Title: Good Approximate Quantum LDPC Codes from Spacetime Circuit Hamiltonians

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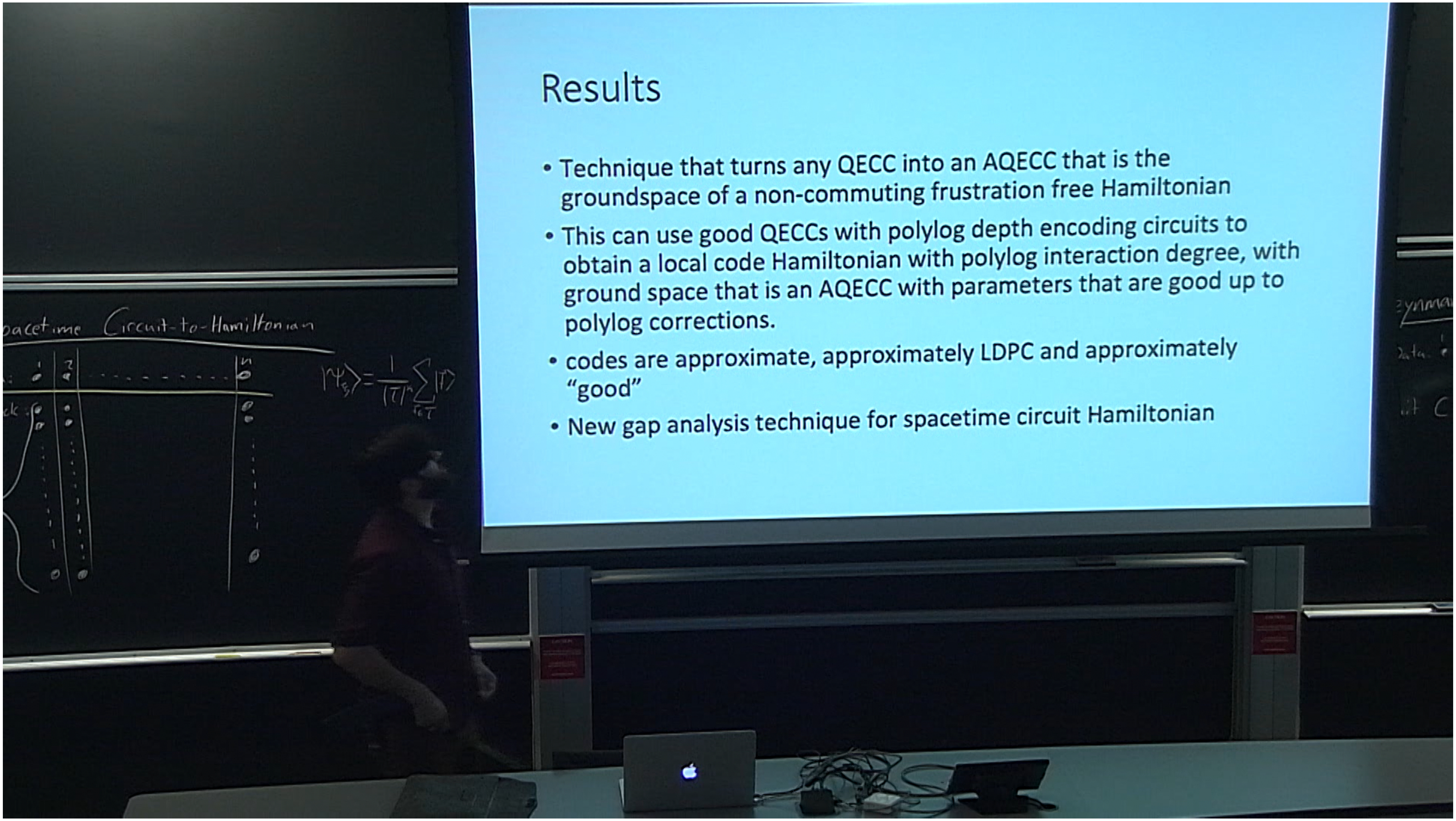
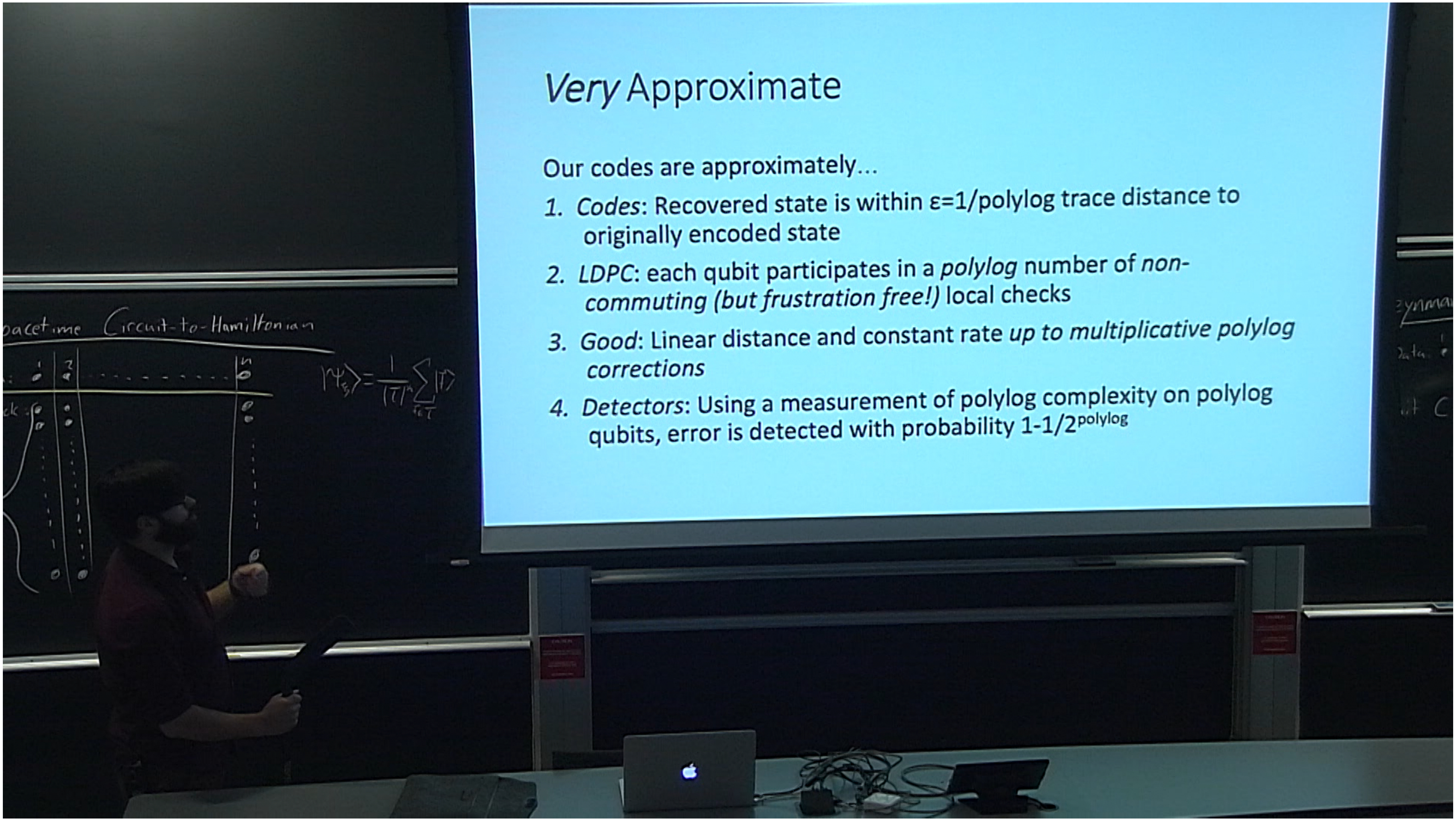
Abstract: We study approximate quantum low-density parity-check (QLDPC) codes, which are approximate quantum error-correcting codes specified as the ground space of a frustration-free local Hamiltonian, whose terms do not necessarily commute. Such codes generalize stabilizer QLDPC codes, which are exact quantum error-correcting codes with sparse, low-weight stabilizer generators (i.e. each stabilizer generator acts on a few qubits, and each qubit participates in a few stabilizer generators). Our investigation is motivated by an important question in Hamiltonian complexity and quantum coding theory: do stabilizer QLDPC codes with constant rate, linear distance, and constant-weight stabilizers exist? We show that obtaining such optimal scaling of parameters (modulo polylogarithmic corrections) is possible if we go beyond stabilizer codes: we prove the existence of a family of [[N,k,d,Îµ]] approximate QLDPC codes that encode k = Î©(N/polylog N) into N physical qubits with distance d = Î©(N/polylog N) and approximation infidelity Îµ = 1/polylog N. We prove the existence of an efficient encoding map, and we show that arbitrary Pauli errors can be locally detected by circuits of polylogarithmic depth. Finally, we show that the spectral gap of the code Hamiltonian is Î©(N^(-3.09)) (up to polylog(N) factors) by analyzing a spacetime circuit-to-Hamiltonian construction for a bitonic sorting network architecture that is spatially local in polylog(N) spatial dimensions. (Joint work with Elizabeth Crosson, Chinmay Nirkhe, and Henry Yuen, arXiv:1811.00277)


