Title: Generalized Gibbs Ensemble and KdV charges in 2d CFTs

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Abstract: 2D CFTs have an infinite set of commuting conserved charges, known as the quantum KdV charges. There is a generalised Gibbs ensemble for these theories where we turn on chemical potentials for these charges. I will describe some partial results on calculating this partition function, both in the limit of large charges and perturbatively in the chemical potentials.

GGE and KdV charges in 2d CFTs

with Alex Maloney, Gim-Seng Ng & Ioannis Tsiares, 1810.11053, 1810.11054

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Plan

2d CFTs live on the border between integrable and chaotic:

- Rational CFTs ($c < 1$) are integrable.
- Irrational CFTs ($c > 1$) are expected to be chaotic.

Interesting environment to study thermal physics.

Virasoro algebra leads to an infinite set of conserved charges
Thermal ensemble (β) $\rightarrow$ Generalised Gibbs Ensemble (β, μ_i).

Relevant to thermalization for general initial states Cardy

Study GGE

- at infinite c: saddle point
- perturbatively in μ_i for finite c.

KdV charges

Integrable systems perspective on Virasoro symmetry:

There is an infinite sequence of *commuting* conserved charges, formed from products of Viraroso generators:

$$I_1 = \oint T(z) = L_0 - k,$$

$$I_3 = \oint (TT)(z) = \sum_{n=1}^{\infty} L_{-n}L_n + L_0^2 - (k + \frac{1}{6})L_0 + k(k + \frac{11}{60})$$

etc, where $k = \frac{c}{24}$ (and similarly for $\bar{T}$).

- I_{2m-1} is the zero mode of a current $J_{2m} = T^m + \ldots$
- Mutually commuting: $[I_{2m-1}, I_{2n-1}] = 0$
- Act within a given level in a Virasoro representation.
- Generate the KdV hierarchy at $c \to -\infty$:
  $\delta_{I_3} T(z) = \partial_z^3 T(z) - \frac{3}{2k} T(z)\partial_z T(z)$, etc.

GGE

Generalized Gibbs ensemble with chemical potentials for these charges:

$$Z_{GGE}(\beta, \mu_3, \mu_5, \ldots) = \mathrm{Tr}[e^{-\beta H + \mu_3 I_3 + \mu_5 I_5 + \ldots}]$$

Aim: to understand the structure of the GGE

- Modular transformation properties
- Statistics of KdV charges

ETH

Eigenstate thermalization hypothesis: for simple operators

$$\langle E_i|\mathcal{O}|E_j\rangle = f_{\mathcal{O}}(E_i)\delta_{ij} + O(e^{-S(E_i)/2})$$

Consider quasiprimary descendents of the vacuum O_{2m}: Basu, Das, Datta & Pal
At large T and large c, $\langle E_i|O_{2m}|E_i\rangle \approx \langle O_{2m}\rangle_\beta$ but fails at finite c.

Datta, Kraus & Michel: This was for $|E_i\rangle$ primary, not the typical state.
We will see variance around thermal average is small at large T:
$\langle E_i|O_{2m}|E_i\rangle \approx \langle O_{2m}\rangle_\beta$ for *typical* states even at finite c.

General initial states thermalize to GGE: supported on the subset of states which are typical in GGE.

Infinite c See also Dymarsky & Pavlenko

- For large temperatures, typical states in thermal ensemble have $h \sim \frac{c}{\beta^2}$, $n \sim \frac{1}{\beta^2}$.
- At large c, can ignore descent contributions. In terms of entropy,

$$\rho(E) = e^{2\pi\sqrt{\frac{c}{6}E}}, \rho_P(E) = e^{2\pi\sqrt{\frac{(c-1)}{6}E}} \to \frac{S_P}{S} = \sqrt{1-\frac{1}{c}},$$

  so neglecting descendents will lead to an $O(1/c)$ error in $\log Z$.
- $I_{2n-1} = L_0^n + \ldots$, so at high energies $I_{2n-1} \approx h^n$.

