Title: Equivariant SYZ mirror construction

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Abstract: Strominger-Yau-Zaslow explained mirror symmetry via duality between tori.  There have been a lot of recent developments in the SYZ program, focusing on the non-equivariant setting.  In this talk, I explain an equivariant construction and apply it to toric Calabi-Yau manifolds.  It has a close relation to the equivariant open GW invariants found by Aganagic-Klemm-Vafa and studied by Katz-Liu, Graber-Zaslow and many others.

# Equivariant SYZ mirror construction

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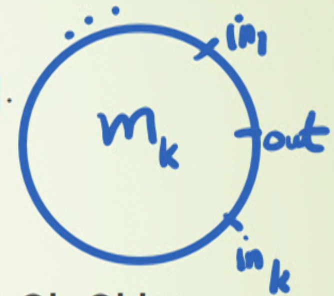
# SYZ mirror symmetry

# Strominger-Yau-Zaslow: Mirror symmetry is T-duality

- Konsevich's **homological mirror symmetry conjecture**:
$$DCoh(\check{M}) \cong DFuk(M).$$
- $O_p$ in $\check{M}$ is mirror to a Lagrangian brane $L$ in $M$.
$O_p$ has n dimensional deformation space. Thus expect $h^1(L) = n$.
$Ext^*(O_p, O_p) = \Lambda^* \mathbb{C}^n$. So $L \cong T^n$ cohomologically.
- $O_p$ moves in the whole space.
Thus $L$ should be a leaf of a foliation, or simply a **Lagrangian fibration**.
- Need to complexify. Take $(L, \nabla)$.
$\nabla$ is a flat $U(1)$ connection in $Hom(\pi_1(L), U(1)) = (T^n)^*$.
- $\{\text{flat } U(1) \text{ connections on } L\} \cong T^*$ gives a torus in $\check{M}$.
Thus $\check{M}$ should admit a **dual torus fibration**.
- Special Lagrangians are canonical representatives of Lagrangians. Want special Lagrangian fibrations.

# Quantum corrections

- Need $\boldsymbol{HF^*(L,L)} \cong H^*(T)$ instead of $H^*(L)$. $L$ does not need to be a torus.
- In particular the fibration can have **singular fibers** $L$.
- $HF^*(L,L)$ is defined via counting stable discs [**Fukaya-Oh-Ohta-Ono**].
- $L$ may be obstructed: $HF^*(L,L)$ not defined if
$$m_0^L = W \cdot [L] + \sum h_Y \cdot Y \neq 0 \in C^*(L).$$




- Then have **Landau-Ginzburg model** $\left(\breve{M} := \{L\colon h_Y = 0 \;\forall\, Y\}, W\right)$ [**Fukaya-Oh-Ohta-Ono toric, Cho-Hong-L.** $\boldsymbol{P^1_{a,b,c}}$ using **Seidel Lagrangians**].
- Matrix factorizations instead of $O_p$.
Then $HF^*(L,L)$ don't need to be $H^*(T)$! Even more general.
- May have no solutions to $h_Y = 0 \;\forall\, Y$.
Need to enlarge to noncommutative $\breve{M}$ [**Cho-Hong-L.**].
Ex. Noncommutative resolution of conifold.
- $\boldsymbol{HF^*(L,L)}$ is defined over the Novikov field
$$\Lambda = \left\{ \sum_{i=0}^{\infty} a_i T^{A_i} \colon A_i \to +\infty \right\}.$$
$\check{X}$ defined over $\Lambda$.

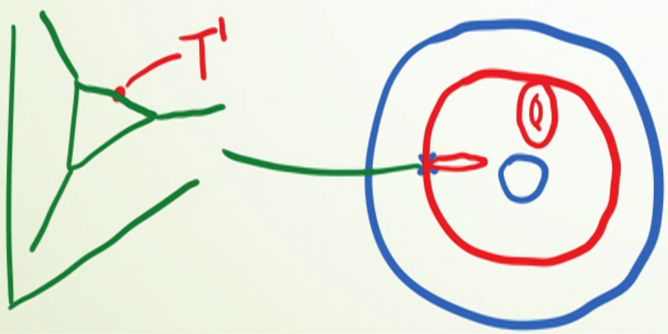
# SYZ for toric CY manifolds

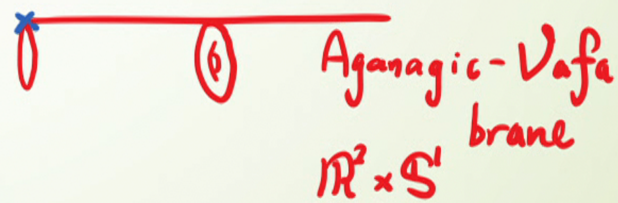
# Toric Calabi-Yau manifolds

- Local building blocks of compact CY. **[Vafa…]**
- Ex. Total spaces of canonical line bundles of toric Fano manifolds. $M = K_{\mathbb{P}^2}$.
- Has a global holomorphic nowhere-zero top form

$$d\zeta_1 \wedge \cdots \wedge d\zeta_n = \nu\, d\log\zeta_1 \wedge \cdots \wedge d\log\zeta_n\,,$$

  $\nu = \zeta_1 \ldots \zeta_n$ is a global holomorphic function.
- Need **Lagrangian torus fibration** for SYZ.
  Symplectic $T^{n-1}$ reduction: curves in complex $\nu$-plane give Lagrangians.
  Take concentric circles. Take away $\nu = 0$ (divisor $D$).
- **Aganagic-Vafa branes** correspond to rays.



