Title: Higher-Order Relative Entropy and Subregion Complexity in the AdS Black Hole Background

Speakers: Kevin Grosvenor

Series: Quantum Fields and Strings

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Abstract: I will discuss the computation of second-order terms in the entanglement entropy and subregion complexity for a spherical entangling region in the AdS black hole background relative to pure AdS. I will suggest an extension of the conjectured relationship between subregion complexity and Fisher information into a relation that is reminiscent of the first law of thermodynamics. By analogy, entanglement and complexity play the roles of heat and work, respectively. Time permitting, I will also discuss the computation of third- and fourth-order terms in the relative entropy.

Relative Entropy and Subregion Complexity in the AdS Black Hole Background

Kevin T. Grosvenor

Julius-Maximilians Universität
Würzburg, Germany

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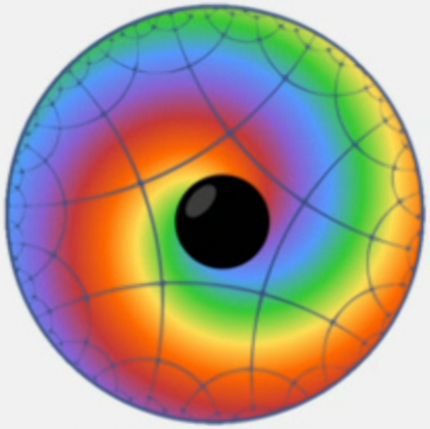


Collaborators

Aranya Bhattacharya and Shibaji Roy

Saha Institute of Nuclear Physics
Calcutta, India

and

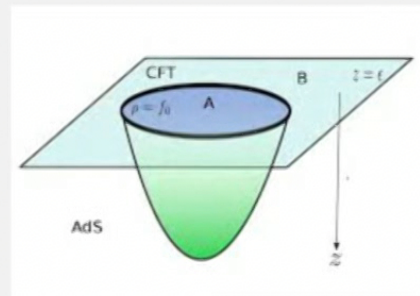
Homi Bhabha National Institute Training School Complex
Mumbai, India

Outline

1. Background Information
2. The Calculations of ΔS and ΔC
3. Analogy with the First Law
4. Outlook

Geometry and Information in AdS/CFT

- Ryu-Takayanagi (RT) '06: $S = \frac{2\pi A}{\ell_{\mathrm{P}}^{d-1}}$
- Blanco-Casini-Hung-Myers (BCHM) '13
  First law of entanglement: $\Delta\langle H\rangle \geq \Delta S$
- Susskind '14: "Complexity $=$ Volume"
- Alishahiha '16: "Complexity $=$ RT Volume"
  $C = \frac{V}{L\ell_{\mathrm{P}}^{d-1}}$
- Lashkari-Van Raamsdonk (LvR) '16:
  "Fisher information $=$ canonical energy"
- Banerjee-Erdmenger-Sarkar (BES) '18:
  "Fisher information $= \Delta$ RT Volume"



First Law of Entanglement

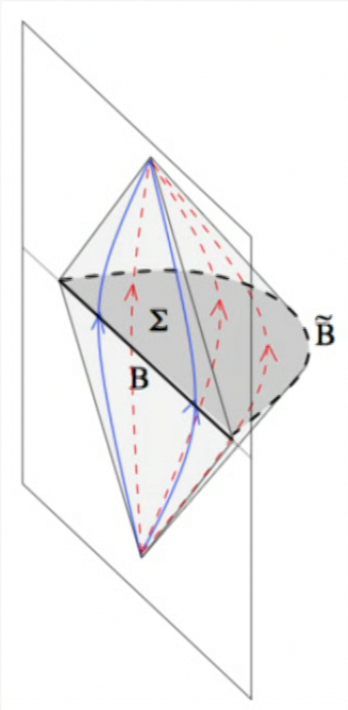
BCHM

- $|\Psi(\lambda)\rangle =$ one-parameter family of CFT states
- $|\Psi(0)\rangle =$ vacuum state
- $\rho(\lambda) =$ density matrix reduced to an entangling region
- Modular Hamiltonian: $\rho(0) = \frac{1}{Z} e^{-H}$
- Entanglement entropy: $S(\rho) = -\mathrm{tr}(\rho \log \rho)$
- $\Delta\langle H\rangle = \mathrm{tr}(H\rho(\lambda)) - \mathrm{tr}(H\rho(0))$
- $\Delta S = S(\rho(\lambda)) - S(\rho(0))$
- Relative entropy: $S_{\mathrm{rel}}(\lambda) = \Delta\langle H\rangle - \Delta S$
- First law of entanglement: $S_{\mathrm{rel}}(\lambda) = \Delta\langle H\rangle - \Delta S \geq 0$

