Title: A Gleason-type theorem for qubits based on mixtures of projective measurements

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Series: Quantum Foundations

Date: May 14, 2019 - 3:30 PM

URL: http://pirsa.org/19050023

Abstract: We derive Bornâ€™s rule and the density-operator formalism for quantum systems with Hilbert spaces of finite dimension. Our extension of Gleasonâ€™s theorem only relies upon the consistent assignment of probabilities to the outcomes of projective measurements and their classical mixtures. This assumption is significantly weaker than those required for existing Gleason-type theorems valid in dimension two.

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Gleason-type theorems

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PI 2019

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This talk

1. A Gleason-type theorem for qubits from mixtures of projective measurements

2. Only general probabilistic theories satisfying the no-restriction hypothesis admit a Gleason-type theorem

Gleason's Theorem

Projective measurements	$\Longrightarrow$	Density operators and Born rule
$A, \{\Pi_1, \Pi_2, \ldots\}$		$\mathrm{Tr}\left(\Pi_j \rho\right)$

Applying Gleason's Theorem

- Postulates:
  - (H): A **quantum system** is associated with a separable **Hilbert space**, $\mathcal{H}$.
  - (M): **Measurements** can be represented by **collections of projections** $\{\Pi_1, \Pi_2, \ldots\}$ which sum to the identity, where each projection corresponds to an outcome of the measurement.

- **States**: specify outcome probabilities
  - $f : \mathcal{P}(\mathcal{H}) \to [0, 1]$
  - $f(\Pi_1) + f(\Pi_2) + \ldots = 1$

Gleason's Theorem

Theorem (Gleason 1957)

Let $\mathcal{H}$ *be a separable Hilbert space with* $\dim(\mathcal{H}) \geq 3$. *A function* $f : \mathcal{P}(\mathcal{H}) \to [0,1]$ *satisfying*

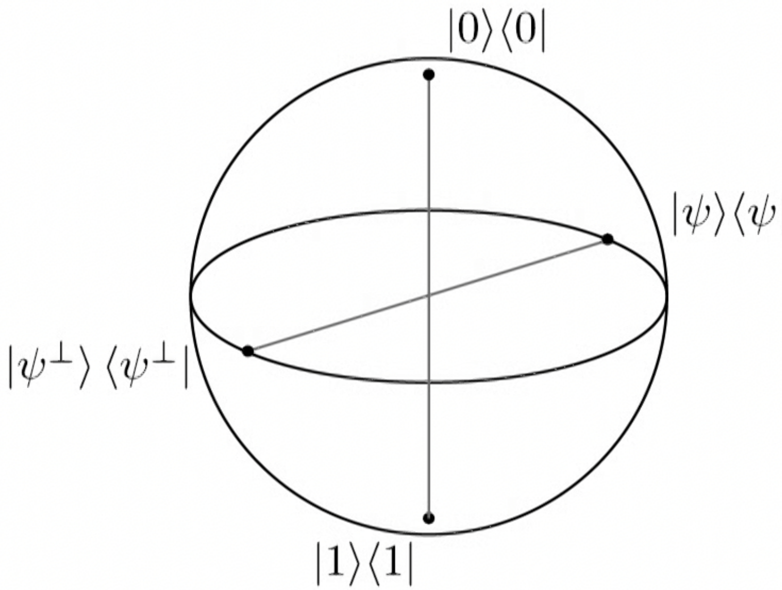
$$f(\Pi_1) + f(\Pi_2) + \ldots = 1$$

for any collection of self-adjoint projections $\{\Pi_1, \Pi_2 \ldots\}$ *onto mutually orthogonal closed subspaces of* $\mathcal{H}$ *admits an expression*

$$f(\Pi) = \mathrm{Tr}(\Pi\rho)$$

for all $\Pi \in \mathcal{P}(\mathcal{H})$ *and some density operator* ρ *on* $\mathcal{H}$.

Dimension two



Each rank-one projection $|\psi\rangle\langle\psi|$ only features in one equation:

$$f\left(|\psi\rangle\langle\psi|\right) + f\left(|\psi^\perp\rangle\langle\psi^\perp|\right) = 1\,.$$

Generalised measurements

- Postulates:
  - (H): A **quantum system** is associated with a separable **Hilbert space**, $\mathcal{H}$.
  - (M): **Measurements** can be represented by **collections of projections** $\{\Pi_1, \Pi_2, \ldots\}$ which sum to the identity, where each projection corresponds to an outcome of the measurement.