# SYZ mirrors of toric CY manifolds

**Theorem** [**Chan-L.-Leung 12**]:

The SYZ mirror of a toric Calabi-Yau manifold $M - D$ takes the form

$$\check{M} = \left\{ (u, v, \vec{z}) \in \mathbb{C}^2 \times \mathbb{C}^{\times^{n-1}} : uv = \left(1 + \delta_0(q)\right) + z_1 + \cdots + z_{n-1} + \sum_{j=n+1}^{m} q^{A_j} \left(1 + \delta_j(q)\right) \vec{z}^{\vec{v_j}} \right\}.$$

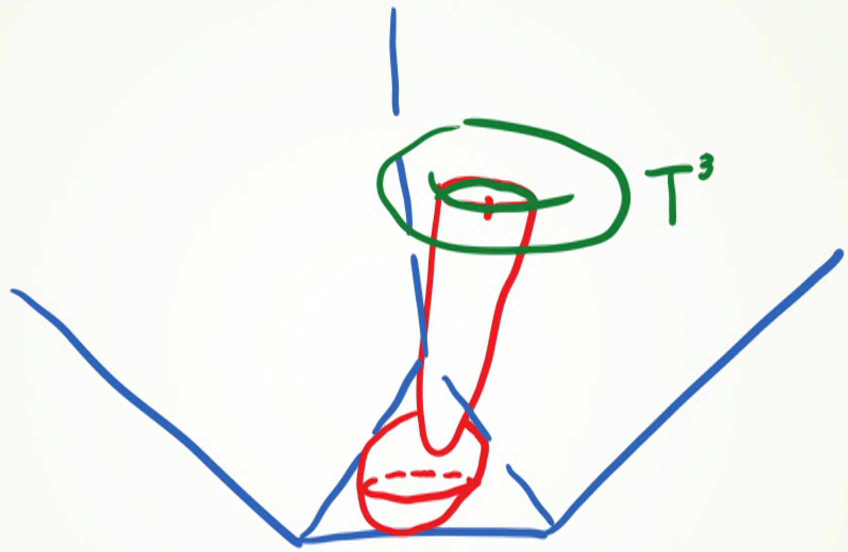
- $q$ are the Kaehler parameters. $z_i$ are variables on $T^*$ parametrizing flat connections.
- $\left(1 + \delta_j(q)\right)$ are generating functions of stable discs bounded by toric fibers.
- Ex.$K_{\mathbb{P}^2}$. $uv = \left(1 - 2q + 5q^2 - 32q^3 + \cdots\right) + z_1 + z_2 + q z_1^{-1} z_2^{-1}$.
- It agrees with the physics result of **[Hori-Iqbel-Vafa]** via the mirror map.

**Theorem** [**Chan-Cho-L.-Leung 16**]:

The generating functions $\left(1 + \delta_j(q)\right)$ equal to the inverse mirror map.

- [**Abouzaid-Auroux-Katzarkov 16**] proved that the SYZ mirrors of $\check{M}$ is the toric CY. In this direction don't have the series $\left(1 + \delta_j(q)\right)$.

# Open GW invariants of SYZ torus fibers

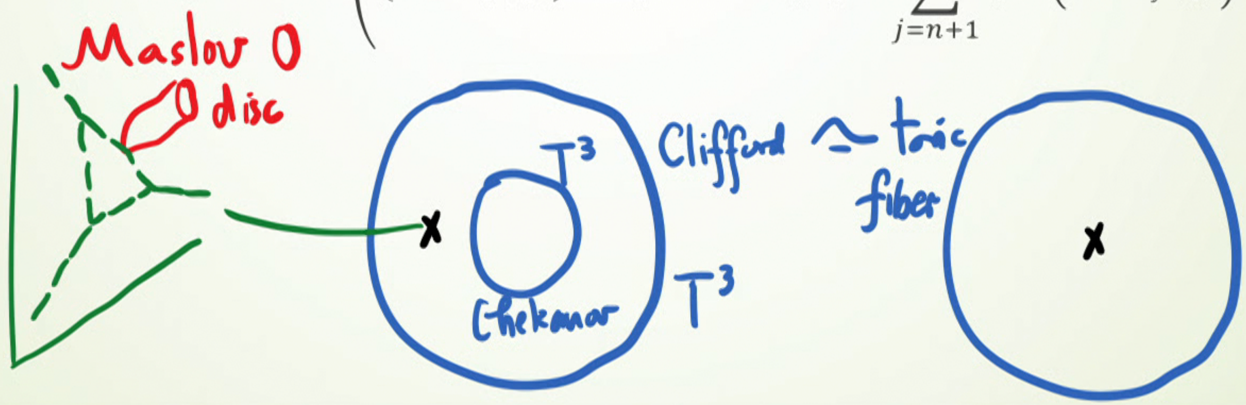



- $CF^*(L, L)$ has quantum corrections by stable discs.
- Non-triviality of disc countings $(1 - 2q + 5q^2 - 32q^3 + \cdots)$: **sphere bubbling**.
- The disc moduli is **obstructed** and hard to compute directly.
- We show that it is isomorphic to the corresponding moduli of rational curves in the compactification. Thus we can compute them systematically and show that they equal to the inverse mirror map.

