Title: Relative Quantum Time

Speakers: Leon Loveridge

Series: Quantum Foundations

Date: May 28, 2019 - 3:30 PM

URL: http://pirsa.org/19050039

Abstract: The need for a time-shift invariant formulation of quantum theory arises from fundamental symmetry principles as well as heuristic cosmological considerations. Such a description then leaves open the question of how to reconcile global invariance with the perception of change, locally. By introducing relative time observables, we are able to make rigorous the Page-Wootters conditional probability formalism to show how local Heisenberg evolution is compatible with global invariance.

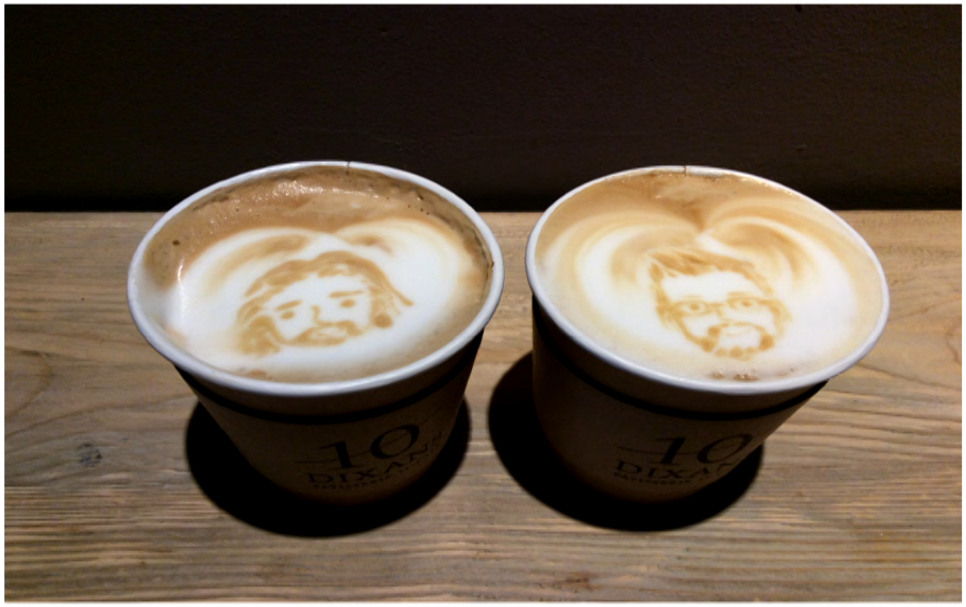
Relative Quantum Time

Leon Loveridge, with Paul Busch and Takayuki Miyadera

University of South-Eastern Norway

May 28, 2019

Dedicated to the memory of Paul Busch, 1955-2018.


10
DIXAN

Ordinary Quantum Framework

- Hilbert space $\mathcal{H}$
- States $\rho \in \mathcal{T}(\mathcal{H})$
- Outcomes $(\Omega, \mathcal{F})$
- Observables $\mathsf{E} : \mathcal{F} \to \mathcal{L}(\mathcal{H})$
- Probabilities $X \mapsto \mathrm{tr}\,[\rho \mathsf{E}(X)]$
- Symmetry $G \mapsto \mathrm{aut}\mathcal{L}(\mathcal{H})$

What is observable under symmetry?

- First guess: invariants
- Problem: (apparently) very little left
- No coherent states (phase shift invariance)
- Very "few" superpositions
- No localised particles (shift invariance)
- No dynamics, etc.
- Ordinary framework very successful... how can it be?
- Look to classical situation

Classical Quantities and Symmetry

Symmetry, relativity, reference frames are related:

- Positions, angles, event times, velocities are *relative*
- $G =$ Galilei group
- Invariant once frame-dependence is accounted for: apply symmetry at composite level

"Absolute" $\sim$ relative:

- Reference frame $=$ inertial frame
- Coordinate system or "suitable" classical particle
- Particle localised with respect to all classical variables
- Can "externalise" particle/RF and work in "absolute" sense
- Full equivalence of pictures!

