Title: Multiple zeta values in deformation quantization

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Series: Mathematical Physics

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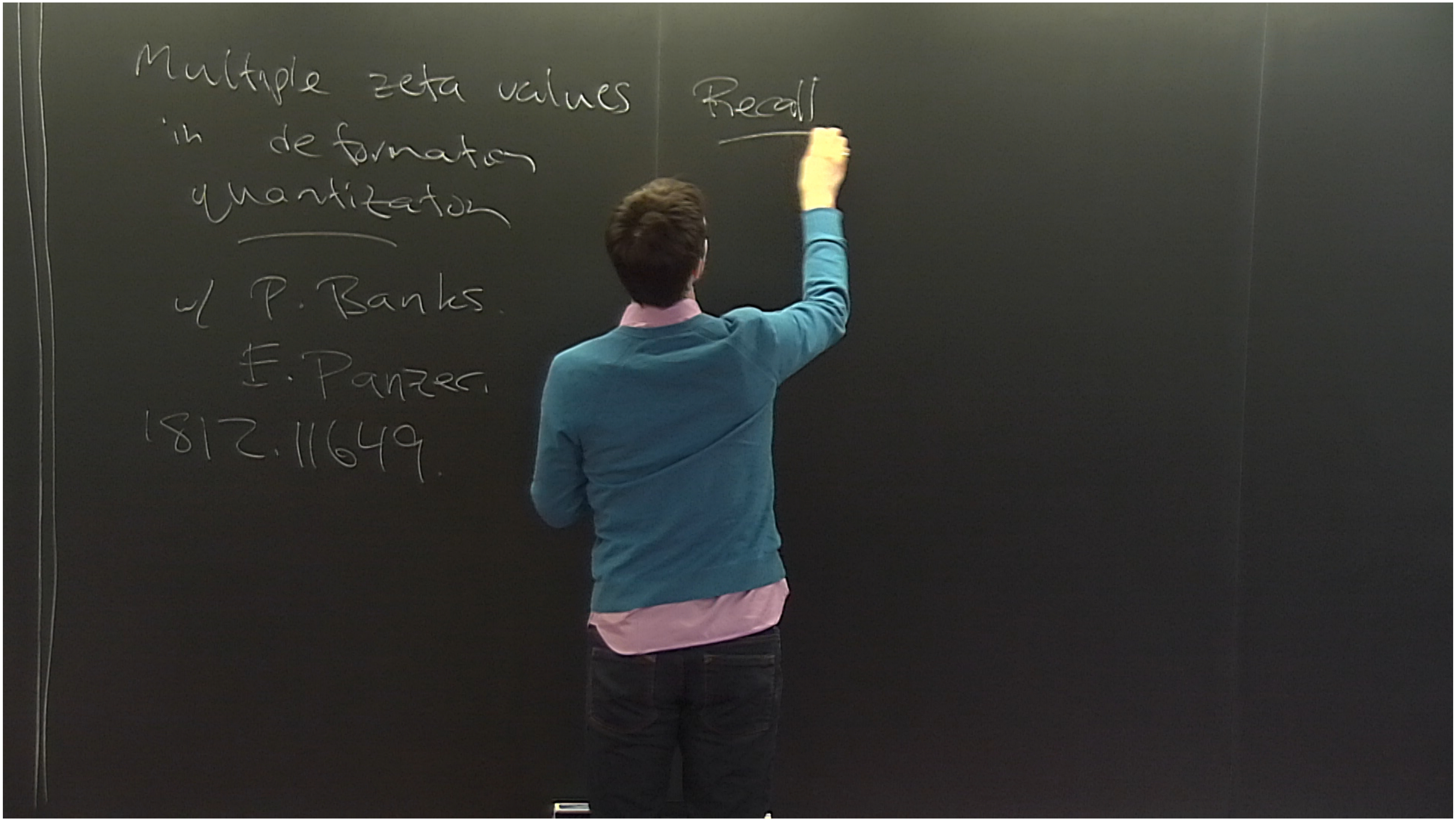
Abstract: In 1997, Kontsevich gave a universal solution to the "deformation quantization" problem in mathematical physics: starting from any Poisson manifold (the classical phase space), it produces a noncommutative algebra of quantum observables by deforming the ordinary multiplication of functions. His formula is a Feynma expansion, involving an infinite sum over graphs, weighted by volume integrals on the moduli space of marked holomorphic disks. The precise values of these integrals are currently unknown. I will describe recent joint work with Banks and Panzer, in which we develop a theory of integration on these moduli spaces via suitable sheaves of polylogarithms, and use it to prove that Kontsevich's integrals evaluate to integer-linear combinations of special transcendental constants called multiple zeta values, yielding the first algorithm for their calculation.


Multiple zeta values
in deformation quantization
w/ P. Banks.
E. Panzer.
1812.11649.
Recall

Recall A deformation quantization of

X – C^∞-mfld

cx affine smooth var.

is deformation of $\mathcal{O}(X)$ as an assoc. alg.

$f * g = fg + \hbar B_1(f,g) + \cdots$

@ 1st order

$f*g - g*f = \hbar\{f,g\} + \cdots$

$\{f,g\}$ Poisson bracket.