# Wall-crossing of disc countings

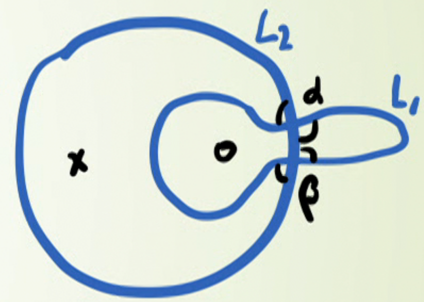
- **[Auroux 07]**: Generating functions of discs under wall-crossing when moving $L$.
- Need to glue the dual tori in a correct way to match these functions.
- Two types of torus fibers: Chekanov and Clifford.
- Chekanov torus: only bound one disc by maximal principle.
- Clifford torus: Lagrangian isotopic to a toric fiber without hitting Maslov-zero discs. Hence have the same disc counting as toric fiber.
- To match them, need to glue $(u, z_1, \dots, z_{n-1}) \in (\mathbb{C}^\times)^n$ <->$(v, z_1, \dots, z_{n-1}) \in (\mathbb{C}^\times)^n$ via

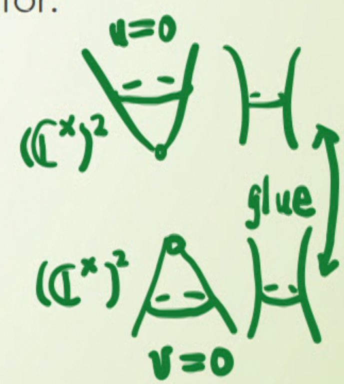
$$v = u^{-1}\left( \left(1 + \delta_0(q)\right) + z_1 + \cdots + z_{n-1} + \sum_{j=n+1}^{m} q^{A_j} \left(1 + \delta_j(q)\right) \vec{z}^{\,\vec{v_j}} \right).$$



# A recent update: gluing via isomorphisms of Lagrangians

- More systematic understanding: $HF^*(L, L) \cong T^*$ varies as we move $L$ around. **Family Floer theory [Fukaya, Tu, Abouzaid].**
- **[Cho-Hong-L.]**: use pseudo-isomorphisms to capture the variation.
- **Isomorphism**: $\alpha \in CF(L_1, L_2)$, $\beta \in CF(L_2, L_1)$ such that
$$m_1(\alpha) = 0, m_1(\beta) = 0; m_2(\alpha, \beta) = \mathbf{1}_{L_2}; m_2(\beta, \alpha) = \mathbf{1}_{L_1}.$$
- Solving this system of equations gives the gluing formula.
- [Cho-Hong-L.]: Use isomorphisms to glue up a well-defined mirror functor.
- **[Seidel]** and **[Pascaleff-Tonkonog]** solved the equation $m_1(\alpha) = 0$ and obtain the gluing formula
$$v = u^{-1}(1 + z)$$
in the exact/monotone setting.
- $u = v = 0$ is missing!

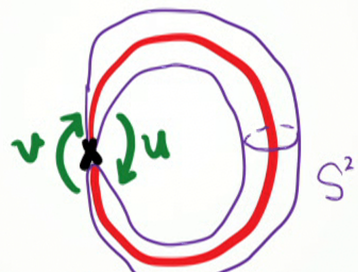


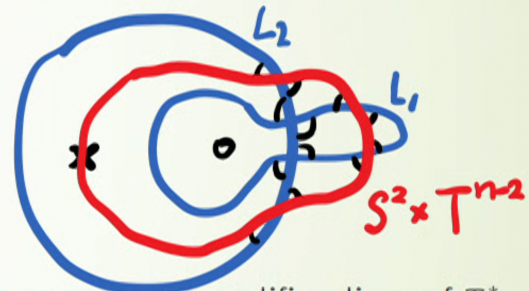



# Glue in immersed Lagrangians to complete the SYZ mirror

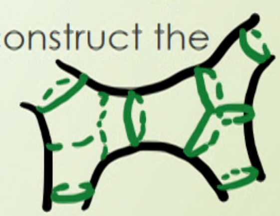
**Theorem [Hong-Kim-L.]**: The formal deformations $u_0 U + v_0 V$ of $S^2$ is unobstructed.

$$(S^2, u_0 U + v_0 V) \cong (T^2_{\text{Clifford}}, \nabla^{u,z}) \cong (T^2_{\text{Chekanov}}, \nabla^{v,z})$$
$$\text{for } u_0 = u; v_0 = v; u_0 v_0 = 1 + z.$$

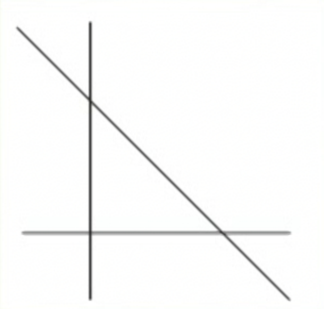



- $\nabla^{u,z}$ on $T$ runs in $T^*_{\mathbb{C}}$.
- The deformation space $\Lambda_0^2 - \{\text{val}(u_0 v_0) = 0\}$ of $S^2$ gives a compactification of $T^*_{\mathbb{C}}$.
- The charts glue up a complete mirror, filling in the missing $u = v = 0$.
- **[Rizell-Ekholm-Tonkonog]** obtained the gluing formula from Legendrian topology.
- We use it to compute the disc potential of immersions in $Gr(2, n)$ and construct the SYZ mirror.
- The gluing method also works for Riemann surfaces with pair-of-pants decompositions to prove HMS **[Cho-Hong-L.]**.