Phase

- Number observables $N_{\mathcal{S}}, N_{\mathcal{R}}, N$, groups $U_{\mathcal{S}}(\theta) := e^{iN_{\mathcal{S}}\theta}$ etc
- "Absolute" phase (POM) F of $\mathcal{S}$ characterised by $U_{\mathcal{S}}(\theta)\mathsf{F}(X)U_{\mathcal{S}}(\theta)^* = \mathsf{F}(X+\theta)$
- $\mathcal{S}$ alone: observables commute with $N_{\mathcal{S}}$ (invariant under phase-shifts)
- ρ and $\tau_{\mathcal{S}_*}(\rho) := \sum_n P_n \rho P_n$ cannot be distinguished
- Coherent and incoherent states are observationally equivalent ($\sim$ class of states)
- c.f. "optical coherence controversy". Isn't coherence "real"?!
- Answer comes with relative quantities, rethinking definition of "coherent".
- System-plus-reference $\mathcal{S}+\mathcal{R}$ represented by $\mathcal{H} \equiv \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}}$

Relativisation, ¥ map

Can relativise "absolute" quantities to give relative/invariant ones:

$$¥(A) = \int_G U_{\mathcal{S}}(g) A U_{\mathcal{S}}(g)^* \otimes \mathsf{F}^{\mathcal{R}}(dg) \tag{1}$$

- Works for any locally compact metrizable group G (finite, $\mathbb{R}$, S_1,...)
- Unital, *-preserving, normal, completely positive...
- * Can choose $\mathcal{H}_{\mathcal{R}} = L^2(G)$, System of Imprimitivity, $(\mathcal{A} \otimes L^2(G))^G \cong G \ltimes \mathcal{A}$, $¥ : \mathcal{A} \hookrightarrow G \ltimes \mathcal{A}$...
- ¥ functions as expected in familiar cases: position, angle, phase etc.
- What is the relationship between "absolute" description (A) and relative description ($¥(A)$)?

Restriction

Compare like with like: "restrict" relative quantities of $\mathcal{S}+\mathcal{R}$ to quantities of $\mathcal{S}$. Precisely:

- Fix a state $\rho_{\mathcal{R}}$ of $\mathcal{R}$
- Define "restriction map" (conditional expectation) $\Gamma_{\rho_{\mathcal{R}}} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}_{\mathcal{S}})$
- $\Gamma_{\rho_{\mathcal{R}}}(A \otimes B) = A\,\mathrm{tr}\left[\rho_{\mathcal{R}} B\right]$, extend by linearity, continuity
- Gives description in terms of $\mathcal{S}$, *contingent* on state $\rho_{\mathcal{R}}$ of $\mathcal{R}$
- Can now find conditions under which A and $(\Gamma_{\rho_{\mathcal{R}}} \circ ¥)(A)$ are close (and conditions under which they are not).

Localisation/delocalisation

Is there $\rho_{\mathcal{R}}$ for which "absolute" A is close (in appropriate sense) to restricted, relativised A? Yes! Subject to a condition.

- Need "norm-1" property for $\mathsf{F}^{\mathcal{R}}$: for any $X \in \mathcal{B}(G)$ for which $\mathsf{F}^{\mathcal{R}}(X) \neq 0$, $\exists (\phi_i) \in \mathcal{H}_{\mathcal{R}}$ s.t. $\lim_{i\to\infty} \langle \phi_i \,|\, \mathsf{F}^{\mathcal{R}}(X)\phi_i \rangle = 1$.
- Localisability condition - satisfied for PVMs, "canonical phase", etc...
- Then (e.g., for phase), choose X "very small set" containing origin, (ϕ_i) localising sequence
- Find that $(\Gamma_{\phi_i} \circ ¥)(A) \xrightarrow{weak} A$ as $i \to \infty$.
- E.g., $\mathsf{F}^{\mathcal{R}}$ "canonical phase", limit taken across a set of high-amplitude coherent states
- Other extreme: take $\tau_{\mathcal{R}*}(\rho_{\mathcal{R}})$ as reference state:

$$(\Gamma_{\tau_{\mathcal{R}*}(\rho_{\mathcal{R}})} \circ ¥)_*(\rho) = \frac{1}{2\pi}\int_{S^1} U(\theta)^* \rho U(\theta) d\theta = \tau_{\mathcal{S}_*}(\rho) \qquad (2)$$

Localisation/delocalisation

Is there $\rho_{\mathcal{R}}$ for which "absolute" A is close (in appropriate sense) to restricted, relativised A? Yes! Subject to a condition.

