Title: Supersymmetric Wilson Loops, Instantons, and Deformed W-algebras

Speakers: Nathan Haouzi

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Abstract: Wilson loops are important observables in gauge theory. In this talk, we study half-BPS Wilson loops of a large class of five dimensional supersymmetric quiver gauge theories with 8 supercharges. The Wilson loops are codimension 4 defects of the quiver gauge theory, and their interaction with self-dual instantons is captured by a 1d ADHM quantum mechanics. We compute the partition function as its Witten index. It turns out that we can understand the 5d physics in 3d gauge theory terms. This comes about from so-called gauge/vortex duality; namely, we study the vortices on the Higgs branch of the 5d theory, and reinterpret the physics from the point of view of the vortices. This perspective has an advantage: it has a dual description in terms of "deformed" Toda Theory on a cylinder, in the Coulomb gas formalism. We show that the gauge theory partition function is equal to a (chiral) correlator of the deformed Toda Theory, with stress tensor and higher spin operator insertions. We derive all the above results from type IIB string theory, compactified on a resolved ADE singularity X times a cylinder with punctures. The 5d quiver gauge theory arises as the low energy limit of a system of D5 branes wrapping various two-cycles of X, the Wilson loops are D1 branes, and the duality to Toda theory emerges after introducing additional D3 branes.

Supersymmetric Wilson Loops, Instantons, and Deformed W-algebras

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Based on:

1905.xxxxx

- Wilson loops make up an important class of observables in gauge theory.

- These are non-local operators in gauge theory, which encode essential aspects of the strongly-coupled regime. A Wilson loop is formulated as the trace of a holonomy matrix, where a quark is parallel transported along a closed curve in spacetime, and the trace is evaluated in a representation R of the gauge group.

- The focus of this talk will be to describe the physics of certain supersymmetric (i.e. BPS) Wilson loops in 5d gauge theories, and give a dual realization in 3d gauge theory.

- The 5d supersymmetric theories that will be relevant to us have N=1 supersymmetry (8 supercharges), and they will be compactified on a circle: $S^1(\widehat{R}) \times \mathbb{C}^2$.

- For a $U(n)$ gauge theory, one can introduce ½-BPS Wilson lines using a one-dimensional fermion field χ, in the fundamental representation of $U(n)$, coupled to the 5d gauge field in the bulk as:

$$S^{1d} = \int dt\, \chi^\dagger \left(\partial_t - iA_t + \Phi + M\right) \chi$$

[Gomis-Passerini '06]

- Above, A_t and Φ are the pullback of the 5d gauge field and adjoint scalar of the vector multiplet. M Is the real mass of the fermions (i.e. the background gauge field for the $U(1)$ symmetry acting on χ); it sets the energy scale for the excitation of the fermions.

- The path integral of the coupled 5d/1d system is a generating function of Wilson loops in the fundamental representations (i.e. the antisymmetric representations) of the gauge group:

$$Z^{5d/1d} = \int D\psi D\chi \, e^{i(S^{5d}[\psi]+S^{1d}[\psi,\chi,M])} = z^{-n/2} \sum_{j=0}^{n} (-z)^j \langle W_{\Lambda^j} \rangle$$

$$\langle W_{\mathbf{R}} \rangle = \mathrm{Tr}_{\mathbf{R}} \, \mathcal{P} \exp\left(i \int dt (A_t^{(a)} + i\, \Phi^{(a)}) \right)$$

- We denoted $z \equiv e^M$ above. In particular, in a trivial instanton background, the partition function is a polynomial of degree n in z.

- What happens when the quark moves in the presence of instantons?

- Instantons are solutions of the self-dual Yang Mills equations; a powerful way to identify such solutions is the so-called ADHM construction. [Atiyah-Drinfeld-Hitchin-Manin ’78]

- The presence of Wilson lines generalizes the ADHM construction, since when an instanton moves in the presence of a quark, it now also experiences a Lorentz force. The endeavor of understanding the dynamics of instantons in this modified background was initiated only recently. [Tong ‘14], [Tong-Wong ‘14], [Nekrasov ‘15], [Kim ‘16]

- Even in a nontrivial instanton background, the partition function of the 5d/1d system is still a polynomial in the fugacity z, of degree n:

$$Z^{5d/1d} = z^{-n/2} \sum_{j=0}^{n} (-z)^j \langle W_{\Lambda^j} \rangle$$

[Nekrasov ‘15], [Kim ‘16]

We will now generalize the previous analysis to a large class of 5d quiver gauge theories, and then reinterpret the results in 3d. Our starting point will be the type IIB string.

Outline

- 5d quiver gauge theory with Wilson Loops
- A duality to 3d
- From 3d gauge theories to deformed W-algebras

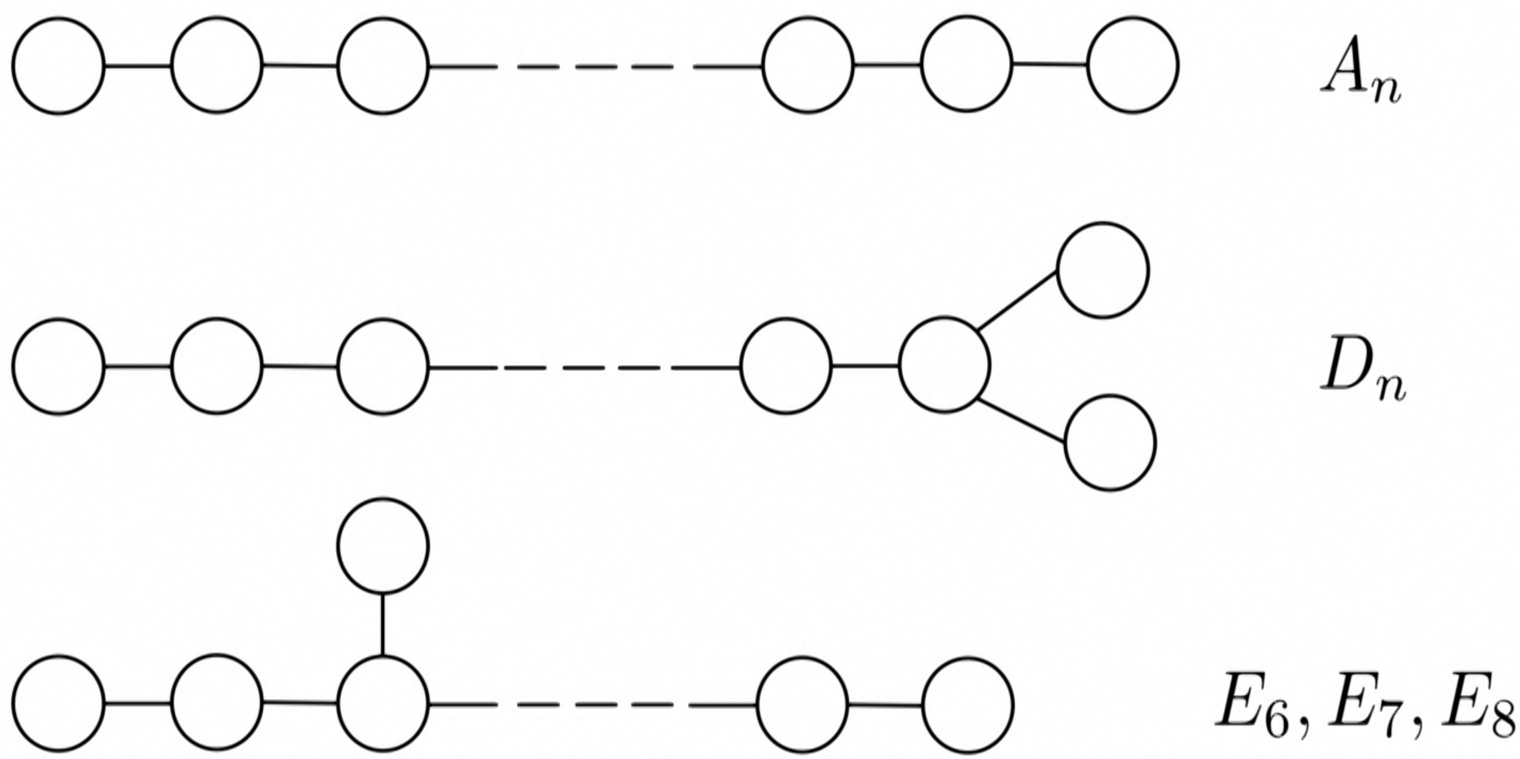
- For definiteness, let us consider:

$$\text{Type IIB on } \widetilde{\mathbb{C}^2/\Gamma_{\mathfrak{g}}} \times \mathcal{C} \times \mathbb{C}^2$$

$\mathcal{C} = S^1(R) \times \mathbb{R}$ is an infinite cylinder.

$\widetilde{\mathbb{C}^2/\Gamma_{\mathfrak{g}}}$ is a resolved $\mathfrak{g}$ = ADE singularity, labeled by a discrete subgroup of $SU(2)$.

