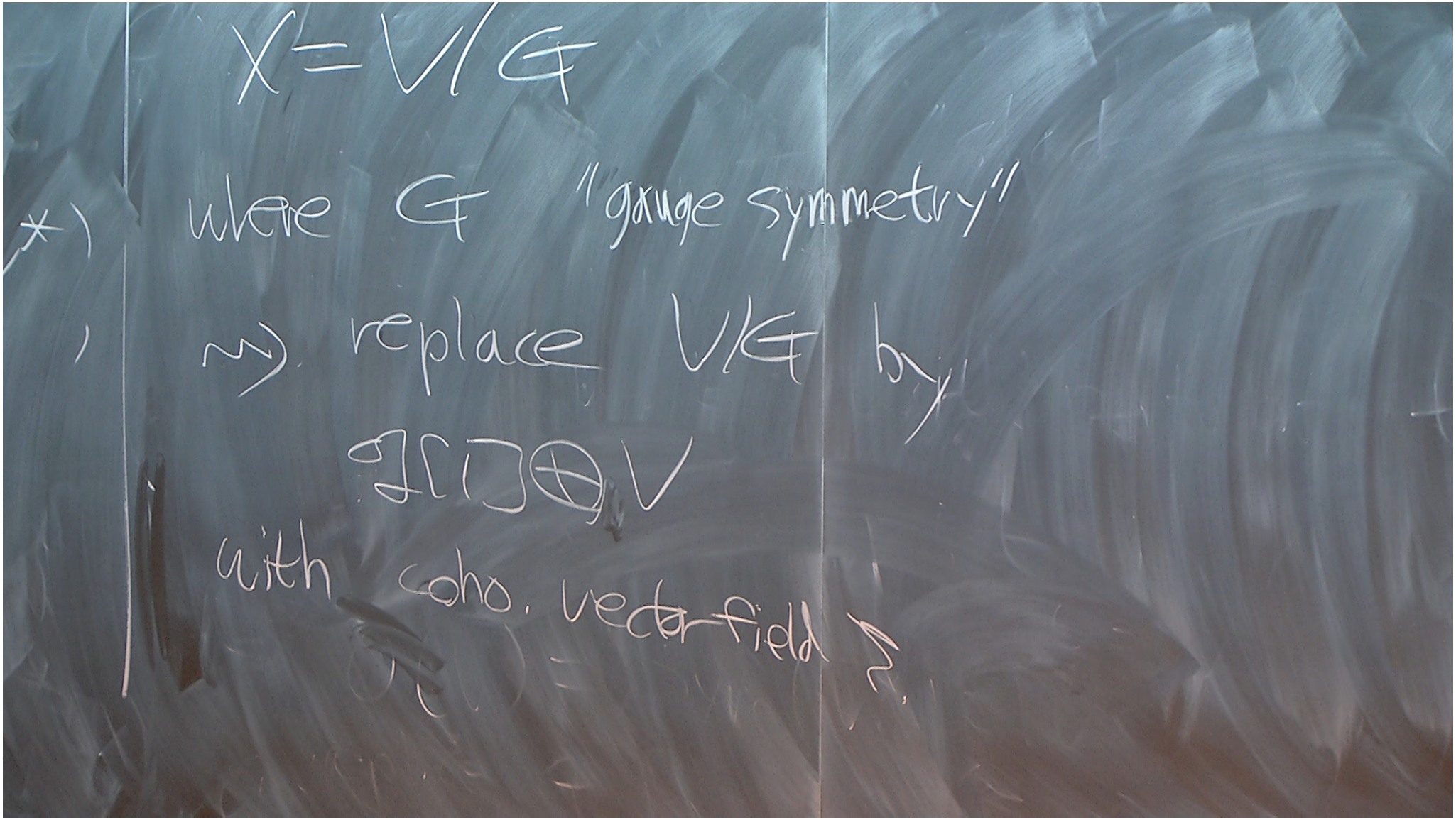

2) Gauge Theory
$X = V/G$ "stack"
We are working with perturbation theory
work with a point in $V$.
$0 \in V$
$\Rightarrow O(V) = \widehat{Sym}(V^*)$
replace $G$ by $\mathbb{Z}$

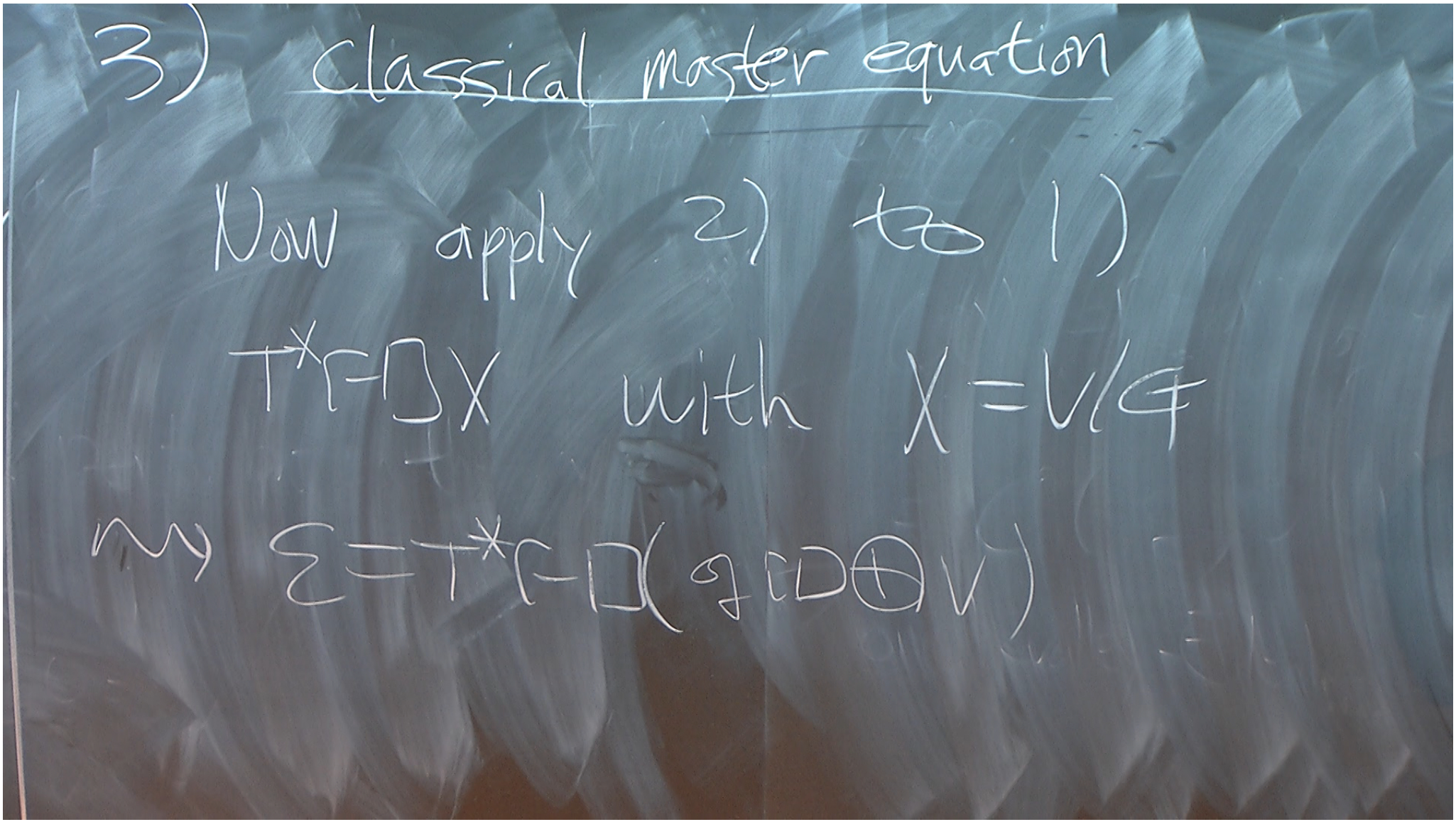

$\mathcal{O}(X) = \mathcal{O}(U)^{\mathbb{G}}$
$\mathcal{O}(X) = R\mathrm{Hom}_{\mathbb{G}}(\mathbb{C}, \mathcal{O}(U))$
$= C^{\bullet}(\mathbb{G}, \mathcal{O}(U))$
Chevalley-Eilenberg cochain