# SYZ for hypertoric manifolds



**Theorem [L.-Zheng 17]:**

Let $M = T^*\mathbb{P}^2$ (or more generally a hypertoric manifold $\mathbb{C}^{2n}///T^k_{\mathbb{R}}$).

$$D = \{z_1 w_1 = 1\} \cup \{z_2 w_2 = 1\}.$$

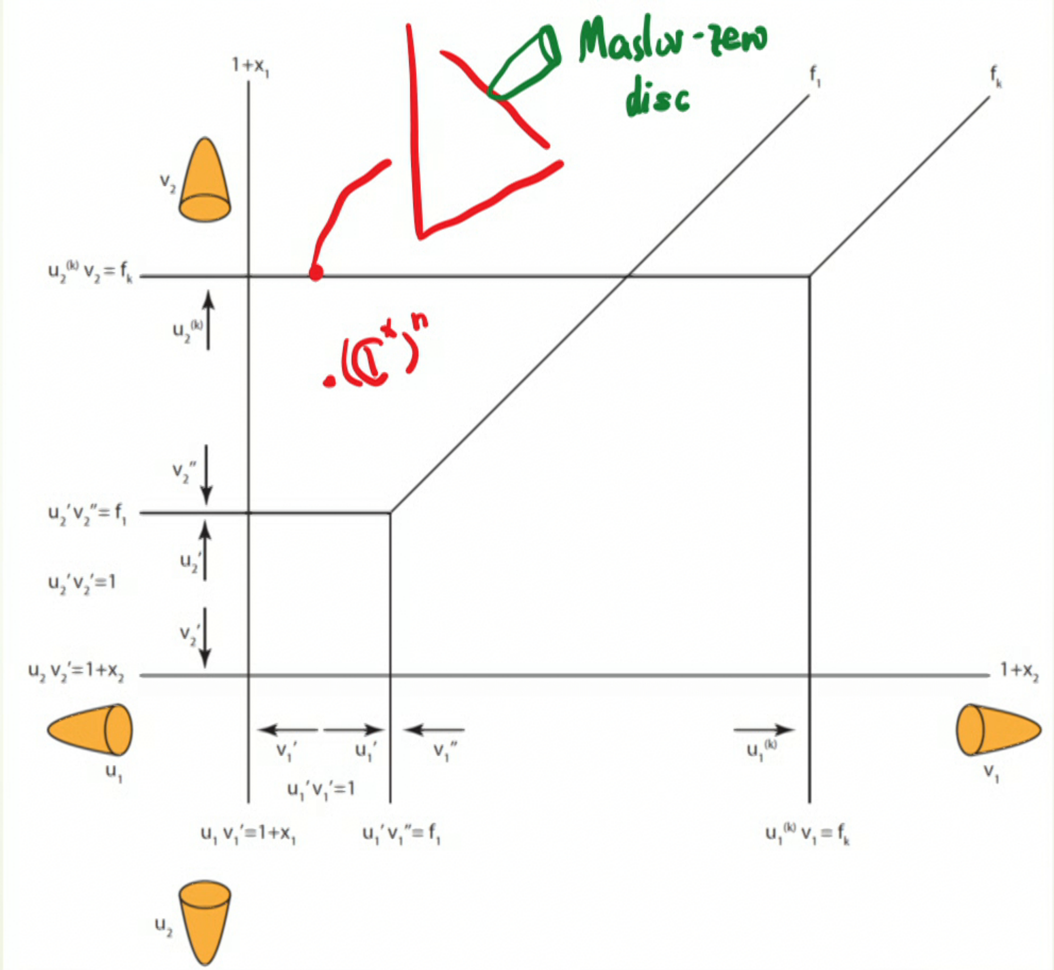
The SYZ mirror of $M - D$ is a certain resolution of

$$u_1 v_1 = (1 + z_1)\left(1 + q z_1^{-1} z_2^{-1}\right);$$
$$u_2 v_2 = (1 + z_2)\left(1 + q z_1^{-1} z_2^{-1}\right)$$

(or similar equations in the general case.)

- Use the method of **[Abouzaid-Auroux-Katzarkov]** to construct Lagrangian fibrations.
- $M^{2d}//T^d \cong \mathbb{C}^d$ has a non-standard reduced symplectic form. Use Moser argument to pull back the standard Lagrangian fibration on $(\mathbb{C}^d, \omega_{std})$.
- The recent work **[Gammage-Mcbreen-Webster]** proved homological mirror symmetry for multiplicative hypertoric varieties.
  They conjecture that it is certain divisor complement of hypertoric varieties.