Results
• Technique that turns any QECC into an AQECC that is the groundspace of a non-commuting frustration free Hamiltonian
• This can use good QECCs with polylog depth encoding circuits to obtain a local code Hamiltonian with polylog interaction degree, with ground space that is an AQECC with parameters that are good up to polylog corrections.
• codes are approximate, approximately LDPC and approximately "good"
• New gap analysis technique for spacetime circuit Hamiltonian
Circuit-to-Hamiltonian


Very Approximate
Our codes are approximately…
1. Codes: Recovered state is within ε=1/polylog trace distance to originally encoded state
2. LDPC: each qubit participates in a polylog number of non-commuting (but frustration free!) local checks
3. Good: Linear distance and constant rate up to multiplicative polylog corrections
4. Detectors: Using a measurement of polylog complexity on polylog qubits, error is detected with probability 1-1/2^polylog
Circuit-to-Hamiltonian

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2. *LDPC*: each qubit participates in a *polylog* number of *non-commuting (but frustration free!)* local checks
3. *Good*: Linear distance and constant rate *up to multiplicative polylog corrections*
4. *Detectors*: Using a measurement of polylog complexity on polylog qubits, error is detected with probability $1\text{-}1/2^{\text{polylog}}$

Motivations

- Quantum LDPC Conjecture: do there exist CSS stabilizer codes with constant rate, linear distance, and local, sparse stabilizer generators? We try and make progress by going *beyond* the paradigm of *exact* codes that are *stabilizer*
- We don't yet know for sure what we can say about possible code parameters, entanglement properties, and connections to condensed matter and holography for codes with *commuting* checks
- Very little is known about the realm of possibilities with codes whose checks do not commute

The state of QLDPC…

Reference	# of logical qubits	Distance	Locality	Notes
Tillich, Zémor '14	$\Omega(N)$	$\Omega(\sqrt{N})$	$\mathcal{O}(1)$	CSS stabilizer code
Freedman, Meyer, Luo, '02	$\mathcal{O}(1)$	$\Omega(\sqrt{N \log N})$	$\mathcal{O}(1)$	CSS stabilizer code
Hastings '17	$\mathcal{O}(1)$	$\Omega(N^{1-\xi})$	$\mathcal{O}(1)$	CSS Stabilizer code (conjectured)
Bacon, Flammia, Harrow, Shi '14	$\Omega(N)$	$\Omega(N^{1-\xi})$	$\mathcal{O}(1)$	Subsystem code, Frustrated H
(this work)	$\tilde{\Omega}(N)$	$\tilde{\Omega}(N)$	$\mathcal{O}(1)$	Approx. code, frustration-free noncommuting H

Our codes do not…

- Have explicit example constructions
- Have any known efficient recovery process
- Appear to be useful for quantum fault-tolerance in any practical sense

Approximate QLDPC Code

Like a standard QLDPC code except:

1. Codewords are recovered with fidelity 1-ε
2. Codespace is ground space of frustration free local Hamiltonian whose terms need not commute
3. Qubit interaction degree should be at worst polylogarithmic rather than constant

Starting Point

- Nirkhe, Vazirani + Yuen (2018) showed that you can create an AQECC by taking any QECC and feeding its encoding circuit into the Feynman-Kitaev circuit-to-Hamiltonian construction (Kitaev 1999)
- The constructed Hamiltonian H_{circuit} has a ground space spanned by states of the form

$$|\psi_\xi\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |unary(t)\rangle \otimes U_t ... U_1 |\xi\rangle |0^{n-k}\rangle$$

- ξ is the k-qubit state that we are encoding into an n-qubit codeword via the encoding circuit $C = U_T \ldots U_1$

$$|\psi_\xi\rangle = \frac{1}{\sqrt{T+1}}\sum_{t=0}^{T}|unary(t)\rangle \otimes U_t ... U_1|\xi\rangle|0^{n-k}\rangle$$

- This is a codeword of the *approximate* code, and is made up of *n+T* qubits (where *n* is number of qubits of the underlying code, *T* is size of encoding circuit)
- Even superposition over all stages of "partial encoding" of ξ
- Only the final term of the superposition is the actual codeword, meaning high error!
- Idea: pad the encoding circuit with identity gates, then most of the weight will be on the actual desired codeword!