Thus for high temperatures and $c \gg 1$,

$$Z_{GGE} \approx \int dh e^{2\pi\sqrt{4kh}-\beta h+\mu_3 h^2+\mu_5 h^3 \ldots}$$

Infinite c

$$Z_{GGE} \approx \int dh e^{2\pi\sqrt{4kh}-\beta h+\mu_3 h^2+\mu_5 h^3\ldots}$$

Saddle-point analysis:

Set $h = \frac{(2\pi)^2 k}{\beta^2}\bar{h}$, $\mu_{2m-1} = \frac{\beta^{2m-1}}{(2\pi)^{2m-2}k^{m-1}}\tilde{\mu}_{2m-1}$:

$$Z_{GGE} \approx \int d\bar{h} e^{\frac{(2\pi)^2 k}{\beta}[\sqrt{\bar{h}}-\bar{h}+\tilde{\mu}_3\bar{h}^2+\tilde{\mu}_5\bar{h}^3\ldots]}$$

Take $\beta \to 0$ at fixed $\tilde{\mu}$: saddle-point $h_\star = \frac{(2\pi)^2 k}{\beta^2}\bar{h}_\star(\tilde{\mu})$,

$$\log Z_{GGE} \approx \frac{(2\pi)^2 k}{\beta}F(\tilde{\mu}) = \frac{(2\pi)^2 k}{\beta}\left[1+\sum_n \sum_{m_1\ldots m_n} a_{m_1,\ldots m_n}\tilde{\mu}_{m_1}\cdots\tilde{\mu}_{m_n}\right],$$

$$a_{m_1,\ldots,m_n} = 2\frac{(2M-n)!}{(2M+2-2n)!n!}\prod_{i=1}^{n} m_i.$$

Expansion in μ_i

Coefficients in $\log Z_{GGE}$ are connected correlation functions of the I's. E.g.,

$$\log Z_{GGE}(\beta, \mu_3) = \log Z(\beta) + \langle I_3 \rangle_{\beta,c} \mu_3 + \frac{1}{2} \langle I_3^2 \rangle_{\beta,c} \mu_3^2 + \dots .$$

Previous results:

$$\langle I_3 \rangle_\beta = \frac{(2\pi)^4 k^2}{\beta^4}, \quad \langle I_3^2 \rangle_{\beta,c} = \frac{8(2\pi)^6 k^3}{\beta^7}, \quad \langle I_3^3 \rangle_{\beta,c} = \frac{144(2\pi)^8 k^4}{\beta^{10}}, \quad \dots$$

See variance small: $\langle I_3^2 \rangle_{\beta,c} \ll \langle I_3 \rangle_\beta^2$. Thus, typical states contributing to thermal ensemble have $\langle \psi | I_3 | \psi \rangle \approx \langle I_3 \rangle_\beta$. Datta, Kraus & Michel

Study GGE at finite c by computing connected correlation functions:

- In thermodynamic limit, can calculate thermal correlation fns from plane correlation fns. Lashkari, Dymarsky & Liu
- We instead use modular transformation properties.

Correlation functions are derivatives

By cyclicity and Virasoro algebra, connected correlation functions are derivatives of partition function: E.g.,

$$\langle I_3 \rangle_\beta = \mathrm{Tr}\,[I_3 q^{L_0 - k}] = 2\langle \sum_{n=1}^{\infty} L_{-n} L_n \rangle_\beta + \partial^2 Z - \frac{1}{6}\partial Z + \frac{k}{60} Z,$$

where $q = e^{2\pi i \tau} = e^{-\beta}$, $\partial = q\partial_q$.

$$\langle L_{-n} L_n \rangle = q^n \langle L_n L_{-n} \rangle_\beta = q^n \left(\langle L_{-n} L_n \rangle + 2n\partial Z + 2kn^3 Z \right),$$

which gives

$$\langle I_3 \rangle_\beta = \left[\partial^2 - \frac{1}{6} E_2(q)\partial + \frac{k}{60} E_4(q) \right] Z.$$

$\Rightarrow$ Z_{GGE} has no new information: complicated diff operator on $Z(\beta)$.