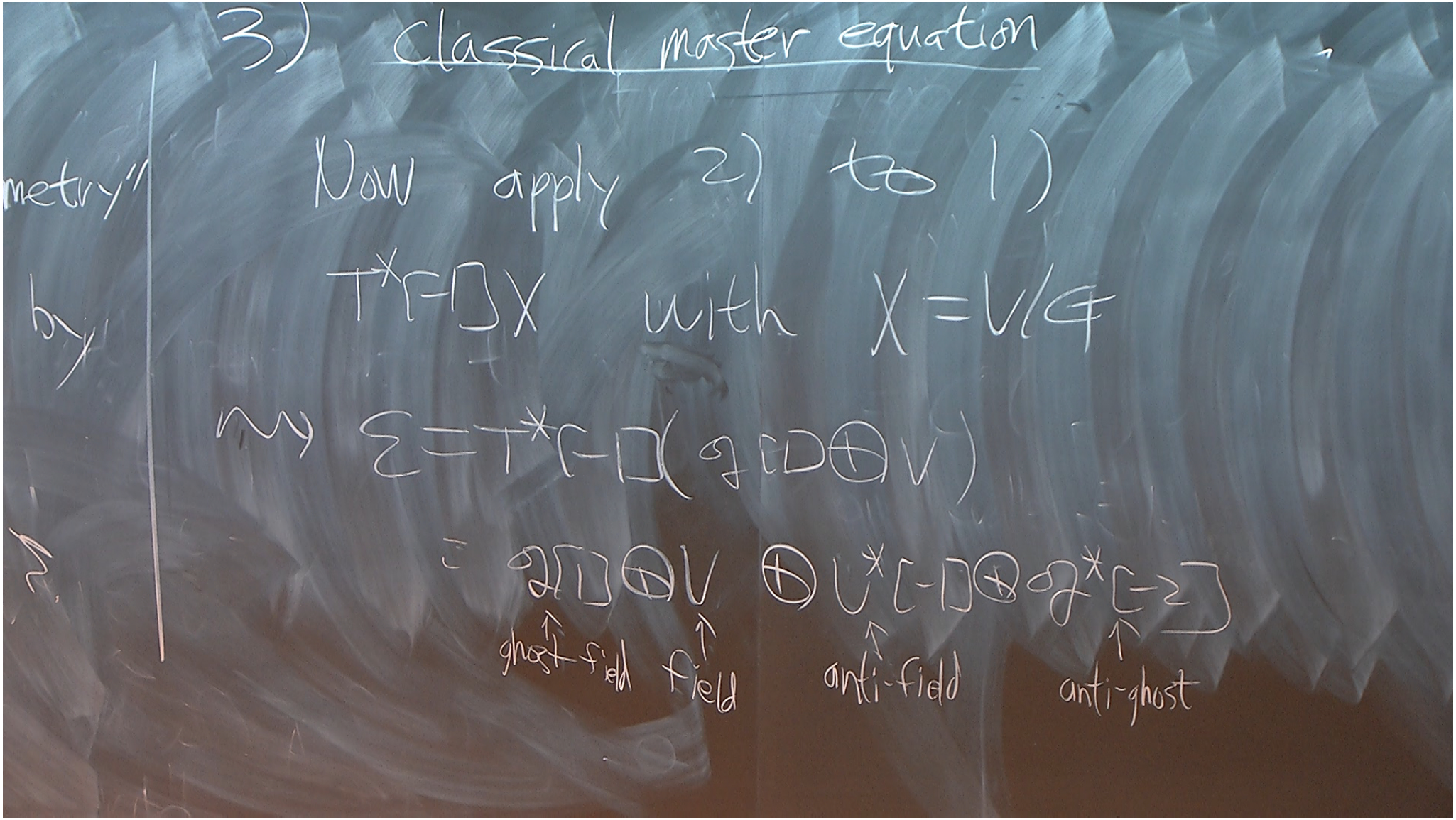

$\mathfrak{g} \subset M$ module
$C^\bullet(\mathfrak{g}, M)$
as a graded vect space
$= \mathrm{Sym}(\mathfrak{g}^*[-1]) \otimes M$
differential $d_{CE}$
$M \to \mathfrak{g}^* \otimes M \hookleftarrow$
$\mathfrak{g}^* \to \mathfrak{g}^* \otimes \mathfrak{g}^* \hookleftarrow$
extend by Leibniz rule
$\mathfrak{g} \otimes M \to M$,
$\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$
$\mathrm{Hom}_{U\mathfrak{g}}(\mathbb{C}, A(V))$
Eilenberg cochain

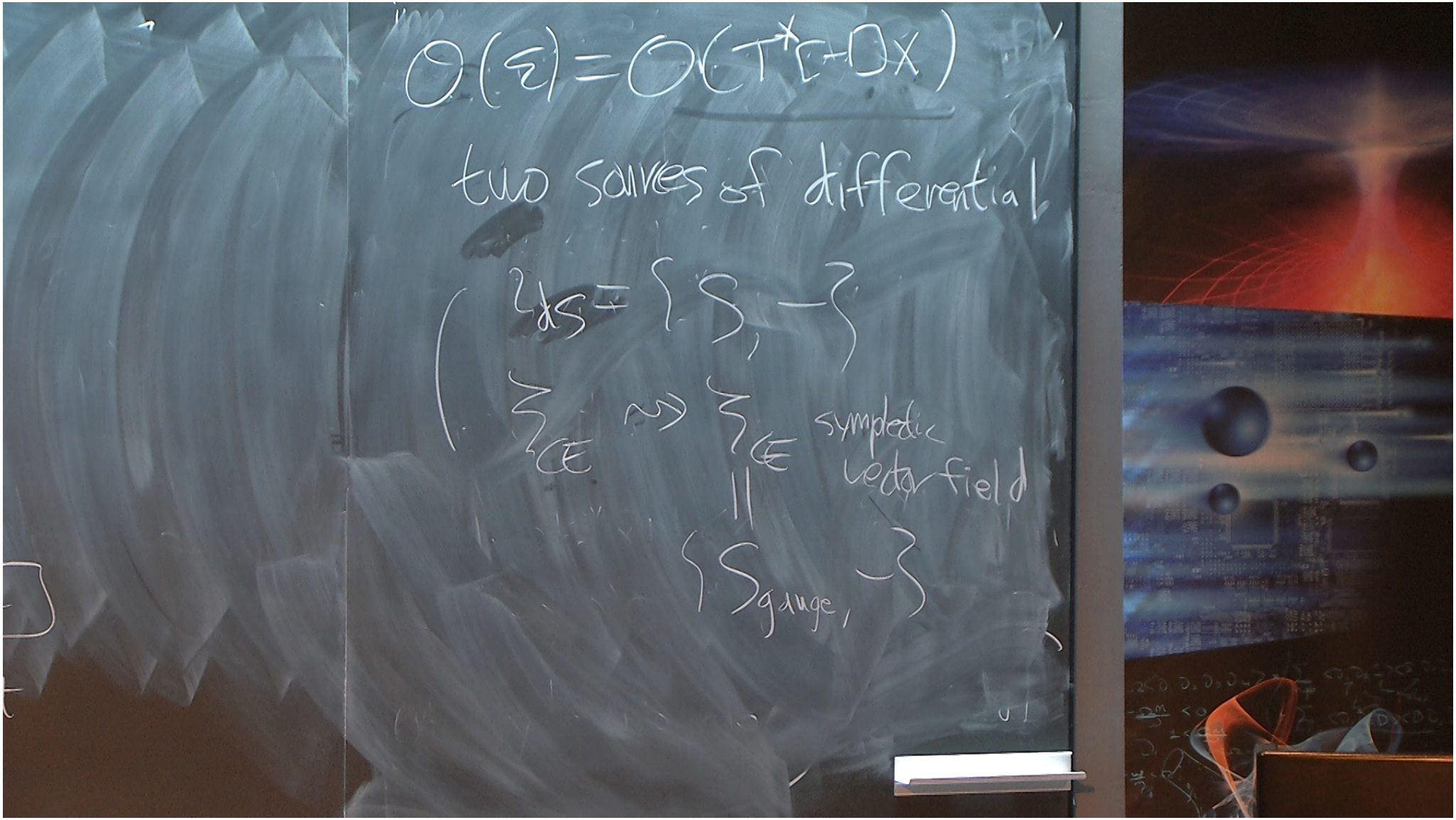

$C^\infty(\mathfrak{g}, \mathcal{O}(V))$
as a graded vector space
$= \mathrm{Sym}(\mathfrak{g}^*[-1] \oplus \mathrm{Sym}(V^*)$
$= \mathrm{Sym}(\mathfrak{g}^*[-1] \oplus V^*)$
$= \mathrm{Sym}((\mathfrak{g}[1] \oplus V)^*)$
$= \mathcal{O}(\mathfrak{g}[1] \oplus V)$
2)
We