Fisher Information = Canonical Energy

LvR

- $\lambda = 0$ is a minimum $\Longrightarrow \frac{d}{d\lambda} S_{\text{rel}}(\lambda) = 0$
- Fisher information: $\mathcal{F} \equiv \frac{d^2}{d\lambda^2} S_{\text{rel}}(\lambda) \geq 0$
- Canonical energy: $\mathcal{E} = \int \xi^a T_{ab} d\Sigma^b$ [Hollands-Wald '13]
- Showed $\mathcal{F} = \mathcal{E}$
- Corroborated by pure CFT calculations [Sárosi-Ugajin '16, '17]



Fisher Information = Δ RT Volume

BES

- Showed $\Delta V^{(1)} = 0$ explicitly
- Conjectured $\mathcal{F} = C_d \Delta V^{(2)}$
- AdS black hole: $C_d = \frac{\pi^{5/2}\Gamma(d+1)}{2^{d-2}(d+1)\Gamma(d+\frac{3}{2})} \frac{1}{L\ell_{\rm P}^{d-1} c_d}$
- Conformal dimension Δ scalar: $C_{d,\Delta} = \cdots$

Complexity = RT Volume

Alishahiha

- AdS$_{d+1}$ black hole of mass m
- Ball entangling region of radius R
- Complexity $C = \frac{V}{L\ell_{\mathrm{P}}^{d-1}}$ where $V =$ RT volume
- $\Delta C = c_d \frac{\Omega_{d-2}}{d-1} \frac{L^{d-1}}{\ell_{\mathrm{P}}^{d-1}} m^2 R^{2d}$ where "$c_2 = 0, c_3 = \frac{1}{128}, \ldots$"

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Setup

AdS Black Hole

$$ds^2 = \frac{L^2}{z^2}\left[-(1-mz^d)dt^2 + \frac{dz^2}{1-mz^d} + dr^2 + r^2 d\Omega_{d-2}^2\right]$$

SETUP

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RT embedding: $z = z(r) = z_0(r) + mR^d z_1(r) + m^2 R^{2d} z_2(r) + \cdots$

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Dimensionless variables: $x \equiv \frac{r}{R}$ $\quad y(x) \equiv \frac{z(r)}{R}$ $\quad \lambda \equiv mR^d$

Varying the Area Functional

Area Functional

$$s \equiv \frac{A}{\Omega_{d-2} L^{d-1}} = \int_0^1 dx \frac{x^{d-2}}{y(x)^{d-1}} \sqrt{1 + \frac{y'(x)^2}{1 - \lambda y(x)^d}}$$

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Embedding function

$y(x) = y_0(x) + \lambda y_1(x) + \lambda^2 y_2(x) + \cdots$ where $y_0(x) = \sqrt{1 - x^2}$

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Riemann-Papperitz Equation

$y_n'' + p y_n' + q y_n = \sigma_n$ where $p = \frac{d-2-2x^2}{x(1-x^2)}$ and $q = -\frac{d-1}{(1-x^2)^2}$

Boundary conditions: $y_n(1) = 0$ and $y_n'(0) = 0$

The Embedding Function

1st Order

$$y_1(x) = -\frac{1}{2(d+1)}(1 - x^2)^{\frac{d-1}{2}}(2 - x^2)$$

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The Embedding Function

1st Order

$$y_1(x) = -\frac{1}{2(d+1)}(1-x^2)^{\frac{d-1}{2}}(2-x^2)$$

2nd Order

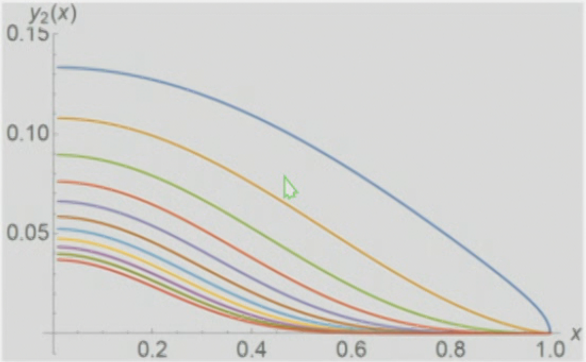
Expression involving ${}_2F_1$'s and ${}_3F_2$'s

