Title: Zeta-regularized vacuum expectation values

Speakers: Tobias Hartung

Series: Tensor Network

Date: April 18, 2019 - 11:00 AM

URL: http://pirsa.org/19040123

Abstract: Computing vacuum expectation values is paramount in studying Quantum Field Theories (QFTs) since they provide relevant information for comparing the underlying theory with experimental results. However, unless the ground state of the system is explicitly known, such computations are very difficult and Monte Carlo simulations generally run months to years on state-of-the-art high performance computers. Additionally, there are various physically interesting situations, in which most numerical methods currently in use are not applicable at all (e.g., the early universe or setting requiring Lorentzian backgrounds). Thus, new algorithms are required to address such problems in QFT. In recent joint work with K. Jansen (NIC, DESY Zeuthen), I have shown that zeta-functions of Fourier integral operators can be applied to regularize vacuum expectation values with Euclidean and Lorentzian backgrounds and that these zeta-regularized vacuum expectation values are in fact physically meaningful. In order to prove physicality, we introduced a discretization scheme which is accessible on a quantum computer. Using this discretization scheme, we can efficiently approximate ground states on a quantum device and henceforth compute vacuum expectation values. Furthermore, the Fourier integral operator ζ-function approach is applicable to Lattice formulations in Lorentzian background.

ζ-regularized vacuum expectation values

Tobias Hartung

King's College London

In collaboration with K. Jansen (NIC, DESY Zeuthen).

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Feynman's alternative formulation of quantum mechanics

probability density $P = \left|\sum_j \Phi_j\right|^2 = \left|\sum_{p\in\text{paths}} \Phi_p\right|^2$

Theorem (Feynman; 1948)

Let S_{cl} be the classical action. Then,

$$\forall p \in \text{paths}: \quad \Phi_p = \exp\left(\frac{i}{\hbar} S_{\text{cl}}(p)\right)$$

Feynman's alternative formulation of quantum mechanics

- lattice → dense
- continuous paths
- ⇝ inductive limit

Feynman path integral (Feynman; 1948)

Propagator of a particle moving from (t_0, x_0) to (t_1, x_1):

$$K(t_1, x_1; t_0, x_0) = \int_{p \in \text{paths}((t_0,x_0)\to(t_1,x_1))} \exp\left(\frac{i}{\hbar} S_{\text{cl}}(p)\right) \mathcal{D}p$$

Let Ω be a discretized observable. Then, the vacuum expectation value $\langle\Omega\rangle_L$ with Lorentzian background is given by

$$\langle\Omega\rangle_L = \frac{\int_{\mathbb{R}^N} \Omega(p)\exp\big(\frac{i}{\hbar}S_{\mathrm{cl}}(p)\big)dp}{\int_{\mathbb{R}^N} \exp\big(\frac{i}{\hbar}S_{\mathrm{cl}}(p)\big)dp}.$$

Let Ω be a discretized observable. Then, the vacuum expectation value $\langle\Omega\rangle_E$ with Euclidean background is given by

$$\langle\Omega\rangle_E = \frac{\int_{\mathbb{R}^N} \Omega(p)\exp\big(-\frac{1}{\hbar}S_{\mathrm{cl}}(p)\big)dp}{\int_{\mathbb{R}^N} \exp\big(-\frac{1}{\hbar}S_{\mathrm{cl}}(p)\big)dp}.$$

time evolution and vacuum expectation values

- C_0-group "time evolution" $(U(t,t'))_{t,t'\in\mathbb{R}}$ on a Hilbert space $\mathcal{H}$:

$$K(t,x;t',x') = \langle \delta_x, U(t,t')\delta_{x'}\rangle_{\mathcal{H}}$$

- Hamiltonian H: (texp = "time-ordered exponential")

$$U(t,t') = \operatorname{texp}\left(-\frac{i}{\hbar}\int_{t'}^{t} H(s)ds\right)$$

- Given an observable Ω (operator on $\mathcal{H}$), its vacuum expectation is given by

$$\langle\Omega\rangle := \lim_{T\to\infty+i0^+} \frac{\operatorname{tr}\ U(T,0)\Omega}{\operatorname{tr}\ U(T,0)}.$$

ζ-regularization

- Let $\mathcal{A}$ be an operator algebra and $\mathcal{A}_0 \subseteq \mathcal{A}$ a subalgebra.
- Let $\tau: \mathcal{A}_0 \to \mathbb{C}$ be a continuous trace.
- To define $\tau(A)$ for $A \in \mathcal{A} \setminus \mathcal{A}_0$, consider $\varphi: \mathbb{C} \to \mathcal{A}$ holomorphic such that $\varphi(0) = A$ and

$$\{z \in \mathbb{C};\ \varphi(z) \in \mathcal{A}_0\} \text{ has connected non-empty interior.}$$

- The maximal meromorphic extension of $\tau \circ \varphi$ is $\zeta(\varphi)$.
- Is $\zeta(\varphi)$ holomorphic in neighborhood of 0?
- Does $\varphi(0) = \psi(0)$ imply $\zeta(\varphi)(0) = \zeta(\psi)(0)$?