Thm (Kontsev

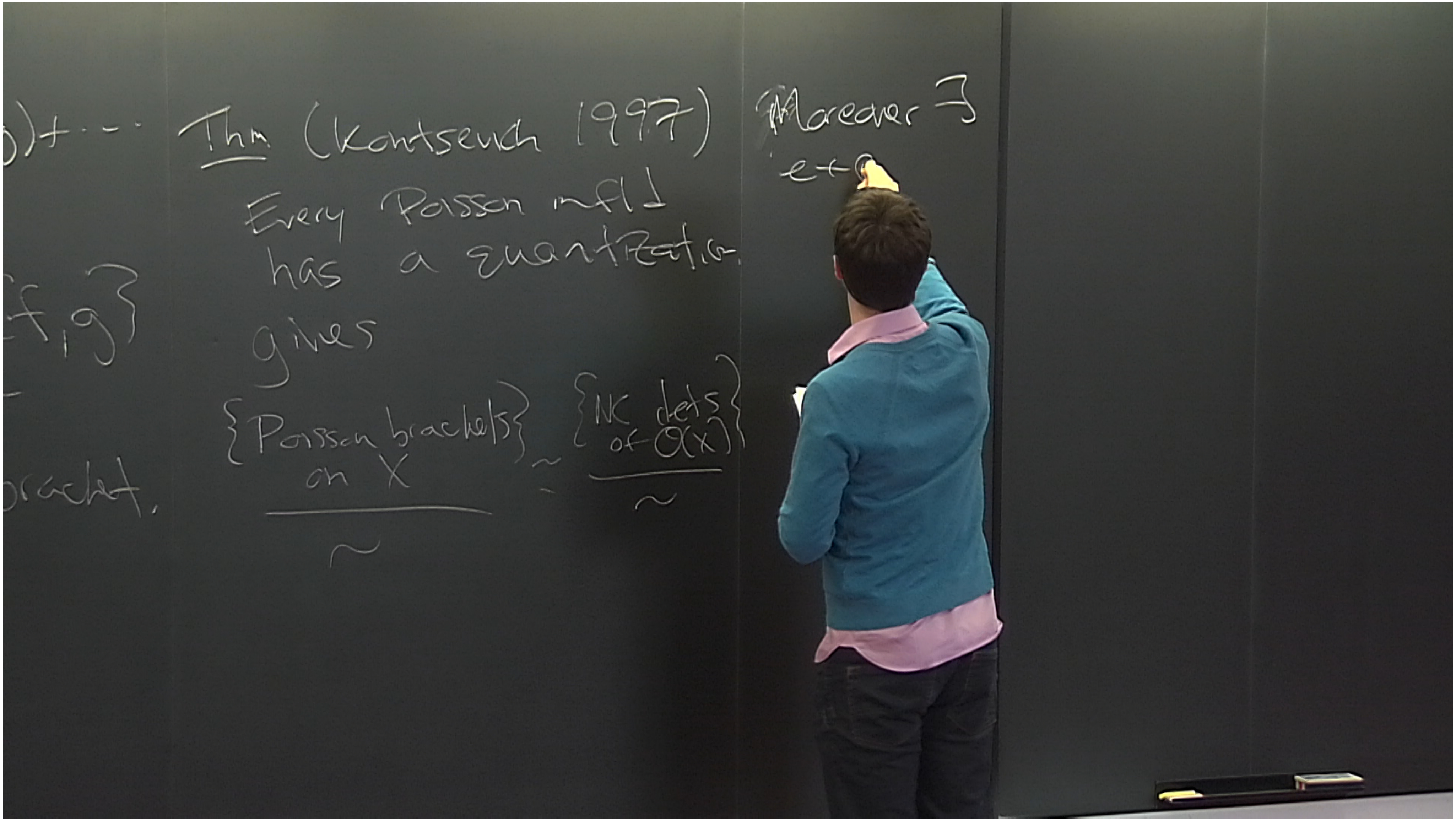
Every Poisso

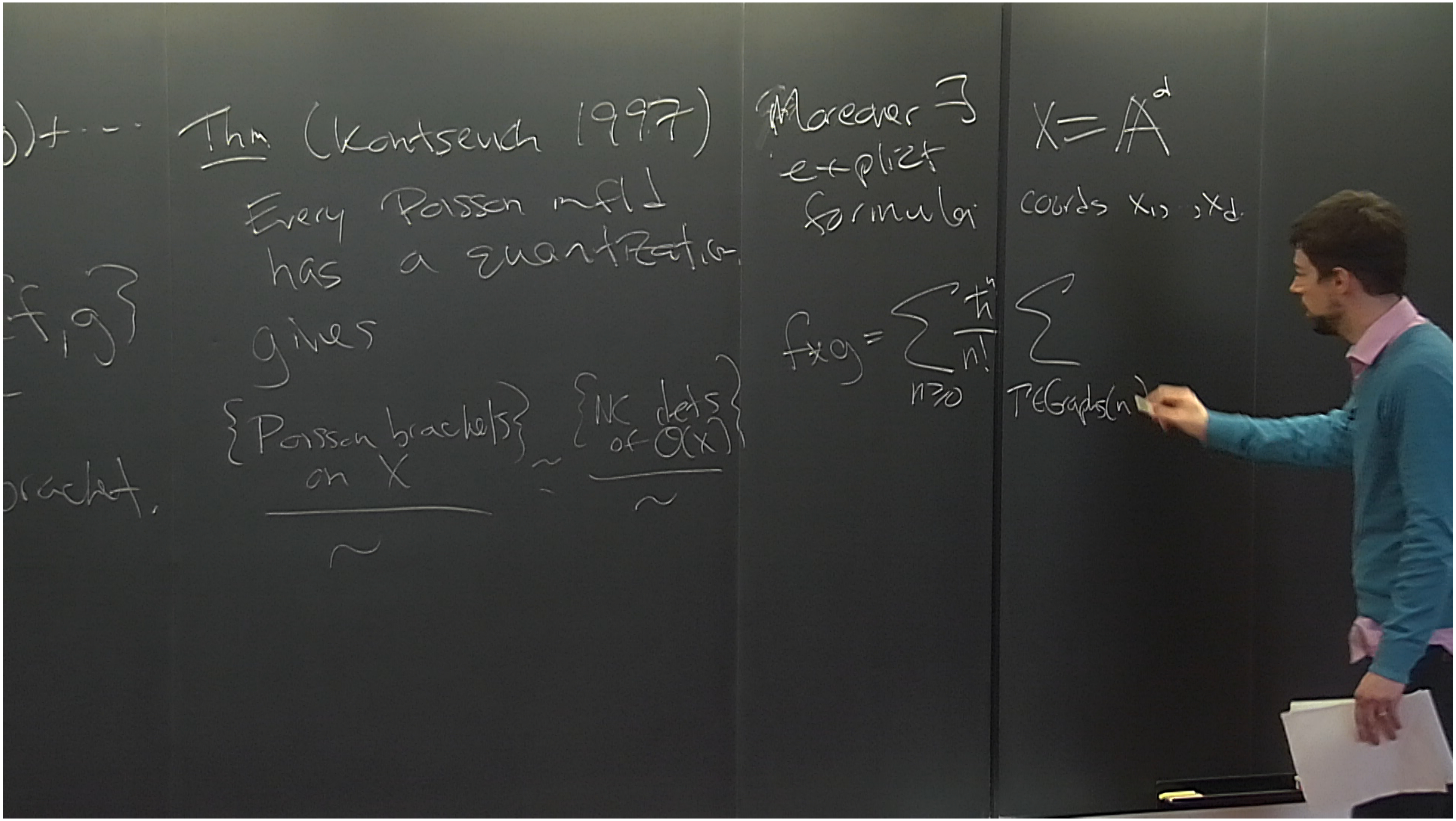
has a

gives

{Poisson brackets on X}

$\sim$


Thm (Kontsevich 1997)
Every Poisson mfld
has a quantization
gives
{Poisson brackets on X} / ~ ≅ {NC defs of O(X)} / ~
Moreover ∃
bracket.


)+ · · ·
{f, g}
bracket.
Thm (Kontsevich 1997)
Every Poisson mfld has a quantization
gives
{Poisson brackets on X} / ~ ≃ {NC defs of O(X)} / ~
Moreover ∃ explicit formula
$f \star g = \sum_{n \geq 0} \frac{\hbar^n}{n!} \sum_{\Gamma \in \mathrm{Graphs}(n)}$
$X = \mathbb{A}^d$
coords $x_1, \ldots, x_d$.

$)+ \cdots$

