Title: Quantum geometry of moduli spaces of local systems

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Abstract: Let G be a split semi-simple algebraic group over Q. We introduce a natural cluster structure on moduli spaces of G-local systems over surfaces with marked points. As a consequence, the moduli spaces of G-local systems admit natural Poisson structures, and can be further quantized. We will study the principal series representations of such quantum spaces. It will recover many classical topics, such as the q-deformed Toda systems, quantum groups, as well as the modular functor conjecture for such representations, which should lead to new quantum invariants of threefolds. This talk will mainly be based on joint work with A.B. Goncharov.

Quantum Geometry of moduli spaces of
local systems

J.w   A. Goncharov

(will appear on Arxiv)
tomorrow.

1. Notations

G a semisimple alg gp over $\mathbb{C}$
with trivial center (e.g. $G = PGL_n$)

$B = \{$ Borel subgps

$\mathcal{B} = \{ \text{Borel subgps in } G \}$

$= \text{flag variety}$

- A pair $B, B' \in \mathcal{B}$ is

  $\underline{\text{generic}}$ if $B \cap B' := T$ is Abelian

  $\left(\text{e.g} \quad G = PGL_n \right.$

  $B = \{ \text{upper trian matrices} \}$

  $\left. B' = \{ \text{lower } \text{---} \} \right)$

$PGL_n)$

$I = \{1, \cdots, r\}$ parametrizes

Simple positive coroots $\alpha_1^{\vee}, \cdots, \alpha_r^{\vee}$

$\forall i \in I, \exists$ a natural

$\alpha_i^{\vee}: \mathbb{G}_m \longrightarrow T$

- The datum

  $p = (B, B', x_i, y_i, i \in I)$

  is called a pinning over a

  generic pair (B, B') if $\forall i$

  $\exists \gamma_i: SL_2 \longrightarrow G$

  $\gamma_i \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = x_i(a)$

  $\gamma_i \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} = \alpha_i^{\vee}(a)$

  $\gamma_i \begin{pmatrix} 1 & \\ a & 1 \end{pmatrix} = y_i(a)$

Example

$$G = PGL_n$$

B_1, B_2 generic

$\Longleftrightarrow$ a decomposition

$$\mathbb{C}^n = \ell_1 \oplus \ell_2 \oplus \cdots \oplus \ell_n$$

s.t.
$$B_1 = (\ell_1 \subset \ell_1 \oplus \ell_2 \subset \cdots \subset \ell_1 \oplus \cdots \oplus \ell_n)$$

$$B_2 = (\ell_n \subset \ell_n \oplus \ell_{n-1} \subset \cdots \subset \ell_n \oplus \cdots \oplus \ell_1)$$

a pinning over $(B_1, B_2) \iff$ a choice of a line

$\ell \subset \mathbb{C}^n$ that is

generic

$y_i(a)$

i.e. $l \in l_1 \oplus \hat{l_i} \oplus \cdots \oplus l_n$

$$\forall i = 1, \ldots, n$$

$= l_1 \oplus \cdots \oplus l_n \,)$

$\subset l_n^{\oplus \cdots \oplus l_i} \,)$

a choice of a line

$l \subset \mathbb{C}^n$ that is

2. A toy example

Def The moduli space $P_{a.t}$ parametrizes G-orbits of

$$(B_1, B_2, B_3, P_{12}, P_{23}, P_{31})$$

where

- B_1, B_2, B_3 are pairwisely generic
- P_{ij} is a pinning over (B_i, B_j)
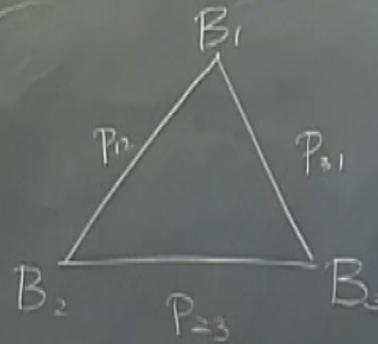
The cyclic group $\mathbb{Z}/3$ acts on $P_{a.t}$

Thm The

Thm. The space $P_{G,t}$ is a smooth affine variety

It admits a natural _cluster_

Poisson structure invariant under $\mathbb{Z}/3$

i.e

1) $P_{G,t}$ is a Poisson variety

2) $P_{G,t}$ admits a collection C of local coordinate charts that are related by cluster Poisson transf.