- **States**: specify outcome probabilities
  - $f : \mathcal{P}(\mathcal{H}) \to [0, 1]$
  - $f(E_1) + f(E_2) + \ldots = 1$

Generalised measurements

- Postulates:
  - (H): A **quantum system** is associated with a separable **Hilbert space**, $\mathcal{H}$.
  - ~~(M): **Measurements** can be represented by **collections of projections** $\{\Pi_1, \Pi_2, \ldots\}$ which sum to the identity, where each projection corresponds to an outcome of the measurement.~~
  - (GM): **Measurements** can be represented by **collections of effects** $\{E_1, E_2, \ldots\}$ which sum to the identity, where each effect corresponds to an outcome of the measurement.
- **States**: specify outcome probabilities
  - $f : \cancel{\mathcal{P}(\mathcal{H})} \mathcal{E}(\mathcal{H}) \to [0, 1]$
  - $f(E_1) + f(E_2) + \ldots = 1$

Gleason-type Theorem

Theorem (Busch 2003)

Let $\mathcal{H}$ be a separable Hilbert space with $\dim(\mathcal{H}) \geq 2$*. A function* $f : \mathcal{E}(\mathcal{H}) \to [0,1]$ *satisfying*

$$f(E_1) + f(E_2) + \ldots = 1$$

for any collection of effects $\{E_1, E_2 \ldots\}$ *on* $\mathcal{H}$ *satisfying* $\sum_j E_j = \mathrm{I}$ *necessarily admits an expression*

$$f(E) = \mathrm{Tr}(E\rho)$$

for all $E \in \mathcal{E}(\mathcal{H})$ *and some density operator* ρ *on* $\mathcal{H}$*.*

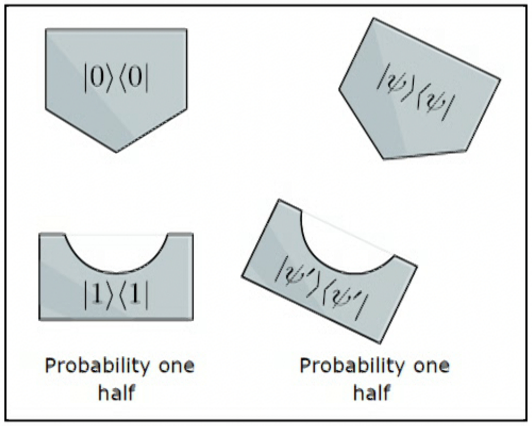
Smaller sets of measurements

- Does there exist a smaller set of measurements with this property?
  - i.e. a set of POMs $\mathbf{M}$ such that, if $f(E_1) + f(E_2) + \ldots = 1$ for all POMs $\{E_1, E_2, \ldots\} \in \mathbf{M}$ then f is necessarily of the form

$$f(E) = \mathrm{Tr}(E\rho)?$$

- Yes!

Projective simulable measurements[1]



Probability of observing zero outcome=

$$\frac{1}{2}\mathrm{Tr}\left(|0\rangle\langle 0|\,\rho\right) + \frac{1}{2}\mathrm{Tr}\left(|\psi\rangle\langle\psi|\,\rho\right)$$
$$= \mathrm{Tr}\left(\left(\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|\psi\rangle\langle\psi|\right)\rho\right)$$

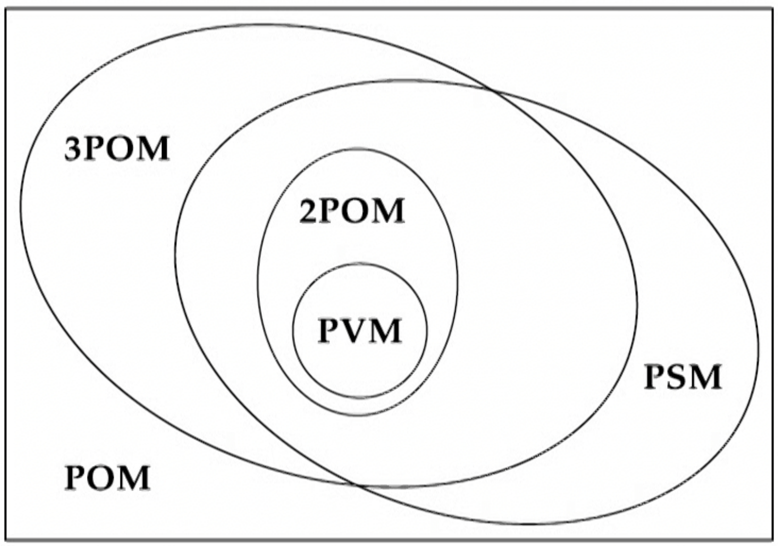
The statistics of the box are the same as any other implementation of the POM