Modular properties

Dijkgraaf

One-point functions are modular forms of weight $2m$:

$$\langle I_{2m-1}\rangle(q) = \langle \oint J_{2m}\rangle(q) = \langle \int J_{2m}\rangle(q)$$

- Constrains differential operator, fixing form up to coefficients:

$$\langle I_{2m-1}\rangle(q) = D^m Z + c_1 E_4 D^{m-2} Z + \dots$$

E.g.,

$$\langle I_3\rangle_\beta = \left[D^2 + \frac{k}{60} E_4\right] Z$$

- Modular transformation relates high temperature to low temperature:

$$\langle I_{2m-1}\rangle(q) = \frac{1}{\tau^{2m}} \langle I_{2m-1}\rangle(\hat{q}),$$

where $\hat{\tau} = -\tau^{-1}$. E.g.,

$$\langle I_3\rangle_\beta = \frac{(2\pi)^4}{\beta^4}\langle 0|I_3|0\rangle = \frac{(2\pi)^4}{\beta^4} k(k + \frac{11}{60}).$$

Modular properties

Dijkgraaf

Higher-point functions are quasi-modular, due to contact terms:

$$\langle \int J_{2m} \int J_{2n} \rangle = \langle \oint J_{2m} \oint J_{2n} \rangle + \frac{1}{2\tau_2} \langle \oint [J_{2m} J_{2n}]_2 \rangle,$$

Differential operator involves E_2 to reproduce inhomogeneous transformations:

$$\begin{aligned} \langle I_{2m-1} I_{2n-1} \rangle &= D^{m+n} Z + c_1 E_4 D^{m+n-2} Z + \ldots \\ &\quad + E_2 (D^{m+n-1} Z + d_1 E_4 D^{m+n-3} Z + \ldots), \end{aligned}$$

E.g.,

$$\begin{aligned} \langle I_3^2 \rangle &= \left[D^2 \langle I_3 \rangle + \left(\frac{k}{60} + \frac{1}{18} \right) E_4 \langle I_3 \rangle - \frac{60k+11}{1080} E_6 DZ + \frac{k(60k+11)}{3780} E_4^2 Z \right] \\ &\quad + E_2 \left[\frac{2}{3} D \langle I_3 \rangle + \frac{(60k+11)}{1080} E_4 DZ - \frac{k(60k+11)}{3780} E_6 Z \right]. \end{aligned}$$

Modular properties

Dijkgraaf

High temperature connected correlation function is determined by the inhomogeneous part:

$$\langle I_{2m-1} I_{2n-1} \rangle_{\beta,c} = -i \frac{1}{\tau^{2(m+n)-1}} \frac{\langle \oint [J_{2m} J_{2n}]_2 \rangle (\hat{q})}{Z}$$

Cancellation of leading terms reproduces temperature dependence found in infinite c limit.
E.g.,

$$\langle I_3^2 \rangle_{\beta,c} = \frac{8(2\pi)^6}{\beta^7} k(k + \frac{11}{60})(k + \frac{37}{84}).$$

Minimal models

Bazhanov, Lukyanov & Zamalodchikov

Note that $\langle I_3 \rangle_\beta = \langle I_3^2 \rangle_\beta = 0$ when $k = -\frac{11}{60}$. This value corresponds to a non-unitary minimal model, the Lee-Yang model. This is the case $m = 2$ of the $(2m+1, 2)$ minimal models, with central charge

$$c = 1 - 3\frac{(2m-1)^2}{2m+1}.$$

- I_{2m-1} is the zero mode of the null current at level $2m$ in the $(2m+1, 2)$ minimal model. Thus, I_{2m-1} vanishes as an operator for this value of c.
- In fact, I_{2m-1} vanishes as an operator in the $(2n+1, 2)$ minimal model for all $2m-1$ divisible by $2n-1$. So I_9, I_{15} etc also vanish in the Lee-Yang model.

A useful cross-check on the coefficients in correlation functions.