$\{f, g\}$

bracket.

Thm (Kontsevich 1997)

Every Poisson mfld has a quantization.

gives

$$\frac{\{\text{Poisson brackets on } X\}}{\sim} \simeq \frac{\{\text{NC defs of } \mathcal{O}(X)\}}{\sim}$$

Moreover $\exists$ explicit formula

$X = \mathbb{A}^d$

coords $x_1, \ldots, x_d$.

$$f \star g = \sum_{n \geq 0} \frac{\hbar^n}{n!} \sum_{\Gamma \in \mathrm{Graphs}(n)} c_\Gamma B_\Gamma(\{x_i, x_j\}, f, g)$$

$)+\cdots$

$\{f,g\}$

bracket.

Thm (Kontsevich 1997)
Every Poisson mfld
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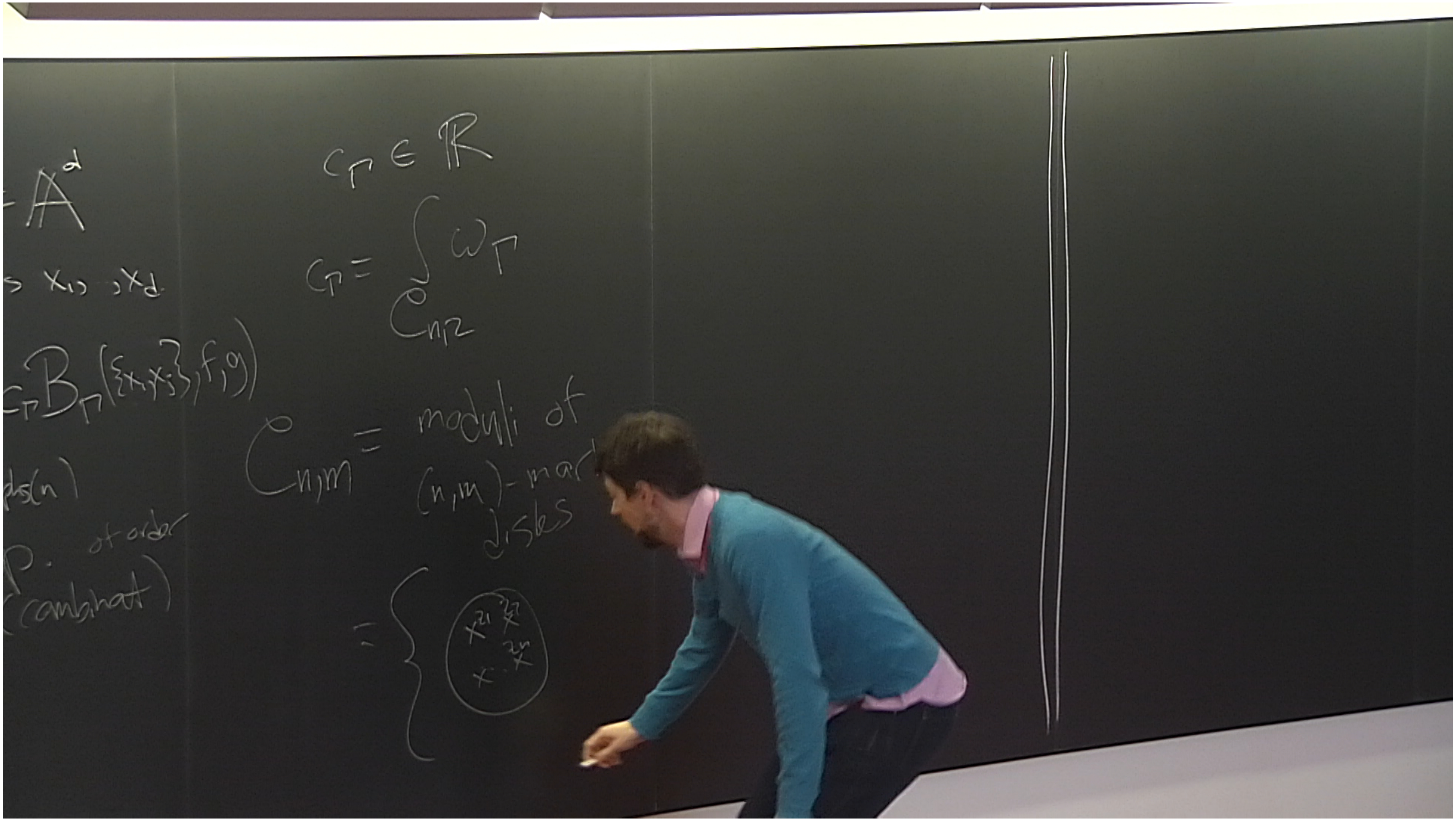
B_Γ = diff op. of order n (combinat)



