Title: Higher-Order Relative Entropy and Subregion Complexity in the AdS Black Hole Background

Speakers: Kevin Grosvenor

Series: Quantum Fields and Strings

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Abstract: I will discuss the computation of second-order terms in the entanglement entropy and subregion complexity for a spherical entangling region in the AdS black hole background relative to pure AdS. I will suggest an extension of the conjectured relationship between subregion complexity and Fisher information into a relation that is reminiscent of the first law of thermodynamics. By analogy, entanglement and complexity play the roles of heat and work, respectively. Time permitting, I will also discuss the computation of third- and fourth-order terms in the relative entropy.

Relative Entropy and Subregion Complexity in the AdS Black Hole Background

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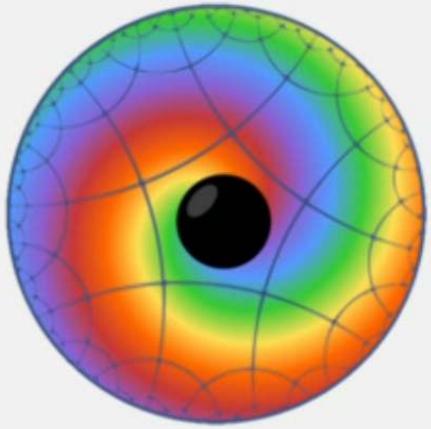

Quantum Fields and Strings Seminar
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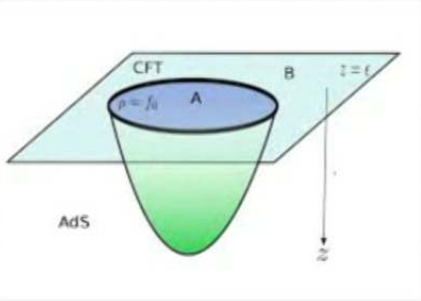
Outline

1. Background Information
2. The Calculations of ΔS and ΔC
3. Analogy with the First Law
4. Outlook

Geometry and Information in AdS/CFT

- Ryu-Takayanagi (RT) '06: $S = \frac{2\pi A}{\ell_{\mathrm{P}}^{d-1}}$
- Blanco-Casini-Hung-Myers (BCHM) '13
  First law of entanglement: $\Delta\langle H\rangle \geq \Delta S$
- Susskind '14: "Complexity $=$ Volume"
- Alishahiha '16: "Complexity $=$ RT Volume"
  $C = \frac{V}{L\ell_{\mathrm{P}}^{d-1}}$
- Lashkari-Van Raamsdonk (LvR) '16:
  "Fisher information $=$ canonical energy"
- Banerjee-Erdmenger-Sarkar (BES) '18:
  "Fisher information $=\Delta$ RT Volume"



First Law of Entanglement

BCHM

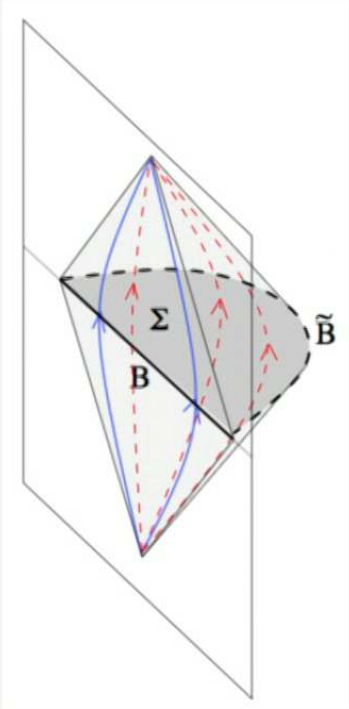
- $|\Psi(\lambda)\rangle =$ one-parameter family of CFT states
- $|\Psi(0)\rangle =$ vacuum state
- $\rho(\lambda) =$ density matrix reduced to an entangling region
- Modular Hamiltonian: $\rho(0) = \frac{1}{Z} e^{-H}$
- Entanglement entropy: $S(\rho) = -\mathrm{tr}(\rho \log \rho)$
- $\Delta\langle H \rangle = \mathrm{tr}(H\rho(\lambda)) - \mathrm{tr}(H\rho(0))$
- $\Delta S = S(\rho(\lambda)) - S(\rho(0))$
- Relative entropy: $S_{\mathrm{rel}}(\lambda) = \Delta\langle H \rangle - \Delta S$