Example

- Consider $|\partial|$ on $\mathbb{R}/2\pi\mathbb{Z}$. Then

$$\operatorname{tr}|\partial| \text{``}=\text{''} \sum_{n\in\mathbb{Z}} |n| = 2\sum_{n\in\mathbb{N}} n.$$

- Let $\varphi(z) := |\partial|^{1+z}$. Then, for $\Re(z) < -2$,

$$\operatorname{tr}\varphi(z) = 2\sum_{n\in\mathbb{N}} n^{1+z} = 2\zeta_R(-z-1).$$

- Hence, $\zeta(\varphi)(z) = 2\zeta_R(-z-1)$ and

$$\operatorname{tr}|\partial| := \zeta(\varphi)(0) = 2\zeta_R(-1) = -\frac{1}{6}.$$

the operators $U(T,0)\tilde{\Omega}$ with $\tilde{\Omega} \in \{1, \Omega\}$

The operator $U(T,0)\tilde{\Omega}$ on a Cauchy surface X is an integral operator with kernel

$$k(x,y) = \int_{\mathbb{R}_{>0}} \int_{\partial B_{\mathbb{R}^N}} e^{ih_2(x,y,\xi)r^2 + ih_1(x,y,\xi)r} a(x,y,r,\xi) d\xi dr.$$

If X is a compact orientable C^∞-manifold without boundary, then the trace formally evaluates to

$$\begin{aligned} \operatorname{tr}\left(U(T,0)\tilde{\Omega}\right) &= \int_X k(x,x) d\mathrm{vol}_X(x) \\ &= \int_X \int_{\mathbb{R}_{>0}} \int_{\partial B_{\mathbb{R}^N}} e^{ih_2(x,x,\xi)r^2 + ih_1(x,x,\xi)r} a(x,x,r,\xi)\ d\xi\ dr\ d\mathrm{vol}_X(x) \end{aligned}$$

gauging $U(T,0)\tilde{\Omega}$

Let $\mathfrak{g}$ be a holomorphic family such that

$$\mathfrak{g}(0) = 1 \quad \wedge \quad \mathfrak{g}(z)(x,r,\xi) = r^z \mathfrak{g}_\partial(z)(x,\xi).$$

Then, there exists $R \in \mathbb{R}$ such that for every $z \in \mathbb{C}_{\Re(\cdot)<R}$ the operator $\varphi(z)$ with kernel

$$k(z)(x,y) = \int_{\mathbb{R}_{>0}} \int_{\partial B_{\mathbb{R}^N}} e^{ih_2(x,y,\xi)r^2 + ih_1(x,y,\xi)r} a(x,y,r,\xi)\mathfrak{g}(z)(x,y,r,\xi) d\xi dr.$$

is of trace class and satisfies

$$\begin{aligned} \operatorname{tr}\varphi(z) &= \int_X k(z)(x,x) d\mathrm{vol}_X(x) \\ &= \int_X \int_{\mathbb{R}_{>0}} \int_{\partial B_{\mathbb{R}^N}} e^{ih_2(x,x,\xi)r^2 + ih_1(x,x,\xi)r} (a\mathfrak{g}(z))(x,x,r,\xi) d\xi dr d\mathrm{vol}_X(x). \end{aligned}$$

gauging $U(T,0)\tilde{\Omega}$

- $h_2 = 0$ on diagonal $\Rightarrow$ FIO-ζ-function.
- $|h_2| > 0$ on diag. $\Rightarrow \mathbb{C}_{\Re(\cdot)<R} \ni z \mapsto \operatorname{tr}\varphi(z) \in \mathbb{C}$ has a holomorphic extension $\zeta(\varphi)$.
- $\zeta(\varphi)(0)$ depends only on $U(T,0)\tilde{\Omega} = \varphi(0)$ and is tracial.
- Thus, we can interpret $\operatorname{tr}U(T,0)\tilde{\Omega}$ as

$$\operatorname{tr}U(T,0)\tilde{\Omega} := \zeta(\varphi)(0).$$

- The fraction

$$\langle\Omega\rangle_T = \frac{\operatorname{tr}U(T,0)\Omega}{\operatorname{tr}U(T,0)}$$

is "almost always" independent of the choice of gauge.

Definition (ζ-regularized vacuum expectation values - H; J Math Phys 2017)

Let $(U(t,s))_{t,s\in\mathbb{R}}$ be the time evolution of your favorite QFT and $\mathfrak{G}$ be a gauged family of operators with $\mathfrak{G}(0) = 1$ such that the kernels $k(z)$ of $U(T,0)\mathfrak{G}(z)$ are of the form

$$k(z)(x,y) = \int_{\mathbb{R}_{>0}} \int_{\partial B_{\mathbb{R}^N}} e^{ih_2(x,y,\xi)r^2 + ih_1(x,y,\xi)r} (a\mathfrak{g}(z))(x,y,r,\xi) d\xi dr.$$