$c_\Gamma \in \mathbb{R}$
$c_\Gamma = \int_{C_{n,2}} \omega_\Gamma$
$C_{n,m}$ = moduli of
(n,m)-
disks

$\in \mathbb{A}^d$

s $x_1, \dots, x_d$

$c_\Gamma B_\Gamma(\{x_i, x_j\}, f, g)$

ds(n)

P. of order

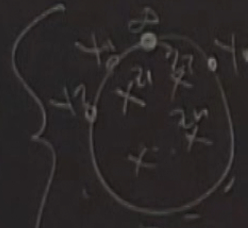
(combinat)

$c_\Gamma \in \mathbb{R}$

$$c_\Gamma = \int_{C_{n,2}} \omega_\Gamma$$

$C_{n,m}$ = moduli of (n,m)-marked disks

$$= \left\{ \right. $$

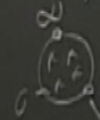


$$\left. \right\} / \sim$$

e.g. $C_{n,2} = \{(z_1, \dots, z_n) \in \mathbb{H}^n, \ z_i \neq z_j \ i \neq j\}$

$y_1 \mapsto 0$

$y_2 \mapsto 1$



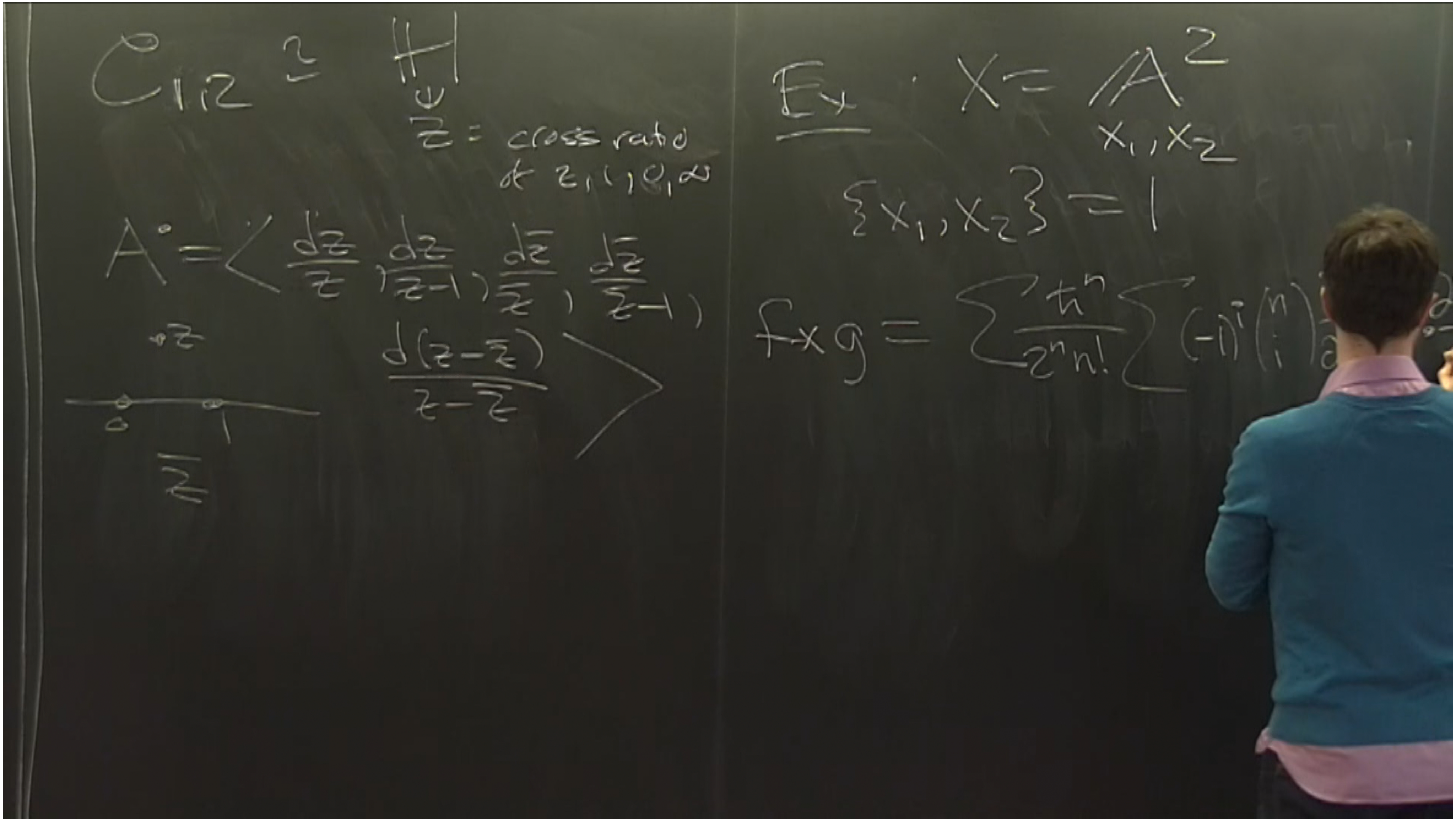
$\cong \mathbb{H} \subset \mathbb{C}$

$\in z_j$

ω_Γ a volume form lying in

$$A^\bullet(C_{n,m}) \subset \Omega^\bullet(C_{n,m})$$

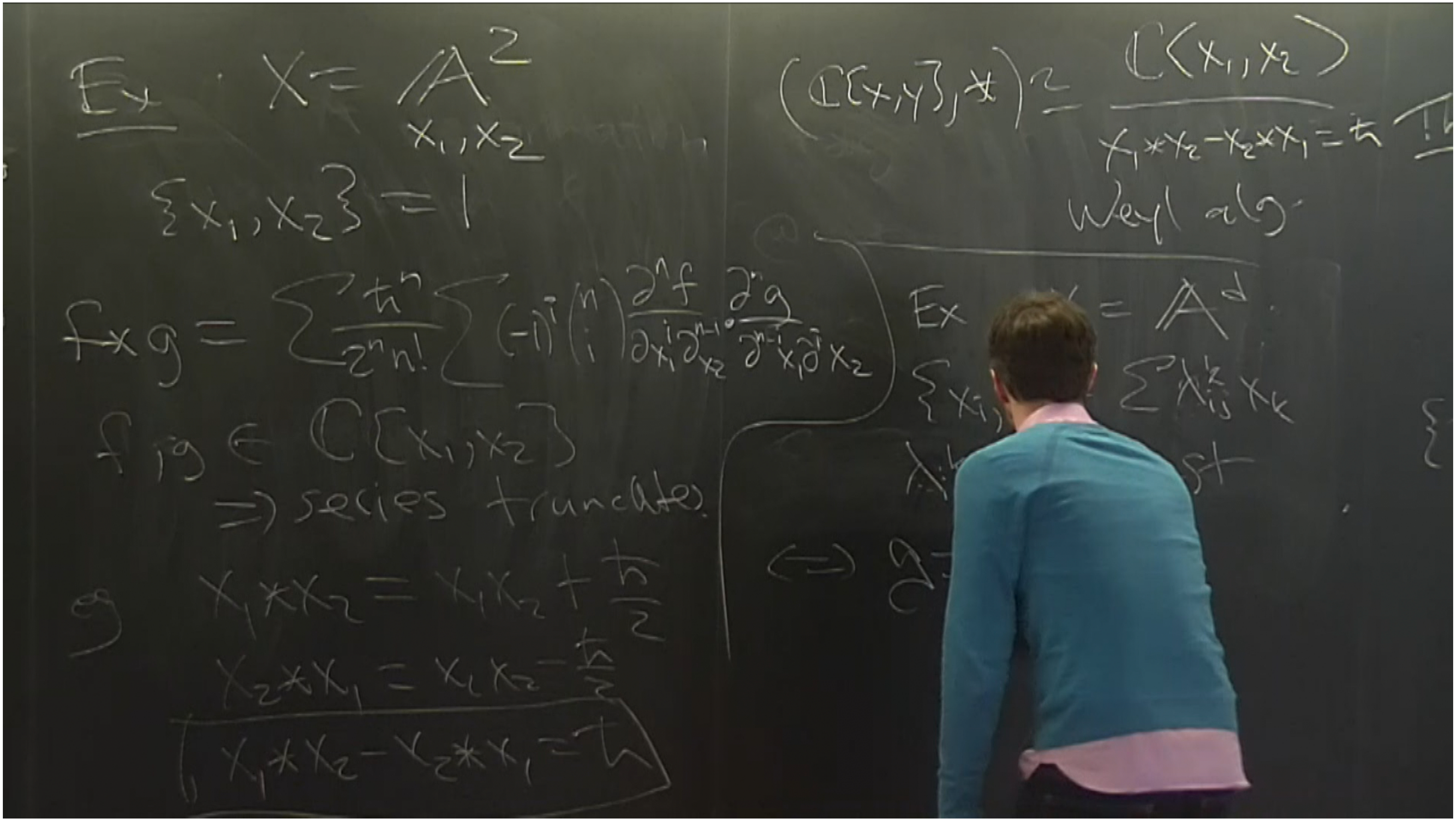
$$= \langle d\log(\text{crossratios}) \rangle$$


C1,2 ≅ H
z = cross ratio
of z, 1, 0, ∞
A° = ⟨ dz/z , dz/(z−1) , dz̄/z̄ , dz̄/(z̄−1) ⟩
d(z−z̄)/(z−z̄)
0 1
z̄
Ex: X = A2
x1 , x2
{x1 , x2} = 1
f×g = Σ ħn/(2n n!) Σ (−1)i (n i)

Ex: $X = \mathbb{A}^2_{x_1, x_2}$

$$\{x_1, x_2\} = 1$$