Limitations…

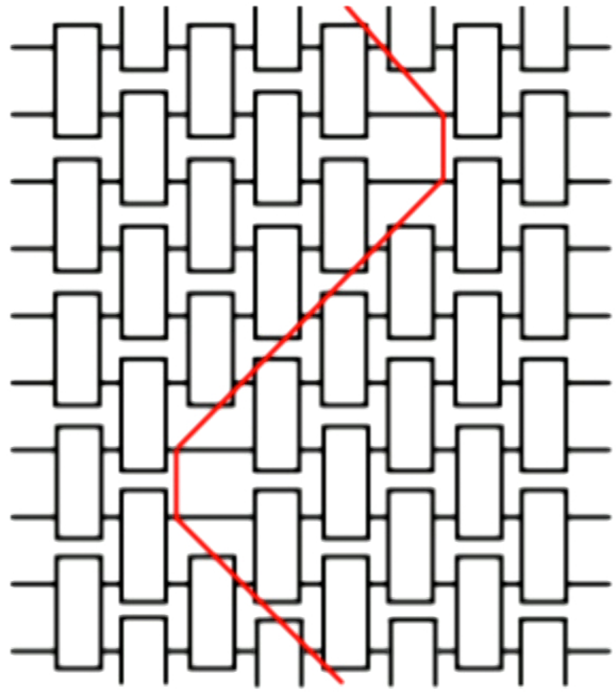
- The NVY construction has constant weight checks (terms of the Hamiltonian H_{circuit}) even if the underlying code does not, which is nice! But each qubit still participates in a large number of checks.
- The number of checks acting on each qubit scales with the depth of the encoding circuit for the underlying code
- Also, we don't have spatial locality

But…

- Now all qubits interact with all other qubits and the construction has absolutely no spatial locality
- To be clear: spatial locality was an issue before, but this is even worse now that all qubits talk to each other
- We need a few more modifications…

Spacetime Circuit to Hamiltonian Construction

- Breuckmann and Terhal came up with a helpful generalization of the Feynman-Kitaev construction
- Instead of enforcing a total order in which the gates of the circuit need to be applied, we can use the partial order enforced by causality
- This allows us to use local clocks for each qubit instead of a single global clock
- It's not completely obvious how, but this ends up allowing our checks to be spatially local in log(n) dimensions



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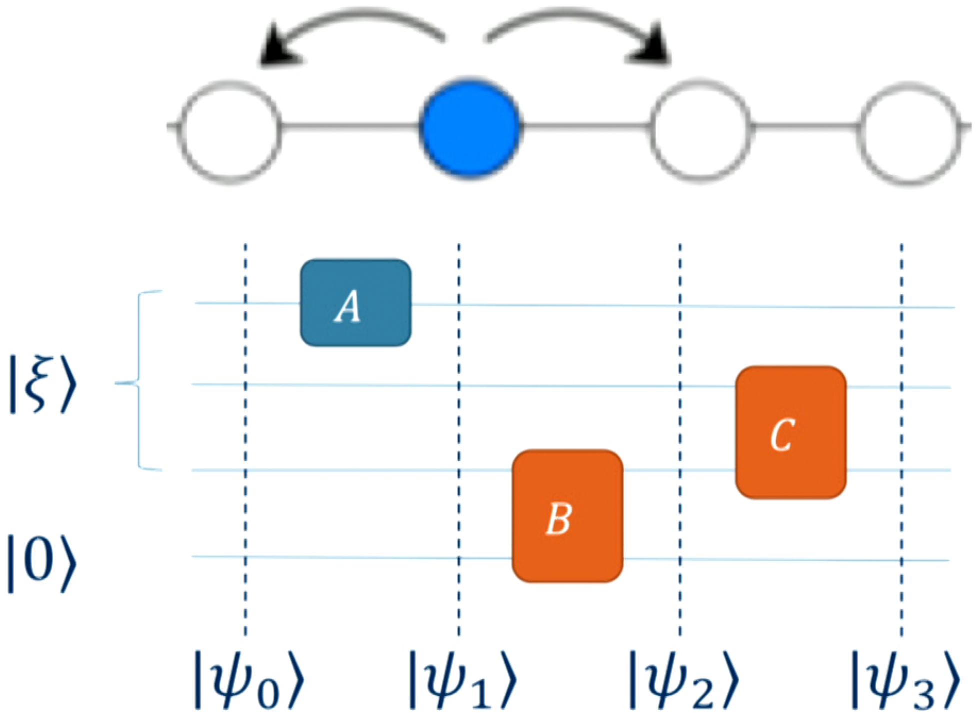
Spacetime Codewords

- Ground states of spacetime circuit Hamiltonian are

$$|\psi_\xi\rangle = \frac{1}{|\mathcal{T}|^{1/2}} \sum_{\vec{t}\in\mathcal{T}} |\vec{t}\rangle \otimes U_{\vec{t}}|\xi\rangle|0^{n-k}\rangle$$

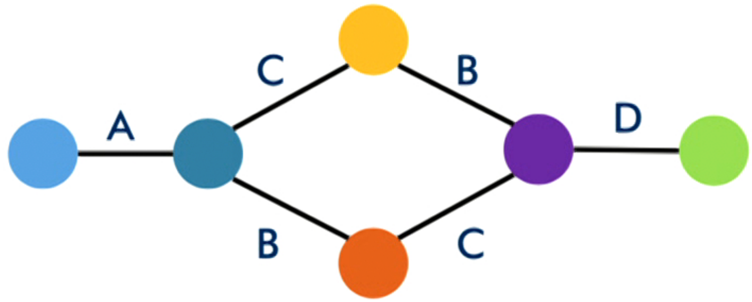
- $\mathcal{T}$ is the set of "valid" circuit configurations $\vec{t} = (t_1, ..., t_n)$ allowed by causality, corresponding to partially applied circuit $U_{\vec{t}}$