Then, we define

$$\langle\Omega\rangle_{\mathfrak{G}}(z) := \lim_{T\to\infty+i0^+} \frac{\zeta(U(T,0)\mathfrak{G}\Omega)}{\zeta(U(T,0)\mathfrak{G})}(z) \quad \text{and} \quad \langle\Omega\rangle := \langle\Omega\rangle_\zeta := \langle\Omega\rangle_{\mathfrak{G}}(0).$$

Free relativistic Fermion (3+1 dimensions)

Hamiltonian:

$$H = \begin{pmatrix} mc^2 & -i\hbar\sigma_k\partial_k \\ -i\hbar\sigma_k\partial_k & mc^2 \end{pmatrix} \sim \begin{pmatrix} mc^2 & \hbar r\sigma_k\xi_k \\ \hbar r\sigma_k\xi_k & mc^2 \end{pmatrix}$$

Gauge:

$$\mathfrak{g}(z)(x,r,\xi) = r^z$$

Observable H (ground state energy)

$$\langle H\rangle = \lim_{T\to\infty+i0^+}\lim_{z\to 0}\frac{\int_{\partial B_{\mathbb{R}^3}}\int_{\mathbb{R}_{>0}}\left(4mc^2\cos(Tr) - 4ir\sin(Tr)\right)r^{z+2}dr\,d\mathrm{vol}_{\partial B_{\mathbb{R}^3}}(\xi)}{\int_{\partial B_{\mathbb{R}^3}}\int_{\mathbb{R}_{>0}}4\cos(Tr)\,r^{z+2}dr\,d\mathrm{vol}_{\partial B_{\mathbb{R}^3}}(\xi)}$$

$$\langle H\rangle = \lim_{T\to\infty+i0^+}\lim_{z\to 0}\frac{\int_{\partial B_{\mathbb{R}^3}}\int_{\mathbb{R}_{>0}}\left(4mc^2\cos(Tr) - 4ir\sin(Tr)\right)r^{z+2}dr\,d\mathrm{vol}_{\partial B_{\mathbb{R}^3}}(\xi)}{\int_{\partial B_{\mathbb{R}^3}}\int_{\mathbb{R}_{>0}}4\cos(Tr)\,r^{z+2}dr\,d\mathrm{vol}_{\partial B_{\mathbb{R}^3}}(\xi)}$$

$$= \lim_{T\to\infty+i0^+}\lim_{z\to 0} mc^2 - \frac{\int_{\mathbb{R}_{>0}} ir\sin(Tr)\,r^{z+2}dr}{\int_{\mathbb{R}_{>0}}\cos(Tr)\,r^{z+2}dr}$$

$$= mc^2 - \lim_{T\to\infty+i0^+}\lim_{z\to 0}\frac{\int_{\mathbb{R}_{>0}}\left(e^{iTr}-e^{-iTr}\right)r^{z+3}dr}{\int_{\mathbb{R}_{>0}}\left(e^{iTr}+e^{-iTr}\right)r^{z+2}dr}$$

$$= mc^2 + \lim_{T\to\infty+i0^+}\lim_{z\to 0}\frac{\left(-e^{-i\frac{\pi(z+3)}{2}} - e^{-3i\frac{\pi(z+3)}{2}}\right)\Gamma(z+4)T^{-z-4}}{i\left(-e^{-i\frac{\pi(z+2)}{2}} + e^{-3i\frac{\pi(z+2)}{2}}\right)\Gamma(z+3)T^{-z-3}}$$