$$f \star g = \sum \frac{\hbar^n}{2^n n!} \sum (-1)^i \binom{n}{i} \frac{\partial^n f}{\partial x_1^i \partial_{x_2}^{n-i}} \cdot \frac{\partial^n g}{\partial^{n-i} x_1 \partial^i x_2}$$


Ex: $X = \mathbb{A}^2_{x_1, x_2}$
$\{x_1, x_2\} = 1$
$f * g = \sum \frac{\hbar^n}{2^n n!} \sum (-1)^i \binom{n}{i} \frac{\partial^n f}{\partial x_1^i \partial^{n-i}_{x_2}} \frac{\partial^n g}{\partial^{n-i} x_1 \partial^i x_2}$
$f, g \in \mathbb{C}[x_1, x_2]$
$\Rightarrow$ series truncates
eg $x_1 * x_2 = x_1 x_2 + \frac{\hbar}{2}$
$x_2 * x_1 = x_1 x_2 - \frac{\hbar}{2}$
$\boxed{x_1 * x_2 - x_2 * x_1 = \hbar}$
$(\mathbb{C}[x,y], *) \cong \frac{\mathbb{C}\langle x_1, x_2 \rangle}{x_1 * x_2 - x_2 * x_1 = \hbar}$
Weyl alg.
Ex $\mathbb{A}^d$
$\{x_i, \quad \sum \lambda_{ij}^k x_k$
st
$\Leftrightarrow$

$\{x_1, x_2\} = 1$

$$f * g = \sum \frac{\hbar^n}{2^n n!} \sum (-1)^i \binom{n}{i} \frac{\partial^n f}{\partial x_1^i \partial_{x_2}^{n-i}} \frac{\partial^n g}{\partial^{n-i} x_1 \partial^i x_2}$$

$f, g \in \mathbb{C}[x_1, x_2]$

$\Rightarrow$ series truncates

eg

$$x_1 * x_2 = x_1 x_2 + \frac{\hbar}{2}$$

$$x_2 * x_1 = x_1 x_2 - \frac{\hbar}{2}$$

$$\boxed{x_1 * x_2 - x_2 * x_1 = \hbar}$$

Weyl alg.