- Need "norm-1" property for $\mathsf{F}^{\mathcal{R}}$: for any $X \in \mathcal{B}(G)$ for which $\mathsf{F}^{\mathcal{R}}(X) \neq 0$, $\exists (\phi_i) \in \mathcal{H}_{\mathcal{R}}$ s.t. $\lim_{i\to\infty} \langle \phi_i \,|\, \mathsf{F}^{\mathcal{R}}(X)\phi_i \rangle = 1$.
- Localisability condition - satisfied for PVMs, "canonical phase", etc...
- Then (e.g., for phase), choose X "very small set" containing origin, (ϕ_i) localising sequence
- Find that $(\Gamma_{\phi_i} \circ \yen)(A) \xrightarrow{weak} A$ as $i \to \infty$.
- E.g., $\mathsf{F}^{\mathcal{R}}$ "canonical phase", limit taken across a set of high-amplitude coherent states
- Other extreme: take $\tau_{\mathcal{R}*}(\rho_{\mathcal{R}})$ as reference state:

$$(\Gamma_{\tau_{\mathcal{R}*}(\rho_{\mathcal{R}})} \circ \yen)_*(\rho) = \frac{1}{2\pi} \int_{S^1} U(\theta)^* \rho U(\theta) d\theta = \tau_{\mathcal{S}*}(\rho) \qquad (2)$$

Story so far

- Observables are invariant quantities (of $\mathcal{S} + \mathcal{R}$)
- "Absolute" quantities of $\mathcal{S}$ represent relative (invariant) observables of $\mathcal{S} + \mathcal{R}$.
- Good representation/approximation comes with good localisation at group identity (i.e., "zero" of phase)
- Good localisation allows externalisation of RF, as in classical case
- *Justifies* use of ordinary quantum framework for calculations
- Bad localisation is bad reference, a quantum restriction arising from uncertainty relation

Hold on...

Some of this smells circular:

- In order to speak of "absolute" quantities and coherent/localized states of $\mathcal{S}$ as representing their invariant counterparts of $\mathcal{S} + \mathcal{R}$, "absolute" quantities and coherent/localized states are presumed for $\mathcal{R}$ (cf superselection rule "debate".)
- Require a fully relational picture
- "Absolute" states of $\mathcal{S}$ represent relative (invariant) states of the form $\tau_*(\rho \otimes P[\phi_i])$
- Localisation and coherence are relational notions
- Properties of *pairs* of systems (cf entanglement)

Relativisation, ¥ map

Can relativise "absolute" quantities to give relative/invariant ones:

$$¥(A) = \int_G U_{\mathcal{S}}(g) A U_{\mathcal{S}}(g)^* \otimes \mathsf{F}^{\mathcal{R}}(dg) \tag{1}$$

- Works for any locally compact metrizable group G (finite, $\mathbb{R}$, S_1,...)
- Unital, *-preserving, normal, completely positive...
- * Can choose $\mathcal{H}_{\mathcal{R}} = L^2(G)$, System of Imprimitivity, $(\mathcal{A} \otimes L^2(G))^G \cong G \ltimes \mathcal{A}$, $¥ : \mathcal{A} \hookrightarrow G \ltimes \mathcal{A}$...
- ¥ functions as expected in familiar cases: position, angle, phase etc.
- What is the relationship between "absolute" description (A) and relative description ($¥(A)$)?

Story so far

- Observables are invariant quantities (of $\mathcal{S} + \mathcal{R}$)
- "Absolute" quantities of $\mathcal{S}$ represent relative (invariant) observables of $\mathcal{S} + \mathcal{R}$.
- Good representation/approximation comes with good localisation at group identity (i.e., "zero" of phase)
- Good localisation allows externalisation of RF, as in classical case
- *Justifies* use of ordinary quantum framework for calculations
- Bad localisation is bad reference, a quantum restriction arising from uncertainty relation

Mutual Coherence

T.f.a.e.:

1. $\exists$ invariant observable E of $\mathcal{S}+\mathcal{R}$ and X such that $\mathrm{tr}\left[(\tau_{\mathcal{S}*}(\rho_{\mathcal{S}})\otimes\rho_{\mathcal{R}})\mathsf{E}(X)\right]\neq\mathrm{tr}\left[(\rho_{\mathcal{S}}\otimes\rho_{\mathcal{R}})\mathsf{E}(X)\right]$
2. $\exists$ invariant observable E of $\mathcal{S}+\mathcal{R}$ and X such that $\mathrm{tr}\left[(\rho_{\mathcal{S}}\otimes\tau_{\mathcal{R}*}(\rho_{\mathcal{R}})\mathsf{E}(X)\right]\neq\mathrm{tr}\left[(\rho_{\mathcal{S}}\otimes\rho_{\mathcal{R}})\mathsf{E}(X)\right]$