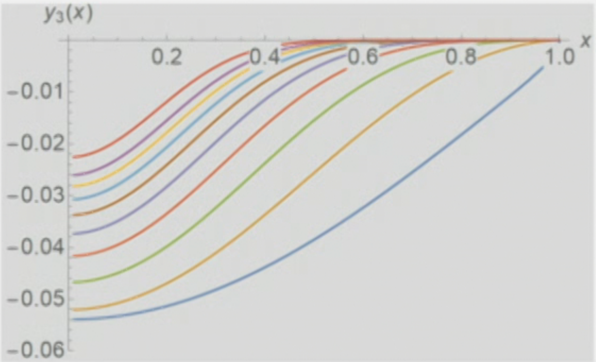
AdS$_3$ and AdS$_4$

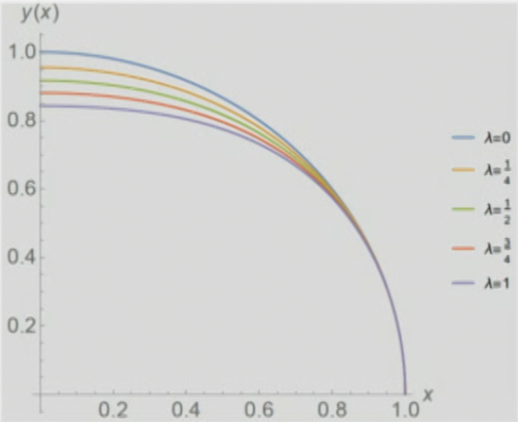
$$y_2^{\mathrm{AdS}_3}(x) = \frac{1}{360}\sqrt{1-x^2}(48-32x^2+3x^4)$$

$$y_2^{\mathrm{AdS}_4}(x) = \frac{1}{4480}\sqrt{1-x^2}(513-771x^2+346x^4-40x^6) + \frac{3}{140}\left(\frac{\ln(1+\sqrt{1-x^2})}{\sqrt{1-x^2}}-1\right)$$

Plots of the Embedding Function







ENTANGLEMENT ENTROPY

Entanglement Entropy (Area)

1st-, 2nd- and 3rd-Order

$$\Delta s^{(1)} = \frac{1}{2(d+1)}$$

$$\Delta s^{(2)} = -\frac{\sqrt{\pi}}{2^{d+4}} \frac{(d-1)\Gamma(d+1)}{(d+1)\Gamma(d+\frac{3}{2})}$$

$$\Delta s^{(3)} = \frac{(9d^2 - 19d + 6)}{192(d+1)^2} \frac{\Gamma(d+1)\Gamma(\frac{d+1}{2})}{\Gamma(\frac{3(d+1)}{2})}$$

A Curiosity

y_n and higher are not needed to get $\Delta s^{(n)}$

For $n \geq 3$, we actually don't need y_{n-1} either!

After integration by parts, y_{n-1}'s contribution looks like

$$\left[f_0(x)(\text{eom for } y_0) + f_1(x)(\text{eom for } y_1)\right] y_{n-1} = 0$$

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```
sfunc12sol01
sfunc12sol = sfunc12sol01 /. y2replace;
S12 = Integrate[sfunc12sol, {x, 0, 1}]
```

$$\frac{x^{-1+d}\left(x\, y2[x] - \left(2 - 3x^2 + x^4\right) y2'[x]\right)}{2\sqrt{1-x^2}}$$

$$\left\{\frac{227}{10240}, \frac{656}{75075}, \frac{101}{30240}, \frac{179744}{142585443}, \frac{597}{1261568}, \frac{3431936}{19361274075}, \frac{1913}{28828800}, \frac{215343104}{8667078323475}, \frac{229}{24600576}, \frac{2883584}{826155958355}\right\}$$