ADE Classification:



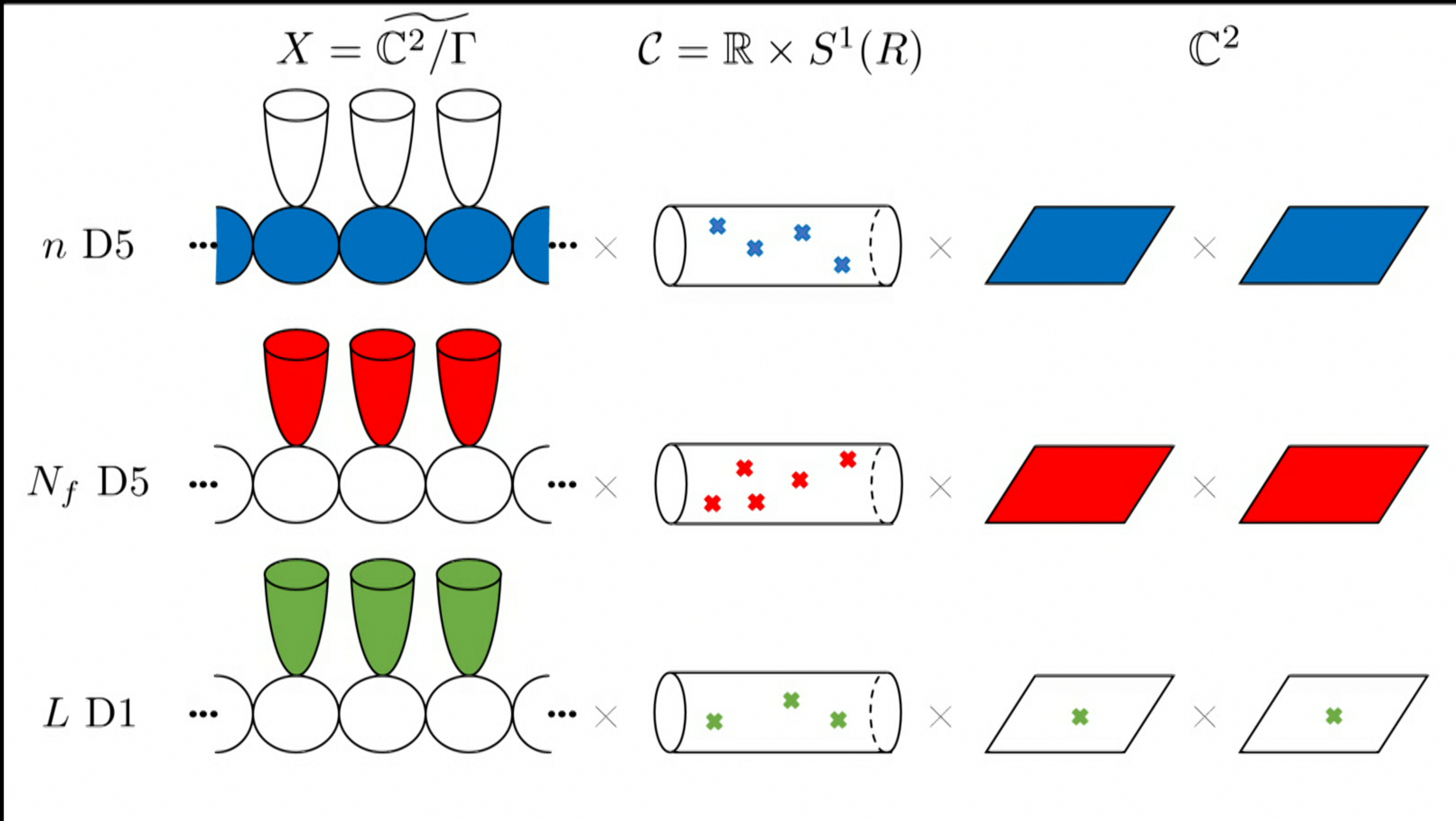
McKay Correspondence:

Representation theory of $\mathfrak{g}$ ⟺ Geometry of $X = \widetilde{\mathbb{C}^2/\Gamma_{\mathfrak{g}}}$

Roots in root lattice Λ ⟺ Compact two-cycles in $H_2(X, \mathbb{Z})$

Weights in weight lattice Λ_* ⟺ Non-compact two-cycles in $H_2(X, \partial X; \mathbb{Z})$

So far the background has 16 supercharges. Since we are ultimately interested in the dynamics of five dimensional gauge theories with 8 supercharges, we need to break supersymmetry further. A simple way to achieve this in string theory is to add D-branes.



$X = \widetilde{\mathbb{C}^2/\Gamma}$
$\mathcal{C} = \mathbb{R} \times S^1(R)$
$\mathbb{C}^2$
n D5
N_f D5
L D1

The D5 brane configuration preserves 8 supercharges.

After introducing the D1 branes, the number goes down to 4.

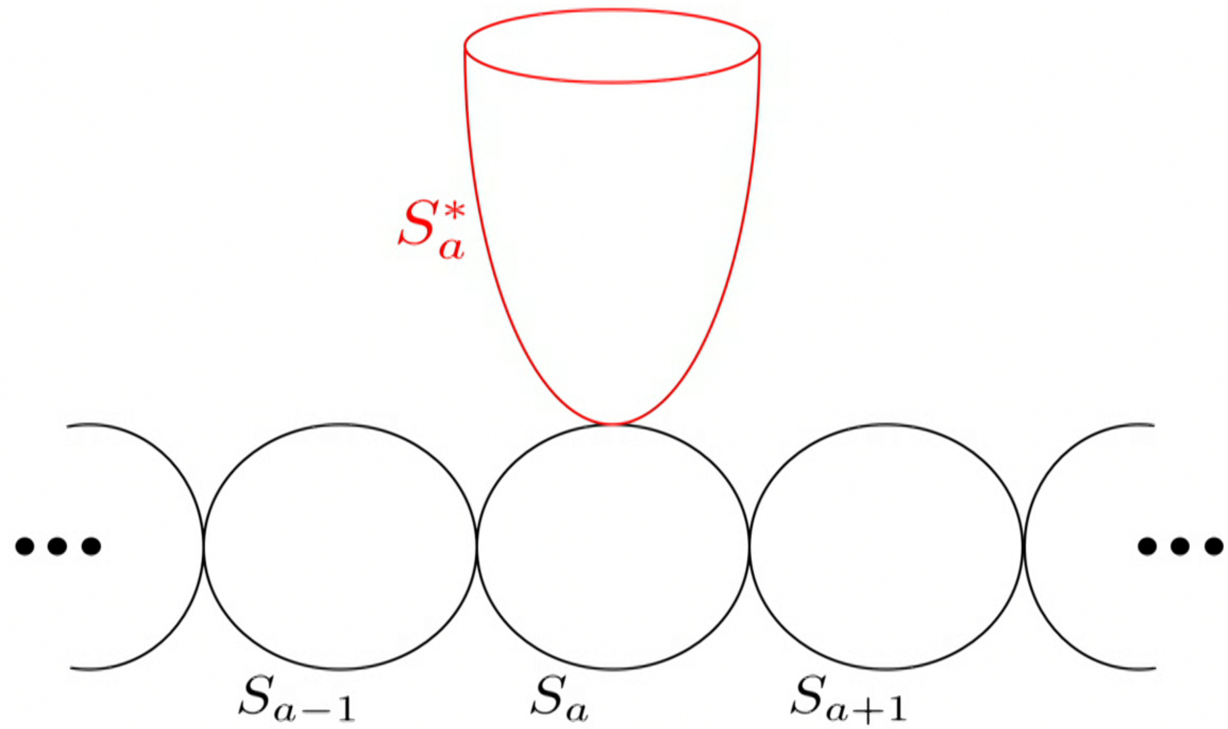
- We choose a class in $H_2(X;\mathbb{Z})$ to represent the brane charge due to the D5 branes wrapping the compact two-cycles, expanded in terms of positive simple roots:

$$[S] = -\sum_{a=1}^{m} n^{(a)}\,\alpha_a \qquad\qquad (\,[S_a] \equiv \alpha_a\,)$$