# Wall-crossing in hypertoric manifolds



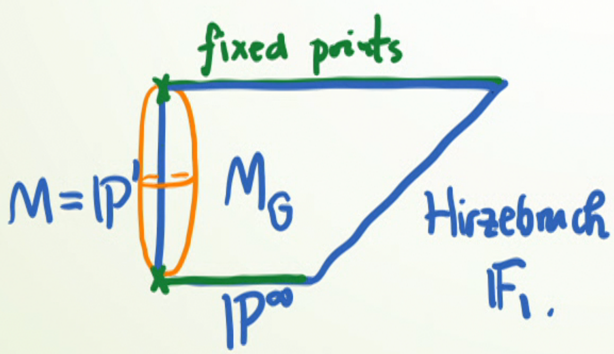
# Equivariant construction

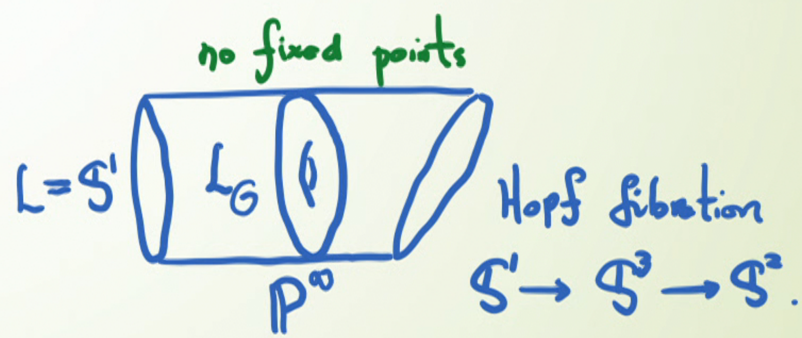
# Motivation

- In **[McBreen-Shenfeld 13]**, equivariant quantum cohomology of a hypertoric manifold is computed by the periods of
$$\Omega = e^{\hbar \sum_{i=1}^{n} \log(1+Z_i) + \sum_{i=1}^{d} \lambda_i \log z_i}\, d\log z_1 \wedge \cdots \wedge d\log z_d\,.$$
- $\hbar$ and $\lambda_i$ are equivariant parameters for certain $\mathbb{C}^\times$ actions.
- Want to understand the equivariant terms $\hbar \sum_{i=1}^{n} \log(1+Z_i) + \sum_{i=1}^{d} \lambda_i \log z_i$.
- For toric Calabi-Yau manifold, the Aganagic-Vafa branes have very interesting $\mathbb{S}^1$-equivariant open Gromov-Witten invariants. **[Katz-Liu, Graber-Zaslow, Fang-Liu-Zong, Fang-Liu-Tseng…]**
- Want to understand them from the SYZ perspective.
- For compact toric manifold, equivariant quantum cohomology is computed by
$$W = W^{\text{non-equiv.}}(z_1, \dots, z_n) + \sum_{i=1}^{n} \lambda_i \log z_i\,.$$
- $W^{\text{non-equiv.}}(z_1, \dots, z_n)$ equals to the disc potential **[Cho-Oh, Fukaya-Oh-Ohta-Ono]**. How about the log terms?

# Equivariant cohomology

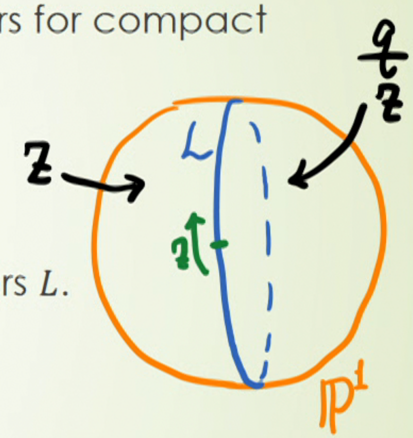
- If G acts freely on M, $H_G(M) = H(M/G)$.
- In general, enlarge the space: $M \times EG$ so that $G$ acts freely.
- $EG$: a contractible space that admits a free G-action.
- $BG := EG/G$ is known as the classifying space.
  $M_G := (M \times EG)/G$ replaces $M/G$.
- $H_G(M) := H(M_G)$.
- $M \to M_G \to BG$ fiber bundle.



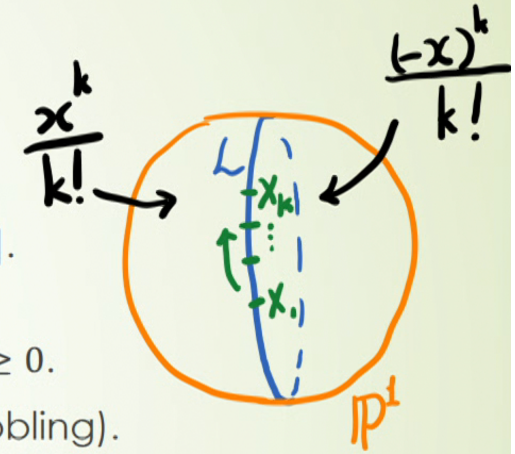
# Equivariant superpotential for compact toric varieties

- **[Givental]** and **[Hori-Vafa]** found the Landau-Ginzburg mirrors for compact toric varieties.
- Important for computing quantum cohomology.
- Ex. $\mathbb{P}^1$. $W = z + \frac{q}{z}$. $W_T = z + \frac{q}{z} + \lambda \log z$.
- **[Cho-Oh]** classified holomorphic discs bounded by toric fibers $L$. $W$ understood as a weighted sum of holomorphic discs.
- $z$: holonomy of flat U(1) connections on $L$.
- Want to understand the term $\lambda \log z$.
- Not a series at $z = 0$! Hard to interpret as counting!