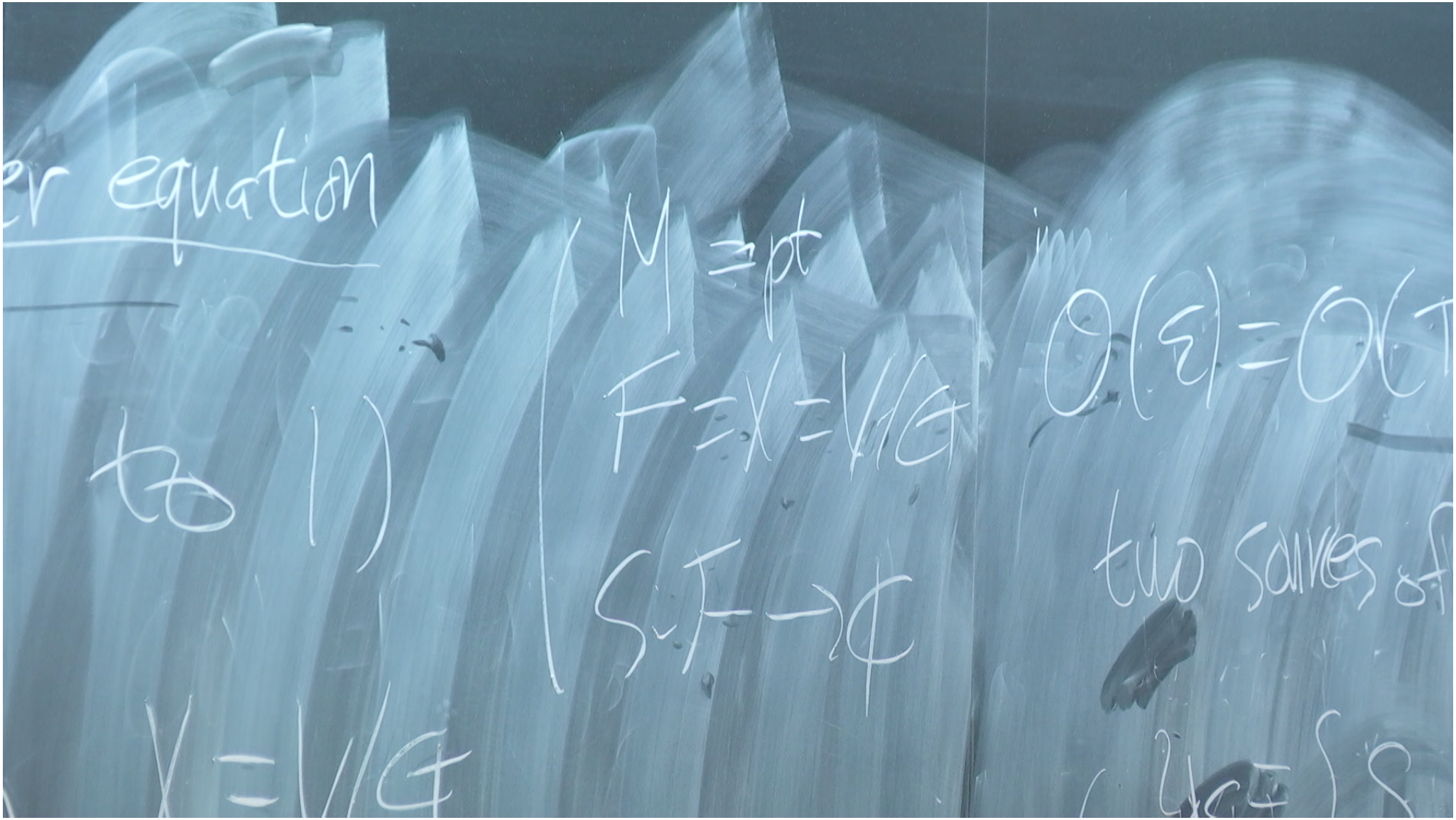

$C^\infty(\mathfrak{g}, \mathcal{O}(V))$
as a graded vector space
$= \mathrm{Sym}(\mathfrak{g}^*[-1] \oplus \mathrm{Sym}(V^*)$
$= \mathrm{Sym}(\mathfrak{g}^*[-1] \oplus V^*)$
$= \mathrm{Sym}((\mathfrak{g}[1] \oplus V)^*)$
$= \mathcal{O}(\mathfrak{g}[1] \oplus V)$
$\Rightarrow \mathcal{O}(V)$
replace
$d_{CE} \rightsquigarrow$ vector field $\Sigma_{CE}$
of ch. deg 1
s.t. $[\Sigma_{CE}, \Sigma_{CE}] = 0$.

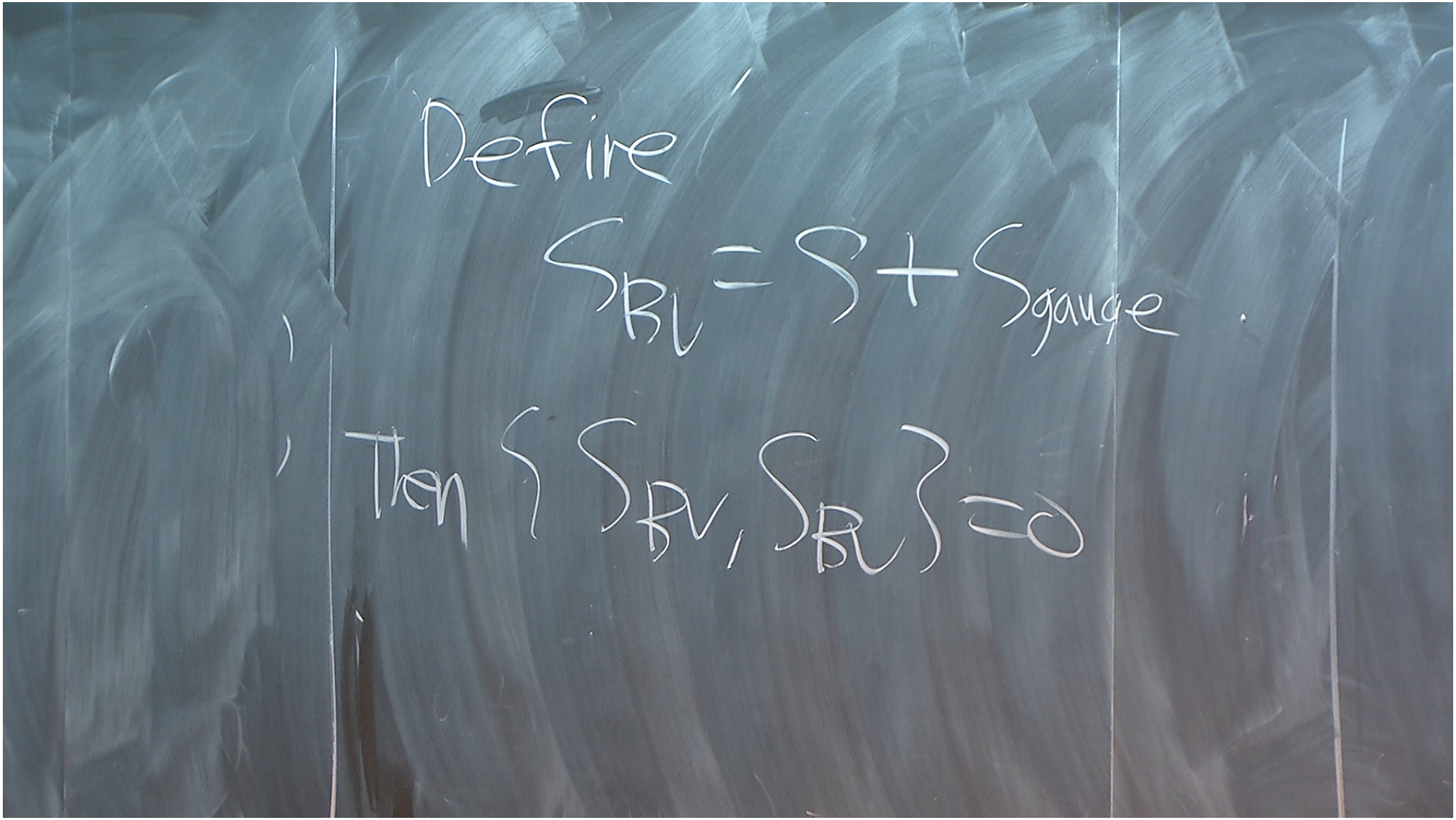

X = V/G
*) where G "gauge symmetry"
~> replace V/G by
with coho. vector field


3) Classical master equation
Now apply 2) to 1)
$T^*[-1]X$ with $X = V/G$
$\mathcal{E} = T^*[-1](g[1] \oplus V)$


