Title: Mathematical hints of 3-d mirror symmetry

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Abstract: This is practice for a talk at Berkeley, so it will involve explaining stuff many people here probably already know. I'll try to summarize what I've learned about 3 dimensional N=4 supersymmetric quantum field theories, their twists and how these manifest in terms of interesting objects in mathematics. If nothing else, hopefully there will be some comedy value in my attempt.

Mathematical hints of 3-d mirror symmetry

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April 15, 2019

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Important question: am I giving this talk as

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reasonably competent mathematician

Important question: am I giving this talk as

reasonably competent mathematician

not so competent physicist

I'll do my best to do a bit of both, but apologies to essentially everyone.

So, why bother at all?

A quantum field theory is a framework that looks very much like differential geometry, but even it is not safe from Grothendieck. A quantum field theory spits out a great number of categories as different types of boundary conditions.

Just topological quantum field theories have had a powerful influence in mathematics, being key in the development of theory of higher categories and quantum knot invariants. But most quantum field theories aren't (purely) topological!

"Twisting" is a technique that allows us to start with a more general quantum field theory and identify topological pieces inside it. Sometimes there is more than one such piece, and we can play them off each other.

My talk is going to be about 3-dimensional $\mathcal{N} = 4$ supersymmetric quantum field theory.

Why $d = 3, \mathcal{N} = 4$?

Briefly: lots of room to have different topological twists whose interaction we can think about.

In each of these twists, $d = 3 \supset d = 1$ is also a natural context in TQFT for commutative algebras deforming to associative ones ("almost commutative algebras").

More specifically, $d = 3, \mathcal{N} = 4$ supersymmetric theories have an action of $\mathrm{Spin}_4(\mathbb{C}) = SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$. There are two twists that privilege these two different factors, attached to the terms "Higgs" and "Coulomb" which give us two almost commutative algebras.

We'll be interested in one specific class of $3d$ $\mathcal{N} = 4$ supersymmetric quantum field theories: the gauge theory attached to a compact group G and representation N over $\mathbb{C}$.

In physics-speak, we couple an adjoint vectormultiplet for G with a hypermultiplet tranforming in the representation N.

I'll only embarass myself when I try to explain what this means precisely, but as a classical theory, there is no question about the answer, in whatever framework you like to use for classical field theories.

The quantum theory is a different story. I think physicists are reasonably happy just saying that you do a path integral...this still leaves them with questions they cannot answer.

One of the key invariants of a QFT is the **moduli space of vacua** which captures the topological part of the **algebra of local operators**. This acts on everything in the picture (naturally on boundary conditions, etc.).

For a 3d $\mathcal{N} = 4$ supersymmetric theory, the moduli space of vacua is a (singular) hyperkähler manifold; a choice of a 2d $\mathcal{N} = (2, 2)$ supersymmetric boundary condition fixes a prefered complex structure.

Classically, the moduli space of vacua is given by the equations:

$$[\vec{\phi}, \vec{\phi}] = 0 \qquad (\vec{\phi} + \vec{m}) \cdot (X, Y) \qquad \vec{\mu}(X, Y) + \vec{t} = 0$$

The Higgs branch is when $\vec{\phi} = \vec{m} = 0$, so the result is a hyperkähler quotient:

$$\mathfrak{M}_H = \{(X, Y) \mid \vec{\mu}(X, Y) + \vec{t} = 0\}/G$$

The Coulomb branch is when $X = Y = 0$, so

$$\mathfrak{M}_C \approx T^* T^\vee_{\mathbb{C}}/W$$

For $T \subset G$ a maximal torus, $T^\vee$ its Langlands dual.

The $\approx$ is because this is the classical answer, and it will be "corrected" when we quantize. The Higgs branch does not have this issue.

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Why as a mathematician am I interested in this story?

Hyperkähler quotients have proven to be very powerful objects in geometric representation theory. Nakajima quiver varieties in particular have shown themselves to be extremely important.

Thus, whenever we find a theory that has an interesting Higgs branch, it's very natural to look at its Coulomb branch to see if it is equally interesting.