$$q = e^{-\mathrm{Area}(\mathbb{P}^1)}.$$

# Boundary deformations vs flat connections

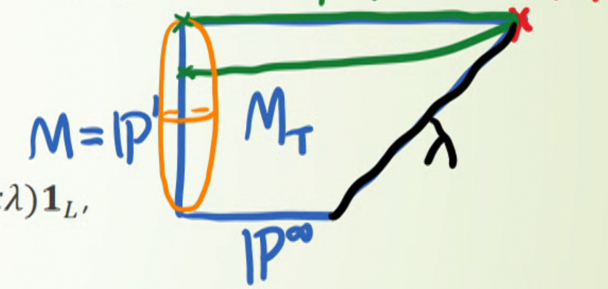


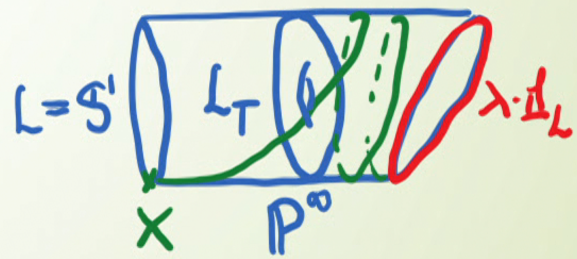

- Instead of decorating $L$ by flat connection $z$,
  Use $xX$: boundary deformations [**Fukaya-Oh-Ohta-Ono**].
- $m_0^{xX} = \sum_{k=1}^{\infty} m_k(xX, \dots, xX) = \sum_{k=1}^{\infty} x^k\, m_k(X, \dots, X)$
  count discs passing through $X$ $k$ times, and sum over $k \geq 0$.
- Have transversality issue for repeated inputs $X$ (disc bubbling).
  Perturb any repeated inputs to distinct positions.
- Have $k!$ choices of ordering. Take all the possibilities and divide by $k!$.
$$W = \sum_{k=0}^{\infty} \frac{x^k}{k!} + q \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} = \exp x + q \exp(-x)\,.$$
- Expect the equivariant disc potential is
$$W_T = \exp x + q \exp(-x) + \lambda x.$$
- What disc does $\lambda x$ corresponds to?

# Motivation from equivariant QH.


$$q = e^{-\mathrm{Area}(\mathbb{P}^1)}.$$

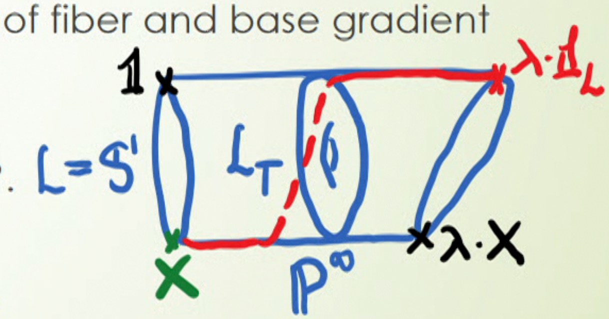
- Example: $\mathrm{H}^*(\mathbb{P}^1) = \langle \mathbf{1}_{\mathbb{P}^1}, [\mathrm{pt}] \rangle$.
- $\mathrm{H}^*_T(\mathbb{P}^1) = \langle \mathbf{1}_T, [\mathrm{pt}]_T \rangle \otimes \mathbb{C}[\lambda]$.
- $[\mathrm{pt}] \cup [\mathrm{pt}] = 0$.
- $[\mathrm{pt}]_T *_q [\mathrm{pt}]_T = q \cdot \mathbf{1}_T + \lambda \cdot [\mathrm{pt}]_T$.
- Want: understand $\left(\mathcal{C}_T(L), m_k^T\right)$ similarly.
- Taking x-derivative of $m_0^T = (\exp x + q\exp(-x) + x\lambda)\mathbf{1}_L$, expect $m_1^T(X) = \left((\mathrm{usual}) + \lambda\right)\mathbf{1}_L$.
- $X$ is understood as a hypertorus of $L$. $\lambda \cdot \mathbf{1}_L$ is understood as the preimage of hyperplane in $\mathbb{P}^\infty$.
- $T$ acts on $L$ freely. $L_T \to BT$ is non-trivial. When we try to take a "section" of $X$, it is not well-defined and "flow" to the whole $\lambda \cdot \mathbf{1}_L$! "Thus" $m_1^T(X) = (\lambda + \cdots)\mathbf{1}_L$.