$$\left\{\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|\psi\rangle\langle\psi|, \frac{1}{2}|1\rangle\langle 1| + \frac{1}{2}|\psi'\rangle\langle\psi'|\right\}$$

[1]Oszmaniec et al., "Simulating positive-operator-valued measures with projective measurements".

Measurement sets

Not all POMs projective simulable, but all two-outcome POMs are.



Main result

Theorem (Wright and Weigert 2018)

Let $\mathcal{H} = \mathbb{C}^d$*. A function* $f : \mathcal{E}(\mathcal{H}) \to [0,1]$ *satisfying*

$$f(E_1) + f(E_2) + \ldots = 1$$

for any collection of effects $\{E_1, E_2 \ldots\}$ *on* $\mathcal{H}$ *constituting a projective-simulable POM necessarily admits an expression*

$$f(E) = \mathrm{Tr}(E\rho)$$

for all $E \in \mathcal{E}(\mathcal{H})$ *and some density operator* ρ *on* $\mathcal{H}$*.*

Sketch of proof

1. $\{E, \mathbb{I}-E\}$ and $\{E/2, E/2, \mathbb{I}-E\}$
   - $f(E)+f(\mathbb{I}-E)=f(E/2)+f(E/2)+f(\mathbb{I}-E)=1$
   - $f(E)/2=f(E/2)$
2. $\{E/2+E'/2, \mathbb{I}-(E+E')/2\}$ and $\{E/2, E'/2, \mathbb{I}-(E+E')/2\}$
   - $f((E+E')/2)=f(E/2)+f(E'/2)$
   - $f(E+E')=f(E)+f(E')$

New postulates

(H): A quantum system is associated with a separable Hilbert space, $\mathcal{H}$.

(M): Measurements can be represented by collections of projections $\{\Pi_1, \Pi_2 \ldots\}$ which sum to the identity, where each projection corresponds to an outcome of the measurement.

(CM): The classical mixture of measurements $\{\Pi_1, \Pi_2 \ldots\}$ and $\{\Pi'_1, \Pi'_2 \ldots\}$ with probabilities p and $(1-p)$ resp. can be represented by $\{p\Pi_1 + (1-p)\,\Pi'_1, p\Pi_2 + (1-p)\,\Pi'_2 \ldots\}$.

Even smaller sets of measurements

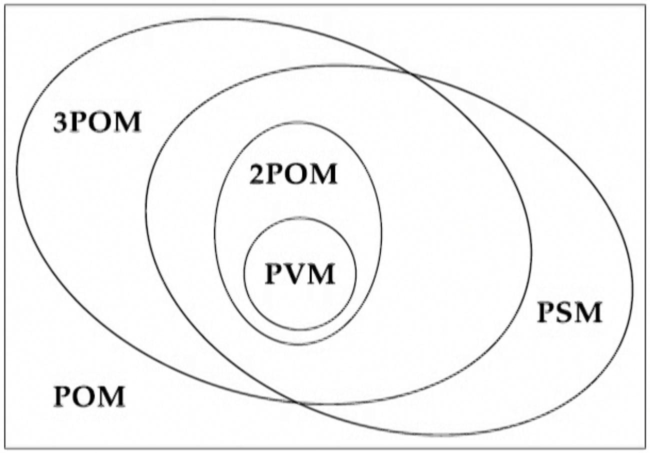
- Is the set of projective-simulable POMs the "smallest" measurement set from which we can prove a Gleason-type theorem?
- No!

A subset of projective simulable measurements

PSM′ consists of projective-simulable POMs of with two outcomes and those of the form

$$\{E/2, E/2, \mathbb{I}-E\} \text{ or } \{E_1/2, E_2/2, \mathbb{I}-(E_1+E_2)/2\}$$

for $E, E_1, E_2 \in \mathcal{E}(\mathcal{H})$ such that $E_1+E_2 \in \mathcal{E}(\mathcal{H})$.