- First law of entanglement: $S_{\mathrm{rel}}(\lambda) = \Delta\langle H \rangle - \Delta S \geq 0$

Fisher Information = Canonical Energy

LvR

- $\lambda = 0$ is a minimum $\Longrightarrow \frac{d}{d\lambda} S_{\rm rel}(\lambda) = 0$
- Fisher information: $\mathcal{F} \equiv \frac{d^2}{d\lambda^2} S_{\rm rel}(\lambda) \geq 0$
- Canonical energy: $\mathcal{E} = \int \xi^a T_{ab} d\Sigma^b$ [Hollands-Wald '13]
- Showed $\mathcal{F} = \mathcal{E}$
- Corroborated by pure CFT calculations [Sárosi-Ugajin '16, '17]



Fisher Information = Δ RT Volume

BES

- Showed $\Delta V^{(1)} = 0$ explicitly
- Conjectured $\mathcal{F} = C_d \Delta V^{(2)}$
- AdS black hole: $C_d = \frac{\pi^{5/2}\Gamma(d+1)}{2^{d-2}(d+1)\Gamma(d+\frac{3}{2})} \frac{1}{L\ell_{\mathrm{P}}^{d-1} c_d}$
- Conformal dimension Δ scalar: $C_{d,\Delta} = \cdots$

Complexity = RT Volume

Alishahiha

- AdS_{d+1} black hole of mass m
- Ball entangling region of radius R
- Complexity $C = \frac{V}{L\ell_P^{d-1}}$ where $V =$ RT volume
- $\Delta C = c_d \frac{\Omega_{d-2}}{d-1} \frac{L^{d-1}}{\ell_P^{d-1}} m^2 R^{2d}$ where "$c_2 = 0, c_3 = \frac{1}{128}, \ldots$"

Setup

AdS Black Hole

$$ds^2 = \frac{L^2}{z^2}\left[-(1-mz^d)dt^2 + \frac{dz^2}{1-mz^d} + dr^2 + r^2 d\Omega_{d-2}^2\right]$$

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RT embedding: $z = z(r) = z_0(r) + mR^d z_1(r) + m^2 R^{2d} z_2(r) + \cdots$

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Dimensionless variables: $x \equiv \frac{r}{R}$ $y(x) \equiv \frac{z(r)}{R}$ $\lambda \equiv mR^d$

Varying the Area Functional

Area Functional

$$s \equiv \frac{A}{\Omega_{d-2} L^{d-1}} = \int_0^1 dx \frac{x^{d-2}}{y(x)^{d-1}} \sqrt{1 + \frac{y'(x)^2}{1 - \lambda y(x)^d}}$$

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Embedding function

$y(x) = y_0(x) + \lambda y_1(x) + \lambda^2 y_2(x) + \cdots$ where $y_0(x) = \sqrt{1 - x^2}$

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Riemann-Papperitz Equation

$y_n'' + p y_n' + q y_n = \sigma_n$ where $p = \frac{d-2-2x^2}{x(1-x^2)}$ and $q = -\frac{d-1}{(1-x^2)^2}$

Boundary conditions: $y_n(1) = 0$ and $y_n'(0) = 0$

The Embedding Function

1st Order

$$y_1(x) = -\frac{1}{2(d+1)}(1-x^2)^{\frac{d-1}{2}}(2-x^2)$$

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2nd Order

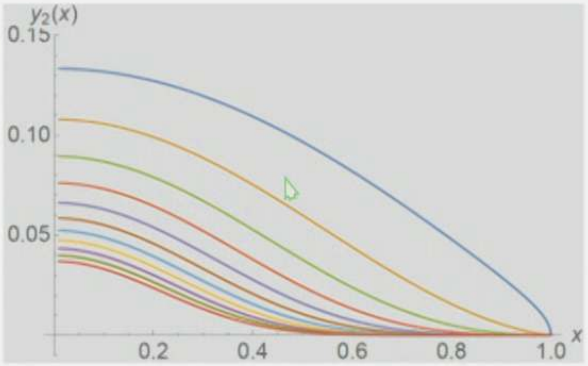
Expression involving ${}_2F_1$'s and ${}_3F_2$'s