# Family Morse for $L_T \to BT$

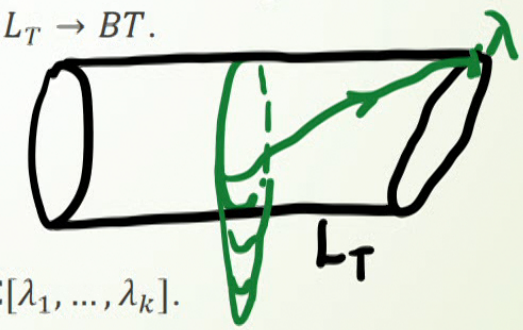
- **[Hutchings]: family Morse theory** for $F \to M \to B$.
- Take a Morse function on $B$. The pull-back to $M$ is not Morse.
- Take a function on $M$, thought as family of functions on fibers. Cannot be Morse on every fiber!
- Assume Morse on fibers over critical points of $B$. Use these critical points to make a chain complex.
- Morse index: the sum of the indices in fiber and base directions.
- Morse differential: counting flow lines of the sum of fiber and base gradient vector fields (with the help of a connection).
- Apply to $L \to L_T \to BT$ for $T = U(1)$ and $L = \mathbb{S}^1$. We take a standard Morse function on $BT = \mathbb{CP}^\infty$.
- Critical points: $\mathbf{1}_L \otimes \lambda^k, X \otimes \lambda^k$ for $k \geq 0$. Equivariant parameters are simply critical points.




- Non-trivial bundle change leads to unique flow line from $X \otimes \mathbf{1}_{BT}$ to $\mathbf{1}_L \otimes \lambda$.
- Thus $m_1^T(X) = (\lambda + \cdots)\mathbf{1}_L$.

# Equivariant Floer theory for $L_T \subset X_T$

- [**Seidel-Smith**] studied $\mathbb{Z}_2$-equivariant Floer theory in the exact setting. They also had a method of studying $HF_{\mathbb{Z}_2}(L_1, L_2)$ by family Morse theory.
- [**Cho-Hong**] studied equivariant Fukaya category for finite groups.
[**Wu**] studied free action by finite groups.
[**Hendricks-Lipshitz-Sarkar**] used homotopy method for finite groups.
- [**Daemi-Fukaya**] is developing $G$-equivariant Floer cohomology for divisor complements for a Lie group $G$.
- Our formulation: count pearl trajectories consisting of flow lines in $L_T$ and stable discs bounded by fibers $L$ of $L_T \to BT$.



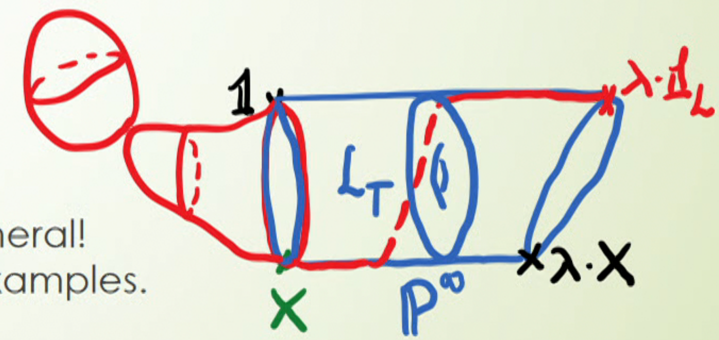

- Get an $A_\infty$ algebra over $H^*(BT) = \mathbb{C}[\lambda_1, \ldots, \lambda_k]$.
- Our method is more direct to compute the equivariant disc potential and construct equivariant mirror. (Finite dimensional in the chain level.)

# Equivariant disc potential for a compact toric manifold

**Theorem [Kim-L.-Zheng in progress]:** For a compact semi-Fano toric manifold, $L_T$ is weakly unobstructed, and

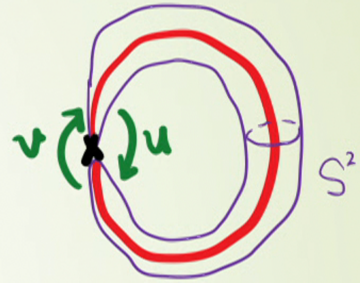
$$W_T = \sum_{i=1}^{m} \left(1 + \delta_i(q)\right) q^{\beta_i} \exp(\vec{x}, \partial\beta_i) + \sum_{j=1}^{n} \lambda_j x_j \,.$$

- **[Chan-L.-Leung-Tseng 15]**:
$\left(1 + \delta_i(q)\right)$ are **generating functions of stable bubbled discs**, and they equal to the **inverse mirror map**.
- The second term is the equivariant contribution. Taking $\lambda \to 0$, it recovers the non-equivariant $W$.
- The first term counts Maslov-two discs.
The second term counts Maslov-zero discs.
They are constant discs in this case.
- We have **non-constant Maslov-zero discs** in general!
**Toric Calabi-Yau manifolds** provide excellent examples.



# Gluing in the equivariant setting

- For toric Calabi-Yau, recall that
$$(S^2, u_0 U + v_0 V) \cong (T^2_{\text{Clifford}}, \nabla^{u,z}) \cong (T^2_{\text{Chekanov}}, \nabla^{v,z})$$
for $u_0 = u; v_0 = v; u_0 v_0 = 1 + z$.




- Simply multiply extra $T^{n-2}$ factors in higher dimensions.
- Admit $T^{n-1}$ action.

**Lemma [Hong-Kim-L.-Zheng in progress]:**
Assume $L$ has minimal Maslov index zero.
If $L$ is weakly unobstructed, then $L_T$ is also weakly unobstructed:
$$m_0^T = W \cdot \mathbf{1}_L + h \cdot \lambda.$$
If $\alpha$ is an isomorphism for $(L_1, L_2)$, then it lifts to an isomorphism $\alpha^T$ for $(L_1^T, L_2^T)$.