Ex $X = \mathbb{A}^d$

$\{x_i, x_j\} = \sum \lambda_{ij}^k x_k$

$\lambda_{ij}^k \in \mathbb{C}$ const

$\Leftrightarrow$ $\mathfrak{g} = \mathrm{span}\{x_i\}$ Lie alg,

NC alg $\cong U(\mathfrak{g}, \hbar)$

$$= \frac{\mathbb{C}\langle x_i \rangle}{x_i x_j - x_j x_i = [x_i, x_j]\hbar}$$

$(\mathbb{C}[\{x,y\},\star))^{?} \cong \dfrac{\mathbb{C}\langle x_1,x_2\rangle}{x_1\star x_2 - x_2\star x_1 = \hbar}$

Weyl alg.

Ex $X = \mathbb{A}^d$

$\{x_i,x_j\} = \sum \lambda_{ij}^k x_k$

$\lambda_{ij}^k \in \mathbb{C}$ const

$\Rightarrow \mathfrak{g} = \mathrm{span}\{x_i\}$

Lie alg.

NC alg $\cong U(\mathfrak{g},\hbar) = \dfrac{\mathbb{C}\langle x_i\rangle}{x_i x_j - x_j x_i = [x_i,x_j]\hbar}$

u_1,u_2 coords on $\mathbb{A}^2$

Ex $\{u_1,u_2\} = u_1u_2$

$u_1\star u_2 = g(\hbar)\,u_1u_2$

$u_2\star u_1 = g(-\hbar)\,u_1u_2$

$g(\hbar)\in\mathbb{C}[[\hbar]]$

computer

$g(\hbar) = 1 + \frac{\hbar}{2} + \frac{\hbar^2}{24} - \frac{\hbar^3}{48} - \frac{\hbar^4}{1440} + \frac{\hbar^5}{480} + \left(\frac{-17}{184320} + \frac{25(\zeta(3))^2}{\pi^6}\right)\hbar^6$

Moreover $\exists$ explicit formula

$f\star g = \sum_{n\geq 0} \frac{\hbar^n}{n!}$

$B_\Gamma = \mathrm{diff}$

...olds on $\mathbb{A}^2$

$u_1 u_2$

u_2

$[[\hbar]]$

$\frac{\hbar^2}{24}$

$+\frac{\hbar^5}{480}$

$\frac{251\,\zeta(3)^2}{\pi^6}\Big)\hbar^6$

$\zeta(3)=\sum_{k\geq 1}\frac{1}{k^3}$

$u_1 * u_2 = q(\hbar)\, u_2 u_1$

$q(\hbar) = e^{\hbar} = \frac{g(\hbar)}{g(-\hbar)} \mod \hbar^7$

<u>Def^n</u>: A normalized <u>multiple zeta value (MZV) of weight N</u> is a number of the form

$$\tilde{\zeta}(n_1,\dots,n_d) = \frac{1}{(2\pi i)^N}\sum_{0<k_1<k_2<\dots<k_d}\frac{1}{k_1^{n_1}\cdots k_d^{n_d}}, \quad n_d\geq 2,$$

$$\in \begin{cases} \mathbb{R} & N \text{ even} \\ i\mathbb{R} & N \text{ odd}. \end{cases}$$

$\zeta(3) = \sum_{k\geq 1} \frac{1}{k^3}$

$u_1 * u_2 = q(\hbar)\, u_2 u_1$

$(\hbar) = e^{\hbar}$

$= \frac{g(\hbar)}{g(-\hbar)}$

mod $\hbar^2$

Def^n A normalized multiple zeta value (MZV) of weight N is a number of the form

$$\tilde{\zeta}(\underbrace{n_1, \dots, n_d}_{d \text{ entries}}) = \frac{1}{(2\pi i)^N} \sum_{0<k_1<k_2<\dots<k_d} \frac{1}{k_1^{n_1} \cdots k_d^{n_d}} \quad n_d \geq 2$$

d: "depth"

$$\in \begin{cases} \mathbb{R} & N \text{ even} \\ i\mathbb{R} & N \text{ odd} \end{cases}$$

$$n_1 + \dots + n_d = N$$

Set $\tilde{Z} = \mathbb{Z}$-linear span of $\{\tilde{\zeta}, \tfrac{1}{2}\}$ $\subset \mathbb{C}$

$\tilde{Z}_N \subset \tilde{Z}$ span of #s of w/ $\leq N$

shuffle product

$\tilde{Z}_N \tilde{Z}_M \subset \tilde{Z}_{N+M}$

$\exists$ conjectural basis
for $\mathcal{Z}_N \otimes \mathbb{Q}$
(Zagier, Hoffman)
known to span
(Goncharov, Terasoma)

N	0	1	2	3	4	5	6
real gens	1	$\frac{1}{2}$	$\frac{-\zeta(2)}{(2\pi i)^2} = \frac{1}{24}$	$\frac{1}{48}$	$\frac{1}{5760}$	$\frac{1}{11520}$	$\frac{1}{2903040}$, $\frac{\zeta(3)^2}{128\pi^6}$
imag gens	/	/	/	$\frac{i\zeta(3)}{8\pi^3}$	$\frac{i\zeta(3)}{16\pi^3}$	...	

Thm (Banks–Panzer–P. building on F. Brown 2006 Goncharov.)