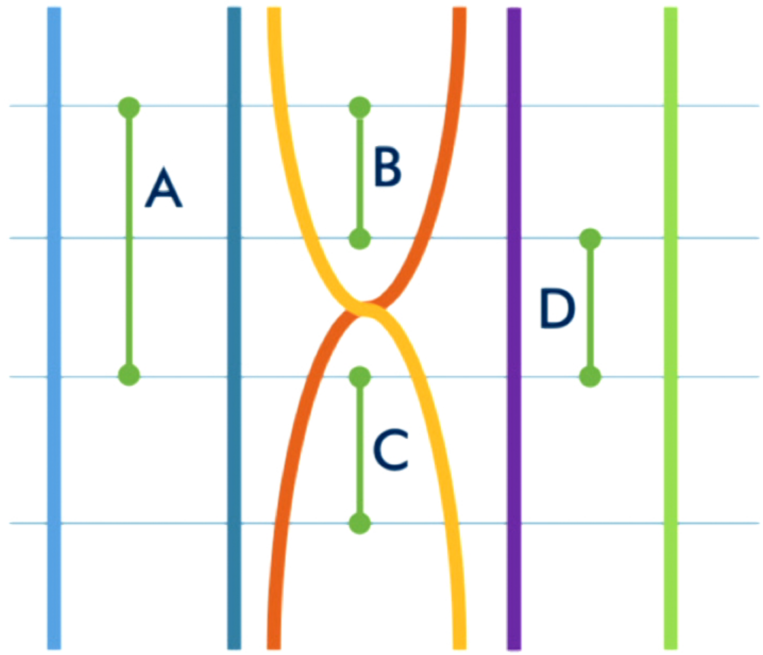
$$|\psi_0\rangle = |\xi\rangle|0\rangle$$
$$|\psi_1\rangle = A|\psi_0\rangle$$
$$|\psi_2\rangle = B|\psi_1\rangle$$
$$|\psi_3\rangle = C|\psi_2\rangle$$

The story so far

- We take a random BF encoding circuit on n qubits, plug it into the *spacetime* circuit-to-Hamiltonian construction
- Obtain an approximate QLDPC code on $N=O(n\log^3 n)$ qubits
- Encodes $\Omega(n)=\Omega(N/\log^3 N)$ qubits (inherited from BF code)
- Has distance $\Omega(n)=\Omega(N/\log^3 N)$ (inherited from BF code)
- We think things are starting to sound good…



$= |2\rangle|2\rangle|1\rangle|1\rangle$



Why?

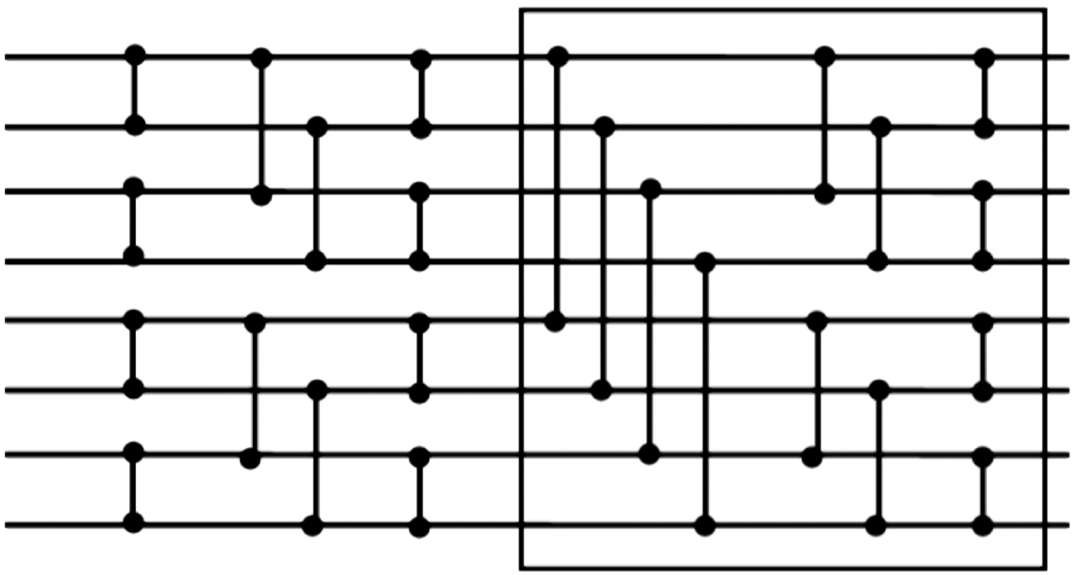
- We analyze the spectral gap of these Hamiltonians by analyzing the spectral gap of a Markov chain on the corresponding graph Laplacian
- We would want to use Cheeger’s Inequality to get a lower bound on the gap
- But with a random 2-qubit gate circuit, the analysis and sheer number of partial configurations of a circuit make it seem like the best that we could hope for is an exponentially small gap ☹
- This would make the code pretty useless for any purpose (it would be exponentially difficult to determine if a state is in the codespace or not)

One more technical trick!

- Analysis would be simpler if exact structure and qubit connectivity of the encoding circuit could be known/assumed
- So we *uniformize* the structure of the random BF encoding circuit using *bitonic sorting networks*

Bitonic Sort

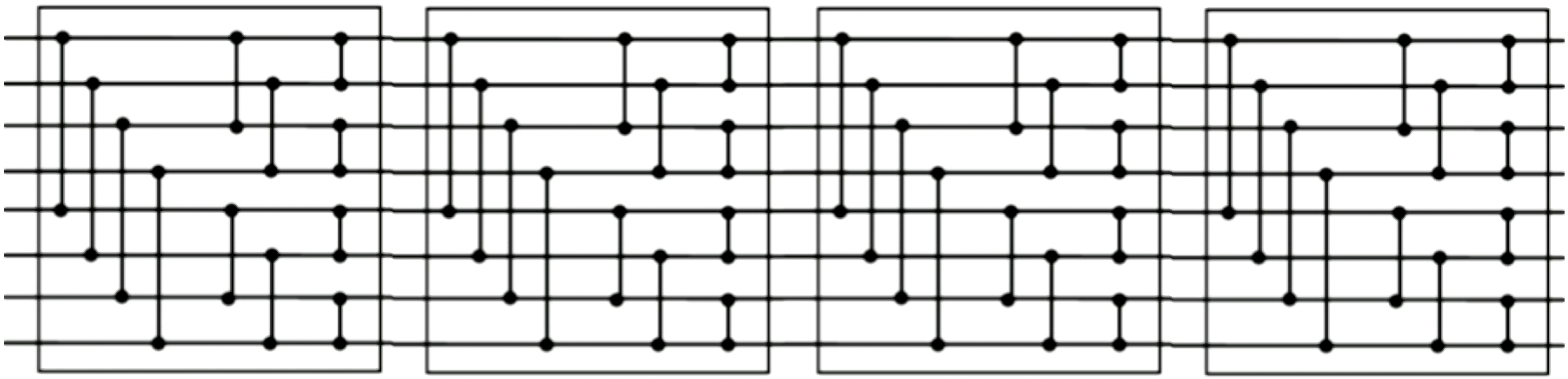
- Bitonic sort (Batcher) is a parallel sorting algorithm that uses a depth $O(\log^2 n)$ circuit made up of swap gates that can enact any permutation on a set of data (like qubit positions)