3) $O(P_{G,t}) = \bigcap_{\alpha \in C} \mathbb{C}[X_{\alpha,1}^{\pm 1}, \dots X_{\alpha,n}^{\pm 1}]$

$\mathbb{Z}/3$ acts

Example

$G = PGL_n$

B_1, B_2 generic

$\Longleftrightarrow$ a decompo

$\mathbb{C}^n = \ell_1 \oplus$
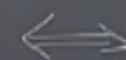
$(\ell_i$

Remarks

1) Cluster algebras are a class of comm algebras introduced by Fomin & Zelevinsky in 2000

2) Most cluster algebras admit natural bases (called $\oplus$-basis) with non-negative structural coefficients (Gross-Hacking-Keel -Kontsevich)

e.g. the $\oplus$-basis of $\mathcal{O}(P_{a,t}) \Longleftrightarrow$ bases of tensor invariants of reps of G

$$\text{i.e } \ell \xleftarrow{} \ell_1 \oplus \ell_i \oplus \cdots \oplus \ell_n$$
$$\forall i = 1, \ldots, n$$

$$\left(V_\lambda \otimes V_\mu \otimes V_\nu \right)^G$$

3) Due to Fock-Goncharov, every cluster Poisson variety can be quantized

For example, there is a $\mathbb{Z}[q^{\pm}]$-algebra $O_q(\mathcal{P}_{G,t})$ whose quasi-classical limit @ $q=1$ recovers $O(\mathcal{P}_{G,t})$ and its Poisson structure

3. Moduli space of G-local systems.

S oriented topological surface with punctures & special pts

smoot

It
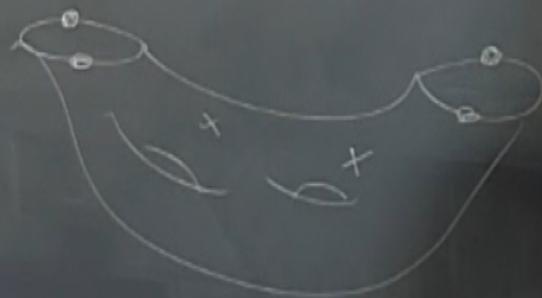
f G-local

topological surface

ures & special pts

$\mathcal{L}$ is a G-local system
over S

$\iff \mathcal{L} \in \mathrm{Hom}(\pi_1(S), G)/G$

$\iff$ principal G-bundle over S

with flat connections

$\mathcal{L}_B = \mathcal{L} \underset{G}{\times} B$ ... associated B-bundle.

e.g.

<u>Def</u> The moduli space P_{GS} parametrizes

$(\mathcal{L}, \beta, P)$, where

· $\mathcal{L}$ is a G-local system.

· β is a flat section of $\mathcal{L}_B$ on every special pt & every circle sounding punctures

· $P = (P_e)$ is a collection of pinning over (B_i, B_j) for every boundary interval

$P = (ij)$

Example

S = once-punctured disk
with 2 special pts

B_1

Pe_1 B^x g Pe_2

B_2
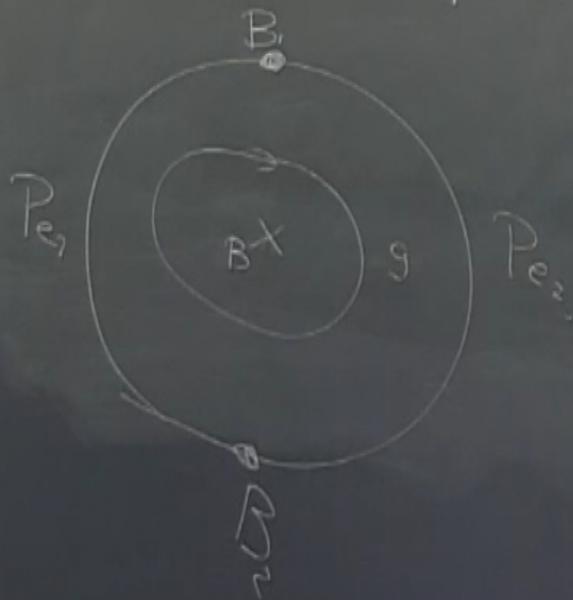
Natural actions on P_Gs

four groups acts on P_Gs

· mapping class group of S

· Outer automorphism of G

$$\left(e.g \quad x. \quad G \to G \atop \qquad\qquad g \mapsto (g^x)^{-1} \right)$$