Unfortunately, for quite a few years, Coulomb branches could only be computed from the quantum perspective for a few special theories, such as when G was abelian, or the theory corresponded to an affine type A quiver variety.

In joint work with Braden, Licata and Proudfoot, we noticed some very interesting features of these examples.

- There seemed to be a bijection between fixed points for natural torus actions on the Higgs and Coulomb branches (known to physicists)
- This bijection seemed to reflect a Koszul duality between category $\mathscr{O}$'s (a surprise to physicists)

The functions on the Higgs and Coulomb branches both have noncommutative deformations A_H and A_C based on the action of S^1 on $\mathbb{R}^3$. Category $\mathscr{O}$ is a category of modules over these quantizations depending on a choice of mass and FI parameters.

These Koszul dualities needed to be checked rather indirectly, especially in the quiver variety case.

Luckily, eventually the issue of defining the Coulomb branch was resolved by Braverman, Finkelberg and Nakajima.

Unfortunately, the answer is complicated enough that it would eat the rest of my talk if I tried to explain it. Let's just say that the affine Grassmannian is involved.

This definition allows one to give a uniform proof of the Koszul duality mentioned earlier:

Theorem

The category $\mathcal{O}$'s attached to the Higgs and Coulomb branches are Koszul dual.

This proof spun off a surprising large number of additional observations that I think are worth discussing:

- Gelfand-Tsetlin modules
- line operators and “monopole operators for paths.”
- noncommutative resolutions of singularities

The proof of Koszul duality for category $\mathcal{O}$ factors through a larger category: the Gelfand-Tsetlin modules for A_C.

The algebra A_C contains a maximal commutative subalgebra S given by invariant polynomials in $\vec{\phi}$. A **Gelfand-Tsetlin** module V over A_C is one on which S acts locally finitely (i.e. $\dim(S \cdot v) < \infty$ for all $v \in V$).

Question to the physicists:

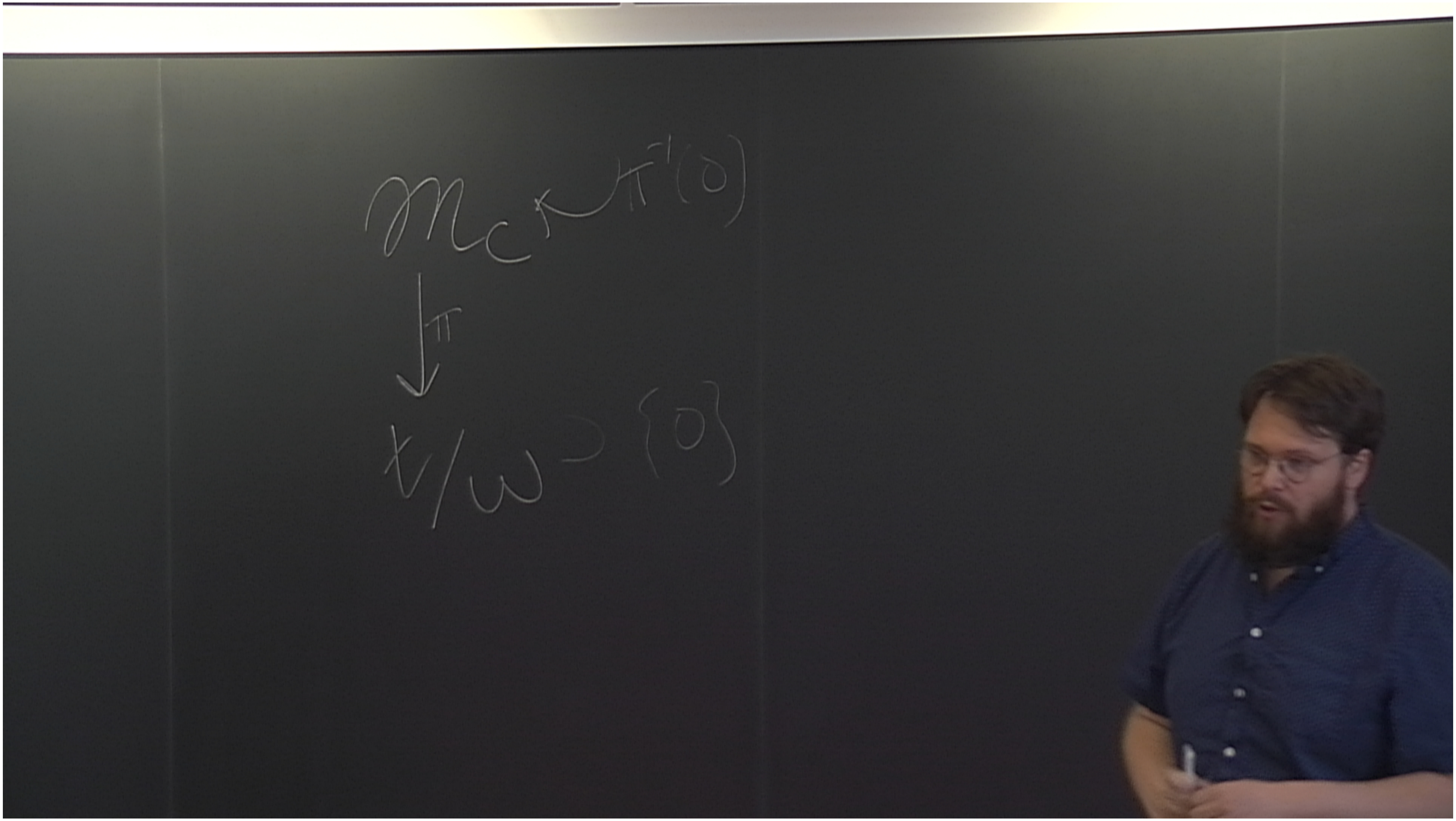
Does this have a natural physical meaning?