```
sfunc012sol01
sfunc012sol = sfunc012sol01 /. y2replace;
S012 = Integrate[sfunc012sol, {x, 0, 1}]
```

$$-\frac{1}{2(1+d)\left(1-x^2\right)^{3/2}} x^{-2+d} \left(\left(-1+d\right)\left(x^4 + d\left(2 - 3x^2 + x^4\right)\right) y2[x] + x\left(-1+x^2\right)\left(2 + (1+d)\, x^2 \left(-2+x^2\right)\right) y2'[x]\right)$$

$$\left\{-\frac{227}{10240}, -\frac{656}{75075}, -\frac{101}{30240}, -\frac{179744}{142585443}, -\frac{597}{1261568}, -\frac{3431936}{19361274075}, -\frac{1913}{28828800}, -\frac{215343104}{8667078323475}, -\frac{229}{24600576}, -\frac{2883584}{826155958355}\right\}$$

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SUBREGION COMPLEXITY

Subregion Complexity (Volume)

Volume Functional

$$c \equiv \frac{(d-1)V}{\Omega_{d-2}L^d} = (d-1)\int_0^1 dx\, x^{d-2} \int_\epsilon^{y(x)} \frac{dy}{y^d\sqrt{1-\lambda y^d}}$$

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1st- and 2nd-Order

$$\Delta c^{(1)} = 0$$

$$\Delta c^{(2)} = \frac{\sqrt{\pi}}{2^{d+2}(d+1)} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}-1\right)}$$

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Volume Functional

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AdS_3 to AdS_{10}

$$\Delta c^{(2)} = 0, \frac{1}{128}, \frac{3\pi}{1280}, \frac{1}{192}, \frac{15\pi}{14336}, \frac{1}{512}, \frac{35\pi}{98304}, \frac{1}{1600}, \ldots$$

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Higher-Order Complexity

3rd-Order

$$\Delta c^{(3)} = -\frac{d(9d-4)(2d-3)(d-1)(d-2)}{192(d+1)^2}\frac{\Gamma\left(d-\frac{3}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{3d}{2}+1\right)}$$

AdS_3 to AdS_{10}