3) Classical master equation
metry"
Now apply 2) to 1)
by
$T^*[-1]X$ with $X = V/G$
$\mathcal{E} = T^*[-1](g[1] \oplus V)$
$= \mathfrak{g}[1] \oplus V \oplus V^*[-1] \oplus \mathfrak{g}^*[-2]$
ghost-field
field
anti-field
anti-ghost


$\Theta(\mathcal{E}) = \Theta(T^*\mathcal{E} \to \mathcal{O}X)$
two sources of differential
$Q_{ds} = \{S, -\}$
$\mathfrak{X}_{CE}$
$\mathfrak{X}_{CE}$
symplectic vector field
$\{S_{gauge}, -\}$


er equation
M = pt
F = X = V/G
two solves of


Define
$S_{BV} = S + S_{gauge}$.
Then $\{S_{BV}, S_{BV}\} = 0$

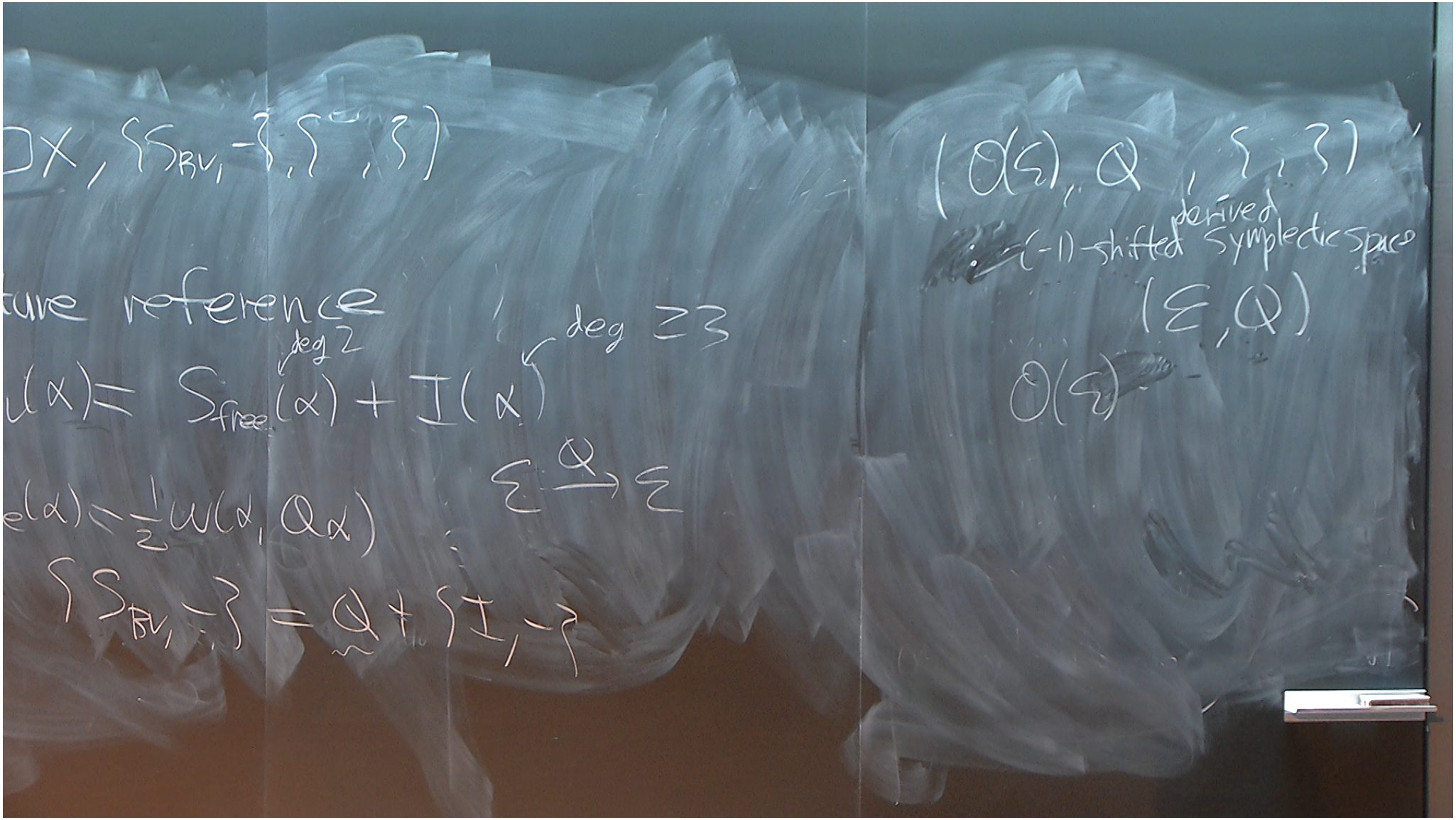
$(\mathcal{O}(T^*[-1]DX), \{S_{BV}, -\}, \bar{\partial}, \int)$