- Likewise, we choose a class in $H_2(X,\partial X;\mathbb{Z})$ to represent the total charge due to the D5 branes wrapping the noncompact two-cycles, expanded in terms of fundamental weights:

$$[S^*] = \sum_{a=1}^{m} N_f^{(a)}\,\lambda_a \qquad\qquad (\,[S^*_a] \equiv \lambda_a\,)$$

$$\#(S_a \cap S^*_b) = \delta_{ab}$$


S_a^*
S_{a-1}
S_a
S_{a+1}

- Likewise, we choose a class in $H_2(X, \partial X; \mathbb{Z})$ to represent the total charge due to the D1 branes wrapping the noncompact two-cycles, expanded in terms of fundamental weights:

$$[L] = \sum_{a=1}^{m} L^{(a)} \lambda_a$$

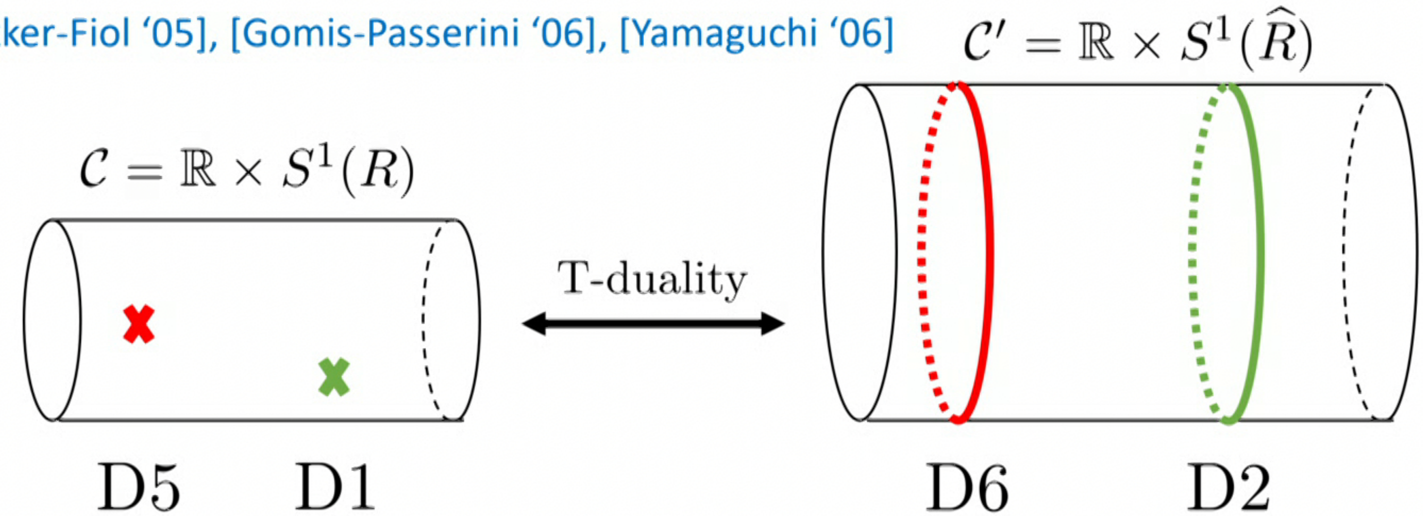
- We decouple gravity by taking the string coupling limit $g_s \to 0$.

- (<u>Sidenote</u>: In effect, what we are then studying is the so-called (2,0) little string on $\mathcal{C} \times \mathbb{C}^2$, with codimension 2 defects (the D5 branes) and point defects (the D1 branes).)

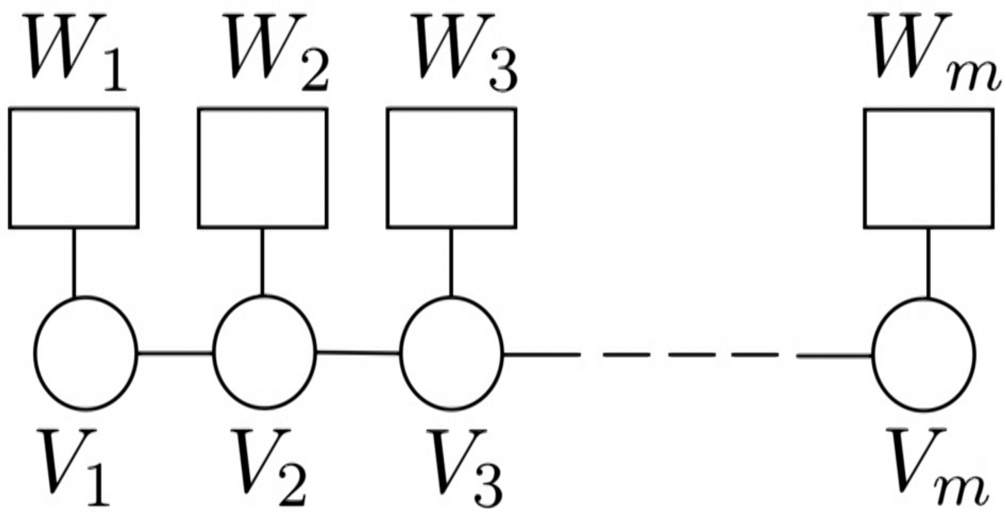
- The low energy theory on the D5 branes is a quiver gauge theory, of shape the Dynkin diagram of $\mathfrak{g}$. [Douglas-Moore '96]

- The low energy theory on the D5 branes has the Poincare invariance of a 4d N=2 theory, but it really is a 5d N=1 theory, on $\mathbb{C}^2 \times S^1(\widehat{R})$. We call this gauge theory T^{5d}.
- Likewise, the D1 brane are really loops wrapping $S^1(\widehat{R})$. These are precisely ½ BPS Wilson loops in string theory.

[Drukker-Fiol '05], [Gomis-Passerini '06], [Yamaguchi '06]