$$-\Delta c^{(3)} = 0, \frac{23}{10080}, \frac{\pi}{1600}, \frac{41}{34749}, \frac{625\pi}{3211264}, \frac{59}{201552}, \frac{10829\pi}{254803968}, \frac{2464}{41788125}, \cdots$$

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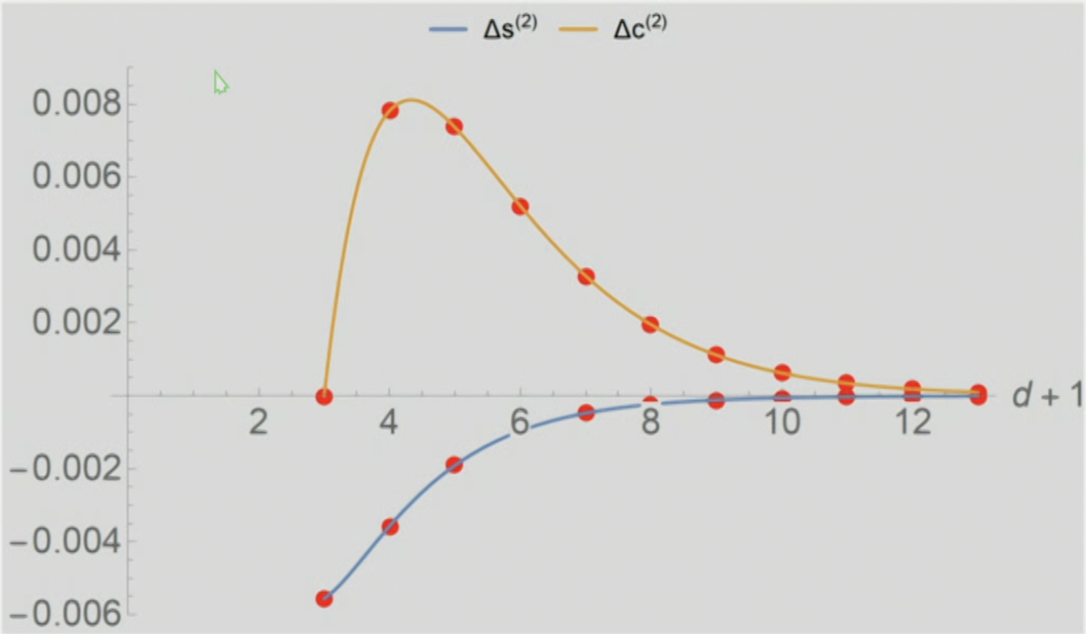
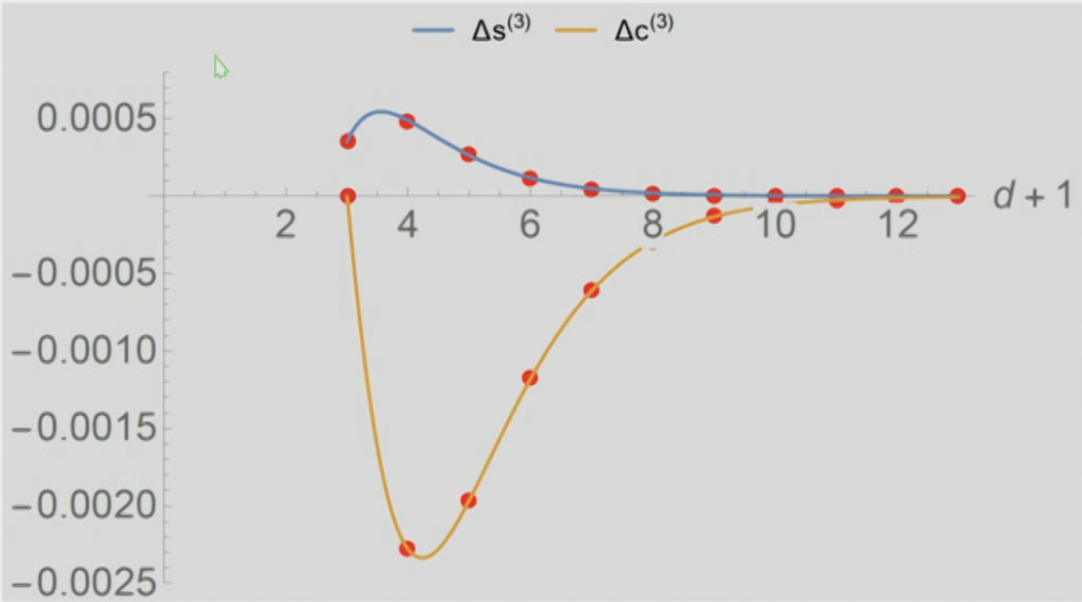
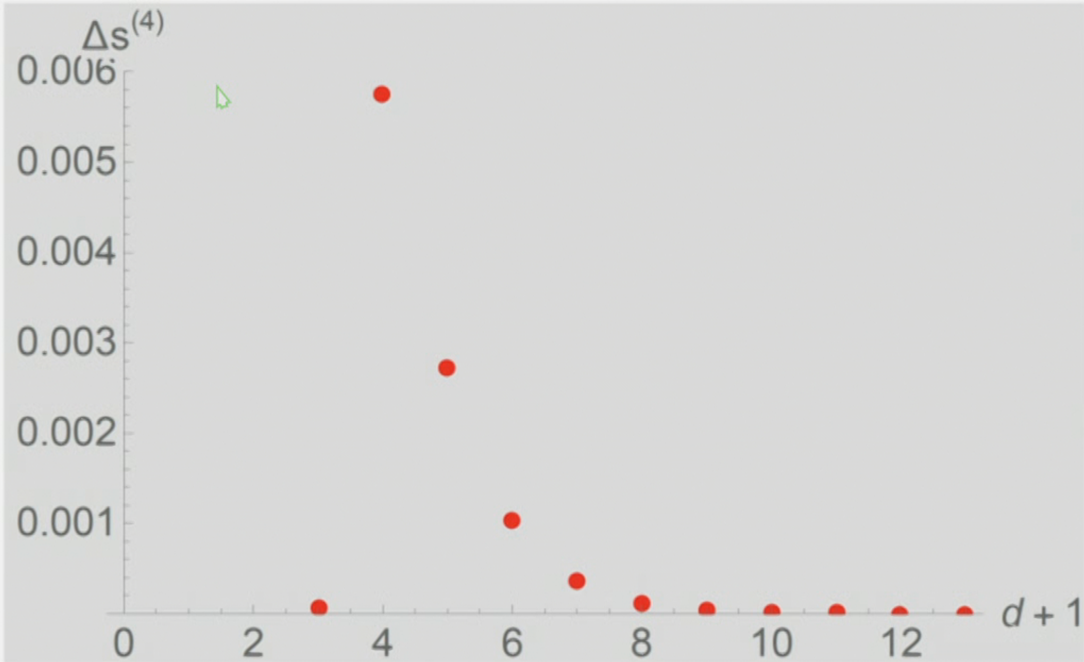
Plots of 2nd-Order Results



Plots of 3rd-Order Results



Plots of 4th-Order Results



AdS$_3$ Aside

Exact results

$$y_{\mathrm{AdS}_3}(x) = \frac{1}{\sqrt{\lambda}}\sqrt{1 - \frac{\cosh^2(x\sqrt{\lambda})}{\cosh^2(\sqrt{\lambda})}}$$

$$\Delta c_{\mathrm{AdS}_3} = 0$$

Analogy with the First Law

Energy and Heat

Change in Energy in the Entangling Region

Stress tensor $T_{00} = \varepsilon$ and $T_{ij} = \frac{\varepsilon}{d-1}\delta_{ij}$ where $\varepsilon = \frac{d-1}{2}\left(\frac{L}{\ell_{\mathrm{P}}}\right)^{d-1}\frac{\lambda}{R^d}$

ENERGY AND HEAT

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$$\Delta E = \frac{1}{2}\Omega_{d-2}\left(\frac{L}{\ell_{\mathrm{P}}}\right)^{d-1}\frac{\lambda}{R}$$

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$\Delta E = \frac{1}{2}\Omega_{d-2}\left(\frac{L}{\ell_{\rm P}}\right)^{d-1}\frac{\lambda}{R}$

Change in Entanglement Entropy at 1st Order

Recall $\Delta S^{(1)} = \frac{1}{2(d+1)}2\pi\Omega_{d-2}\left(\frac{L}{\ell_{\rm P}}\right)^{d-1}\lambda$

Energy and Heat

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Recall $\Delta S^{(1)} = \frac{1}{2(d+1)}2\pi\Omega_{d-2}\left(\frac{L}{\ell_\mathrm{P}}\right)^{d-1}\lambda$

First Law at 1st Order

$\Delta E = T\Delta S^{(1)}$ where $T = \frac{d+1}{2\pi R}$

Work and the First Law

The BES+A Conjecture