$$= mc^2$$

```python
import sympy as smp

z,T = smp.symbols("z,T")
J,xi = smp.symbols("J,xi",real=True)

h = xi**2 / (2*J)
Q = T*xi/(2*smp.pi*J)
g = smp.Abs(xi)**z

num = smp.fourier_transform(smp.exp(-smp.I*T*h)*g*Q**2,xi,0)
den = smp.fourier_transform(smp.exp(-smp.I*T*h)*g,xi,0)

chi_top = smp.limit(smp.limit(num/(-smp.I*T*den),z,0),T,smp.oo)
energy_gap = 2 * smp.pi**2 * chi_top

print("\n\n\nenergy gap = "+str(energy_gap)+"\n\n\n")
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In [7]:
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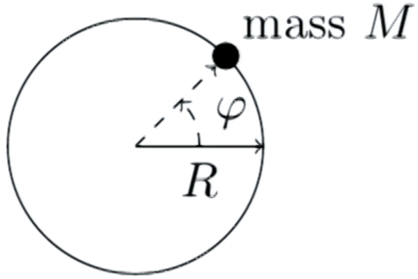
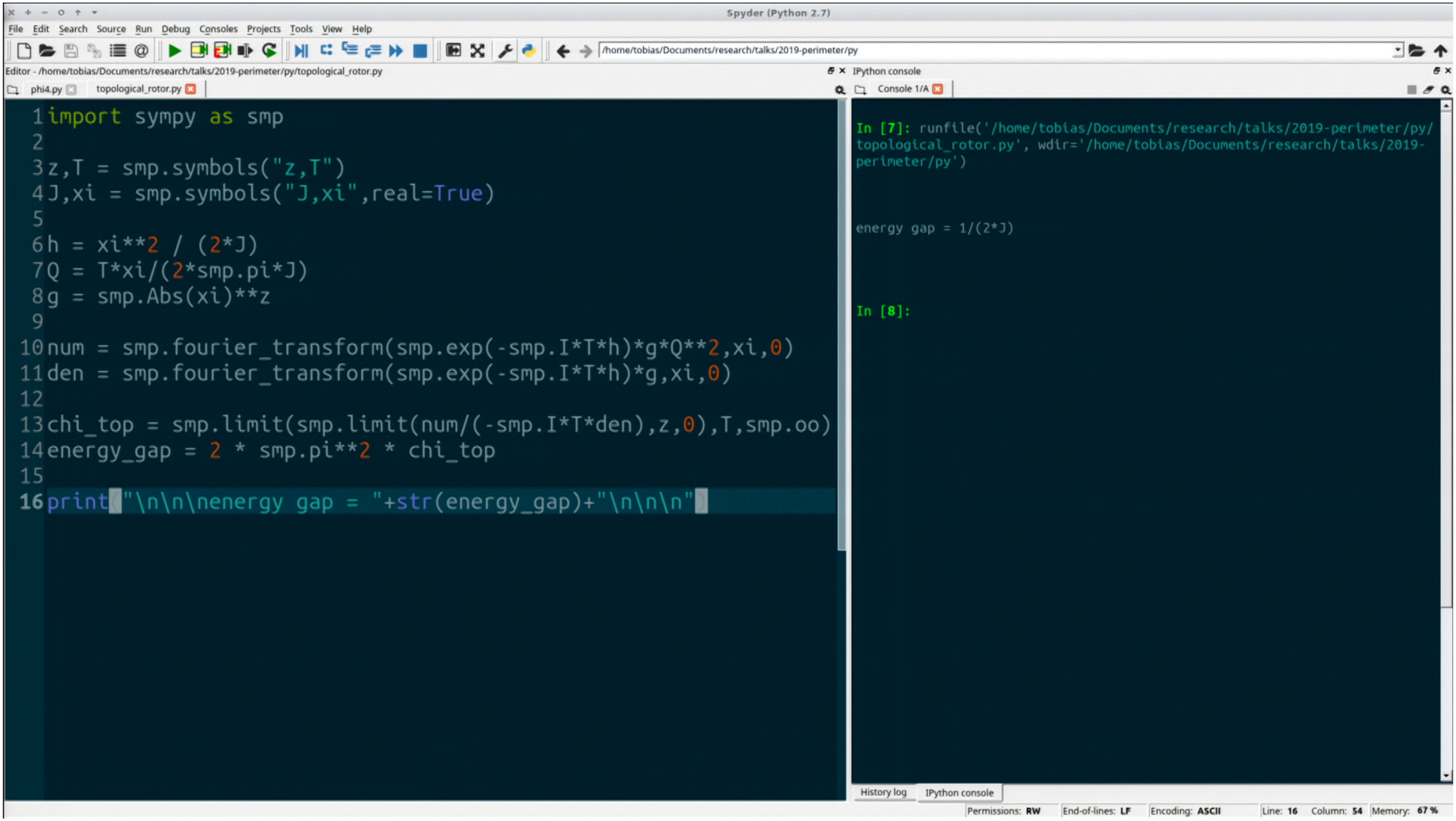
Topological rotor



- moment of inertia: $J := MR^2$
- generalized momentum: $p = J\partial_0\varphi$
- Hamiltonian: $H = \frac{p^2}{2J}$
- topological charge: $Q = \frac{1}{2\pi}\int_0^T \frac{p}{J}$
- topological susceptibility:
  $\chi_{\text{top}} = \lim_{T\to\infty} \frac{\langle Q^2\rangle_T}{-iT}$

Energy gap: $$\Delta E = 2\pi^2\chi_{\text{top}} = \frac{1}{2J}$$


Spyder (Python 2.7)
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Editor - /home/tobias/Documents/research/talks/2019-perimeter/py/topological_rotor.py
phi4.py
topological_rotor.py
import sympy as smp
z,T = smp.symbols("z,T")
J,xi = smp.symbols("J,xi",real=True)
h = xi**2 / (2*J)
Q = T*xi/(2*smp.pi*J)
g = smp.Abs(xi)**z
num = smp.fourier_transform(smp.exp(-smp.I*T*h)*g*Q**2,xi,0)
den = smp.fourier_transform(smp.exp(-smp.I*T*h)*g,xi,0)
chi_top = smp.limit(smp.limit(num/(-smp.I*T*den),z,0),T,smp.oo)
energy_gap = 2 * smp.pi**2 * chi_top
print("\n\n\nenergy gap = "+str(energy_gap)+"\n\n\n")
IPython console
Console 1/A
In [7]: runfile('/home/tobias/Documents/research/talks/2019-perimeter/py/topological_rotor.py', wdir='/home/tobias/Documents/research/talks/2019-perimeter/py')
energy gap = 1/(2*J)
In [8]:
History log
IPython console
Permissions: RW
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Line: 16
Column: 54
Memory: 67 %

Spontaneous Symmetry Breaking

- scalar fields $\varphi^1, \ldots, \varphi^k$
- Hamiltonian: $H(\varphi^1, \ldots, \varphi^k)$
- partition function: $\mathcal{Z}_T(\varphi) = \zeta\left(\exp\left(-\frac{i}{\hbar}\int_0^T H(\varphi)(t)dt\right)\mathfrak{g}\right)(0)$