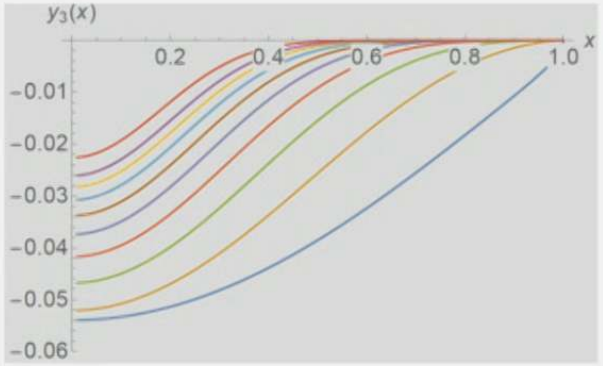
AdS_3 and AdS_4

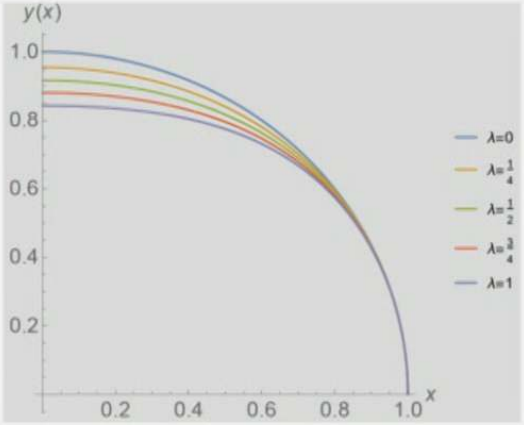
$$y_2^{\mathrm{AdS}_3}(x) = \frac{1}{360}\sqrt{1-x^2}(48-32x^2+3x^4)$$

$$y_2^{\mathrm{AdS}_4}(x) = \frac{1}{4480}\sqrt{1-x^2}(513-771x^2+346x^4-40x^6) + \frac{3}{140}\left(\frac{\ln(1+\sqrt{1-x^2})}{\sqrt{1-x^2}}-1\right)$$

Plots of the Embedding Function







Entanglement Entropy

Entanglement Entropy (Area)

1st-, 2nd- and 3rd-Order

$$\Delta s^{(1)} = \frac{1}{2(d+1)}$$

$$\Delta s^{(2)} = -\frac{\sqrt{\pi}}{2^{d+4}} \frac{(d-1)\Gamma(d+1)}{(d+1)\Gamma(d+\frac{3}{2})}$$

$$\Delta s^{(3)} = \frac{(9d^2-19d+6)}{192(d+1)^2} \frac{\Gamma(d+1)\Gamma(\frac{d+1}{2})}{\Gamma(\frac{3(d+1)}{2})}$$

A Curiosity

y_n and higher are not needed to get $\Delta s^{(n)}$

For $n \geq 3$, we actually don't need y_{n-1} either!

After integration by parts, y_{n-1}'s contribution looks like

$$\big[f_0(x)(\text{eom for } y_0) + f_1(x)(\text{eom for } y_1)\big] y_{n-1} = 0$$

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```
sfunc12sol01
sfunc12sol = sfunc12sol01 /. y2replace;
S12 = Integrate[sfunc12sol, {x, 0, 1}]
```

$$\frac{x^{-1+d}\left(x\, y2[x] - \left(2 - 3x^2 + x^4\right) y2'[x]\right)}{2\sqrt{1-x^2}}$$

$$\left\{\frac{227}{10\,240}, \frac{656}{75\,075}, \frac{101}{30\,240}, \frac{179\,744}{142\,585\,443}, \frac{597}{1\,261\,568}, \frac{3\,431\,936}{19\,361\,274\,075}, \frac{1913}{28\,828\,800}, \frac{215\,343\,104}{8\,667\,078\,323\,475}, \frac{229}{24\,600\,576}, \frac{2\,883\,584}{826\,155\,958\,355}\right\}$$