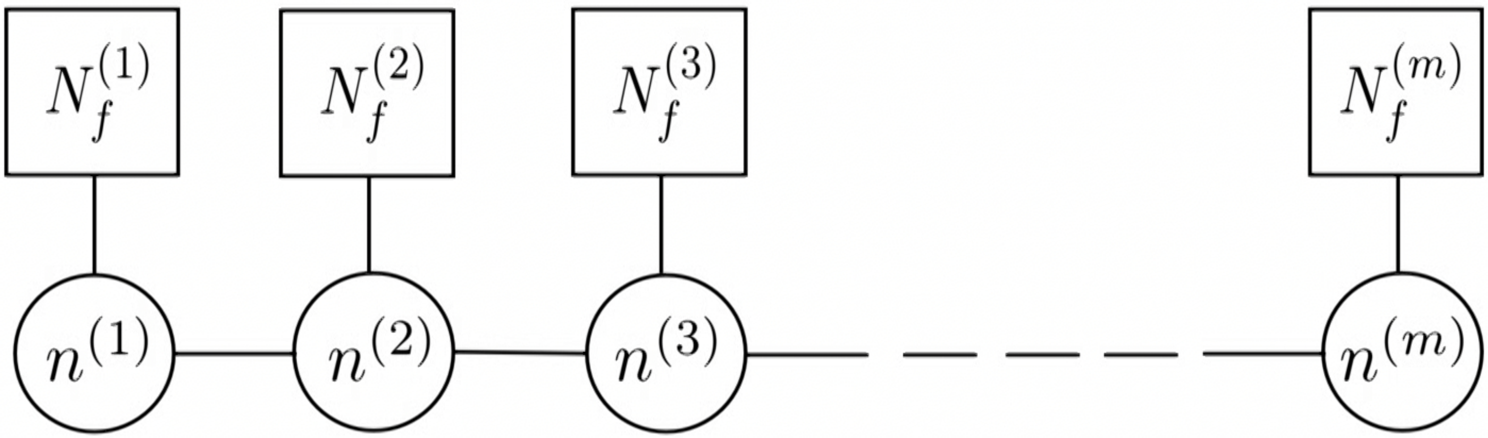


T^{5d} example: $\mathfrak{g} = A_m$

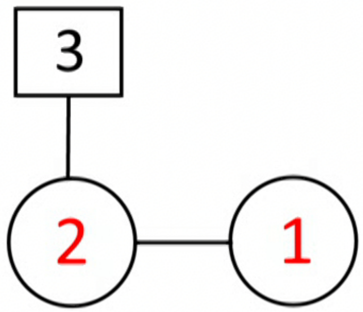


$$\dim(V_a) = n^{(a)} \qquad \dim(W_a) = N_f^{(a)} \qquad a = 1, \ldots, m$$

T^{5d} example: $\mathfrak{g} = A_m$



T^{5d} example: $\mathfrak{g} = A_2$



$$[S^*] = 3\,[S_1^*]$$

$$[S] = 2\,[S_1] + [S_2]$$

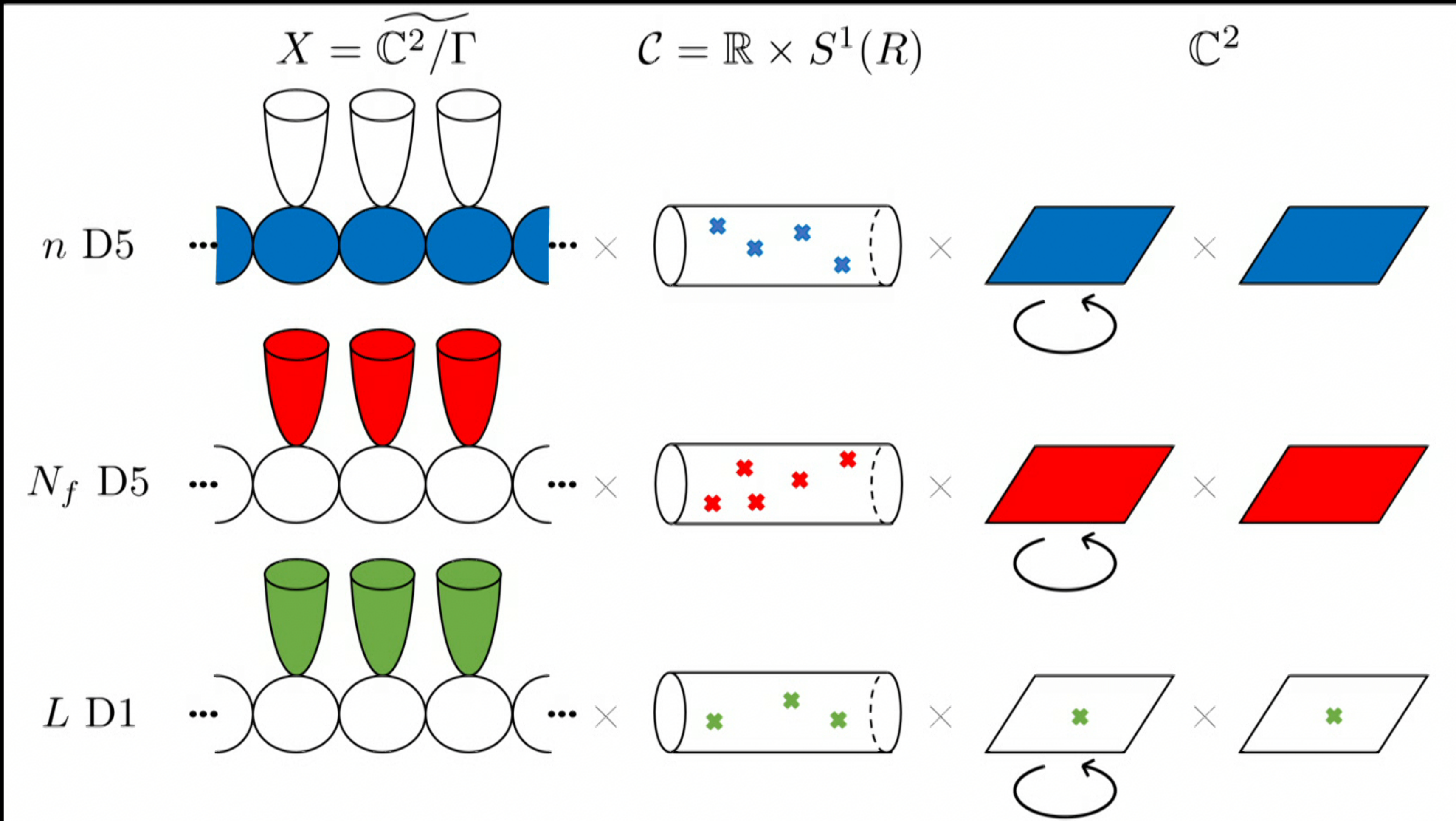
- Positions of non-compact D5 branes on C are mass parameters of the low energy quiver gauge theory T^{5d}.

- Positions of compact D5 branes on C are Coulomb moduli of T^{5d}.

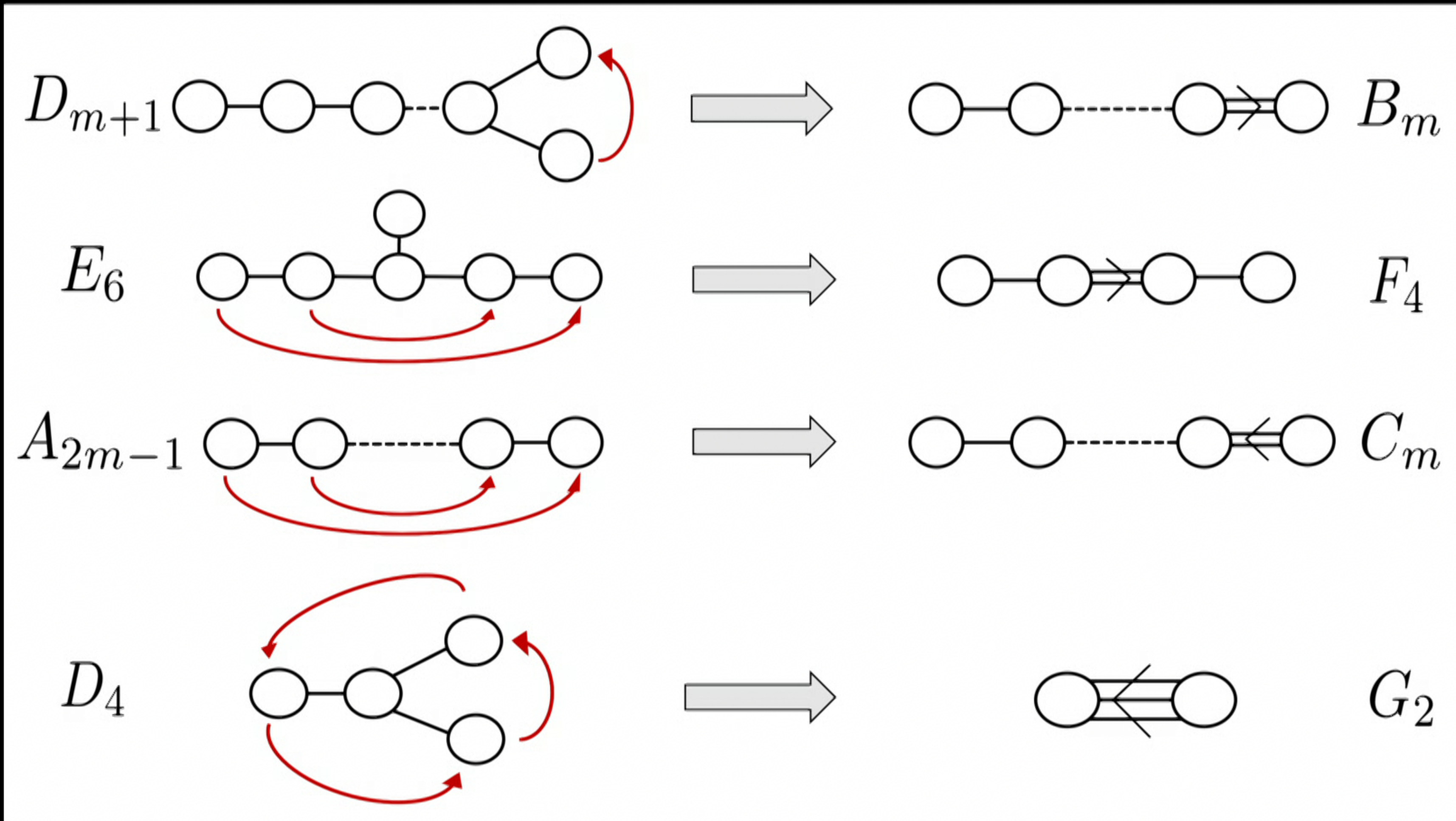
- The gauge couplings and F.I. parameters come from various moduli of the metric on $X = \widetilde{\mathbb{C}^2/\Gamma}$ and spacetime fields.

- Positions of the D1 branes on C are the fermion masses $z_i \equiv e^{M_i}$; the fermions arise from quantizing the D1/D5 strings.

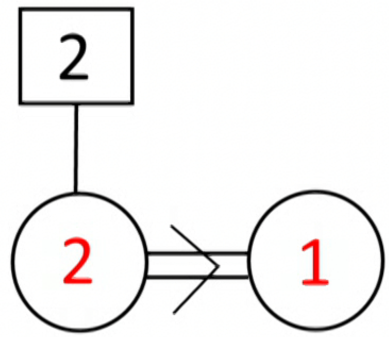
- From the type IIB framework we presented, there is a way to define and construct 5d “non simply-laced quiver gauge theories.” [Aspinwall-Gross ‘96], [Kimura-Pestun ‘17]
- As we go around the origin of one of the complex planes, say $\mathbb{C}_1$, we let the resolved ADE singularity X come back to itself up to the action of a generator of the outer automorphism group of the ADE Lie algebra. We look at the D5 and D1 branes left invariant under this group action.
- The various fields on the branes are only well defined on the r-fold cover of $\mathbb{C}_1$, where r is the order of the outer automorphism group.
- The D5 and D1 brane charges should now be understood in terms of coroots and coweights of the algebra.