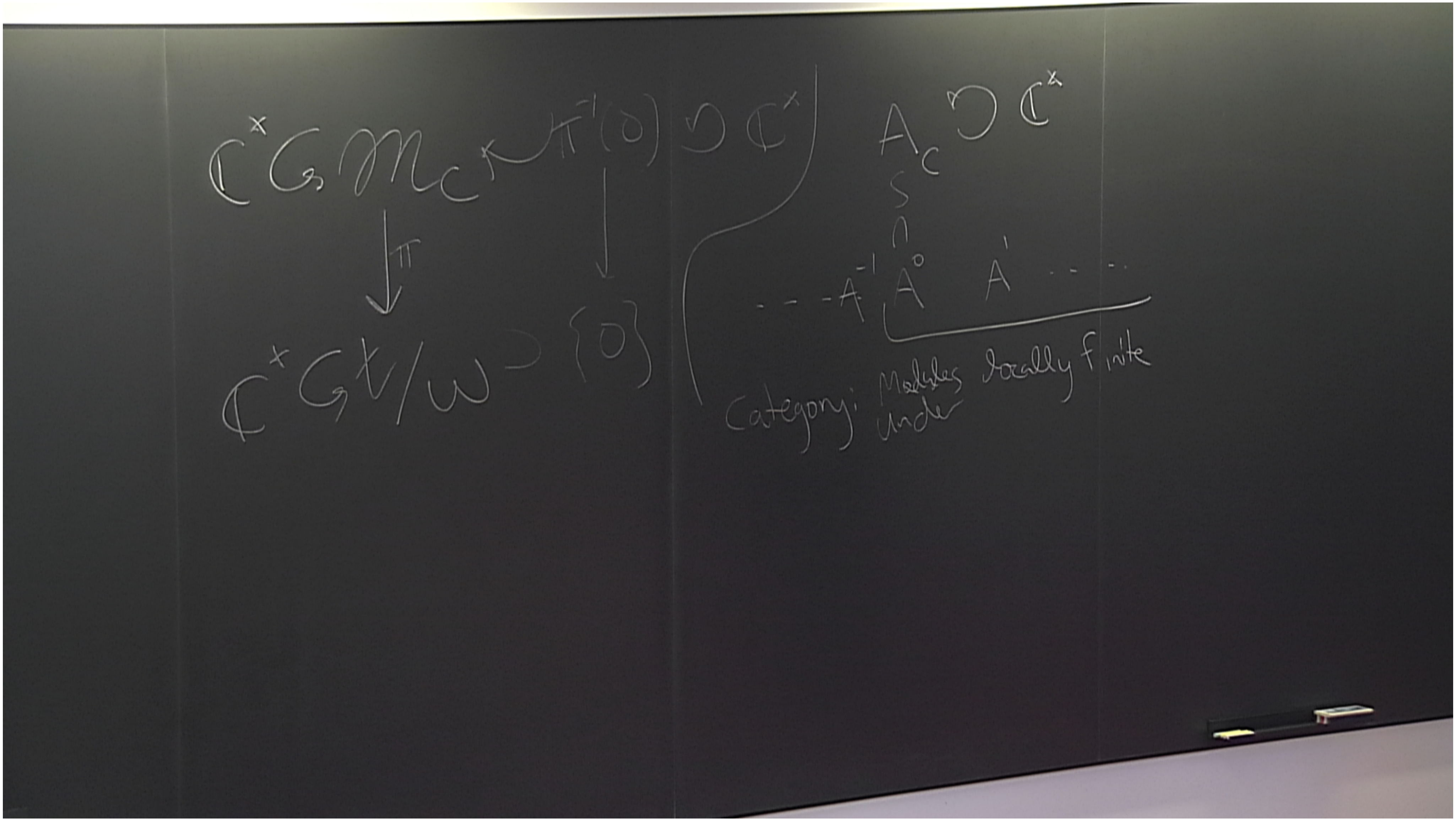
Theorem (W.)