```
sfunc012sol01
sfunc012sol = sfunc012sol01 /. y2replace;
S012 = Integrate[sfunc012sol, {x, 0, 1}]
```

$$-\frac{1}{2(1+d)\left(1-x^2\right)^{3/2}} x^{-2+d}\left((-1+d)\left(x^4 + d\left(2 - 3x^2 + x^4\right)\right) y2[x] + x\left(-1+x^2\right)\left(2 + (1+d)\, x^2\left(-2+x^2\right)\right) y2'[x]\right)$$

$$\left\{-\frac{227}{10\,240}, -\frac{656}{75\,075}, -\frac{101}{30\,240}, -\frac{179\,744}{142\,585\,443}, -\frac{597}{1\,261\,568}, -\frac{3\,431\,936}{19\,361\,274\,075}, -\frac{1913}{28\,828\,800}, -\frac{215\,343\,104}{8\,667\,078\,323\,475}, -\frac{229}{24\,600\,576}, -\frac{2\,883\,584}{826\,155\,958\,355}\right\}$$

Subregion Complexity

Subregion Complexity (Volume)

Volume Functional

$$c \equiv \frac{(d-1)V}{\Omega_{d-2}L^d} = (d-1)\int_0^1 dx\, x^{d-2} \int_\epsilon^{y(x)} \frac{dy}{y^d\sqrt{1-\lambda y^d}}$$

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1st- and 2nd-Order

$$\Delta c^{(1)} = 0$$

$$\Delta c^{(2)} = \frac{\sqrt{\pi}}{2^{d+2}(d+1)} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}-1\right)}$$

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AdS_3 to AdS_{10}

$$\Delta c^{(2)} = 0, \frac{1}{128}, \frac{3\pi}{1280}, \frac{1}{192}, \frac{15\pi}{14336}, \frac{1}{512}, \frac{35\pi}{98304}, \frac{1}{1600}, \ldots$$

Higher-Order Complexity

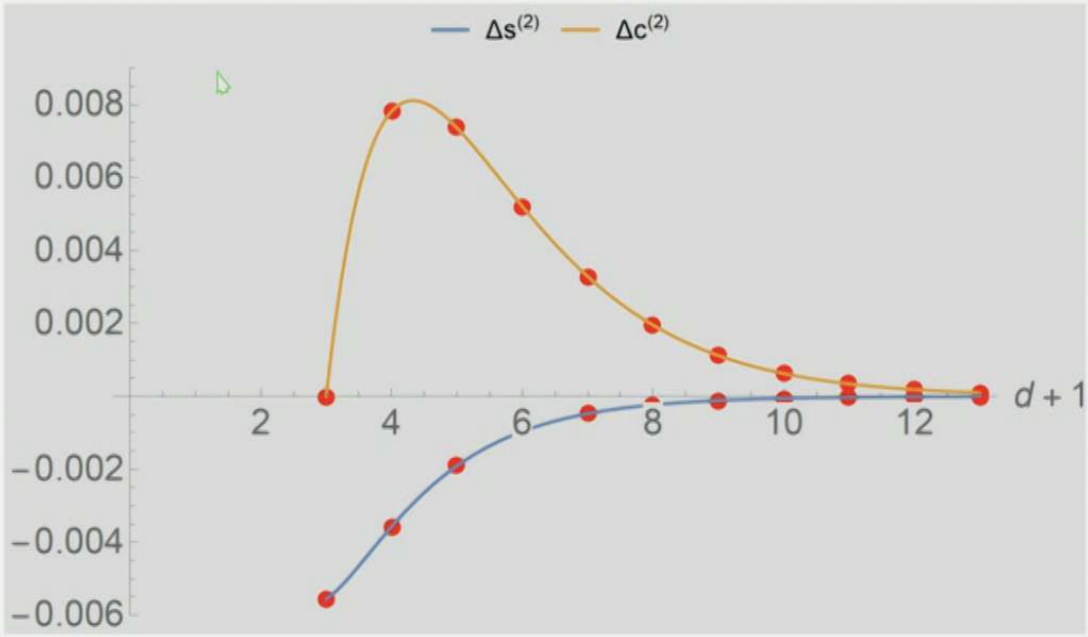
3rd-Order

$$\Delta c^{(3)} = -\frac{d(9d-4)(2d-3)(d-1)(d-2)}{192(d+1)^2}\frac{\Gamma\left(d-\frac{3}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{3d}{2}+1\right)}$$

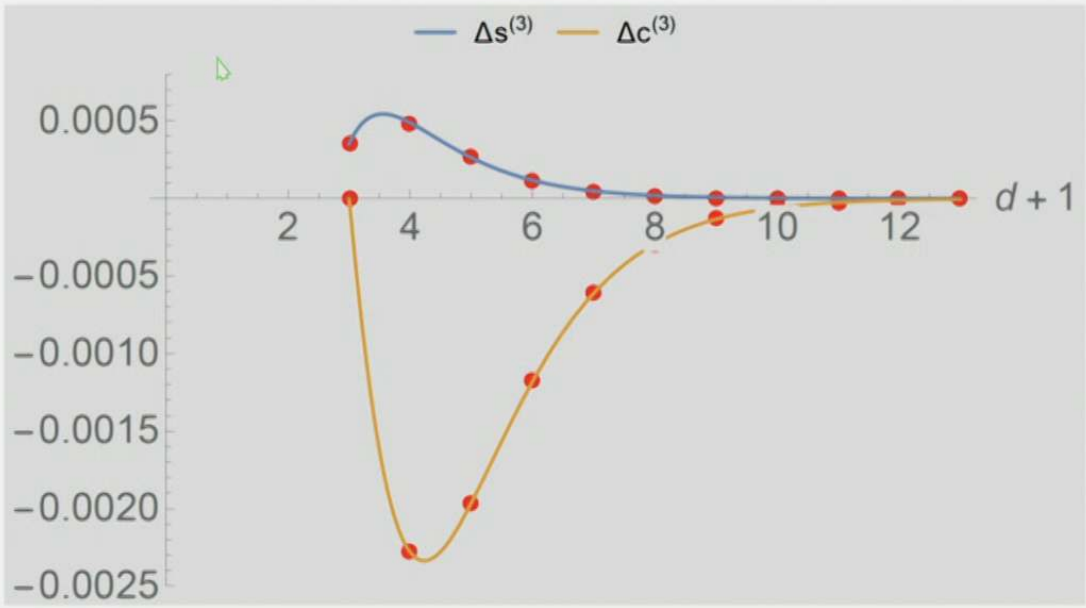
AdS_3 to AdS_{10}