Bitonic Sorting Architecture

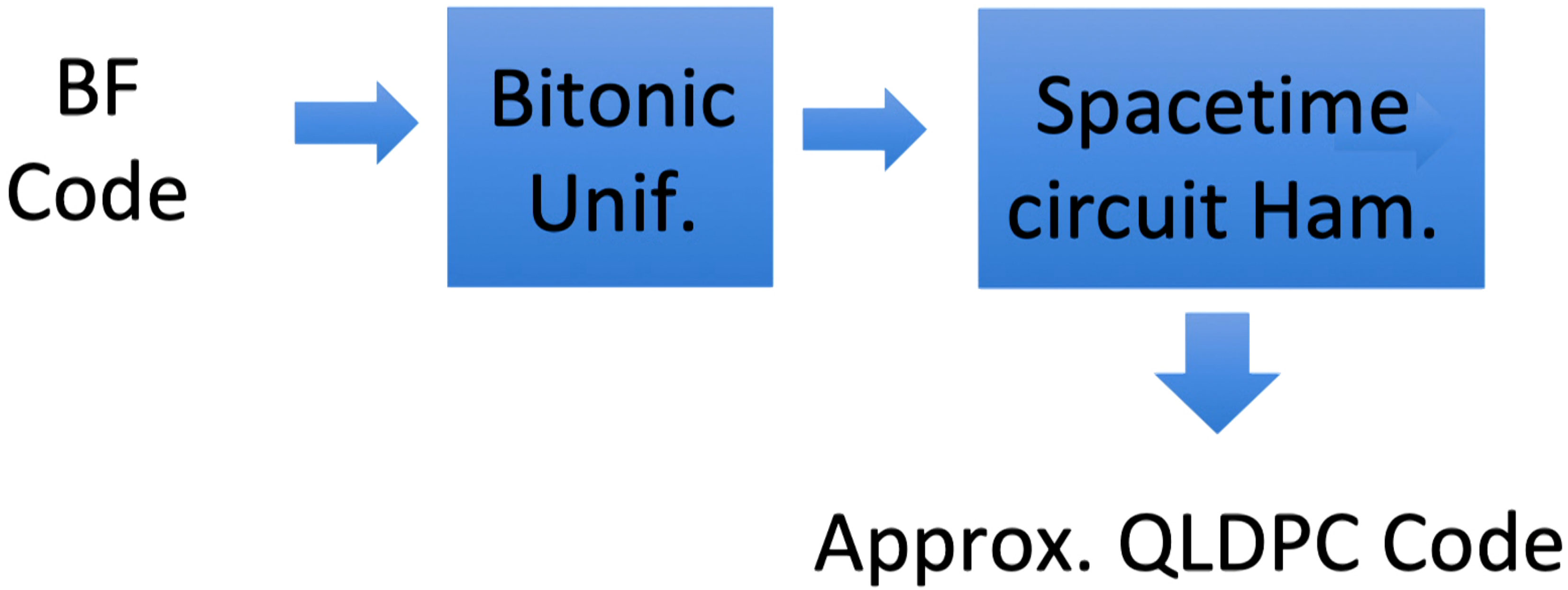
- In this circuit architecture, each qubit interacts with exactly log(n) other qubits (connectivity is that of a hypercube in log(n) dimensions)
- SO: Use a bitonic sorting circuit between each layer of random gates of the BF encoding circuit so that the non-trivial gates always act on the *same* pairs of qubits

Uniformized BF Circuit



- Encoding circuit now has depth $O(\log^5 n)$, but has a structure amenable to tractable analysis of soundness
- The log(n) connectivity of each qubit also allows us to make the construction spatially local in log(n) dimensions!

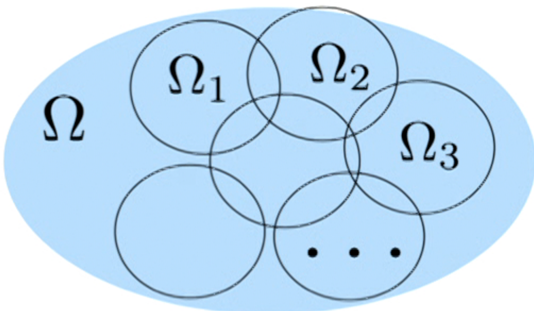
Schematically…



Soundness Methods

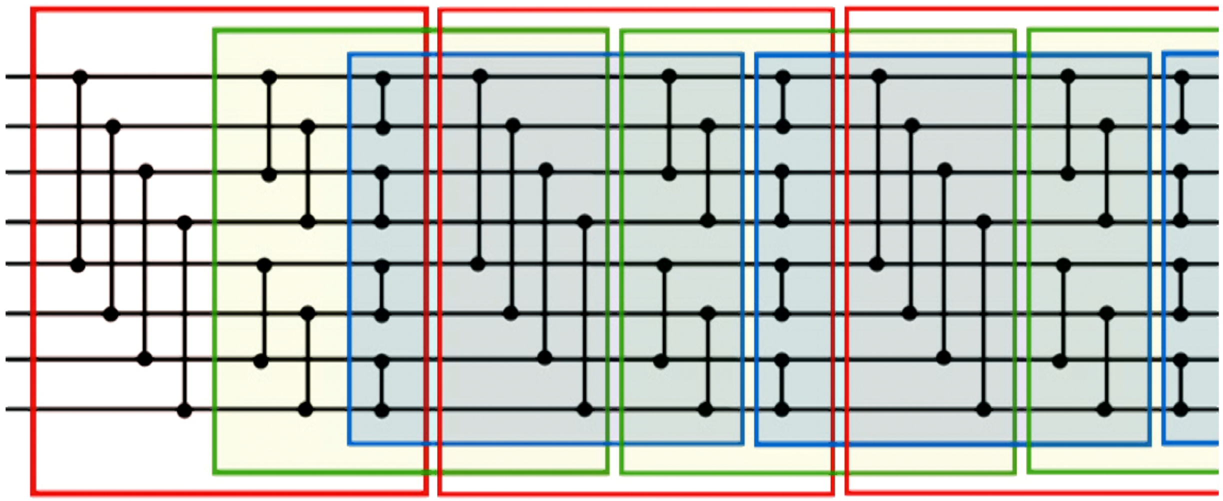
- Let (π, P, Ω) be a reversible Markov Chain, with state space subsets Ω_i
- We can create an *aggregate* Markov chain on the subsets Ω_i:

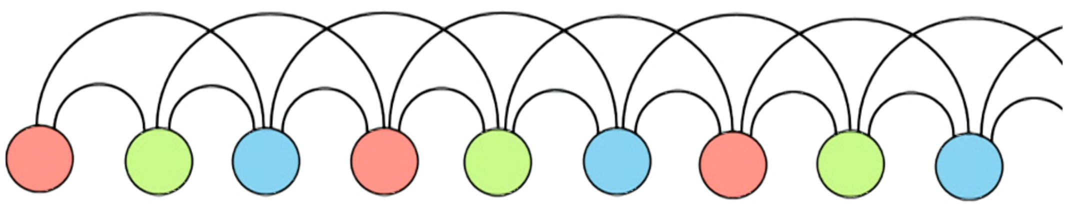
$$\bar{P}\left(\Omega_i \to \Omega_j\right) = \frac{\pi\left(\Omega_i \cap \Omega_j\right)}{B\pi(\Omega_i)}$$



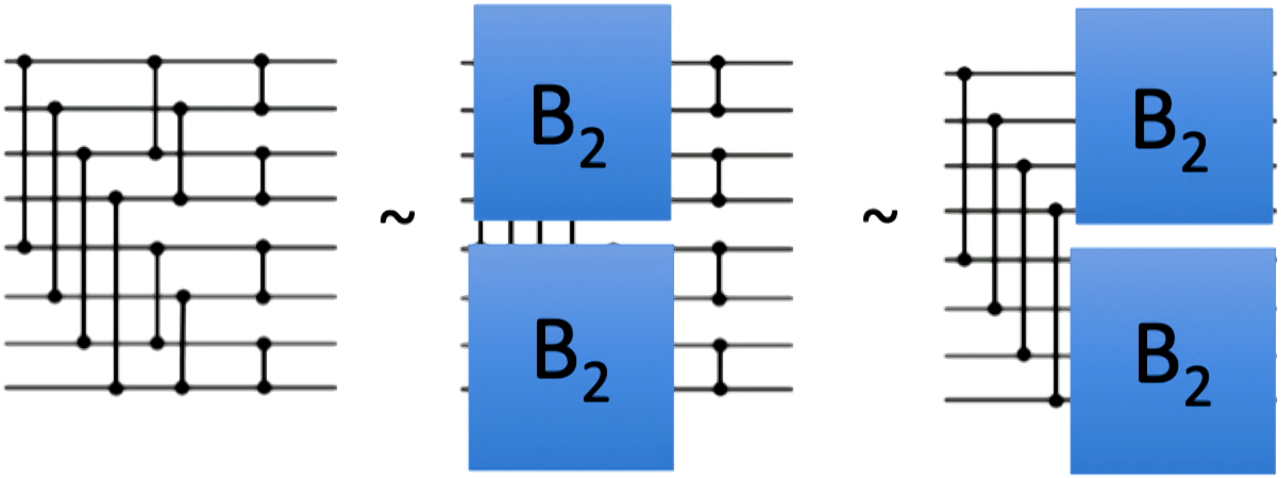
- Define restricted Markov Chains on the subsets, $P_{\Omega i}$, by removing transitions that leave Ω_i
- Markov chain literature (Madras + Randall) says

$$gap(P) \geq \frac{1}{2} gap(\bar{P}) \min_{i=1,\ldots,m} gap\left(P_{\Omega_i}\right)$$





Let's Count Configurations of B_3



- If b_r counts configurations of B_r, then b_3 has no more than $2b_2^2$ configurations

Let's Count Configurations of B_3

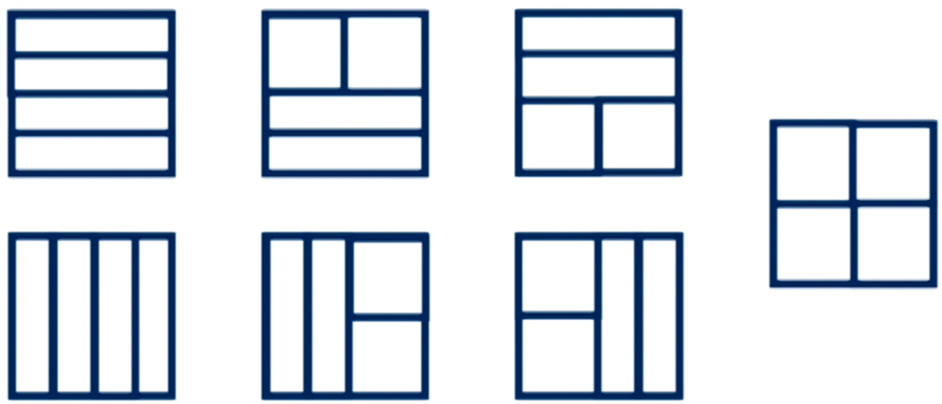
- But we overcounted! We overcounted by the number of configurations that live in the *overlap* of the two B_2 decompositions
- In this case, the overlap is just the second layer of B_3, which is isomoprhic to *four* copies of B_1!
- So,