The category of G-T A_C-modules is Koszul dual to a category of "N-character D-modules" on the quotient N/G.

In particular, the category of G-T modules has a graded lift where the dimensions of S-weight spaces can be computed by a Kazhdan-Lusztig type algorithm.

This tells us something new and interesting about well-known algebras.


$\mathcal{M}_C \hookleftarrow \pi^{-1}(0)$
$\downarrow \pi$
$\mathfrak{t}/W \supset \{0\}$



ℂ^× ↻ 𝔐_C ⊃ π^{-1}(0) ↻ ℂ^×
π
ℂ^× ↻ t/W ⊃ {0}
A_C ↻ ℂ^×
⊆
- - - A^{-1} A^0 A^1 - - -
Category: Modules locally finite under

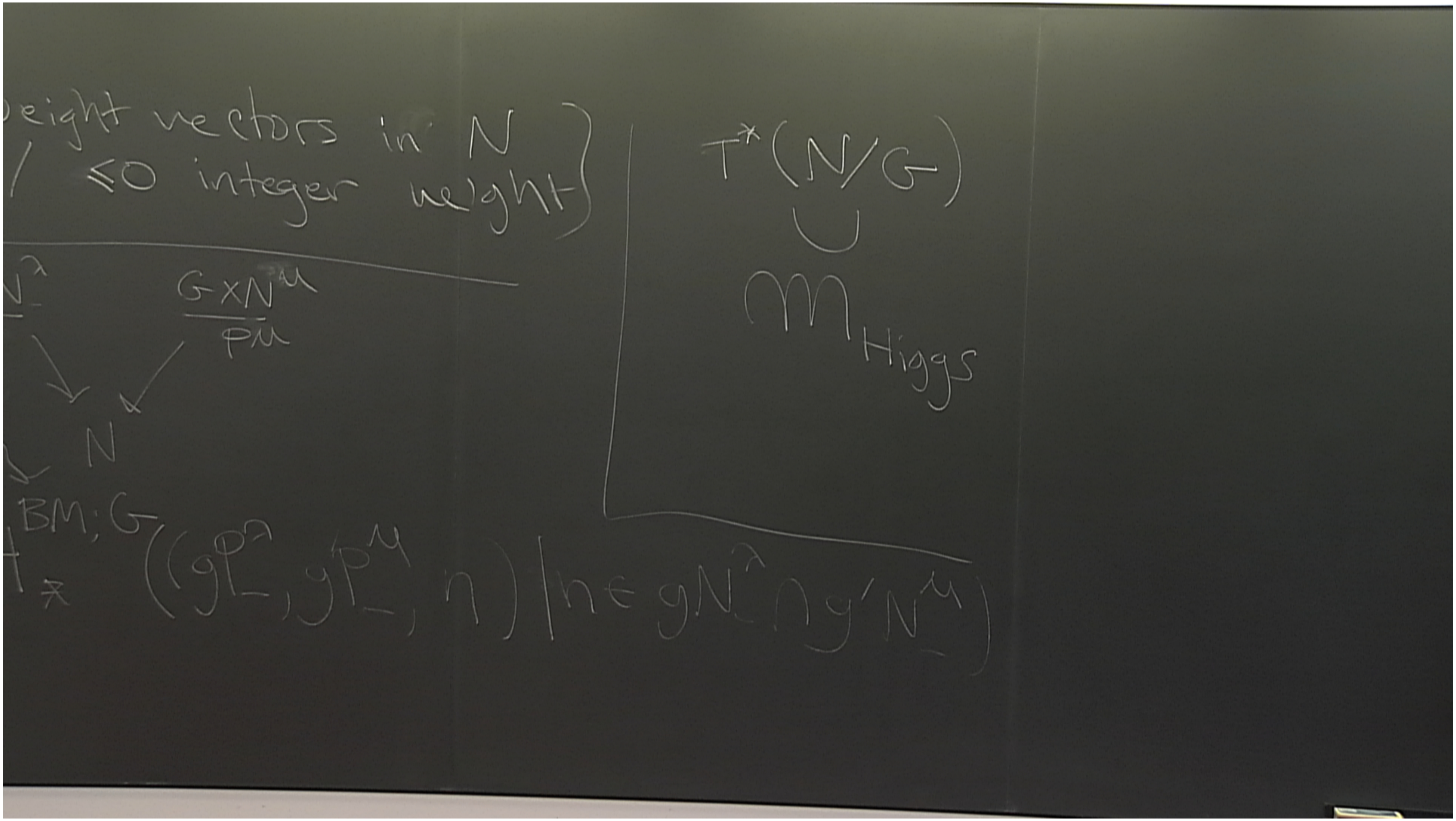
$$\frac{G \times N_-^{\lambda}}{P_-^{\lambda}} = \left\{ \begin{array}{l} \text{weight vectors in } N \\ \text{w/ } \leq 0 \text{ integer weight} \end{array} \right\}$$

$$\downarrow$$

$$G \circlearrowright N$$

$$\frac{G \times N_-^{\lambda}}{P_-^{\lambda}} \longrightarrow N \longleftarrow \frac{G \times N_-^{\mu}}{P_-^{\mu}}$$