BES says $\Delta S^{(2)} = -\frac{1}{T} B_d \Delta C^{(2)}$ for some B_d

Then, $\Delta E = T(\Delta S^{(1)} + \Delta S^{(2)}) + B_d \Delta C^{(2)}$

WORK AND THE FIRST LAW

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First Law to all orders?

ΔC appears to be of opposite sign compared with ΔS

Suggests $W = B_d \Delta C$

Then, $\Delta E = T\Delta S + W$

"Entanglement $\approx$ Heat" and "Complexity $\approx$ Work"

3RD-ORDER TEST

2nd-Order Coefficient

$$B_d^{(2)} = \frac{d-1}{2R}\frac{\Gamma(d+2)\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(d+\frac{3}{2}\right)}$$

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3rd-Order Coefficient

However, $B_d^{(3)} \neq B_d^{(2)}$

Curious fact: $\lim_{d\to\infty} B_d^{(2)}/B_d^{(3)} = \sqrt{3}$

Other information-theoretic quantities?

Recap and Outlook

Recap

- Computed ΔS and ΔC for a spherical entangling region in the AdS black hole background
- Produced analytic formulas for ΔS and ΔC to 3rd order
- For $n \geq 3$, only need up to y_{n-2} to get $\Delta S^{(n)}$
- $\Delta S^{(2)} < 0 < \Delta S^{(3)}$ and $\Delta C^{(2)} \geq 0 \geq \Delta C^{(3)}$
- Extended the BES conjecture to a “first law" in which entanglement $\approx$ heat and complexity $\approx$ work
- Non-universal relationship between complexity and work

Outlook

- Is the fact that we only need up to y_{n-2} to get $\Delta s^{(n)}$ for $n \geq 3$ special to the AdS black hole background?
- What else kicks in at 3rd order and higher order?
- Numerics instead of analytics
- Connections to QNEC
- Connections to Dowling-Nielsen cost function

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THANKS FOR YOU ATTENTION!