Subleading $1/c$ corrections

Dymarsky & Pavlenko

$$I_{2m-1} = L_0^m + \tilde{I}_{2m-1} + \mathcal{O}(c^{m-2})$$

In the basis of descendents $|m_i, h\rangle = L_{-m_1} \dots L_{-m_k}|h\rangle$, $m_1 \geq m_2 \geq \dots \geq m_k$, $\tilde{I}_{2m-1}$ lower triangular.
Eigenvalues

$$h^m + \sum_{p=0}^{m-1} m_i^{2p+1} k^p h^{m-1-p} \xi_m^p + \mathcal{O}(c^{m-2});$$

using our results, find

$$\xi_m^p = \frac{(2m-1)\Gamma(m+1)\Gamma(1/2)}{2\Gamma(p+3/2)\Gamma(m-p)}$$

(checked up to $m = 7$).

Subleading $1/c$ corrections

Dymarsky & Pavlenko

Gives

$$\log Z = \frac{(2\pi)^2}{\beta^2}\left[\frac{c-1}{24} f_0(\tilde{\mu}) + f_1(\tilde{\mu}) + \mathcal{O}(1/c)\right],$$

where

$$f_0(\tilde{\mu}) = F(\tilde{\mu}) = \sum_{m=1}^{\infty} \tilde{\mu}_{2m-1}(2m-1)\sigma^m,$$

$$f_1(\tilde{\mu}) = -\frac{1}{2\pi}\int_0^{\infty} \kappa \log(1 - e^{-2\pi\kappa\gamma(\tilde{\mu},\kappa)}),$$

$$\gamma(\tilde{\mu},\kappa) = \sum_{m=1}^{\infty} \tilde{\mu}_{2m-1} m(2m-1)\sigma^m {}_2F_1(1, 1-m, 3/2, -\kappa^2/\sigma)$$

$$\sqrt{\sigma} = \sum_{m=1}^{\infty} \tilde{\mu}_{2m-1} m\sigma^m.$$

Statistics of KdV charges

- $1/c$ corrections to the correlation functions come from contributions of non-primary states.
- We can study these more directly by considering trace over a single Verma module.
- At high temperatures, dominated by contributions at a level $n \sim \beta^{-2}$:
  $\Rightarrow$ Correlation functions $\langle I^n_{2m-1} \rangle_h$ tell us about statistics of KdV charges at high level.

Given by the same differential operator acting on the character:

$$\langle I_{2m_1-1} \dots I_{2m_n-1} \rangle_h = (D^M + E_4 D^{M-2} + \dots)\chi_h.$$

Statistics of KdV charges

Evaluate at high temperature by using modular transformation:

$$\chi_h = \frac{q^{h-\frac{c-1}{24}}}{\eta} \approx e^{\frac{\pi^2}{6\beta}-\beta(h-k+\frac{1}{24})}\sqrt{\frac{\beta}{2\pi}},$$

$$E_2 \approx -\left(\frac{2\pi}{\beta}\right)^2 + \frac{12}{\beta}, \quad E_4 \approx \left(\frac{2\pi}{\beta}\right)^4, \quad E_6 \approx -\left(\frac{2\pi}{\beta}\right)^6.$$

We find the same temperature dependence as in canonical ensemble: e.g.

$$\langle I_3\rangle_h \sim \frac{k}{\beta^4}, \quad \langle I_3^2\rangle_{h,c} \sim \frac{k^2}{\beta^7}, \quad \langle I_3^3\rangle_{h,c} \sim \frac{k^3}{\beta^{10}}$$

This implies distribution is very sharply peaked.

Discussion

- Calculated high temperature GGE at infinite c.
- At finite c, calculate thermal correlation functions of KdV charges: formal expansion of GGE in μ_j.
- Thermal correlation functions of KdV charges are quasi-modular forms, derivatives of partition function: finite set of undetermined coefficients in each correlation function.
- Coefficients for low order charges can be fixed from q expansion.
- Conjectured form for first subleading $1/c$ corrections to all orders.

Dymarsky & Pavlenko

Discussion

Further directions:

- Understanding structure of $1/c$ corrections.
- Relation to holography de Boer & Engelhardt
- Generalization to other chiral algebras