- effective potential: $\mathcal{Z}_T(\varphi) = \exp\left(-i\int_{[0,T]\times X} V_e^T(\varphi)d(t,x)\right)$
- we are looking for constant φ_0^j that locally minimize

$$V_e^T(\varphi_0) = \frac{\mathcal{Z}_T(\varphi)}{-iT\mathrm{vol}(X)} = \frac{\zeta\left(\exp\left(\frac{-i}{\hbar}\int_0^T H(\varphi_0)(t)dt\right)\mathfrak{g}\right)(0)}{-iT\mathrm{vol}(X)}$$

- φ_0^j are vacuum expectation values of the φ^j (in the limit $T \to \infty$)
- $(\partial_i\partial_j V_e^T(\varphi_0))_{i,j}$ self-adjoint $\rightsquigarrow$ eigenvalues are squared field masses (in the limit $T \to \infty$)

Spontaneous Symmetry Breaking - the φ^4 model

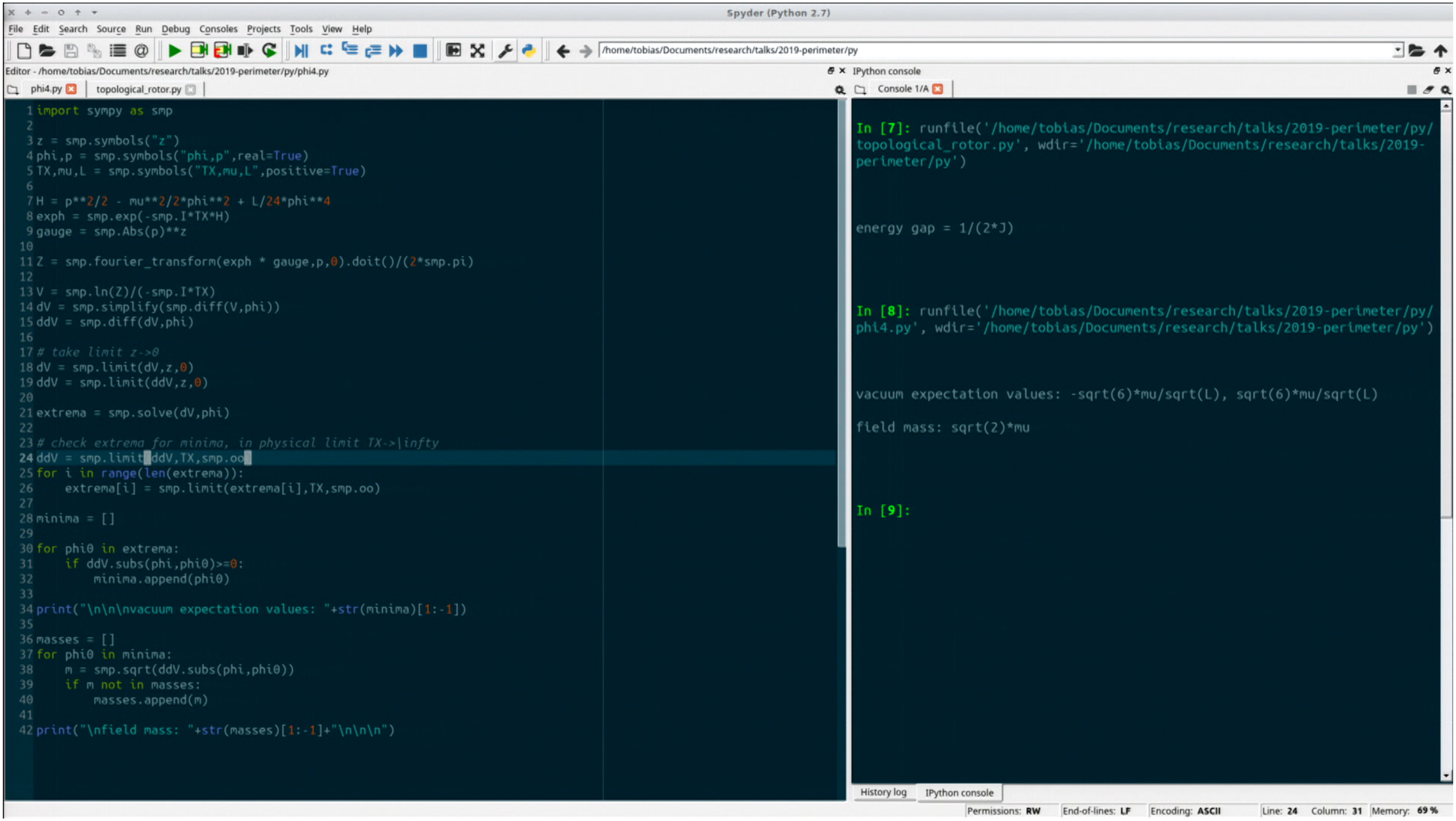
- Hamiltonian: $H = \int_X \frac{p^2}{2} - \frac{\varphi\Delta\varphi}{2} - \frac{\mu^2}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4 dx$
- vacuum expectation values: $\varphi_0 = \pm\sqrt{\frac{6}{\lambda}}\mu$
- field mass: $\sqrt{2}\mu$

gauge dependence of $\mathcal{Z}_T$

The partition function

$$\mathcal{Z}_T(\varphi) = \zeta\left(\exp\left(\frac{-i}{\hbar}\int_0^T H(\varphi)(t)dt\right)\mathfrak{g}\right)(0),$$

and thus V_e^T, is independent of the particular choice of gauge $\mathfrak{g}$.


Spyder (Python 2.7)
File Edit Search Source Run Debug Consoles Projects Tools View Help
/home/tobias/Documents/research/talks/2019-perimeter/py
Editor - /home/tobias/Documents/research/talks/2019-perimeter/py/phi4.py
phi4.py
topological_rotor.py
import sympy as smp
z = smp.symbols("z")
phi,p = smp.symbols("phi,p",real=True)
TX,mu,L = smp.symbols("TX,mu,L",positive=True)
H = p**2/2 - mu**2/2*phi**2 + L/24*phi**4
exph = smp.exp(-smp.I*TX*H)
gauge = smp.Abs(p)**z
Z = smp.fourier_transform(exph * gauge,p,0).doit()/(2*smp.pi)
V = smp.ln(Z)/(-smp.I*TX)
dV = smp.simplify(smp.diff(V,phi))
ddV = smp.diff(dV,phi)
# take limit z->0
dV = smp.limit(dV,z,0)
ddV = smp.limit(ddV,z,0)
extrema = smp.solve(dV,phi)
# check extrema for minima, in physical limit TX->\infty
ddV = smp.limit(ddV,TX,smp.oo)
for i in range(len(extrema)):
extrema[i] = smp.limit(extrema[i],TX,smp.oo)
minima = []
for phi0 in extrema:
if ddV.subs(phi,phi0)>=0:
minima.append(phi0)
print("\n\n\nvacuum expectation values: "+str(minima)[1:-1])
masses = []
for phi0 in minima:
m = smp.sqrt(ddV.subs(phi,phi0))
if m not in masses:
masses.append(m)