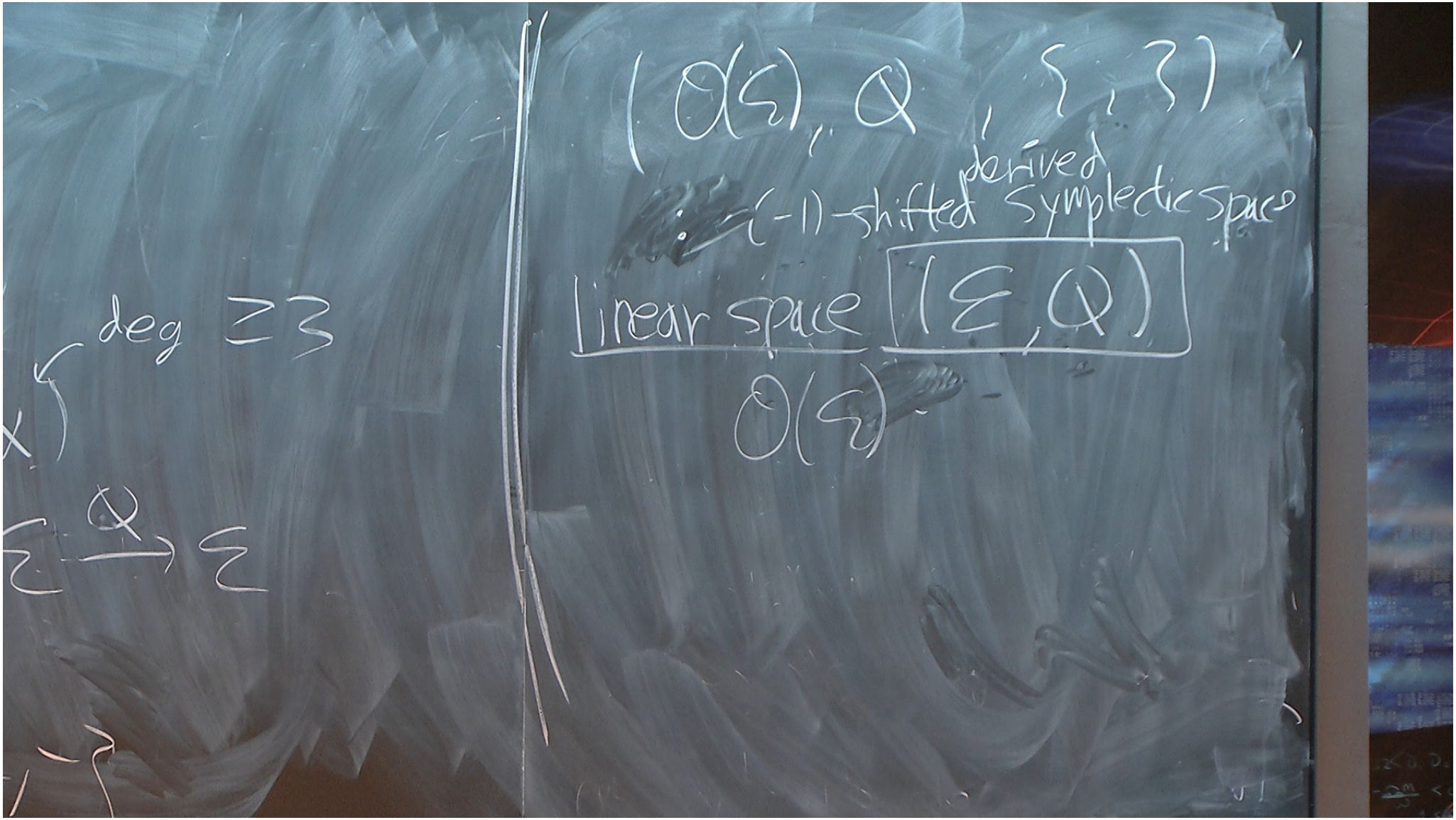
For future reference

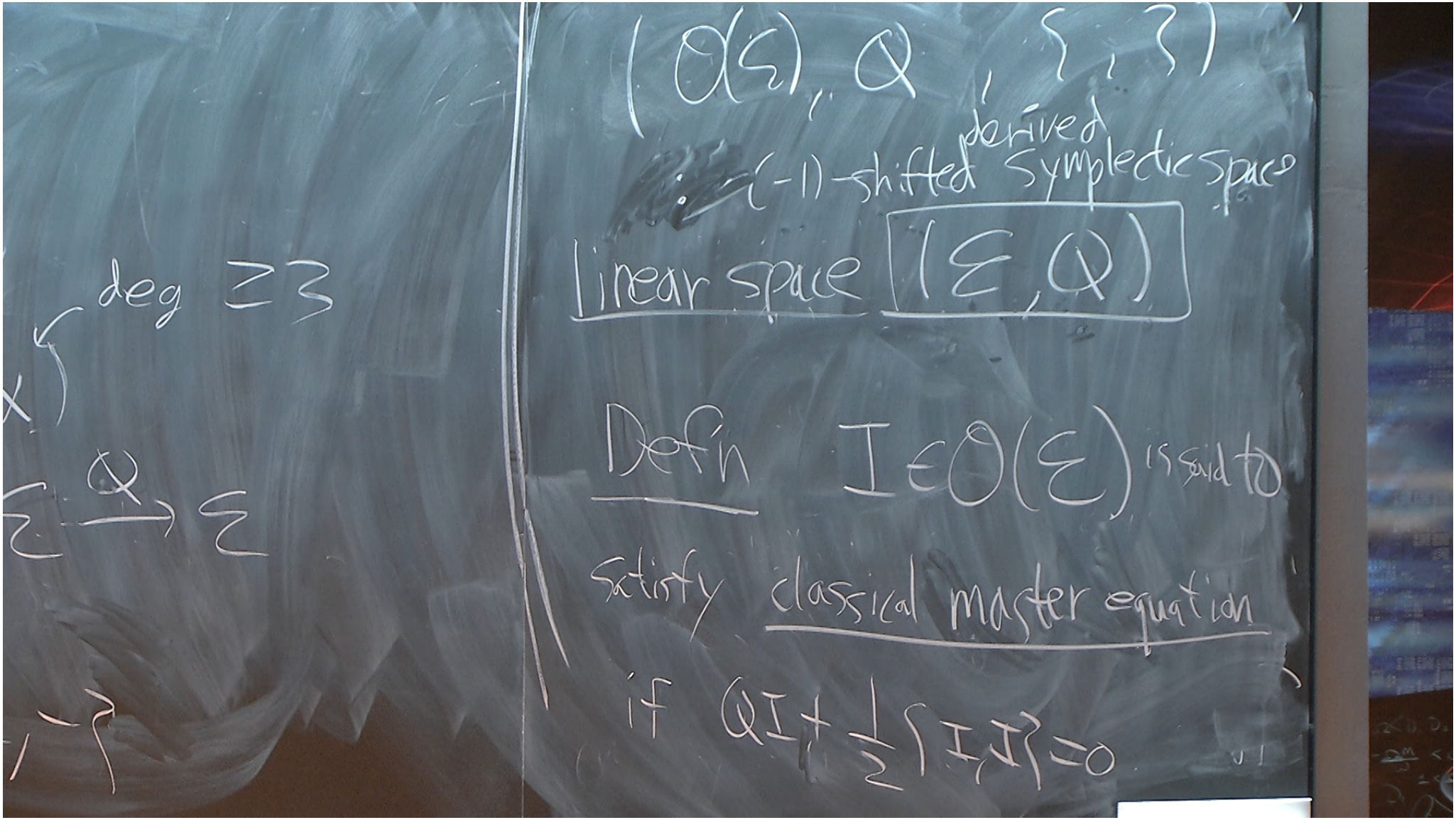
write $S_{BV}(\alpha) = S_{free}(\alpha) + I(\alpha)$ (deg 2; deg $\geq 3$)

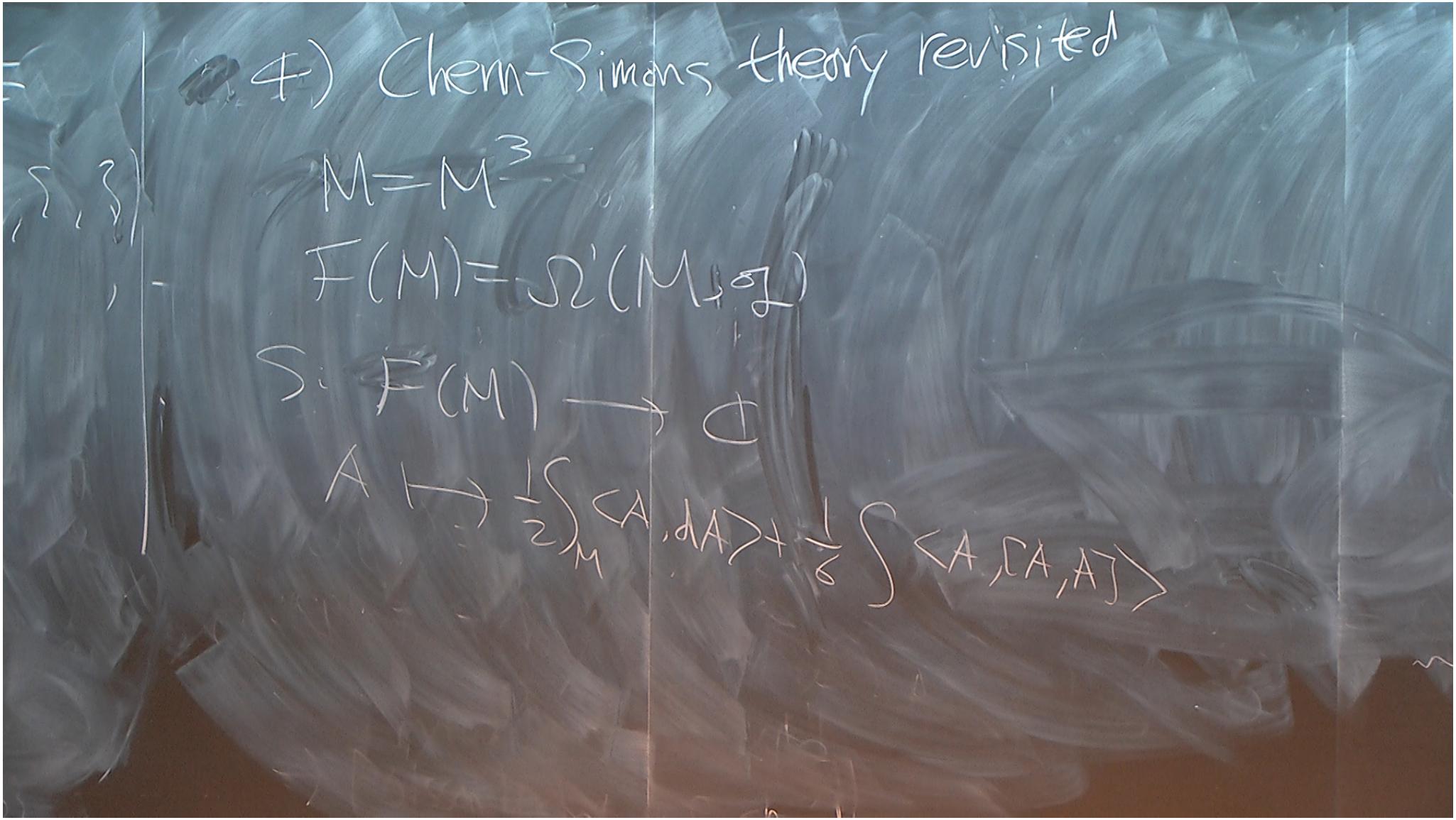
if $S_{free}(\alpha) = \frac{1}{2}\omega(\alpha, Q\alpha)$