Therefore,

- $\rho_{\mathcal{S}}$ is coherent "relative to" $\rho_{\mathcal{R}}$ if and only if $\rho_{\mathcal{R}}$ is coherent "relative to" $\rho_{\mathcal{S}}$
- Better: $(\rho_{\mathcal{S}},\rho_{\mathcal{R}})$ coherent if (1) (or 2) holds
- Truly relational: depends on $\mathcal{S}$ and $\mathcal{R}$
- Same for localisation
- $$\lim_{i\to\infty}\yen_*(\tau_*(\rho_{\mathcal{S}}\otimes P[\phi_i])=\rho_{\mathcal{S}} \tag{3}$$
- Any state of $\mathcal{S}$ can be approximated by invariant states of $\mathcal{S}+\mathcal{R}$.

Is a laser beam coherent?

Consider for $\mathcal{S}$

- "Absolute" phase observable $\mathsf{F}^{\mathcal{S}}$
- Coherent state $|\beta\rangle = \sum_n c_n |n\rangle$

Construct relative phase observable $\mathsf{F}^{T} = ¥ \circ \mathsf{F}^{\mathcal{S}}$,

$$\begin{aligned} \left\langle \beta \,|\, \mathsf{F}^{\mathcal{S}}(X)\beta \right\rangle &= \lim_{i\to\infty} \left\langle \beta \otimes \phi_i \,|\, (¥ \circ \mathsf{F}^{\mathcal{S}})(X)\beta \otimes \phi_i \right\rangle \\ &= \lim_{i\to\infty} \mathrm{tr}\left[\mathsf{F}^{T}(X)\tau_*(P[\beta \otimes \phi_i])\right] \end{aligned}$$

- Limit across set of coherence states $\{\phi_i\}$ (high amplitude)
- $\mathsf{F}^{\mathcal{S}}$ can be "measured" in homodyne detection experiments
- RF a local oscillator in a high-amplitude coherent state
- $(|\beta\rangle, \phi_i)$ mutually coherent pair
- Relational coherence takes on the appearance of "absolute" coherence of a laser in the state $|\beta\rangle$ in the large amplitude limit of the (ϕ_i)

Is a laser beam coherent?

Consider for $\mathcal{S}$

- "Absolute" phase observable $\mathsf{F}^{\mathcal{S}}$
- Coherent state $|\beta\rangle = \sum_n c_n |n\rangle$

Construct relative phase observable $\mathsf{F}^{\mathcal{T}} = \yen \circ \mathsf{F}^{\mathcal{S}}$,

$$\langle \beta | \mathsf{F}^{\mathcal{S}}(X) \beta \rangle = \lim_{i \to \infty} \langle \beta \otimes \phi_i | (\yen \circ \mathsf{F}^{\mathcal{S}})(X) \beta \otimes \phi_i \rangle$$
$$= \lim_{i \to \infty} \operatorname{tr} \left[\mathsf{F}^{\mathcal{T}}(X) \tau_*(P[\beta \otimes \phi_i]) \right]$$

- Limit across set of coherence states $\{\phi_i\}$ (high amplitude)
- $\mathsf{F}^{\mathcal{S}}$ can be "measured" in homodyne detection experiments
- RF a local oscillator in a high-amplitude coherent state
- $(|\beta\rangle, \phi_i)$ mutually coherent pair
- Relational coherence takes on the appearance of "absolute" coherence of a laser in the state $|\beta\rangle$ in the large amplitude limit of the (ϕ_i)

Is a laser beam coherent?

Consider for $\mathcal{S}$

- "Absolute" phase observable $\mathsf{F}^{\mathcal{S}}$
- Coherent state $|\beta\rangle = \sum_n c_n |n\rangle$

Construct relative phase observable $\mathsf{F}^{T} = ¥ \circ \mathsf{F}^{\mathcal{S}}$,

$$\left\langle \beta \,\middle|\, \mathsf{F}^{\mathcal{S}}(X)\beta \right\rangle = \lim_{i\to\infty} \left\langle \beta\otimes\phi_i \,\middle|\, (¥\circ\mathsf{F}^{\mathcal{S}})(X)\beta\otimes\phi_i \right\rangle$$
$$= \lim_{i\to\infty} \mathrm{tr}\left[\mathsf{F}^{T}(X)\tau_*(P[\beta\otimes\phi_i])\right]$$

- Limit across set of coherence states $\{\phi_i\}$ (high amplitude)
- $\mathsf{F}^{\mathcal{S}}$ can be "measured" in homodyne detection experiments
- RF a local oscillator in a high-amplitude coherent state
- $(|\beta\rangle, \phi_i)$ mutually coherent pair
- Relational coherence takes on the appearance of "absolute" coherence of a laser in the state $|\beta\rangle$ in the large amplitude limit of the (ϕ_i)

Time?