$$b_3 = 2b_2^2 - b_1^4$$

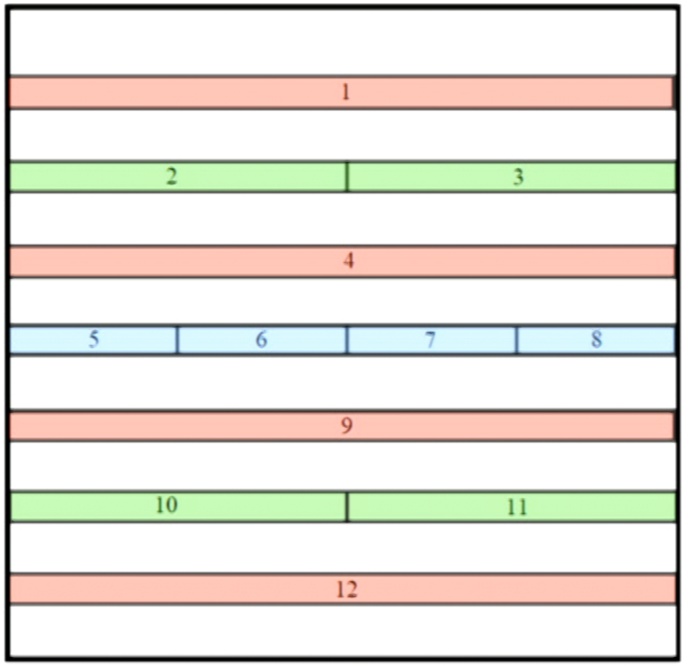
Meanwhile…

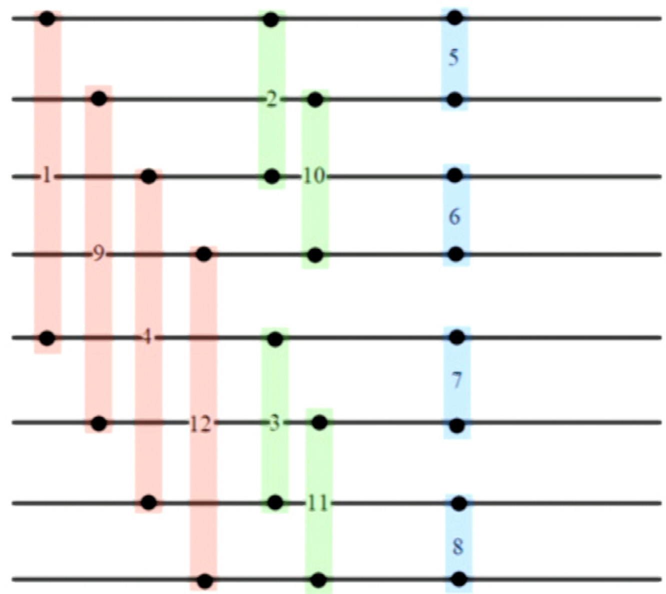
- Dyadic tiling of the unit square by dyadic rectangles:

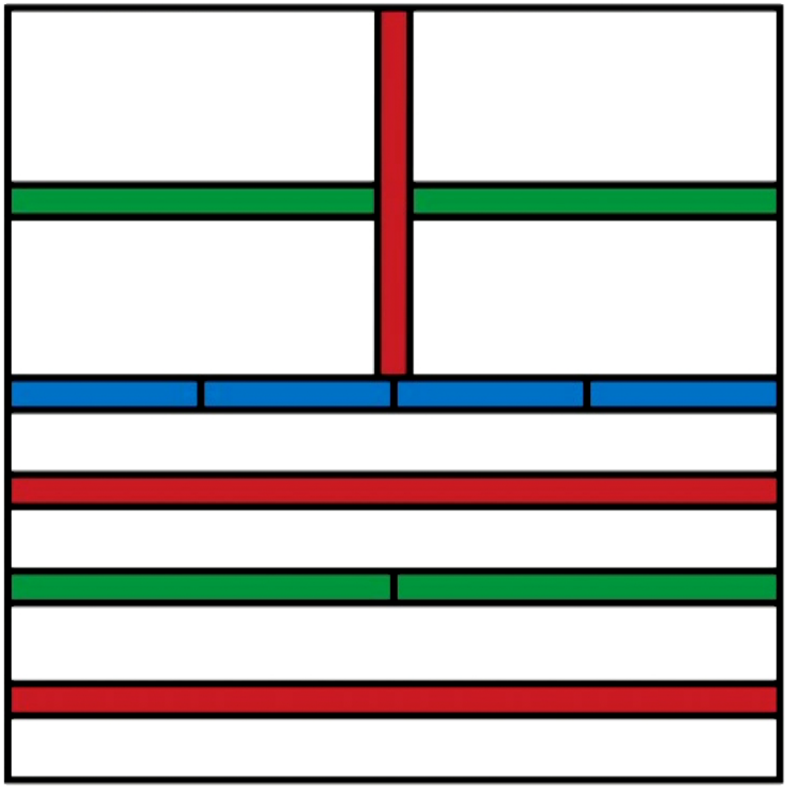
$$[a2^{-s},(a+1)2^{-s}]\times[b2^{-t},(b+1)2^{-t}]\ \ ,\ \ a,b,s,t\in\mathbb{Z}_{\geq 0}$$