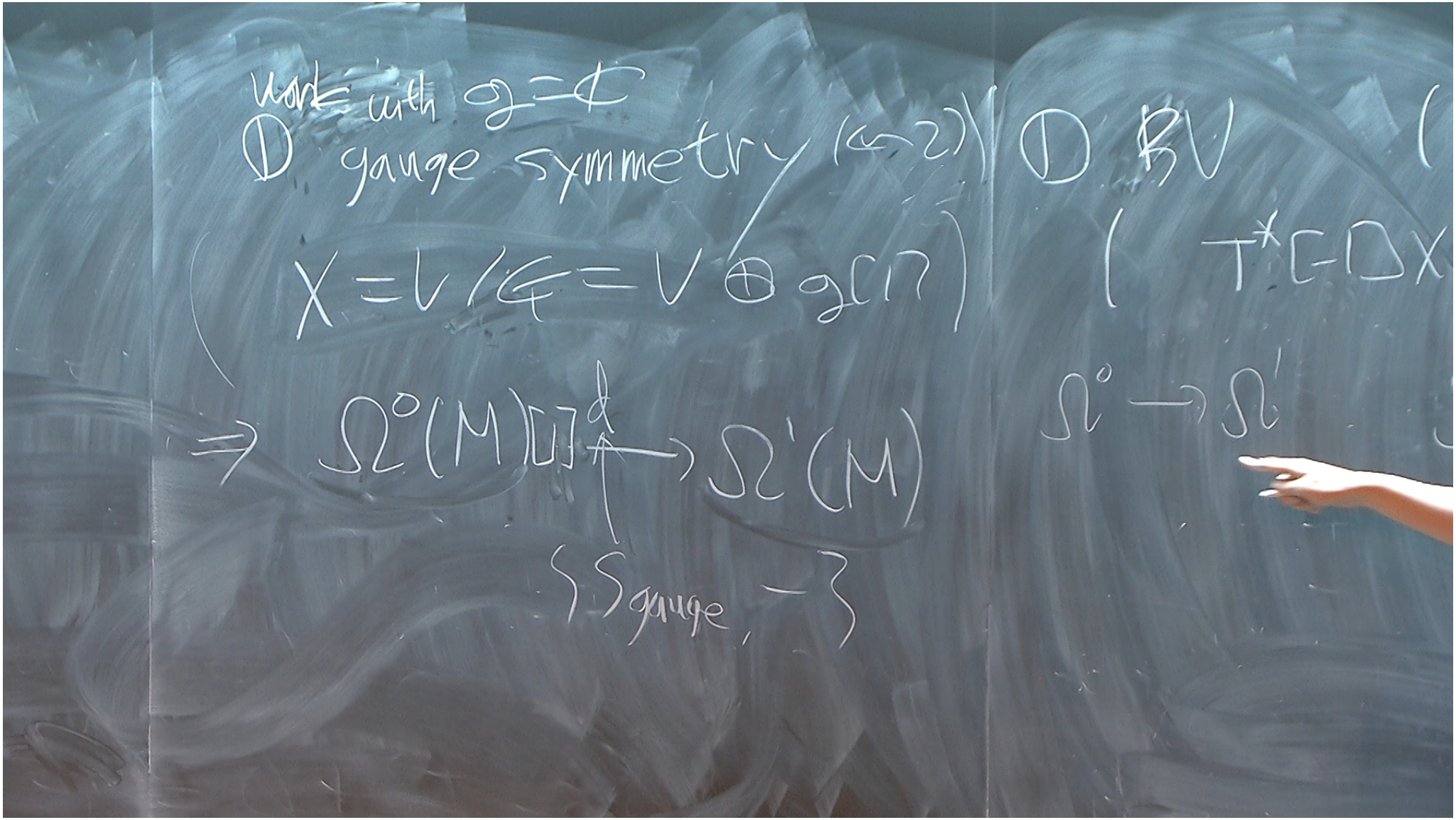
then $\{S_{BV}, -\} = Q + \{I, -\}$

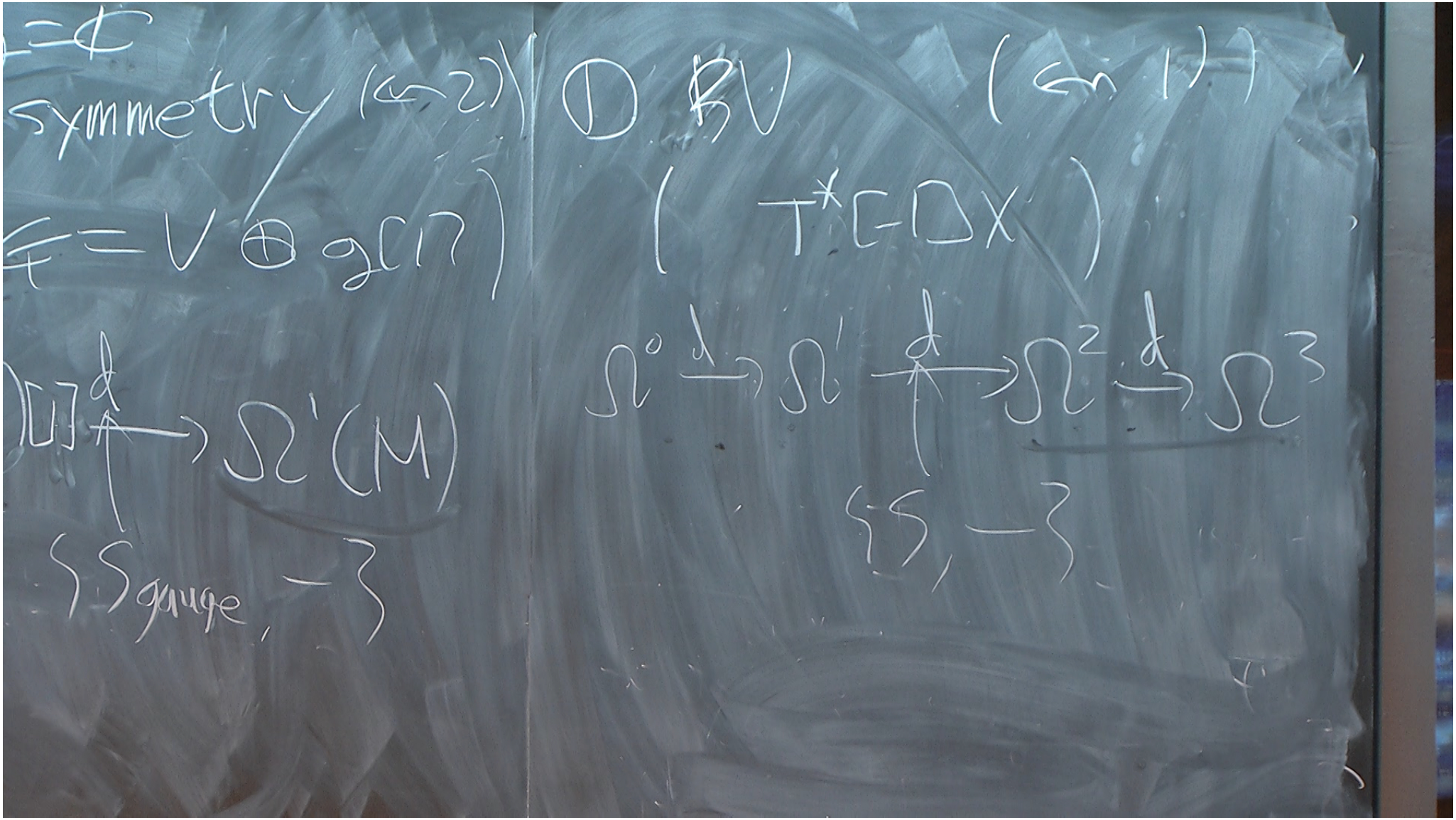

$(\mathcal{E}, \{S_{BV}, -\}, \mathcal{Q}, \int)$
ture reference
deg 2
deg $\geq 3$
$S(\alpha) = S_{free}(\alpha) + I(\alpha)$
$\mathcal{E} \xrightarrow{Q} \mathcal{E}$
$\{S_{BV}, -\} = Q + \{I, -\}$
$(\mathcal{E}, Q, \{,\}, \int)$
(-1)-shifted derived symplectic space
$(\mathcal{E}, Q)$
$\mathcal{E}(\mathcal{E})$