If $\omega \in A^{top}(\mathcal{C}_{n,m})$ is such that

$$I := \int_{\mathcal{C}_{n,m}} \omega$$

converges absolutely

Then

$$I \in \begin{cases} \widetilde{Z}_{n+m-2} & m>0 \\ \widetilde{Z}_{n-1} & m=0 \end{cases}$$

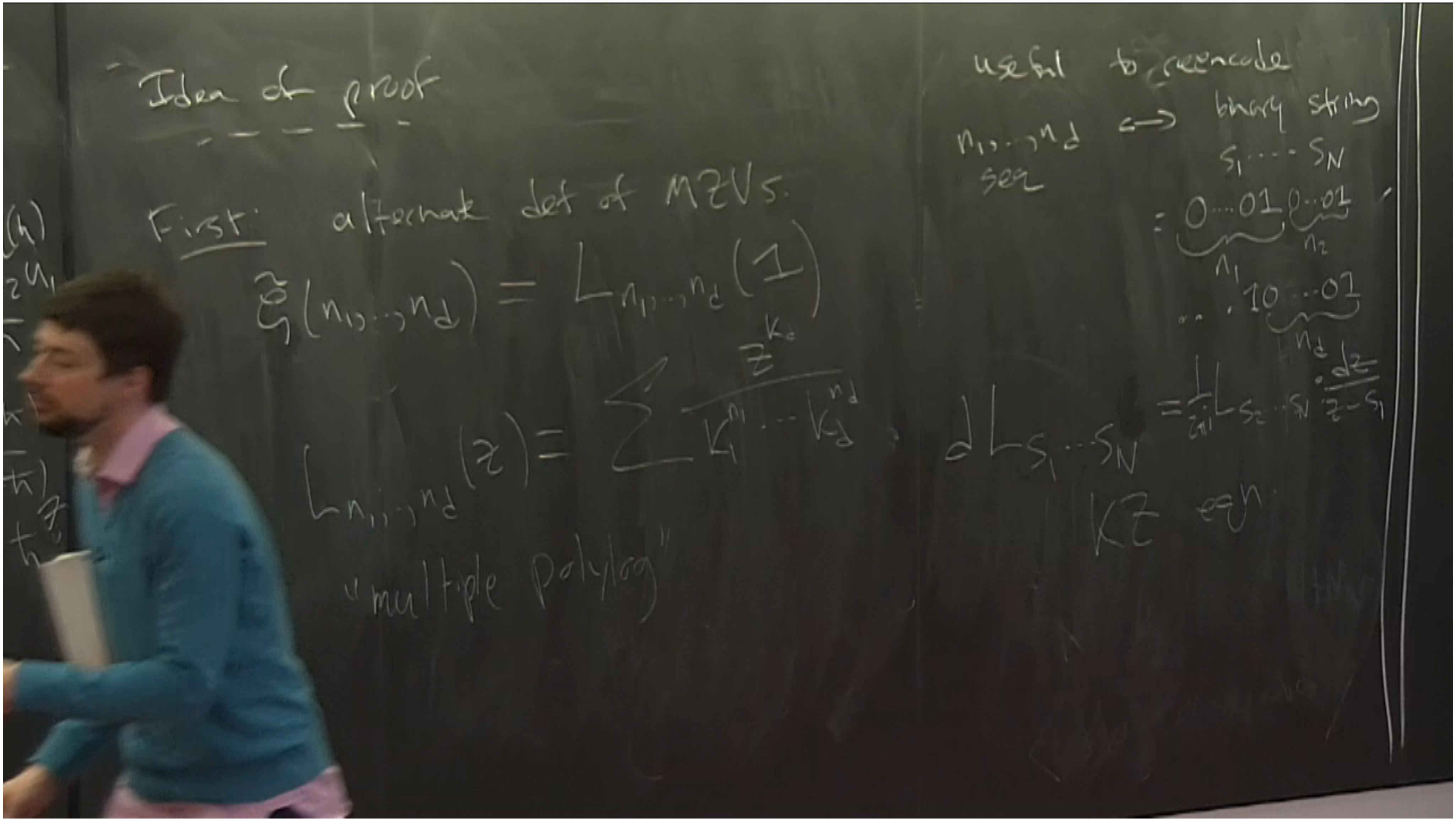
Cor coeff of kontsevich $\star$ product at $\hbar^n$ lies $\frac{1}{4^n}\widetilde{Z}_n \cap \mathbb{R}$

$$\zeta(3) = \sum_{k\geq 1} \frac{1}{k^3}$$

$$u_1 \star u_2 = q(\hbar)\, u_2 u_1$$

$$q(\hbar) = e^{\hbar} = \frac{g(\hbar)}{g(-\hbar)}$$

mod $\hbar^7$


Idea of proof
First: alternate def of MZVs.
ζ(n1,...,nd) = L n1,...,nd (1)
L n1,...,nd (z) = Σ z^kd / k1^n1 ... kd^nd
"multiple polylog"
useful to reencode
n1,...,nd seq ⟷ binary string
s1 ... sN
= 0...01 0...01 ... 10...01
n1 n2 nd
d L s1...sN = L s2...sN dz / z − s1
KZ eqn.

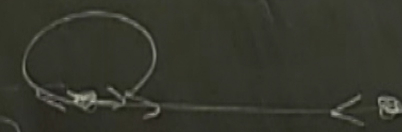
$$L_{s_1 \cdots s_N} = \frac{1}{(2\pi i)^N} \int_0^{z} \frac{dz_N}{z_N - s_N} \int_0^{z_N} \frac{dz_{N-1}}{z_{N-1} - s_{N-1}} \cdots \int_0^{z_2} \frac{dz_1}{z_1 - s_1}$$

"iterated integral
in sense of Chen

$$\zeta = \int_0^1 \underbrace{s_1 \cdots s_N}$$

$$\frac{1}{2\pi i^N} \left(\frac{dz_1}{z - s_1} \cdots \frac{dz_N}{z_N - s_N} \right)$$

extend to all strings $s_1 \cdots s_N$
using "regularization"



$$\frac{1}{2\pi i} \int \frac{dz}{z} = \frac{\pi i}{2\pi i} = \frac{1}{2}$$

...o all
...ys $s_1 \cdots s_N$,
"regularization"

$\leftarrow$

$= \frac{1}{2}$

Goal: use basic operations (e.g. Stokes, Fubini) to reduce $\int_{\mathcal{C}_{n,n}} \omega$ to $\tilde{\zeta}$

Introduce a subsheaf $\mathcal{V}_{n,m} \subset \mathcal{O}_{\mathcal{C}_{n,m}}$ w/ weight filtration $W_0\mathcal{V} \subset W_1\mathcal{V} \subset \cdots$

"sheaf of polylogs"

f

elts $f \in V_{n,m}$ gen by

iterated integrals.