$X = \widetilde{\mathbb{C}^2/\Gamma}$
$\mathcal{C} = \mathbb{R} \times S^1(R)$
$\mathbb{C}^2$
n D5
N_f D5
L D1


D_{m+1}
B_m
E_6
F_4
A_{2m-1}
C_m
D_4
G_2

T^{5d} example: $\mathfrak{g} = B_2$



$$[S^*] = 2\,[S_1^{*\,\vee}]$$

$$[S] = 2\,[S_1^\vee] + [S_2^\vee]$$

- We will now compute the K-theoretic partition function $[\chi^{\mathfrak{g}}]^{5d}$ of the 5d quiver gauge theory in the presence of the Wilson loop.
- This can done by equivariant localization, with respect to:

$$(\mathbb{C}^*)^2 \times \prod_{a=1}^{m} U(n^{(a)}) \times \prod_{a=1}^{m} U(N_f^{(a)}) \times \prod_{a=1}^{m} U(M^{(a)})$$

$$q, t \qquad e_{a,I} \qquad f_{a,i} \qquad z_{a,j} = e^{M_{a,j}}$$

- Then, $[\chi^{\mathfrak{g}}]^{5d}$ can be expressed as a sum over the fixed points in instanton moduli space.

- Let us assume there are $L^{(a)}$ D1 branes wrapping S_a^* . We find:

$$\boxed{[\chi^{\mathfrak{g}}]^{5d}_{(L^{(1)},\ldots,L^{(m)})}\left(z_\rho^{(a)}\right) = \sum_{\omega\in V(\lambda)} \prod_{b=1}^{m} (\tilde{\mathfrak{q}}_b)^{d_b^\omega}\ c_{d_b^\omega}(q,t)\ \left(Q_b(z_{*\rho}^{(a)})\right)^{d_b^\omega}\ \left[\widetilde{Y}^{5d}(z_\rho^{(a)})\right]_\omega}$$

- In the above, ω runs over the weights of the finite dimensional irreducible representation $V(\lambda)$ of $U_q(\widehat{{}^L\mathfrak{g}})$, with highest weight λ.
- d_b^ω is a positive integer figured out from: $\omega = \lambda - \sum_{b=1}^m d_b^\omega\, \alpha_b$.
- $c_{d_s^\omega}(q,t)$ are coefficients depending only on q and t .
- $\tilde{\mathfrak{q}}_b$ are the 5d gauge couplings.
- $Q_b(z_{*\rho}^{(a)}) = \prod_{i=1}^{N_f^{(b)}} \left(1 - f_{b,i}/z_{*\rho}^{(a)}\right)$ is a fundamental hypermultiplet factor.

- The bracket $\left[\widetilde{Y}^{5d}(z_\rho^{(a)})\right]_\omega$ is a product of operators $\left\langle \prod_a \left[Y_a^{5d}\right]^{\pm 1}(z_a)\right\rangle$:

$$\left\langle \left[Y_a^{5d}\right]^{\pm 1}(z)\right\rangle = \sum_{\{R\}} \left[\tilde{\mathfrak{q}}^{|R|} \prod_{I=1}^{n^{(a)}} Z^{5d}(q,t,e_{a,I},f_{a,i}) \prod_{k=1}^{\infty} \left(\frac{1 - t\, x'_{a,I,k}/z}{1 - x'_{a,I,k}/z} \right)^{\pm 1} \right]$$

- The sum is over m-tuples of Young diagrams, one for each $U(1)$ factor

$e_{a,I}$ of the gauge group $\prod_{a=1}^{m} U(n^{(a)})$. So we have:

$$\{R\} = \{R_{a,i}\}\,, \qquad a = 1,\ldots,m \quad I = 1,\ldots,n^{(a)}$$

- The bracket $\left[\widetilde{Y}^{5d}(z_\rho^{(a)})\right]_\omega$ is a product of operators $\left\langle \prod_a \left[Y_a^{5d}\right]^{\pm 1}(z_a) \right\rangle$:

$$\left\langle \left[Y_a^{5d}\right]^{\pm 1}(z) \right\rangle = \sum_{\{R\}} \left[\tilde{\mathfrak{q}}^{|R|} \prod_{I=1}^{n^{(a)}} Z^{5d}(q,t,e_{a,I},f_{a,i}) \prod_{k=1}^{\infty} \left(\frac{1 - t\, x'_{a,I,k}/z}{1 - x'_{a,I,k}/z} \right)^{\pm 1} \right]$$

- The infinite product is the contribution of the Wilson loop. We have defined the variables:

$$x'_{a,I,k} = e_{a,I}\, q^{r_a(R_{a,I})_k}\, t^{-k} \quad .$$

- The factor $Z^{5d}(q,t,e_{a,I},f_{a,i})$ contains all the contributions from the vector and hyper multiplets in the absence of a defect.

A few remarks are in order:

$$[\chi^{\mathfrak{g}}]^{5d}_{(L^{(1)},\dots,L^{(m)})}\left(z_\rho^{(a)}\right) = \sum_{\omega\in V(\lambda)} \prod_{b=1}^{m} (\tilde{\mathfrak{q}}_b)^{d_b^\omega} \; c_{d_b^\omega}(q,t) \left(Q_b(z_{*\rho}^{(a)})\right)^{d_b^\omega} \left[\widetilde{Y}^{5d}(z_\rho^{(a)})\right]_\omega$$

- When there is only one D1 brane at z, the above partition function turns out to be a finite polynomial in z.
- Adding more D1 branes produces "higher" characters (characters for higher finite dimensional irreducible representations of $U_q(\widehat{^L\mathfrak{g}})$).
- When the Lie algebra is of ADE type, the above object was nicknamed a qq-character in [Nekrasov '15].

- Example: A_1

$$\left[\chi^{A_1}\right]^{5d} = \langle Y(z)\rangle + \tilde{\mathfrak{q}} \left\langle \frac{Q(z\,v^{-1})}{Y(z\,v^{-2})} \right\rangle \qquad v \equiv \sqrt{q/t}$$

- Example: A_2

$$\mu \equiv \text{bifundamental mass}$$

$$\left[\chi^{A_2}\right]^{5d}_1 = \langle Y_1(z)\rangle + \tilde{\mathfrak{q}}_1 \left\langle \frac{Q_1(z\,v^{-1})\,Y_2(z\,\mu)}{Y_1(z\,v^{-2})} \right\rangle + \tilde{\mathfrak{q}}_1\tilde{\mathfrak{q}}_2 \left\langle \frac{Q_1(z\,v^{-1})\,Q_2(z\,\mu\,v^{-1})}{Y_2(z\,\mu\,v^{-2})} \right\rangle$$

$$\left[\chi^{A_2}\right]^{5d}_2 = \langle Y_2(z)\rangle + \tilde{\mathfrak{q}}_2 \left\langle \frac{Q_2(z\,v^{-1})\,Y_1(z\,\mu^{-1}v^{-2})}{Y_2(z\,v^{-2})} \right\rangle + \tilde{\mathfrak{q}}_2\tilde{\mathfrak{q}}_1 \left\langle \frac{Q_2(z\,v^{-2})\,Q_1(z\,\mu^{-1}v^{-3})}{Y_1(z\,\mu^{-1}v^{-4})} \right\rangle$$

- Example: G_2

$$v \equiv \sqrt{q/t}$$

$$\mu \equiv \text{bifundamental mass}$$

$$\left[\chi^{G_2}\right]_1^{5d} = \langle Y_1(z)\rangle + \tilde{\mathfrak{q}}_1 \left\langle \frac{Q_1(z\,v^{-1})\,Y_2(z\,\mu)}{Y_1(z\,v^{-2})} \right\rangle + \tilde{\mathfrak{q}}_1\tilde{\mathfrak{q}}_2 \left\langle \frac{Q_1(z\,v^{-1})\,Q_2(z\,\mu\,v^{-1}q^{-2})\,Y_1(z\,v^{-2}q^{-2})}{Y_2(z\,\mu\,v^{-2}q^{-2})} \right\rangle + \ldots \qquad \text{(total of 7 terms)}$$

$$\left[\chi^{G_2}\right]_2^{5d} = \langle Y_2(z)\rangle + \ldots \qquad \text{(total of 15 terms)}$$

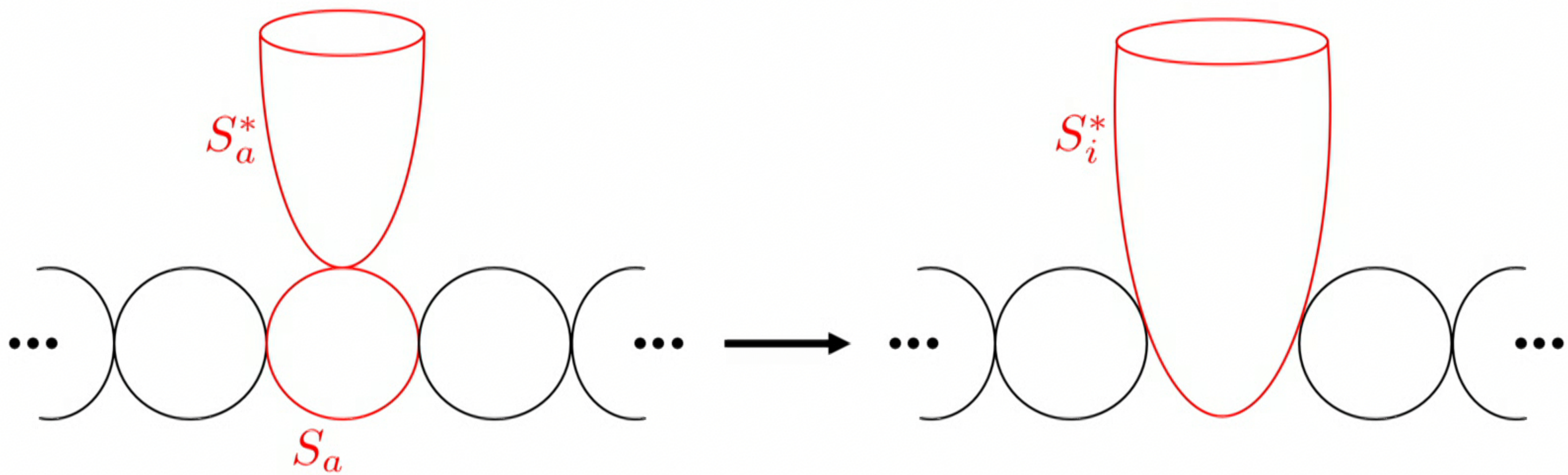
Outline

- 5d quiver gauge theory with Wilson Loops
- A duality to 3d
- From 3d gauge theories to deformed W-algebras

- The Duality to three dimensions comes from turning on vortex flux on the D5 branes. This is known as gauge/vortex duality.