print("\nfield mass: "+str(masses)[1:-1]+"\n\n\n")
IPython console
Console 1/A
In [7]: runfile('/home/tobias/Documents/research/talks/2019-perimeter/py/topological_rotor.py', wdir='/home/tobias/Documents/research/talks/2019-perimeter/py')
energy gap = 1/(2*J)
In [8]: runfile('/home/tobias/Documents/research/talks/2019-perimeter/py/phi4.py', wdir='/home/tobias/Documents/research/talks/2019-perimeter/py')
vacuum expectation values: -sqrt(6)*mu/sqrt(L), sqrt(6)*mu/sqrt(L)
field mass: sqrt(2)*mu
In [9]:
History log
IPython console
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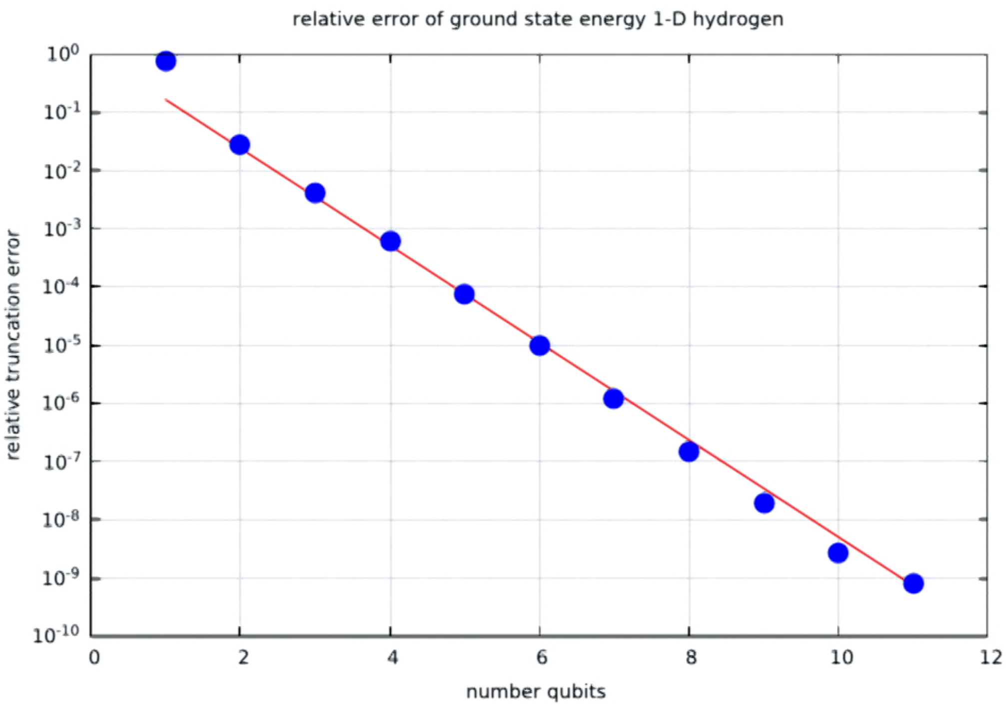
It makes sense!

Theorem (H - Jansen; arXiv:1808.06784)

Let the observable Ω and gauged time-evolution $U(T,0)\mathfrak{G}$ be sufficiently nice. Furthermore, let ψ be the vacuum of the QFT with time evolution U. Then, there exists a discretization scheme disc *such that the continuum limit*

$$\langle\psi|\,\Omega\,|\psi\rangle = \text{continuum-lim}\ \langle\Omega_{\text{disc}}\rangle = \langle\Omega\rangle_\zeta .$$

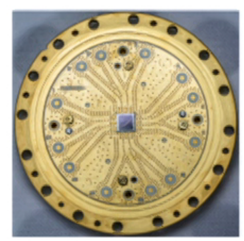
Error of ground state energy $\langle H \rangle_\zeta$ - 1-D hydrogen atom - QVM

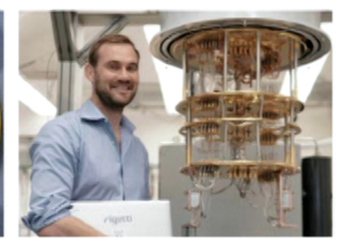


Regression:

- q = number qubits
- $E(q)$ = rel. error of ground state energy

$$E(q) \approx 1.14 \cdot e^{-1.92q}$$





The discretization scheme disc

- Let $\{e_j;\ j\in\mathbb{N}\}$ an orthonormal basis of $\mathcal{H}$, $\mathcal{H}_0 := \mathrm{lin}\{e_j;\ j\in\mathbb{N}\}$.
- Let $\mathcal{H}_1$ densely embedded with $\mathcal{H}_0 \subseteq \mathcal{H}_1 \subseteq \mathcal{H}$ such that $\mathfrak{G}(z)\Omega, \mathfrak{G}(z) \in L(\mathcal{H}_1,\mathcal{H})$ for all z in an open neighborhood of $\mathbb{C}_{\Re(\cdot)\le 0}$.
- Let P_n, Q_n the orthoprojections onto $\mathrm{lin}\{e_1,\ldots,e_n\}$ in $\mathcal{H}$ and $\mathcal{H}_1$ resp.
- Discretization of observable Ω: $\Omega_n := P_n \Omega Q_n$
- Discretization of time evolution $U(T,0)$: $U_n := \mathrm{texp}\left(-\frac{i}{\hbar}\int_0^T P_n H(s) Q_n ds\right)$
- Discretized vacuum ψ_n: minimizer of $\varphi \mapsto \langle\varphi, H_n\varphi\rangle_{\mathcal{H}}$ in $P_n[\mathcal{H}]$ with $\|\varphi\|_{\mathcal{H}} = 1$

Variational Quantum Eigensolver