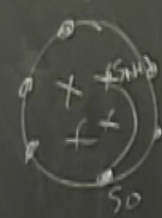
i.e. choose a string in

marked pts & conjugate

$s_1 \cdots s_N$

choose endpts s_0, s_{N+1}

$\mathcal{C}_{n,m}$



$\longmapsto \int_{s_0}^{s_{N+1}} s_1 \cdots s_N$

depends on htpy class of path

useful to ...

$n_1, \ldots, n_d \longleftrightarrow$ ...

seq

$= 0 \ldots$

$d L_{s_1 \cdots s_N} = \frac{1}{2\pi i} L \ldots$

KZ eq

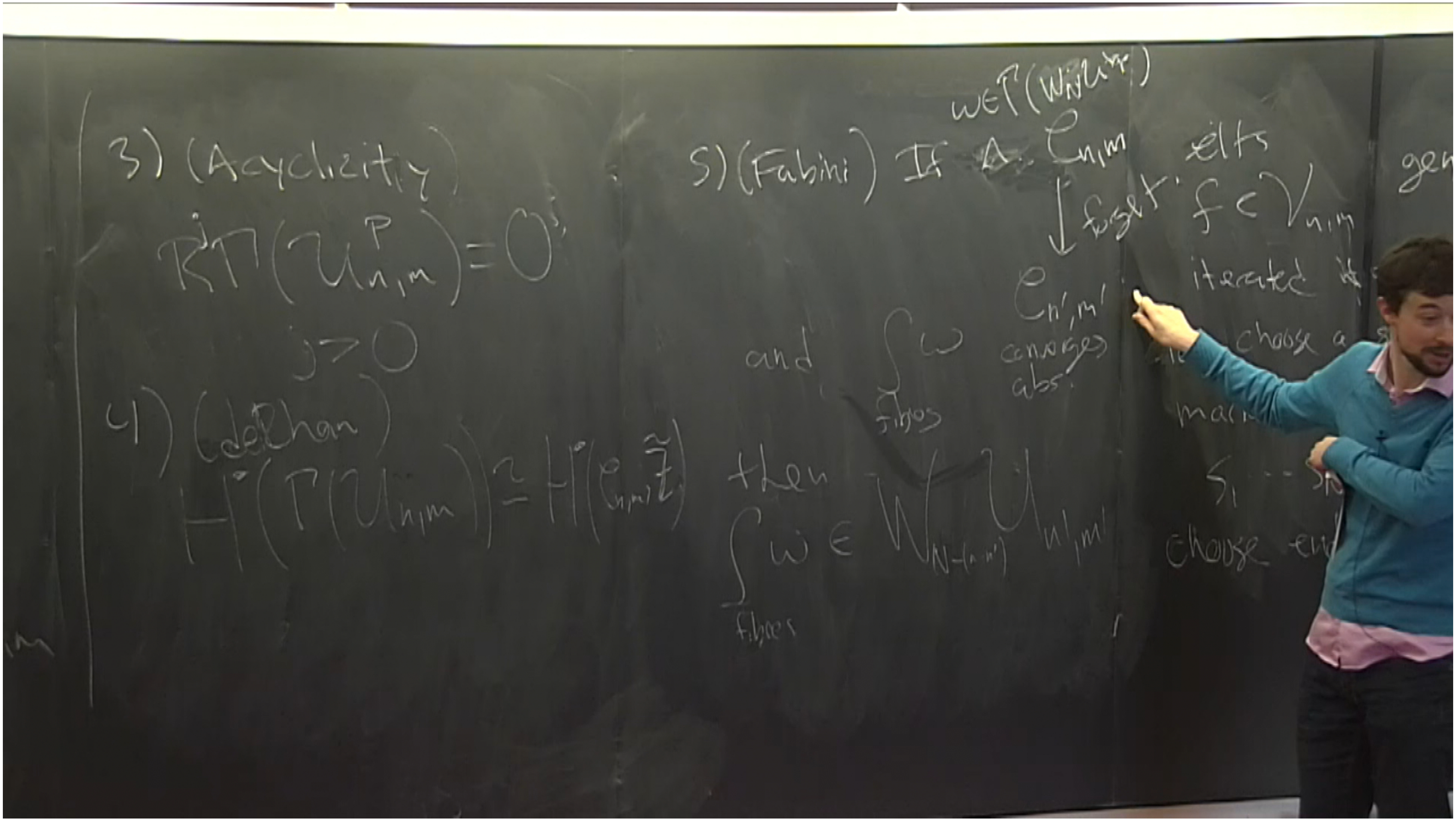
Thm 1) (Picard-Fuchs / KZ eqn)
Brown, Goncharov / $\mathbb{Q}$

$$d W_N \mathcal{V}_{n,m} \subset W_{N-1}\mathcal{V}_{n,m} \otimes A^1(\mathcal{C}_{n,m})$$

2) (Poincaré Lemma)

$$\underline{\tilde{Z}} \xrightarrow[\varepsilon \to 0]{} (\mathcal{U}_{n,m}, d)$$

$$\mathcal{U}_{n,m} = A^\bullet(\mathcal{C}_{n,m}) \otimes \mathcal{V}_{n,m}$$


3) (Acyclicity)
$R^j\Gamma(\mathcal{U}^p_{n,m}) = 0$
$j>0$
4) (deRham)
$H^\bullet(\Gamma(\mathcal{U}_{n,m})) \cong H^\bullet(\mathcal{C}_{n,m}, \tilde{\mathbb{Z}})$
5) (Fubini) If $\omega \in \Gamma(W_N \mathcal{U}_{n,m})$
$\mathcal{C}_{n,m} \xrightarrow{\text{forget}} \mathcal{C}_{n',m'}$
and $\int_{\text{fibres}} \omega$ converges abs.
then $\int_{\text{fibres}} \omega \in W_{N-(n',m')} \mathcal{U}_{n',m'}$
elts $f \in V_{n,m}$
iterated
choose a
choose