- In the Higgs phase of the 5d theory, all the Coulomb moduli are frozen to points where hypermultiplets can get nonzero expectation values. First, let us go to the place in the moduli space where the Coulomb and Higgs branches meet, i.e. the root of the Higgs branch.

- What happens in the string theory picture? At the root of the Higgs branch, the branes have to bind, meaning the positions of the non-compact and compact branes must coincide.
- The branes then recombine to form D5 branes wrapping a collection of non-compact cycles S_i^*, whose homology classes are elements of the weight lattice $\Lambda_* = H_2(X, \partial X; \mathbb{Z})$:



- Having described the root of the Higgs branch, we turn on $N_{a,I}$ units of vortex flux for each $U(1)$ G-equivariant parameter. This induces a shift for the corresponding Coulomb moduli:

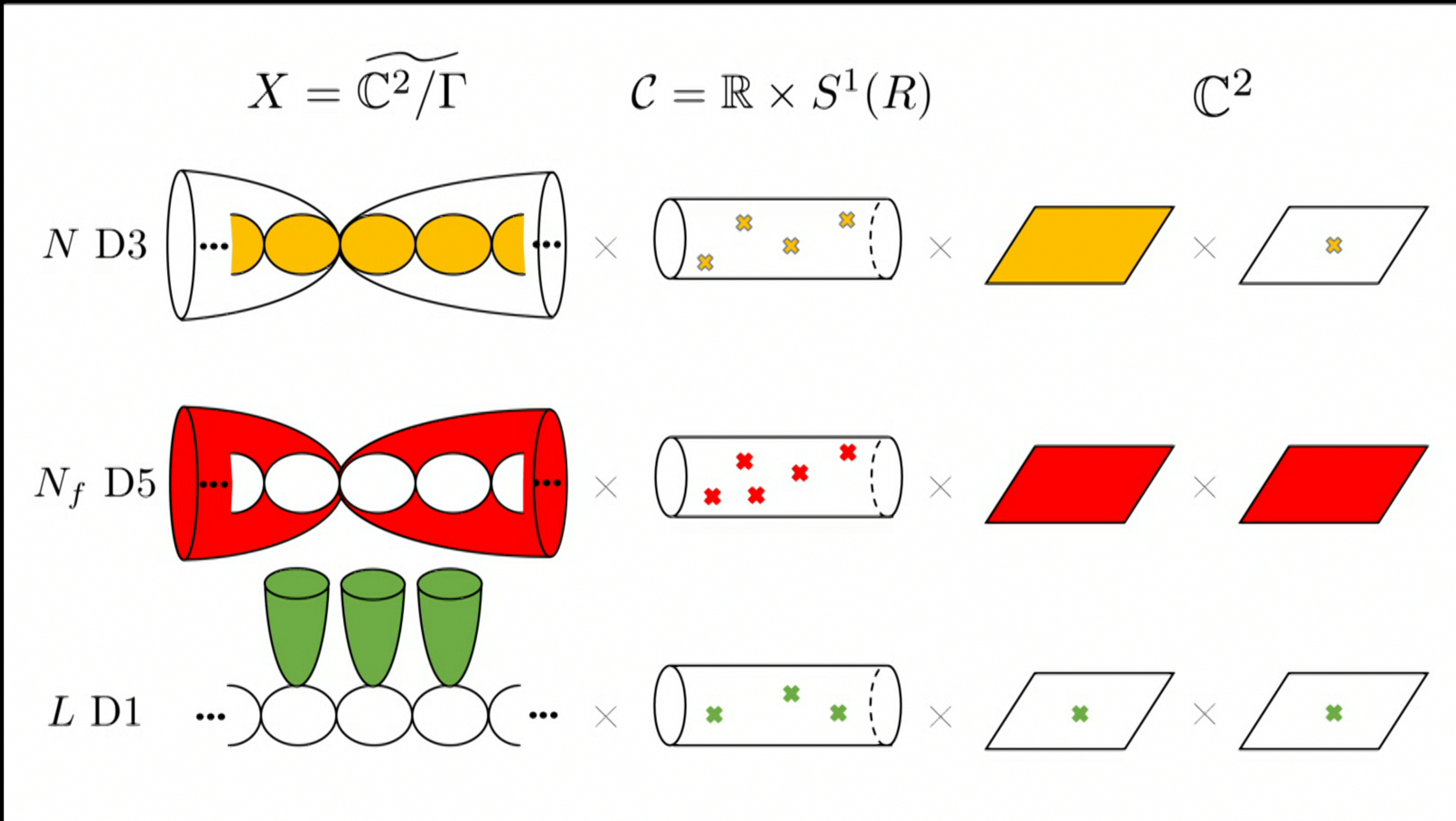
$$e_{a,I} = f_{a,i}\, t^{N_{a,I}}\,.$$

- In effect, we are back on the Coulomb branch, but only on an integer lattice.

- This shift is due to the Omega background (q, t) in 5d.

  [Nekrasov-Witten'10]

Turning on N units of vortex flux on the D5 branes means in string theory that we are introducing N D3 branes ending on them. These D3 branes are wrapping the compact two-cycles of the resolved singularity X.


$X = \widetilde{\mathbb{C}^2/\Gamma}$
$\mathcal{C} = \mathbb{R} \times S^1(R)$
$\mathbb{C}^2$
N D3
N_f D5
L D1

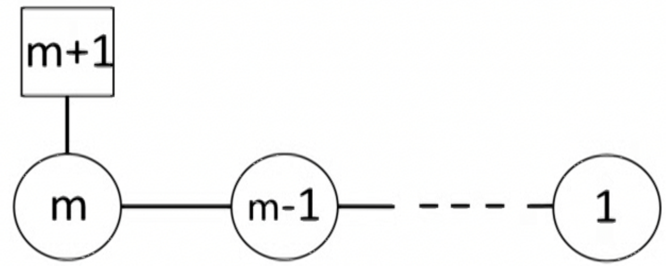
- If only D3 branes are present, the gauge theory G^{3d} on them has 3d N=4 supersymmetry.

- Adding D5 branes breaks supersymmetry by half (there will be chiral multiplets of N=2 SUSY), which makes G^{3d} a 3d N=2 theory.

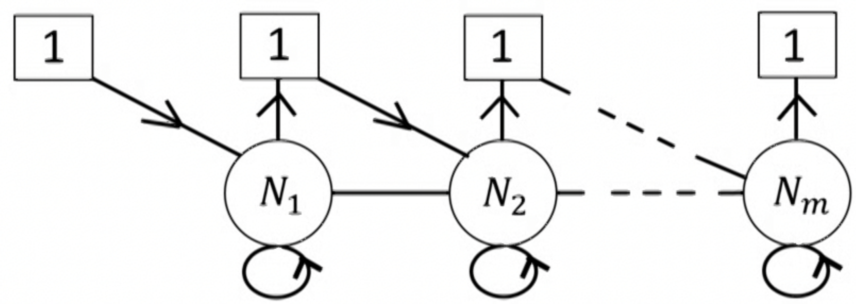
- To determine G^{3d}, one can proceed starting from geometry, like we did for the 5d theory. As before, one finds that G^{3d} is a quiver gauge theory of shape the Dynkin diagram of $\mathfrak{g}$, on $\mathbb{C}_q \times S^1(\widehat{R})$.

(The plane $\mathbb{C}_t$ is transverse to the D3 branes)

5d Quiver T^{5d}:



3d Quiver G^{3d}:



- Adding the D1 branes gives point defects from the point of view of the D3 brane theory G^{3d}. By the same argument we used before in 5d, these are ½-BPS Wilson loops. The configuration with all branes present preserves only 2 supercharges.