· Weyl group W^n , n = # punctures

· Braid group B^m , m = # boundary circles

$\mathcal{L}$
or

$\Longleftrightarrow$

$\Longleftrightarrow$

$\mathcal{L}$

s

ts on $P_{G,S}$

roup of S

ism of G

$\to G$

$\to (g^t)^{-1}$

n

$n = \#$ punctures

$m = \#$ boundary circles

Example

There is a Tits-distance map

$$d: B \times B \to W$$

$\forall$ generic pair B_1, B_2 $\forall w \in W$

$\exists! B'$. s.t

$$d(B_1, B') = w, \quad d(B', B_2) = w^{-1} w_0$$

Def The mod

$(L, \beta$

.s a

There is a $\text{list}\ldots$

$$d: B \times B \longrightarrow W$$

$\forall$ generic pair B_1, B_2 $\qquad \forall w \in W$

$\exists! B'$, s.t.

$$d(B_1, B') = w, \quad d(B', B_2) = w^{-1} w_0$$

$$G = PGL_n$$

punctures

boundary circles

$$B_1 = (\ell_1 \subset \ell_1 \oplus \ell_2 \subset \ell_1 \oplus \ell_2 \oplus \ell_3 \subset \cdots \subset \ell_1 \oplus \cdots \oplus \ell_n)$$

$$B_2 = (\ell_n \subset \ell_n \oplus \ell_{n-1} \subset \cdots$$

$$w \in S_n$$

$$B' = (\ell_{w(1)} \subset \ell_{w(1)} \oplus \ell_{w(2)} \subset \cdots$$

Example

$S =$ once-punctured disk

with 2 special pts

T_i

$S_i \times W_0$
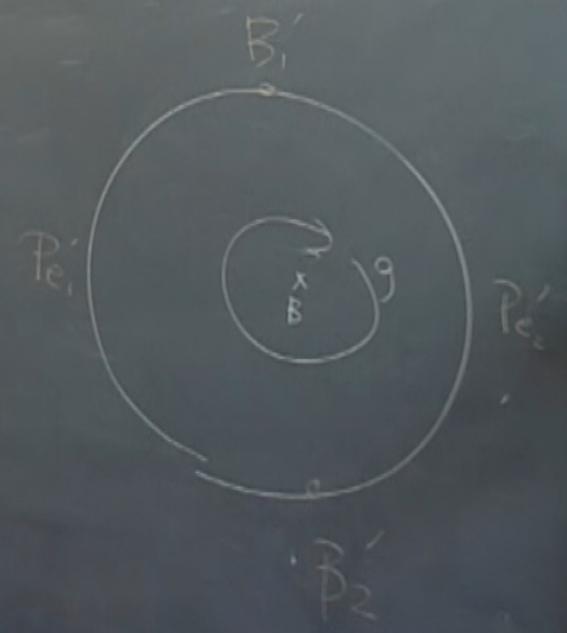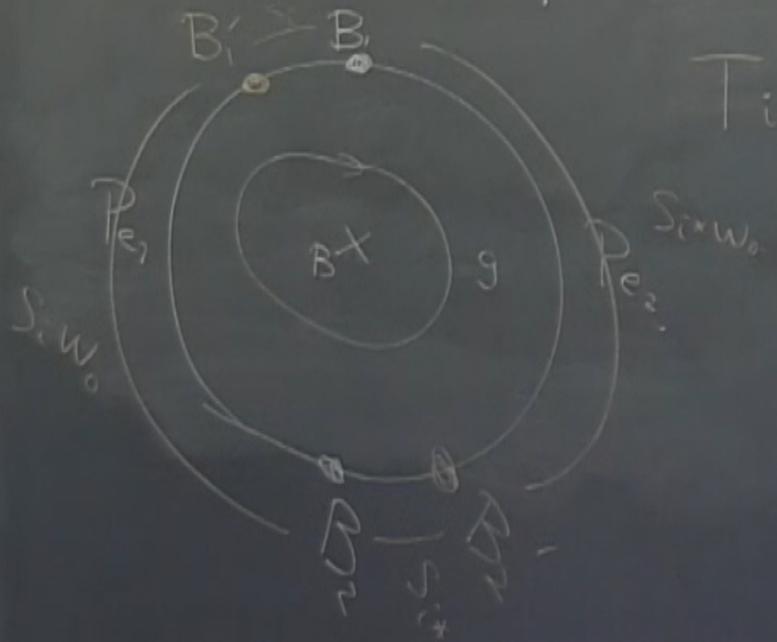
$S_i = W_0 S_i W_0^{-1}$

__Thm__ The space $P_{G,S}$ admits a natural

Cluster Poisson structure invariant under

the above four group actions

__Remarks__

1) These groups act on $\mathcal{O}_q(P_{G,S})$

2) $\mathcal{O}_q(P_{G,S})$ is expected to have

a natural basis (when $q=1$, $G=PGL_n$, this

is proved in [GS, 16]