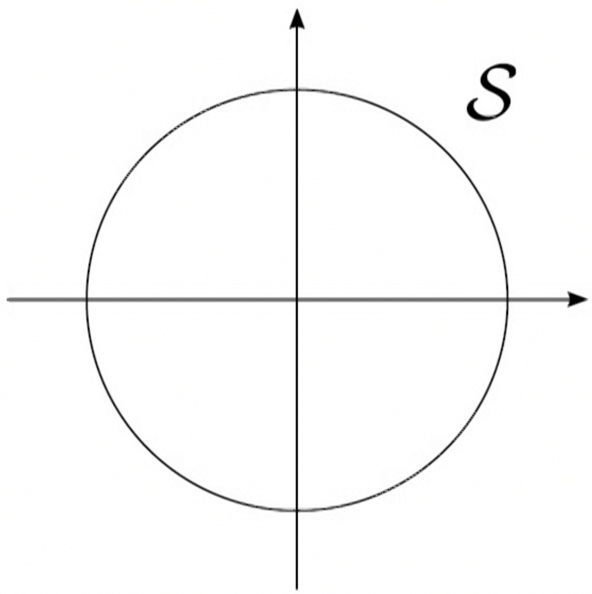


Summary and further work

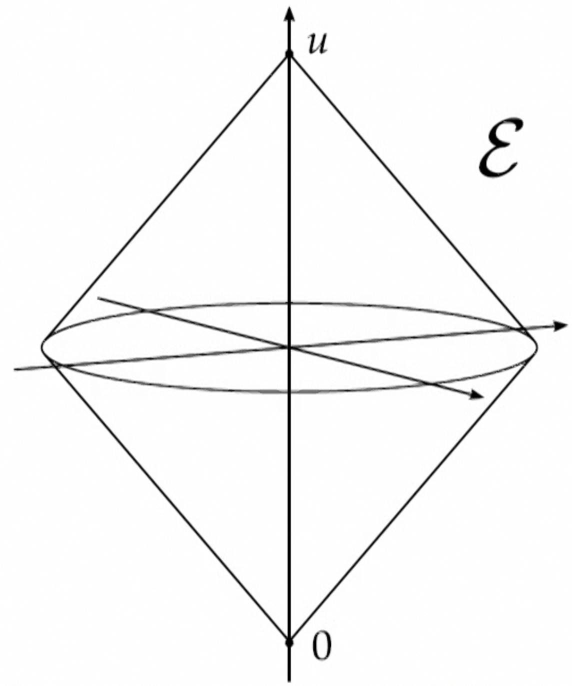
- Projective simulable POMs necessitate the density operator and Born rule formalism
- An additional postulate is identified that may be added to Gleason's original setting to extend the result to dimension two
  - States how classical mixtures of projective measurements are represented
- A smaller set of POMs, **PSM**′, shown to yield Gleason-type theorem
  - Significant weakening of the assumption of all POMs or all three outcome POMs[2].
- Is **PSM**′ minimal?
- Infinite dimensions.

[2] Granström, "Gleason's theorem".

General probabilistic theories



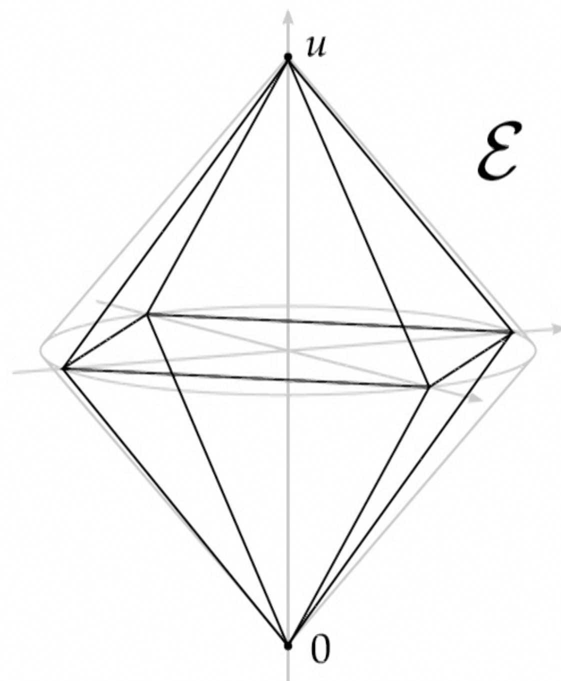
Convex subset of $\mathbb{R}^{d+1}$



A set of vectors $e \in \mathbb{R}^{d+1}$ such that $0 \leq e \cdot w \leq 1$ for all $w \in \mathcal{S}$