- We now compute the partition function $[\chi^{\mathfrak{g}}]^{3d}$ of the 3d theory in the presence of the defects. This can be done once again using localization [Kim-Kim-Kim-Lee '12], [Yoshida-Sugiyama '14]...

- Here, we will take a shortcut and derive the partition function directly from 5d.

- Namely, at the locus $e_{a,I} = f_{a,i}\, t^{N_{a,I}}$, the 5d partition function truncates: it vanishes unless the Young diagrams we sum over have length at most $N_{a,I}$.

- The effect on the partition function is drastic. For instance, there are an infinite number of telescopic cancellations in the Wilson loop factors at this locus:

$$Y^{5d}(z) = \prod_{k=1}^{\infty} \frac{1 - t\, x'_k/z}{1 - x'_k/z} = \underbrace{\prod_{k=1}^{N} \frac{1 - t\, x'_k/z}{1 - x'_k/z}}_{\equiv Y^{3d}(z)} \cdot (1 - f_1/v\, z)$$

$$v \equiv \sqrt{q/t}$$

- The right-hand side should now be understood as the contribution of chiral multiplets of the 3d theory to the partition function.

- Let us assume there are $L^{(a)}$ D1 branes wrapping S_a^* . We find:

$$\boxed{[\chi^{\mathfrak{g}}]^{3d}_{(L^{(1)},\ldots,L^{(m)})}\left(z_\rho^{(a)}\right) = \sum_{\omega\in V(\lambda)} \prod_{b=1}^{m} (\tilde{\mathfrak{q}}'_b)^{d_b^\omega} \; c_{d_b^\omega}(q,t) \; \left(\mathcal{Q}_b(z_{*\rho}^{(a)})\right)^{d_b^\omega} \; \left[\widetilde{Y}^{3d}(z_\rho^{(a)})\right]_\omega}$$

- In the above, ω runs over the weights of the finite dimensional irreducible representation $V(\lambda)$ of $U_q(\widehat{^L\mathfrak{g}})$, with highest weight λ.
- d_b^ω is a positive integer figured out from: $\omega = \lambda - \sum_{b=1}^m d_b^\omega \, \alpha_b$.
- $c_{d_s^\omega}(q,t)$ are coefficients depending only on q and t.
- $\tilde{\mathfrak{q}}_b$ are the 3d F.I. terms.
- $\mathcal{Q}_b(z_{*\rho}^{(a)}) = \prod_{i=1}^{n^{(b)}} \left(1 - f_{b,i}/v\, z_{*\rho}^{(a)}\right)$ is (anti)chiral matter.

- The bracket $\left[\widetilde{Y}^{3d}(z_\rho^{(a)})\right]_\omega$ is a product of operators $\left\langle \prod_a \left[Y_a^{3d}\right]^{\pm 1}(z_a)\right\rangle$:

$$\left\langle \left[Y_a^{3d}\right]^{\pm 1}(z)\right\rangle = \oint d^N \overrightarrow{x} \left[\overrightarrow{x}^{\log \tilde{\mathfrak{q}}'-1}\, Z^{3d}(q,t,\overrightarrow{x},f_{a,i}) \prod_{k=1}^{N_a}\left(\frac{1-t\,x_{a,k}/z}{1-x_{a,k}/z}\right)^{\pm 1}\right]$$

- The partition function is an integral over the Coulomb moduli. In the integrand, the factor Z^{3d} contains all the 3d N=2 chiral and vector multiplets in the absence of a Wilson loop.
- After we specify the contours, we can evaluate the integral by residue. The poles are precisely labeled by m-tuples of Young diagrams:

$$\oint d^N \overrightarrow{x} \to \sum_{\{R\}} \qquad \{R\} = \{R_{a,i}\}\,, \qquad a = 1,\ldots,m \quad I = 1,\ldots,n^{(a)}$$

$$v \equiv \sqrt{q/t}$$

- A_1 example: U(1) with 2 flavors in 5d

$$\left[\chi^{A_1}\right]^{5d} = \langle Y^{5d}(z)\rangle + \tilde{\mathfrak{q}}\,(1 - v\,f_1/z)\,(1 - v\,f_2/z)\left\langle \frac{1}{Y^{5d}(z\,v^{-2})}\right\rangle$$

- A_1 example: U(N) with 1 chiral and 1 anti-chiral in 3d

$$\left[\chi^{A_1}\right]^{3d} = (1 - f_1/v\,z)\langle Y^{3d}(z)\rangle + \tilde{\mathfrak{q}}'\,(1 - v\,f_2/z)\left\langle \frac{1}{Y^{3d}(z\,v^{-2})}\right\rangle$$

- Let $\mathfrak{g}$ be a simple Lie algebra.
- Then the $\mathfrak{g}$- type Toda field theory action can be written down in terms of $n = \mathrm{rk}(\mathfrak{g})$ free bosons in two dimensions with a background charge contribution $Q = \rho\, b + \rho^\vee / b$ and the Toda potential that couples them:

$$S_{Toda} = \int dz d\bar{z}\ \sqrt{g}\ g^{z\bar{z}} [\langle \partial_z \varphi, \partial_{\bar{z}} \varphi \rangle + \langle Q, \varphi \rangle\, R + \sum_{a=1}^{n} e^{\langle \alpha_a, \varphi \rangle / b}]$$

- The Toda CFT has an extended conformal symmetry, which is called a $\mathcal{W}(\mathfrak{g})$-algebra symmetry.

- The deformed W-algebra $\mathcal{W}_{q,t}(\mathfrak{g})$ was defined in Coulomb gas (i.e. free field) formalism in [Frenkel-Reshetikhin '97].

- One start with a deformed Heisenberg algebra $\mathcal{H}_{q,t}(\mathfrak{g})$, with generators

$$\alpha_a[k], \quad k \in \mathbb{Z}, \quad a = 1, \ldots, m$$

which have the following commutators:

$$[\alpha_a[k], \alpha_b[n]] = \frac{1}{k}(q^{\frac{k}{2}} - q^{-\frac{k}{2}})(t^{\frac{k}{2}} - t^{-\frac{k}{2}})\widehat{C}_{ab}(q^{\frac{k}{2}}, t^{\frac{k}{2}})\delta_{k,-n}$$

(In the above, $\widehat{C}_{ab}(q^{\frac{k}{2}}, t^{\frac{k}{2}})$ is a modified Cartan matrix)

- A Fock space representation of the Heisenberg algebra is constructed as follows: for each weight λ of the Cartan subalgebra of $\mathfrak{g}$, the Fock representation is generated by a vector $|\lambda\rangle$ such that

$$\begin{aligned} \alpha_a[0]|\lambda\rangle &= \langle \lambda, \alpha_a \rangle |\lambda\rangle \\ \alpha_a[k]|\lambda\rangle &= 0\,, \qquad\qquad \text{for } k > 0. \end{aligned}$$

- The $\mathcal{W}_{q,t}(\mathfrak{g})$ algebra is then defined as the commutant of the m screening charges $Q_a^\vee$ which act on $|\lambda\rangle$. Namely, it is generated by vertex operators $W_s(z)$ such that

$$[W_s(z), Q_a^\vee] = 0\,, \qquad \text{for all } \; a = 1, \ldots, m, \;\; \text{and} \;\; s = 2, \ldots, m+1\,.$$

There is a deformed Virasoro stress tensor, and higher spin currents.

- The screening charges $Q_a^\vee$ are defined as the integral of screening currents:

$$Q_a^\vee = \int S_a^\vee(x)\, dx \quad , \qquad S_a^\vee(x) = x^{-\alpha_a[0]/r_a} \; : \exp\Big(\sum_{k\neq 0} \frac{\alpha_a[k]}{q^{\frac{k\, r_a}{2}} - q^{-\frac{k\, r_a}{2}}}\, x^k\Big) :$$

- There is no formal definition of primary vertex operators in the literature. We will consider a "generic" vertex operator:

$$V_\beta(u) \equiv : \prod_{i=1}^{N_f} V_{\omega_i}(y_i) :$$

- The ω_i are the (co)weights we defined in gauge theory, with y_i a corresponding point on the cylinder. [Aganagic-Haouzi '15]

- Each individual vertex operator $V_{\omega_i}(y_i)$ in the product is itself the normal ordered product of a "fundamental weight" vertex operator

$$\Lambda_a(y) =: \exp\Big(\sum_{k\neq 0} \frac{w_a[k]}{(q^{\frac{k\,r_a}{2}} - q^{-\frac{k\,r_a}{2}})(t^{\frac{k}{2}} - t^{-\frac{k}{2}})}\, y^k\Big) :$$

and of "simple root" operators:

$$E_a(y) =: \exp\Big(\sum_{k\neq 0} \frac{\alpha_a[k]}{(q^{\frac{k\,r_a}{2}} - q^{-\frac{k\,r_a}{2}})(t^{\frac{k}{2}} - t^{-\frac{k}{2}})}\, y^k\Big) :$$

- The modes $w_a[k]$ are obtained from the Heisenberg generators:

$[\alpha_a[k], w_b[n]] = \frac{1}{k}(q^{\frac{k\,r_a}{2}} - q^{-\frac{k\,r_a}{2}})(t^{\frac{k}{2}} - t^{-\frac{k}{2}})\,\delta_{ab}\,\delta_{k,-n}$, meaning $\alpha_a[k] = \sum_{b=1}^{m} C_{ab}(q^{\frac{k}{2}}, t^{\frac{k}{2}})w_b[k]$.