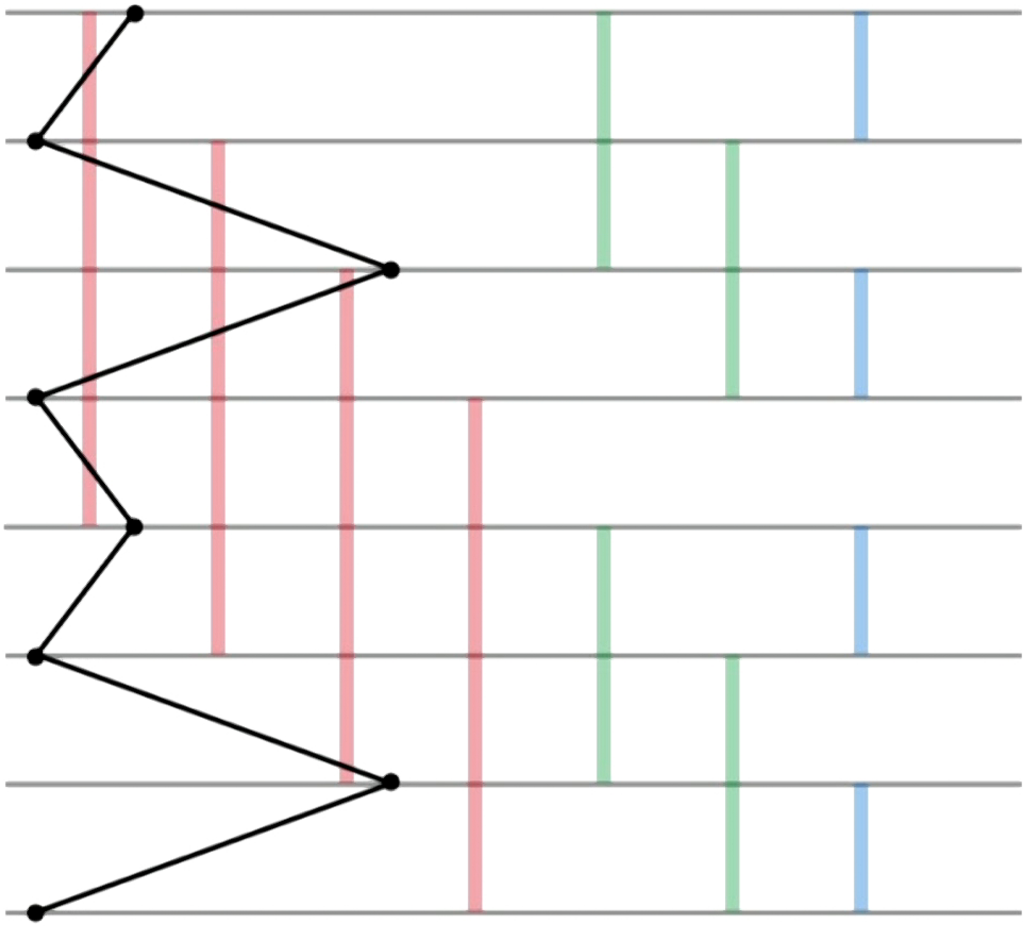


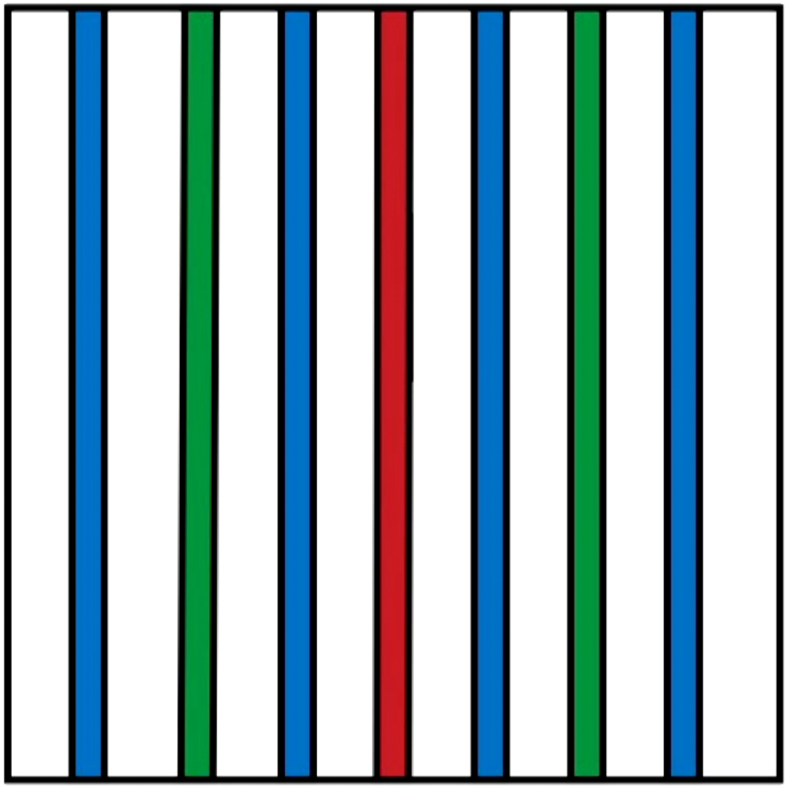
The Isomorphism…

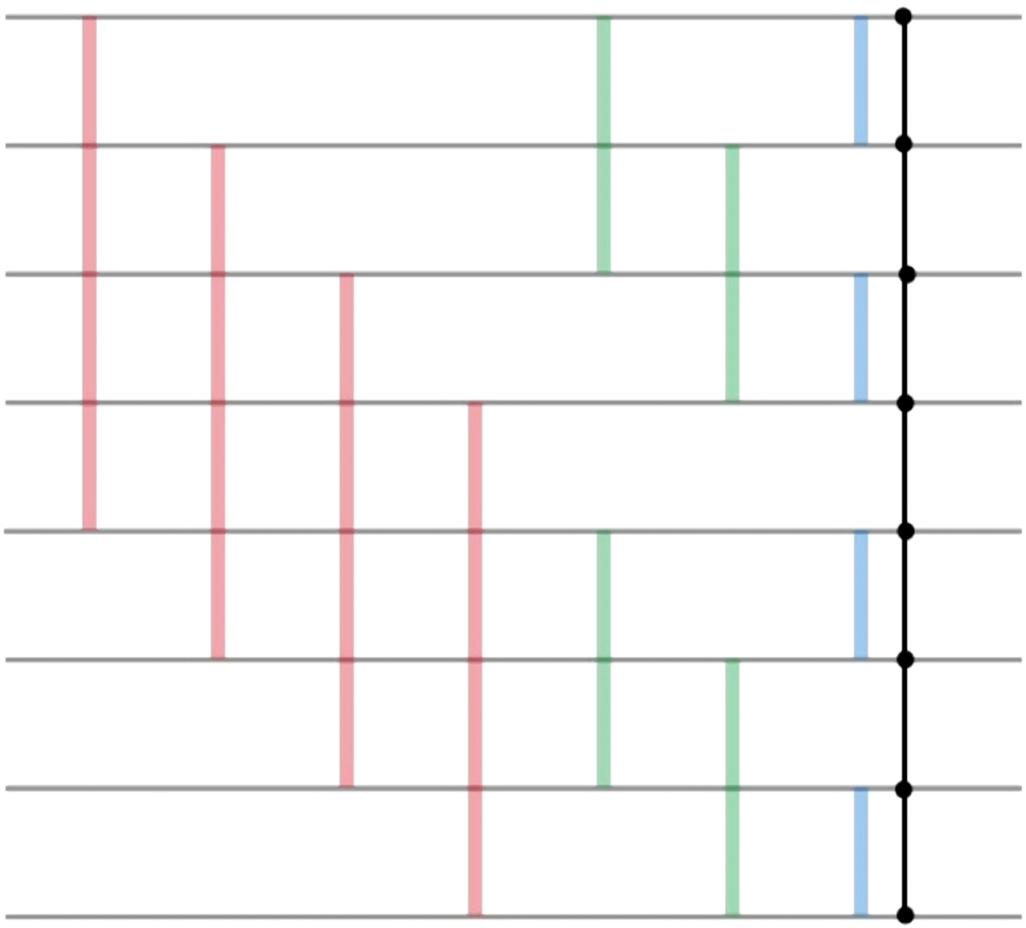












Let's just say…

- Using Cheeger's inequality and similar counting techniques, we can obtain:

$$\Delta_{\overline{P}} \geq \frac{2}{\phi^2 m^2 \log^4 n} = \frac{1}{\mathrm{polylog}(n)}$$

- So for the whole gap:

$$\Delta_H \geq \frac{1}{n^{3.09}\mathrm{polylog}(n)}$$

Encoding + Decoding

- Spacetime codewords can be generated by a poly(n)-size quantum circuit
- Decoding uses the recovery map of the underlying BF code: just throw away the clock qubits and use the recovery map on the data qubits
- Recovered state has error 1/polylog(N)

Summary

- Our “codes” show that if we go beyond the stabilizer paradigm, we can say interesting things about qLDPC!
- New techniques were invented for our analysis: maybe they’ll find application in other problems?
- Many questions about spacetime codes: could we come up with concrete examples? Understand logical operators? Efficient decoding?

Thank you!