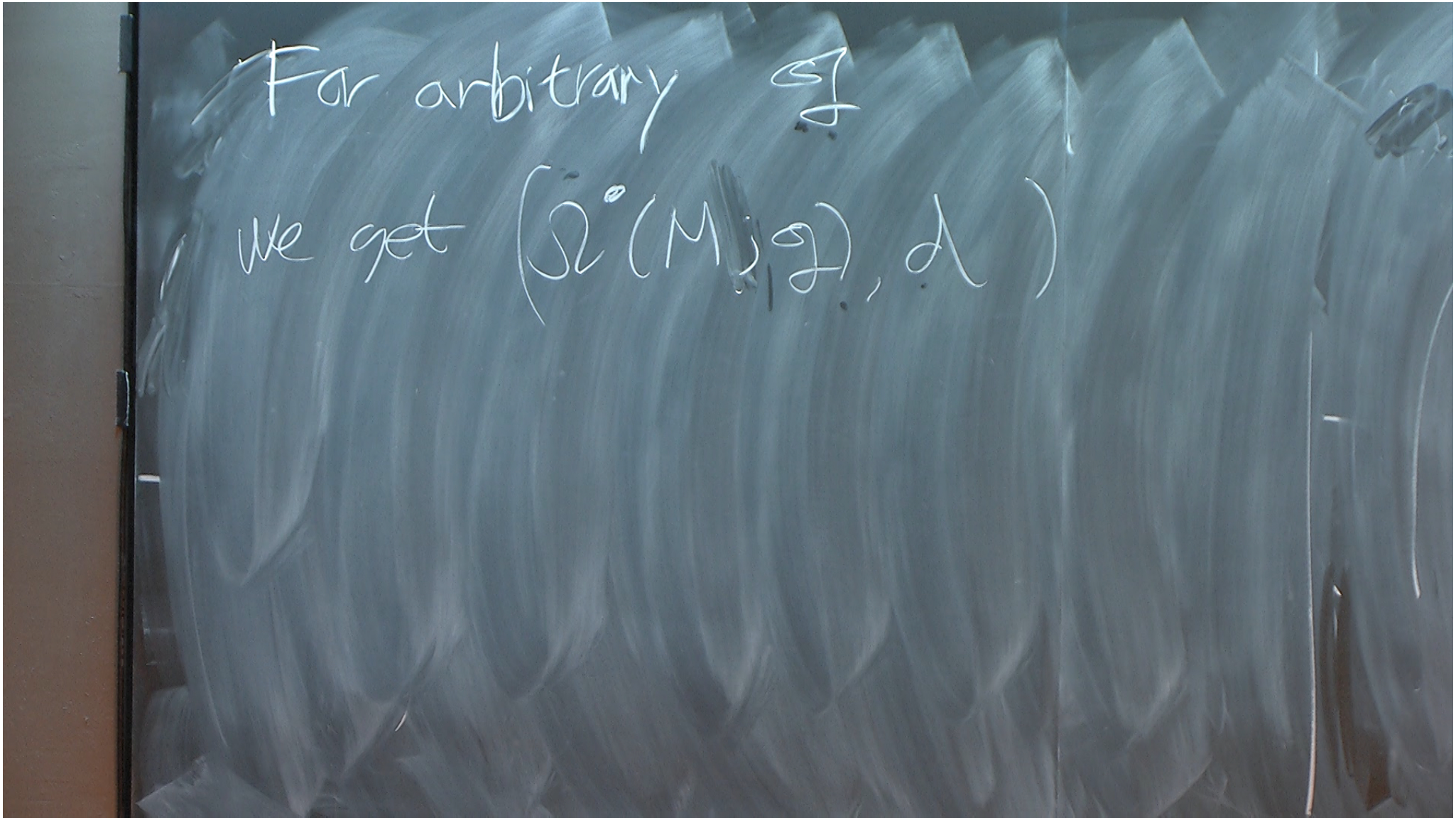

deg ≥ 3
(-1)-shifted
derived
symplectic space
linear space

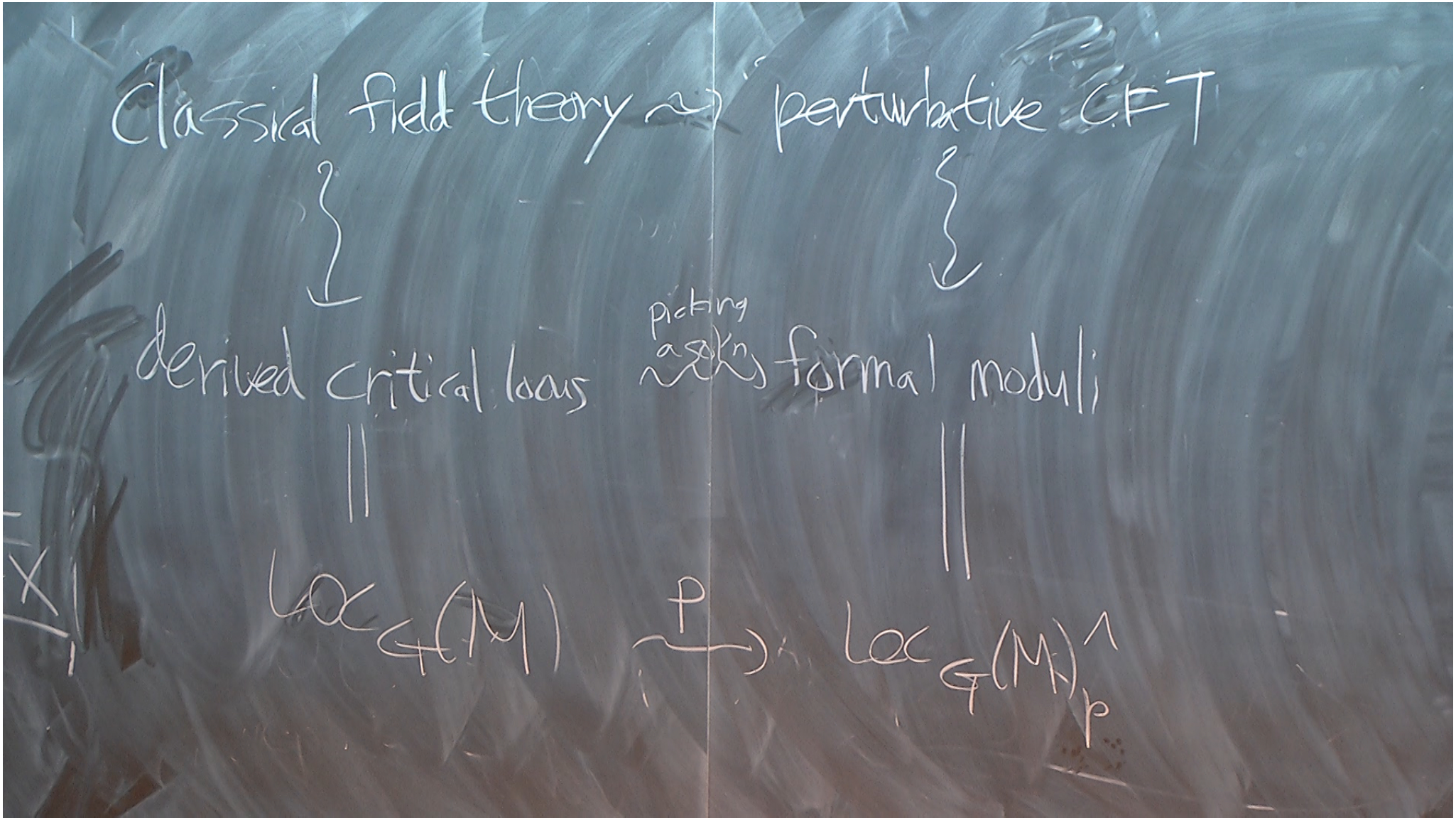

deg ≥ 3
$(\mathcal{O}(\mathcal{E}), Q, \{\,,\,\})$
(-1)-shifted derived symplectic space
linear space $(\mathcal{E}, Q)$
Defn $I \in \mathcal{O}(\mathcal{E})$ is said to
satisfy classical master equation
if $QI + \frac{1}{2}\{I, I\} = 0$