- By the above Lemma,
the gluing formula remains the same in the equivariant setting.
- Equivariant disc potential for Clifford or Chekanov torus is simply
$$W^T = \lambda x.$$

# Equivariant disc potential for $\bar{S}^2 \times T^{n-2}$

**Theorem [Hong-Kim-L.-Zheng in progress]:**

The equivariant disc potential of $\bar{S}^2 \times T^{n-2} \subset$ toric CY $X$ equals to

$$W = \sum_{i=1}^{n-2} \lambda_i x_i + \lambda_{n-1} \log f(uv, \exp x_1, \dots, \exp x_{n-2})$$

where $f = -\exp x_{n-1}$ is a solution to the mirror equation

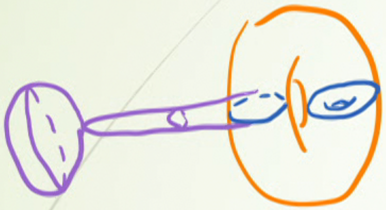
$$uv = \sum_{j=0}^{m} q^{A_j} \left(1 + \delta_j(q)\right) \exp(\vec{\boldsymbol{x}}, v_j).$$

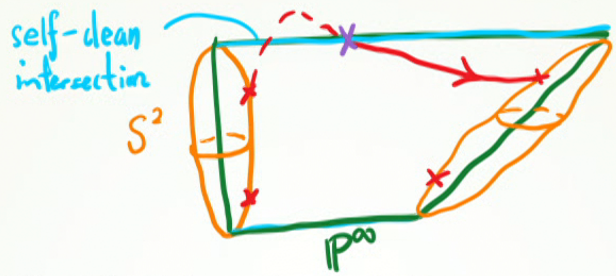
- Ex. $K_{\mathbb{P}^2}$. $uv = (1 - 2q + 5q^2 - 32q^3 + \cdots) + \exp x_1 + \exp x_2 + q \exp(-x_1)\exp(-x_2)$.

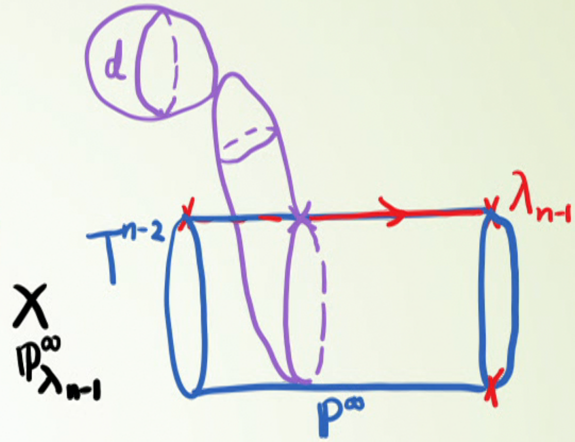
$$f = \frac{\left((1 - 2q + 5q^2 - 32q^3 + \cdots) - uv + \exp x_1\right) + \sqrt{\left((1 - 2q + 5q^2 - 32q^3 + \cdots) - uv + \exp x_1\right)^2 - 4q\exp(-x_1)}}{2}.$$

- Set $u = v = 0$. $\log f = \sum k_{d,p}\, q^d \exp p x_1$ is the equivariant disc potential in $\lambda_2$ direction.
- It equals to the $x_1$-derivative of the physicists potential for Aganagic-Vafa branes.

# What are the counting






```
Series[Log[1 + (((D + z) + Sqrt[1 + 2 D + D^2 - 4 z^(-1) q + 2 z + 2 D z + z^2] - 1) / 2)] /.
  D → -2 q + 5 q^2 - 32 q^3 + 286 q^4 - 3038 q^5 + 35870 q^6 - 454880 q^7, {q, 0, 4}, {z, 0, 4}]
```

$$\left(z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + O[z]^5\right) + \left(-\frac{1}{z} - z + 2z^2 - 3z^3 + 4z^4 + O[z]^5\right) q +$$
$$\left(-\frac{3}{2z^2} + \frac{2}{z} + 5z - \frac{27z^2}{2} + 27z^3 - 47z^4 + O[z]^5\right) q^2 +$$
$$\left(-\frac{10}{3z^3} + \frac{8}{z^2} - \frac{12}{z} - 40z + 122z^2 - \frac{838z^3}{3} + 560z^4 + O[z]^5\right) q^3 +$$
$$\left(-\frac{35}{4z^4} + \frac{30}{z^3} - \frac{65}{z^2} + \frac{104}{z} + 399z - \frac{2597z^2}{2} + 3210z^3 - \frac{28027z^4}{4} + O[z]^5\right) q^4 + O[q]^5$$

- [**Aganagic-Klemm-Vafa**] and [**Graber-Zaslow**] for $K_{\mathbb{P}^2}$. (setting u=v=0.)
- This is $z\partial_z$ of the equivariant potential for Aganagic-Vafa brane.
- [**Fang-Liu**], [**Fang-Liu-Tseng**], [**Fang-Liu-Zong**] formulated the invariants for Aganagic-Vafa brane by localization. They proved the genus-zero formula, and proved the BKMP remodeling conjecture (in all genus).
- Our method is more conceptual, computes discs with corners at immersed points, and works in any dimension.

# Thank you!