- One can turn off the deformation by taking the limit:

$$t = q^{\gamma}\,, \qquad q \to 1$$

- This limit takes $\mathcal{W}_{q,t}(\mathfrak{g})$ to the undeformed algebra $\mathcal{W}_{\gamma}(\mathfrak{g})$, where the central charge depends on γ.
- In particular, our vertex operators

$$V_{\beta}(u) \equiv : \prod_{i=1}^{N_f} V_{\omega_i}(y_i) : \qquad \begin{cases} y_i = u\, q^{-\beta_i} \\ \beta = \sum_{i=1}^{N_f} \beta_i\, \omega_i \end{cases}$$

become the primary vertex operators of $\mathcal{W}_{\gamma}(\mathfrak{g})$.

- The generator $W_2(z)$ becomes the Virasoro stress tensor and so on for the higher spin currents...

- We are interested in computing the following chiral correlators:

$$\langle\lambda'|\prod_{i=1}^{k} V_{\beta_i}(u_i)\ \prod_{s=2}^{m+1}\prod_{p=1}^{L_s} W_s(z_p)\ \prod_{a=1}^{m}(Q_a^\vee)^{N_a}|\lambda\rangle$$

where we recall that the screening charges are integrals,

$$Q_a^\vee = \int S_a^\vee(x)\,dx \ \ ,$$

so one also needs to specify the contour. This can be done explicitly.

- Because of the free field formalism, the evaluation of the correlator can be done using Wick contractions.

- Up to overall normalization, we find the following result:

$$\boxed{\langle \lambda' | \prod_{i=1}^{k} V_{\beta_i}(u_i) \prod_{s=2}^{m+1} \prod_{p=1}^{L_s} W_s(z_p) \prod_{a=1}^{m} (Q_a^{\vee})^{N_a} | \lambda \rangle = [\chi^{\mathfrak{g}}]^{3d}}$$

- Namely, the correlator is the partition function of the 3d gauge theory in the presence of the Wilson loop!

- When there is only one D1 brane present, recall that:

$$[\chi^{\mathfrak{g}}]^{3d}_{(L^{(1)},\ldots,L^{(m)})}(z^{(a)}_{\rho}) = \sum_{\omega\in V(\lambda)} \prod_{b=1}^{m} (\tilde{\mathfrak{q}}'_b)^{d^{\omega}_b}\, c_{d^{\omega}_b}(q,t)\, \left(\mathcal{Q}_b(z^{(a)}_{*\rho})\right)^{d^{\omega}_b} \left[\widetilde{Y}^{3d}(z^{(a)}_{\rho})\right]_{\omega}$$

$$\left\langle \left[Y^{3d}_a\right](z)\right\rangle = \oint d^N\overrightarrow{x}\left[\overrightarrow{x}^{\log\tilde{\mathfrak{q}}'-1}\, Z^{3d}(q,t,\overrightarrow{x},f_{a,i}) \prod_{k=1}^{N_a}\left(\frac{1-t\,x_{a,k}/z}{1-x_{a,k}/z}\right)\right]$$

- When there is only one D1 brane present, recall that:

$$[\chi^{\mathfrak{g}}]^{3d}_{(L^{(1)},\dots,L^{(m)})}\left(z^{(a)}_\rho\right) = \sum_{\omega\in V(\lambda)} \prod_{b=1}^{m} (\tilde{\mathfrak{q}}'_b)^{d^\omega_b}\, c_{d^\omega_b}(q,t) \left(\mathcal{Q}_b\left(z^{(a)}_{*\rho}\right)\right)^{d^\omega_b} \left[\widetilde{Y}^{3d}\left(z^{(a)}_\rho\right)\right]_\omega$$

$\alpha_a[0]|\lambda\rangle$ $\quad\langle V_\beta(u) W_s(z_\rho)\rangle$ $\quad\langle W_s(z_\rho)\, S^\vee_a(x)\rangle$

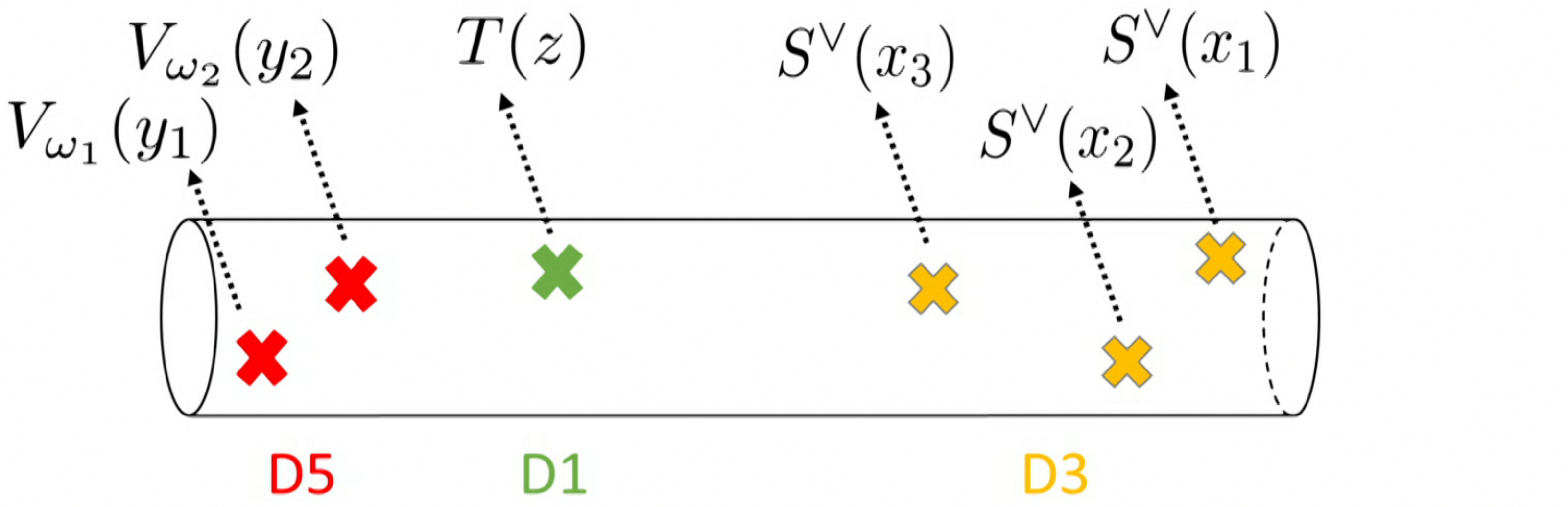
$$\left\langle \left[Y^{3d}_a\right](z)\right\rangle = \oint d^N\vec{x} \left[\vec{x}^{\,\log\tilde{\mathfrak{q}}'-1}\, Z^{3d}(q,t,\vec{x},f_{a,i}) \prod_{k=1}^{N_a}\left(\frac{1-t\,x_{a,k}/z}{1-x_{a,k}/z}\right)\right]$$

$\langle S^\vee_a(x)\, S^\vee_b(x')\rangle$ $\quad\langle S^\vee_a(x)\, V_\beta(u)\rangle$ $\quad\langle S^\vee_a(x)\, S^\vee_a(x')\rangle$

For example, in deformed Liouville $\mathcal{W}_{q,t}(A_1)$:

$$\langle \lambda' | V_\beta(u)\, T(z)\, (Q^\vee)^N | \lambda \rangle$$

$$\beta = \beta_1 \omega_1 + \beta_2 \omega_2$$
$$y_1 = u\, q^{-\beta_1}, \quad y_2 = u\, q^{-\beta_2}$$
$$N = 3$$



- If one actually performs the integrals in the correlator

$$\langle\lambda'|\prod_{i=1}^{k} V_{\beta_i}(u_i)\ \prod_{s=2}^{m+1}\prod_{p=1}^{L_s} W_s(z_p)\ \prod_{a=1}^{m}(Q_a^\vee)^{N_a}|\lambda\rangle\ ,$$

evaluating it by residues gives back the 5d partition function with specialized G-equivariant parameters:

$$\langle\lambda'|\prod_{i=1}^{k} V_{\beta_i}(u_i)\ \prod_{s=2}^{m+1}\prod_{p=1}^{L_s} W_s(z_p)\ \prod_{a=1}^{m}(Q_a^\vee)^{N_a}|\lambda\rangle = [\chi^{\mathfrak{g}}]^{5d}\Big|_{e_{a,I}\,=\,f_{a,i}\,t^{N_{a,I}}}$$

Thank you!