$$-\Delta c^{(3)} = 0, \frac{23}{10080}, \frac{\pi}{1600}, \frac{41}{34749}, \frac{625\pi}{3211264}, \frac{59}{201552}, \frac{10829\pi}{254803968}, \frac{2464}{41788125}, \cdots$$

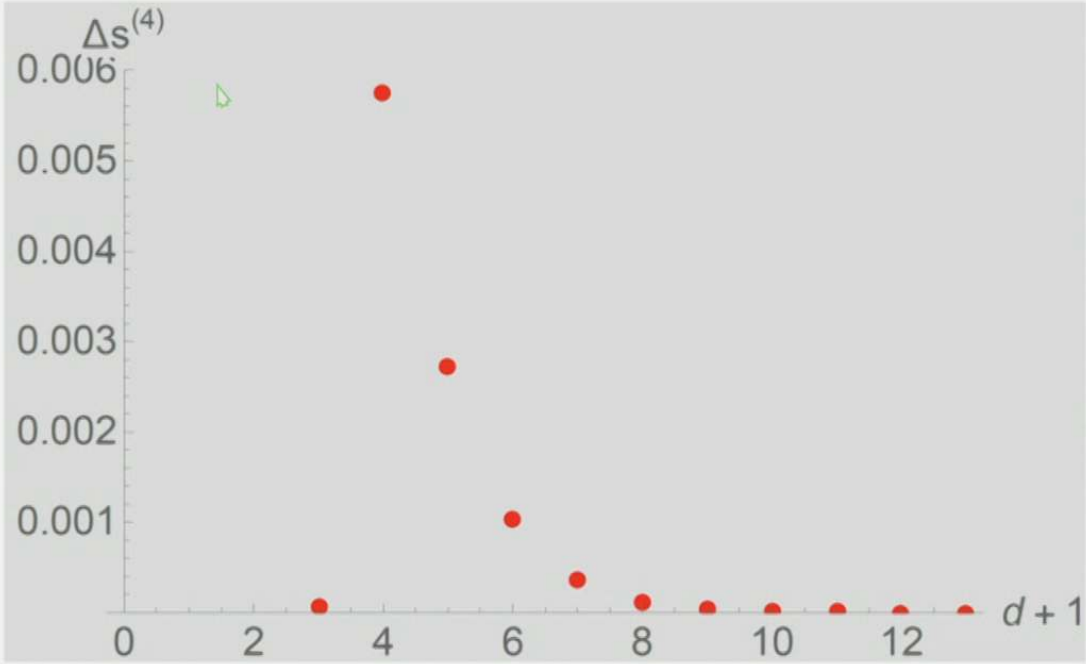
Plots of 2nd-Order Results



Plots of 3rd-Order Results



Plots of 4th-Order Results



AdS$_3$ Aside

Exact results

$$y_{\mathrm{AdS}_3}(x) = \frac{1}{\sqrt{\lambda}}\sqrt{1 - \frac{\cosh^2(x\sqrt{\lambda})}{\cosh^2(\sqrt{\lambda})}}$$

$$\Delta c_{\mathrm{AdS}_3} = 0$$

Analogy with the First Law

Energy and Heat

Change in Energy in the Entangling Region

Stress tensor $T_{00} = \varepsilon$ and $T_{ij} = \frac{\varepsilon}{d-1}\delta_{ij}$ where $\varepsilon = \frac{d-1}{2}\left(\frac{L}{\ell_{\mathrm{P}}}\right)^{d-1}\frac{\lambda}{R^d}$

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$$\Delta E = \frac{1}{2}\Omega_{d-2}\left(\frac{L}{\ell_{\mathrm{P}}}\right)^{d-1}\frac{\lambda}{R}$$

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$$\Delta E = \frac{1}{2}\Omega_{d-2}\left(\frac{L}{\ell_{\mathrm{P}}}\right)^{d-1}\frac{\lambda}{R}$$

Change in Entanglement Entropy at 1st Order

Recall $\Delta S^{(1)} = \frac{1}{2(d+1)}2\pi\Omega_{d-2}\left(\frac{L}{\ell_{\mathrm{P}}}\right)^{d-1}\lambda$

Energy and Heat

Change in Energy in the Entangling Region

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Change in Entanglement Entropy at 1st Order

Recall $\Delta S^{(1)} = \frac{1}{2(d+1)}2\pi\Omega_{d-2}\left(\frac{L}{\ell_{\mathrm{P}}}\right)^{d-1}\lambda$

First Law at 1st Order

$\Delta E = T\Delta S^{(1)}$ where $T = \frac{d+1}{2\pi R}$

Work and the First Law

The BES+A Conjecture