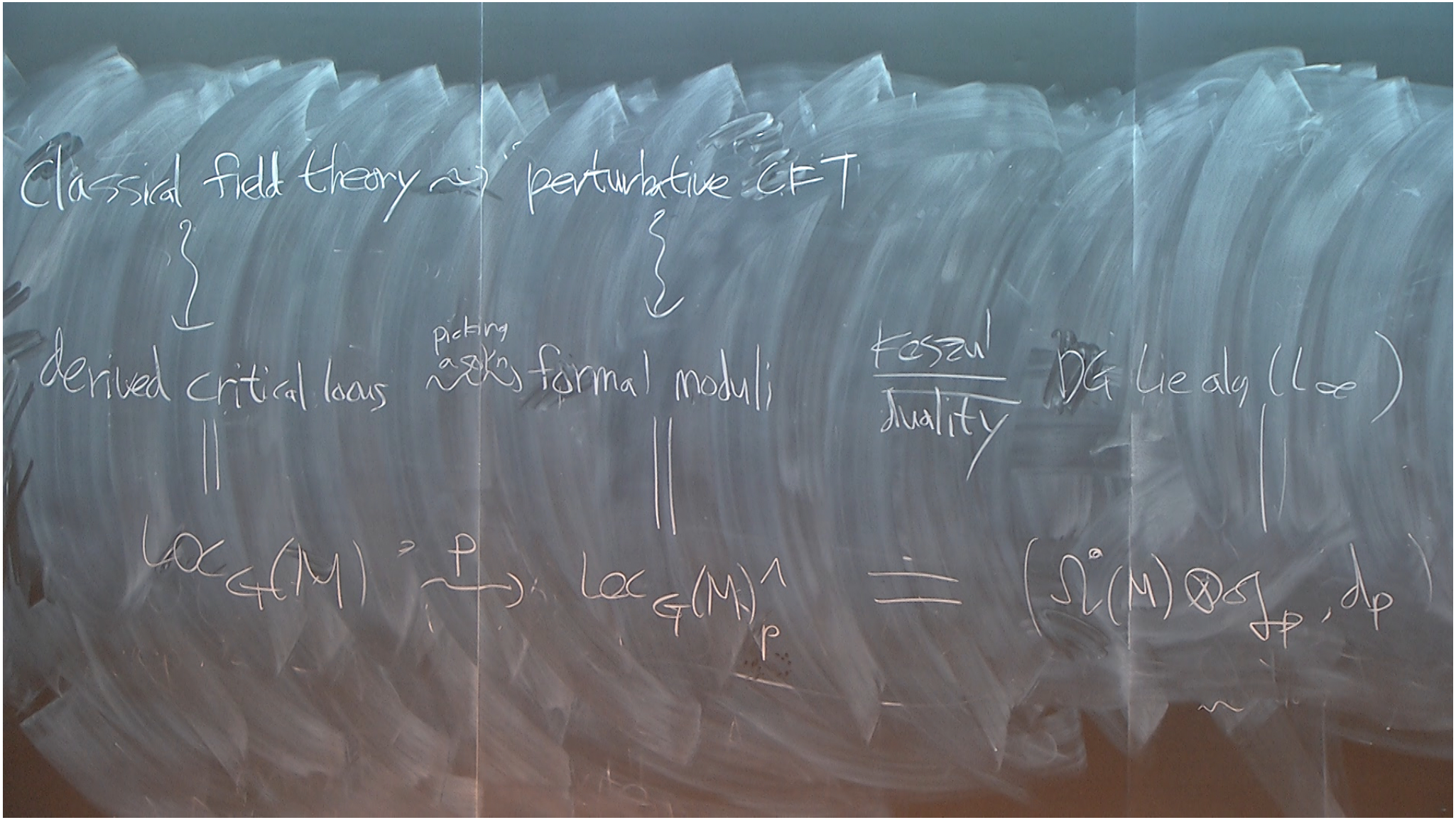

4) Chern-Simons theory revisited
$M = M^3$
$\mathcal{F}(M) = \Omega^1(M; \mathfrak{g})$
$S: \mathcal{F}(M) \longrightarrow \mathbb{C}$
$A \longmapsto \frac{1}{2}\int_M \langle A, dA \rangle + \frac{1}{6}\int \langle A, [A, A] \rangle$

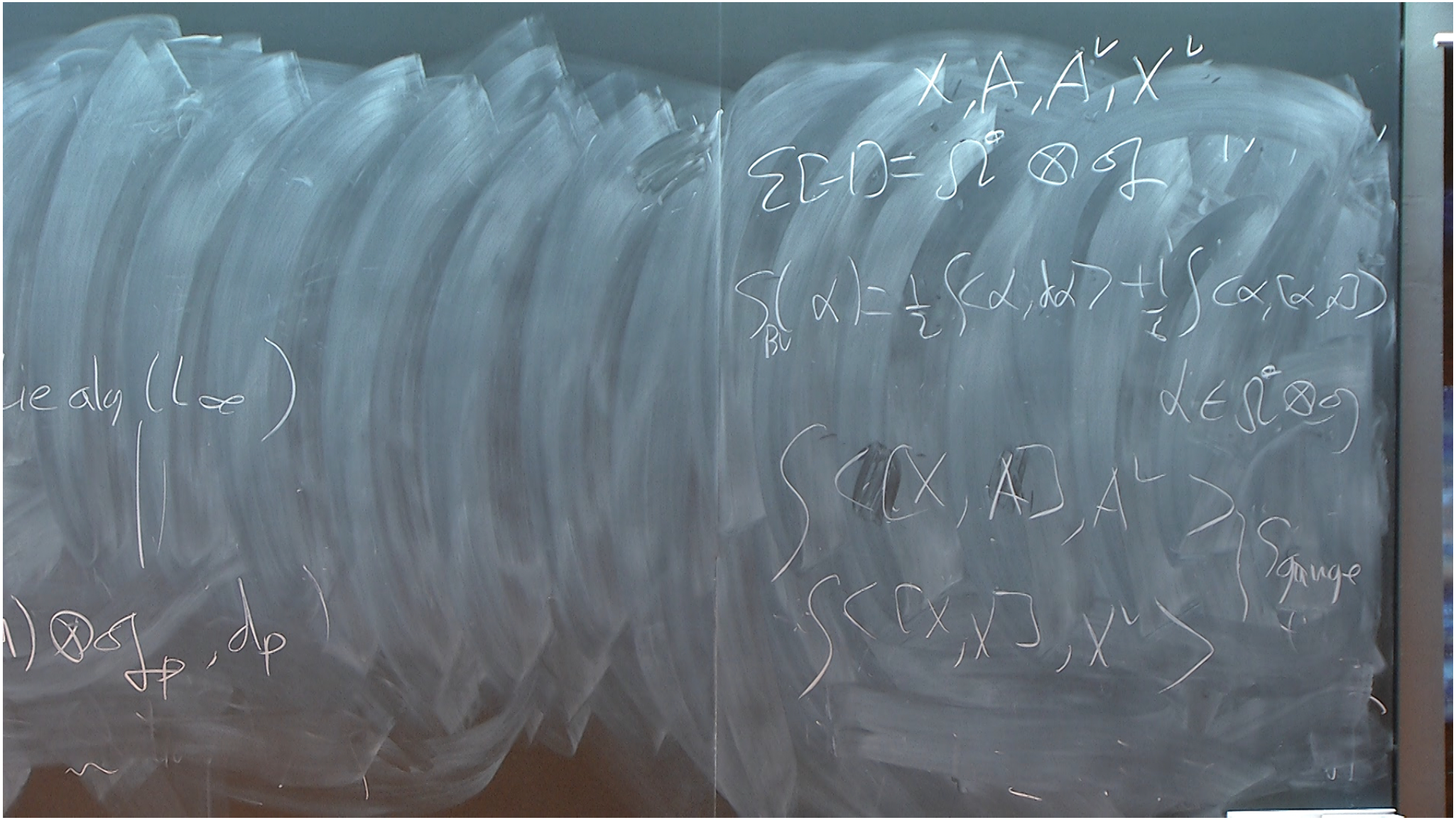

Work with g = ℒ
gauge symmetry (↪2)
X = V/G = V ⊕ g(1)
Ω°(M)[1] —d→ Ω¹(M)
{S_gauge, −}
BV
T*G-DX
Ω° → Ω¹


symmetry
BV
Ω'(M)
Ω⁰ → Ω¹ → Ω² → Ω³


For arbitrary g
we get $(\tilde{\Omega}^\bullet(M;g), d)$


Classical field theory
perturbative QFT
derived critical locus
picking a pt
formal moduli
$\mathrm{Loc}_G(M)$
P
$\mathrm{Loc}_G(M)^{\wedge}_P$


Classical field theory ~ perturbative QFT
derived critical locus
picking a pt
formal moduli
Koszul duality
DG Lie alg (L∞)
Loc_G(M)
Loc_G(M)^
Ω°(M) ⊗ g_P , d_P


X, A, A^∨, X^∨
Σ[-1] = Ω^• ⊗ g
S_BV(α) = ½⟨α, dα⟩ + 1/3! ∫⟨α, [α, α]⟩
α ∈ Ω^• ⊗ g
∫⟨[X, A], A^∨⟩
∫⟨[X, X], X^∨⟩
S_gauge
ie alg (L∞)
A) ⊗ g_p, d_p