- How to reconcile time translation invariance with time evolution of subsystems?
- How to understand classical/external "t" in Schrödinger equation?
- Proposed resolution: Page and Wootters 1983
- Idea: subsystem as "clock", condition on values
- Other system conditionally evolves according to Schrödinger equation
- Criticised on various grounds

Absolute and Relative Time Observables I

- Pauli: no self-adjoint (absolute) time observable in general
- Can be modelled as a POVM $\mathsf{E} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$

$$e^{-iHt}\mathsf{E}(X)e^{iHt} = \mathsf{E}(X + t) \tag{4}$$

- Consider clock $\mathcal{C}$ and reference $\mathcal{R}$
- Hamiltonians $H_{\mathcal{C}}$, $H_{\mathcal{R}}$, $V_{\mathcal{C}}(t)$, $V_{\mathcal{R}}(t)$
- Relative Time Observable Z on $\mathcal{H}_{\mathcal{C}} \otimes \mathcal{H}_{\mathcal{R}}$ defined by:
- $(V_{\mathcal{C}}(t) \otimes V_{\mathcal{R}}(t))^{*}\, \mathsf{Z}(\Delta)\, (V_{\mathcal{C}}(t) \otimes V_{\mathcal{R}}(t)) = \mathsf{Z}(\Delta)$ for all $\Delta \in \mathcal{B}(\mathbb{R})$ (Invariance)
- $V_{\mathcal{C}}(t)^{*}\Gamma_{\rho}(\mathsf{Z}(\Delta))V_{\mathcal{C}}(t) = \Gamma_{\rho}(\mathsf{Z}(\Delta - t))$ for all $\Delta \in \mathcal{B}(\mathbb{R})$ and $\rho \in \mathcal{S}(\mathcal{H}_{\mathcal{R}})$ (Covariance)

Absolute and Relative Time Observables II

Existence established through relativisation:

$$(\yen \circ \mathsf{E}_{\mathcal{C}})(X) = \int_{\mathbb{R}} \mathsf{E}_{\mathcal{C}}(X + t) \otimes \mathsf{E}_{\mathcal{R}}(dt). \tag{5}$$

Makes sense also in discrete setting...

- Replace $\mathbb{R}$ by $\mathbb{Z}_d$
- Discrete and sharp periodic time observables exist in $\mathbb{C}^d$
- Let $T_{\mathcal{C}}$ and $T_{\mathcal{R}}$ be given by $\sum n|n\rangle\langle n|$

Absolute and Relative Time Observables I

- Pauli: no self-adjoint (absolute) time observable in general
- Can be modelled as a POVM $\mathsf{E} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$

$$e^{-iHt}\mathsf{E}(X)e^{iHt} = \mathsf{E}(X+t) \tag{4}$$

- Consider clock $\mathcal{C}$ and reference $\mathcal{R}$
- Hamiltonians $H_{\mathcal{C}}$, $H_{\mathcal{R}}$, $V_{\mathcal{C}}(t)$, $V_{\mathcal{R}}(t)$
- Relative Time Observable Z on $\mathcal{H}_{\mathcal{C}} \otimes \mathcal{H}_{\mathcal{R}}$ defined by:
- $(V_{\mathcal{C}}(t) \otimes V_{\mathcal{R}}(t))^* \, \mathsf{Z}(\Delta) \, (V_{\mathcal{C}}(t) \otimes V_{\mathcal{R}}(t)) = \mathsf{Z}(\Delta)$ for all $\Delta \in \mathcal{B}(\mathbb{R})$ (Invariance)
- $V_{\mathcal{C}}(t)^* \Gamma_\rho(\mathsf{Z}(\Delta)) V_{\mathcal{C}}(t) = \Gamma_\rho(\mathsf{Z}(\Delta - t))$ for all $\Delta \in \mathcal{B}(\mathbb{R})$ and $\rho \in \mathcal{S}(\mathcal{H}_{\mathcal{R}})$ (Covariance)

Absolute and Relative Time Observables II

Existence established through relativisation:

$$(\yen \circ \mathsf{E}_{\mathcal{C}})(X) = \int_{\mathbb{R}} \mathsf{E}_{\mathcal{C}}(X+t) \otimes \mathsf{E}_{\mathcal{R}}(dt). \tag{5}$$

Makes sense also in discrete setting...