$$\rightsquigarrow H_*^{BM;G}\left((gP_-^{\lambda}, g'P_-^{\mu}, n) \mid n \in gN_-^{\lambda} \cap g'N_-^{\mu} \right)$$


weight vectors in N
≤0 integer weight
$T^*(N/G)$
∪
$\mathfrak{M}_{Higgs}$
$\frac{G \times N^\mu}{P^\mu}$
N
BM; G
$H_*((gP^\lambda, g'P^\mu, n) \mid n \in gN^\lambda \cap g'N^\mu)$

In particular, this resolves a long open problem about Gelfand-Tsetlin modules for $U(\mathfrak{gl}_n)$.

Theorem (Kazhdan-Lusztig-Beilinson-Bernstein-Brylinksi-Kashiwara-Soergel-$\cdots$)

If we consider the principal block $\mathcal{O}_0$ of $U(\mathfrak{gl}_n)$, then $K^0(\mathcal{O}_0) \cong (\mathbb{C}^n)^{\otimes n}_{(1,\dots,1)}$, with simple modules matching the dual canonical basis.

Theorem (KTWWY)

The principal block $\mathcal{C}_0$ of simple Gelfand-Tsetlin modules over $U(\mathfrak{gl}_n)$ has $K^0(\mathcal{C}_0) = (U(\mathfrak{n}_-) \otimes (\mathbb{C}^n)^{\otimes n})_{(1,\dots,1)}$ with classes of simples matching dual canonical basis.