Find ψ_n by minimizing $\varphi \mapsto \langle\varphi, H_n\varphi\rangle_{\mathcal{H}}$ in $P_n[\mathcal{H}]$ with $\|\varphi\|_{\mathcal{H}} = 1$.

Lemma

If the QFT has a mass gap and the vacuum ψ in the continuum is non-degenerate, then $\lim_{n\to\infty} \langle \psi_n, \psi \rangle_{\mathcal{H}} \psi_n = \psi$.

Lemma

If the vacuum ψ in the continuum is non-degenerate, $\Omega \in L(\mathcal{H}_1, \mathcal{H})$, Ω^* the adjoint of Ω as an unbounded operator in $\mathcal{H}$, $\psi \in D(\Omega^*)$, and $(\|\Omega_n \psi_n\|_{\mathcal{H}})_{n\in\mathbb{N}}$ bounded, then $\lim_{n\to\infty} \langle \psi_n, \Omega_n \psi_n \rangle_{\mathcal{H}} = \langle \psi | \Omega | \psi \rangle$.

Lemma

For $\Re(z) \ll 0$, $\mathfrak{G}(z)$ *and* $\mathfrak{G}(z)\Omega$ *are trace-class and*

$$\lim_{T\to\infty+i0^+} \frac{\operatorname{tr}(U_n(\mathfrak{G}(z)\Omega)_n)}{\operatorname{tr}(U_n\mathfrak{G}(z)_n)} = \frac{\langle(\mathfrak{G}(z)\Omega)_n\rangle}{\langle\mathfrak{G}(z)_n\rangle}$$

Lemma

$\left(z \mapsto \frac{\langle(\mathfrak{G}(z)\Omega)_n\rangle}{\langle\mathfrak{G}(z)_n\rangle}\right)_{n\in\mathbb{N}}$ *converges compactly to* $\lim_{T\to\infty+i0^+} \frac{\zeta(U(T,0)\mathfrak{G}\Omega)}{\zeta(U(T,0)\mathfrak{G})}$.

Corollary

$$\langle\psi|\,\Omega\,|\psi\rangle = \lim_{n\to\infty}\langle\psi_n, \Omega_n\psi_n\rangle_{\mathcal{H}} = \lim_{n\to\infty} \frac{\langle(\mathfrak{G}(0)\Omega)_n\rangle}{\langle\mathfrak{G}(0)_n\rangle} = \lim_{T\to\infty+i0^+} \frac{\zeta(U(T,0)\mathfrak{G}\Omega)}{\zeta(U(T,0)\mathfrak{G})}(0) = \langle\Omega\rangle_\zeta.$$

Lattice QFT

$$\mathcal{Z}_T^{\Omega} = \int_{\mathbb{R}^n} \exp\left(\frac{i}{\hbar}\int \text{Lagrangian}(p)\ d\text{vol}_{\text{universe}}\right) a^{\Omega}(p)dp$$

Lattice QFT: Wick Rotation

$$\mathcal{Z}_T^{\Omega,\text{Wick}} = \int_{\mathbb{R}^n} \exp\left(-\frac{1}{\hbar}\int \text{Lagrangian}(p)\ d\text{vol}_{\text{universe}}\right) a^{\Omega}(p)dp$$

"ζ-Lattice QFT"

(i) write down your LQFT

$$\mathcal{Z}_T^\Omega = \int_{\mathbb{R}^n} \exp\left(\frac{i}{\hbar}\int \text{Lagrangian}(p)\ d\text{vol}_{\text{universe}}\right) a^\Omega(p)dp$$

(ii) and gauge it

$$\mathcal{Z}_T^\Omega(z) = \int_{\mathbb{R}^n} \exp\left(\frac{i}{\hbar}\int \text{Lagrangian}(p)\ d\text{vol}_{\text{universe}}\right) a^\Omega(z)(p)dp$$