$B_1 = (\ell_1 \subset \ell_1 \oplus$

$B_2 = (\ell_n \subset \ell_n$

$w \in S_n$

$B' = (\ell_{w}$
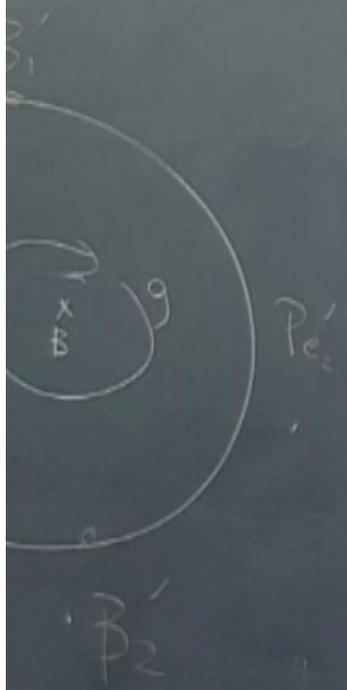
Quantum group

$\mathcal{U}_q(g)$ is a Hopf algebra

generated by

$$\left\{ E_i, F_i, K_i^{\pm 1} \,\middle|\, i \in I \right\}$$

Satisfy a set of relations

$\mathcal{U}_q(b) \subset \mathcal{U}_q(g)$ generated by

$$\left\{ E_i, K_i^{\pm 1} \right\}$$

a Hopf algebra

by

$\left\{ K_i^{\pm 1} \middle| i \in I \right\}$

set of relations

$\mathfrak{h}_2(\mathfrak{g})$ generated by h-distance

± 1

$\forall$ special pt $s \in S$

There exists a collection of natural functions

potential $W_{s,1}, \ldots W_{s,r}$

$K_{s,1}, \ldots K_{s,r}$

([GS, 14])

Thm The space P

Poisson str

above four

$\in S$
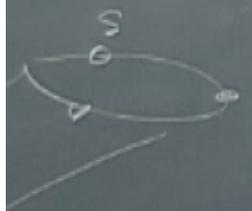
One can lift them to $\mathcal{O}_q(P_{G.s})$

Thm $\exists$ an embedding

$$\mathcal{U}_q(b) \hookrightarrow \mathcal{O}_q(P_{G.s})$$

$$E_i \longmapsto W_{s.i}$$

$$K_i \longmapsto K_{s.i}$$

collection of natural functions

$([GS, 14])$

$/_{s.r}$

$K_{s.r}$

One can lift them to $O_q(P_{G,S})$

Thm $\exists$ an embedding

$$\mathcal{U}_q(b) \hookrightarrow O_q(P_{G,S})$$

$$E_i \longmapsto W_{S,i}$$

$$K_i \longmapsto K_{S,i}$$

functions

)

S_1

$\times 2^W$

S_2

(1) $\mathcal{U}_q(g) \hookrightarrow O_q(P_{G,S})$

Conj $\mathcal{U}_q(g) \stackrel{\simeq}{\longrightarrow} O_q(L_{G,S})$

$O_q(L_{G,S})$ $L_{G,S} :$ Poisson-Lie group

$\parallel$

$O_q(P_{G,S})^W \Big/ K_{S,i} K_{S,i^*} = 1$

$\stackrel{\simeq}{} O_q(L_{G,S})$

(1) is a

S_1

$\times 2^W$

S_2

$\hookrightarrow$

$U_q(\mathfrak{g})$

$O_q(L_{G,S})$

$\mathcal{L}_{G,S}$: Poisson-Lie group

$\parallel$

$O_q(P_{G,S})^W \Big/ K_{S,i} K_{S,i^*} = 1$
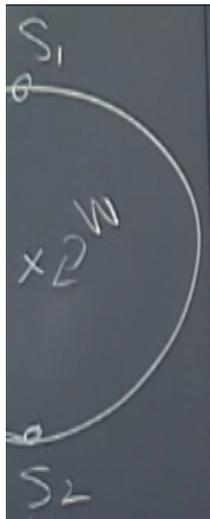
$\cong O_q(L_{G,S})$

(1) is a refinement

of works of

Shapiro-Shrader

& IP

S_1

S_2

$\downarrow$

$S_1 \times S_2$

$$\theta_2 \left(P_{G, S_1 \times S_2} \right) \rightarrow \theta(\ ) \oplus \theta \_\_$$

$\forall$ special pt $s \in S$

There exists a collectio

$V_{S_1}, \cdots W_{S, r}$

$K_{S_1}, \cdots K_{S, r}$