- Replace $\mathbb{R}$ by $\mathbb{Z}_d$
- Discrete and sharp periodic time observables exist in $\mathbb{C}^d$
- Let $T_{\mathcal{C}}$ and $T_{\mathcal{R}}$ be given by $\sum n|n\rangle\langle n|$

Discrete model I

Three systems, $\mathcal{S}$, $\mathcal{C}$, $\mathcal{R}$, arbitrary POVM $\mathsf{A} = \{\mathsf{A}(k)\}_k$ on $\mathcal{S}$

- $H = H_{\mathcal{S}} + P_{\mathcal{C}} + P_{\mathcal{R}}$
- $P_{\mathcal{C}} = \sum m|f_m\rangle\langle f_m|$, with $m \in \mathbb{Z}_d$ and $|f_m\rangle = \frac{1}{\sqrt{d}} \sum_n e^{2\pi imn/d}|n\rangle$.
- Actions $\alpha_k^{\mathcal{S}}(A) = e^{iH_{\mathcal{S}}k} A e^{-iH_{\mathcal{S}}k}$,
  $\alpha_k^{\mathcal{C}}(|n\rangle\langle m|) = e^{iP_{\mathcal{C}}k}|n\rangle\langle m|e^{-iP_{\mathcal{C}}k} = |n-k\rangle\langle m-k|$ etc,
- Relative time observable:

$$\yen(\mathbb{1} \otimes |n\rangle\langle n|) = \sum_m \mathbb{1} \otimes |n+m\rangle\langle n+m| \otimes |m\rangle\langle m|.$$

Discrete model II

- Also relativise A:

$$\yen(\mathsf{A}(k) \otimes \mathbb{1}) = \sum_m \alpha^{\mathcal{S}}_{-m}(\mathsf{A}(k)) \otimes \mathbb{1} \otimes |m\rangle\langle m|.$$

- Conditional probability comes from a joint measurement
- Measure $\{\yen(\mathbb{1} \otimes |n\rangle\langle n|)\}$ and $\{\yen(\mathsf{A}(k) \otimes \mathbb{1})\}$.
- They commute, so can be done
- Joint observable unique since $\{\yen(\mathbb{1} \otimes |n\rangle\langle n|)\}$ is sharp:

$$M(k,n) = \sum_m \alpha^{\mathcal{S}}_{-m}(\mathsf{A}(k)) \otimes |n+m\rangle\langle n+m| \otimes |m\rangle\langle m|. \tag{6}$$

- Joint probability in product state $|\Psi\rangle\langle\Psi| = |\psi^{\mathcal{S}}\rangle\langle\psi^{\mathcal{S}}| \otimes |0\rangle\langle 0| \otimes |\xi\rangle\langle\xi|$:

$$P(k,n) = \left\langle\psi^{\mathcal{S}}\middle|\alpha^{\mathcal{S}}_n(\mathsf{A}(k)\middle|\psi^{\mathcal{S}}\right\rangle |\langle -n|\xi\rangle|^2. \tag{7}$$

Discrete model III

- Marginal

$$P(n) = \sum_k P(k,n) = |\langle -n|\xi\rangle|^2. \tag{8}$$

- Assume these to be non-zero; conditional probability is:

$$P(k|n) = \left\langle \psi^{\mathcal{S}} \middle| \alpha_n^{\mathcal{S}}(\mathsf{A}(k)) \middle| \psi^{\mathcal{S}} \right\rangle. \tag{9}$$

- expectation of the 'Heisenberg-evolved' observable A.

Observations

- A arbitrary
- $|\langle n|\xi\rangle|^2$ non-vanishing for all $n \in \mathbb{Z}_d$ demands $|\xi\rangle$ is broadly spread out in time.
- Simplest choice is $|\xi\rangle = |f_m\rangle$ for some m, i.e., an eigenstate of the reference Hamiltonian.
- $|\Psi\rangle$ is unentangled. There exists an entangled with same distribution.
- Continuous time model shows similar behaviour
- Uses good clock localisation, reference delocalisation

Conclusion: dynamics can emerge out of a time-invariant situation!