(iii) construct distributionally equivalent family of Fourier Integral Operator "traces"

$$\mathcal{Z}_T^\Omega(z) = \sum_{\iota\in I}\int_{\partial B_{\mathbb{R}^n}}\int_{\mathbb{R}_{>0}} e^{ir\vartheta(\xi)} r^{d_\iota+z}\alpha_\iota^\Omega(z)(\xi)\ dr\ d\xi$$

"ζ-Lattice QFT"

(iv) if physics is nice, you're probably in the $|\vartheta| > 0$ case

(v) compute Laplace transform

$$\int_{\mathbb{R}_{>0}} e^{ir\vartheta(\xi)} r^{d_\iota + z} dr = \frac{\Gamma(d_\iota + z + 1)}{(-i\vartheta(\xi))^{d_\iota + z + 1}}$$

(vi) and enjoy the remaining (fully regular) integrals

$$\mathcal{Z}_T^\Omega(0) = \sum_{\iota \in I} \Gamma(d_\iota + 1) \int_{\partial B_{\mathbb{R}^n}} \frac{\alpha_\iota^\Omega(0)(\xi)}{(-i\vartheta(\xi))^{d_\iota + 1}} d\xi$$

"ζ-Lattice QFT"

If physics is not nice, then the k^{th} Laurent coefficient LC_k of $\mathcal{Z}_T^\Omega$ in zero is given by $\forall k \in \mathbb{Z}_{\le -2}:\ \mathrm{LC}_k = 0$ and using $I_0 := \{\iota \in I;\ d_\iota = -n\}$

$$\mathrm{LC}_{-1} = \sum_{\iota\in I_0} \int_{\partial B_{\mathbb{R}^n}} e^{i\vartheta(\xi)} \alpha_\iota^\Omega(0)(\xi)\ d\mathrm{vol}_{\partial B_{\mathbb{R}^n}}(\xi)$$

$$\forall k \in \mathbb{N}_0:\ \mathrm{LC}_k = \frac{\text{p.v.} \int_{B_{\mathbb{R}^n}(0,1)} e^{ir\vartheta(\xi)} \partial^k \left(z \mapsto r^{d_\iota + z} \alpha_\iota^\Omega(z)(\xi)\right)\ d(r\xi)}{k!}$$
$$+ \sum_{\iota \in I\setminus I_0} \sum_{j=0}^{k} \frac{(-1)^{j+1} j! \int_{\partial B_{\mathbb{R}^n}} e^{i\vartheta} \left(\partial^{k-j}\alpha_\iota^\Omega\right)(0)\ d\mathrm{vol}_{\partial B_{\mathbb{R}^n}}}{k!(n+d_\iota)^{j+1}}$$
$$- \sum_{\iota\in I_0} \frac{\int_{\partial B_{\mathbb{R}^n}} e^{i\vartheta} \left(\partial^{k+1}\alpha_\iota^\Omega\right)(0)\ d\mathrm{vol}_{\partial B_{\mathbb{R}^n}}}{(k+1)!}$$

What have we got?

- A mathematically rigorous continuum formulation of vacuum expectation values with Lorentzian background.
- ζ-Lattice QFT for Lorentzian backgrounds
- A discretization scheme that is accessible using quantum computing

What do we need?

- Better quantum processing units (more qubits, less noise)
- Better quantum and hybrid algorithms (QPU)
- Better integration techniques on high-dimensional spheres (ζ-LQFT)