No-restriction hypothesis

- **No-restriction**: $\mathcal{E} = \left\{ e \in \mathbb{R}^{d+1} | 0 \leq e \cdot w \leq 1 \forall w \in \mathcal{S} \right\}$



- **Restriction**: $\mathcal{E}$ is a subset of this set
  - including 0 and u
  - and $u - e$ for all $e \in \mathcal{E}$

Gleason-type theorem

Theorem (Wright and Weigert (in preparation))

Let $\mathcal{S}$ and $\mathcal{E}$ be the state and effect spaces of a GPT. A function $f : \mathcal{E} \to [0, 1]$ satisfying

$$f(e_1) + f(e_2) + \ldots = 1$$

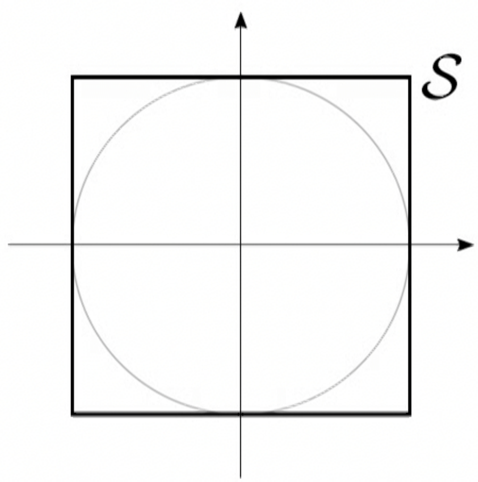
for any collection of effects $\{e_1, e_2 \ldots\}$ satisfying $e_1 + e_2 + \ldots = u$ necessarily admits an expression

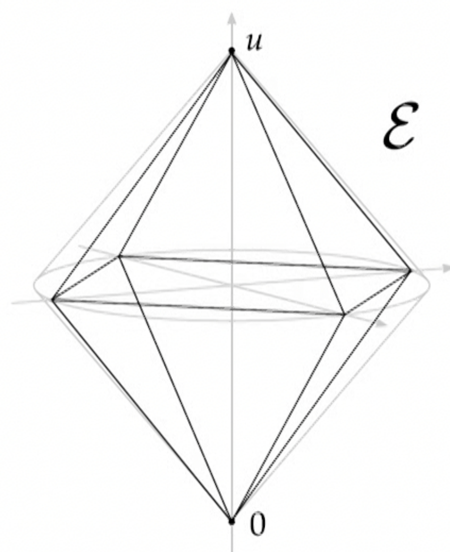
$$f(e) = e \cdot w$$

for all $e \in \mathcal{E}$ and some state $w \in \mathcal{S}$ if and only if the GPT satisfies the no-restriction hypothesis.

Comments

- Theorem reduces to Busch's Gleason type theorem if quantum state and effect spaces are chosen
- Restricted effect spaces lead to larger state spaces:





Summary

- GPTs satisfying the no-restriction hypothesis yield Gleason-type theorems
- "Restricted" GPTs do not
- Requiring the existence of a Gleason-type theorem rules out such theories
- Sainz et al. showed GPTs which produce the set of almost quantum correlations must violate the no-restriction hypothesis
  - and therefore do not yield a Gleason-type theorem
- Why might one require the existence of a Gleason-type theorem?

Thank you for listening!

Wright, V. J. and Weigert S. *A Gleason-type theorem for qubits based on mixtures of projective measurements*. J. Phys. A (2018).

Wright, V. J. and Weigert S. *Only general probabilistic theories satisfying the no-restriction hypothesis admit a Gleason-type theorem*. In preparation.

Good question

Assumptions in GPT framework:

- Countable set of fiducial measurement outcomes $\implies$ convex state space $\mathcal{S}$
- Effects are *linear functionals* on vector space containing $\mathcal{S}$

Our assumptions:

- Effect space $\mathcal{E}$
- States are *additive* maps on $\mathcal{E}$

$$f(E_1+E_2)=f(E_1)+f(E_2)$$

$$\sum_i f(E_i)=1$$