BES says $\Delta S^{(2)} = -\frac{1}{T} B_d \Delta C^{(2)}$ for some B_d

Then, $\Delta E = T(\Delta S^{(1)} + \Delta S^{(2)}) + B_d \Delta C^{(2)}$

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First Law to all orders?

ΔC appears to be of opposite sign compared with ΔS

Suggests $W = B_d \Delta C$

Then, $\Delta E = T \Delta S + W$

"Entanglement $\approx$ Heat" and "Complexity $\approx$ Work"

3rd-Order Test

2nd-Order Coefficient

$$B_d^{(2)} = \frac{d-1}{2R} \frac{\Gamma(d+2)\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(d+\frac{3}{2}\right)}$$

3rd-Order Test

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3rd-Order Coefficient

However, $B_d^{(3)} \neq B_d^{(2)}$

Curious fact: $\lim_{d\to\infty} B_d^{(2)}/B_d^{(3)} = \sqrt{3}$

Other information-theoretic quantities?

Recap and Outlook

Recap

- Computed ΔS and ΔC for a spherical entangling region in the AdS black hole background
- Produced analytic formulas for ΔS and ΔC to 3rd order
- For $n \geq 3$, only need up to y_{n-2} to get $\Delta S^{(n)}$
- $\Delta S^{(2)} < 0 < \Delta S^{(3)}$ and $\Delta C^{(2)} \geq 0 \geq \Delta C^{(3)}$
- Extended the BES conjecture to a “first law" in which entanglement $\approx$ heat and complexity $\approx$ work
- Non-universal relationship between complexity and work

Outlook

- Is the fact that we only need up to y_{n-2} to get $\Delta s^{(n)}$ for $n \geq 3$ special to the AdS black hole background?
- What else kicks in at 3rd order and higher order?
- Numerics instead of analytics
- Connections to QNEC
- Connections to Dowling-Nielsen cost function

THANKS FOR YOU ATTENTION!